CHAPTER XXXIII.

DETERMINANTS.

485. THE present chapter is devoted to a brief discussion of determinants and their more elementary properties. The slight introductory sketch here given will enable a student to avail himself of the advantages of determinant notation in Analytical Geometry, and in some other parts of Higher Mathematics; fuller information on this branch of Analysis may be obtained from Dr Salmon's *Lessons Introductory to the Modern Higher Algebra*, and Muir's *Theory of Determinants*.

486. Consider the two homogeneous linear equations

$$a_1x + b_1y = 0,$$

$$a_2x + b_2y = 0;$$

multiplying the first equation by b_2, the second by b_1, subtracting and dividing by x, we obtain

$$a_1b_2 - a_2b_1 = 0.$$

This result is sometimes written

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

and the expression on the left is called a **determinant**. It consists of two rows and two columns, and in its expanded form each term is the product of two quantities; it is therefore said to be of the *second order*.

The letters a_1, b_1, a_2, b_2 are called the *constituents* of the determinant, and the terms a_1b_2, a_2b_1 are called the *elements*.

487. Since

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

it follows that *the value of the determinant is not altered by changing the rows into columns, and the columns into rows.*

488. Again, it is easily seen that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}, \text{ and } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix};$$

that is, *if we interchange two rows or two columns of the determinant, we obtain a determinant which differs from it only in sign.*

489. Let us now consider the homogeneous linear equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0.$$

By eliminating x, y, z, we obtain as in Ex. 2, Art. 16,

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0,$$

or $$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This eliminant is usually written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

and the expression on the left being a determinant which consists of three rows and three columns is called a determinant of the *third order.*

490. By a rearrangement of terms the expanded form of the above determinant may be written

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1),$$

or $$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix};$$

hence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix};$$

that is, *the value of the determinant is not altered by changing the rows into columns, and the columns into rows.*

491. From the preceding article,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \;\ldots\ldots(1).$$

Also from Art. 489,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \;\ldots(2).$$

We shall now explain a simple method of writing down the expansion of a determinant of the third order, and it should be noticed that it is immaterial whether we develop it from the first row or the first column.

From equation (1) we see that the coefficient of any one of the constituents a_1, a_2, a_3 is that determinant of the second order which is obtained by omitting the row and column in which it occurs. These determinants are called the **Minors** of the original determinant, and the left-hand side of equation (1) may be written

$$a_1 A_1 - a_2 A_2 + a_3 A_3,$$

where A_1, A_2, A_3 are the minors of a_1, a_2, a_3 respectively.

Again, from equation (2), the determinant is equal to

$$a_1 A_1 - b_1 B_1 + c_1 C_1,$$

where A_1, B_1, C_1 are the minors of a_1, b_1, c_1 respectively.

492. The determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$
$$= -b_1(a_2c_3 - a_3c_2) - a_1(c_2b_3 - c_3b_2) - c_1(b_2a_3 - b_3a_2);$$

hence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}.$$

Thus it appears that *if two adjacent columns, or rows, of the determinant are interchanged, the sign of the determinant is changed, but its value remains unaltered.*

If for the sake of brevity we denote the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by $(a_1b_2c_3)$, then the result we have just obtained may be written

$$(b_1a_2c_3) = -(a_1b_2c_3).$$

Similarly we may shew that

$$(c_1a_2b_3) = -(a_1c_2b_3) = +(a_1b_2c_3).$$

493. *If two rows or two columns of the determinant are identical the determinant vanishes.*

For let D be the value of the determinant, then by interchanging two rows or two columns we obtain a determinant whose value is $-D$; but the determinant is unaltered; hence $D = -D$, that is $D = 0$. Thus we have the following equations,

$$a_1A_1 - a_2A_2 + a_3A_3 = D,$$
$$b_1A_1 - b_2A_2 + b_3A_3 = 0,$$
$$c_1A_1 - c_2A_2 + c_3A_3 = 0.$$

494. *If each constituent in any row, or in any column, is multiplied by the same factor, then the determinant is multiplied by that factor.*

For
$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix}$$
$$= ma_1 \,.\, A_1 - ma_2 \,.\, A_2 + ma_3 \,.\, A_3$$
$$= m(a_1A_1 - a_2A_2 + a_3A_3)\,;$$
which proves the proposition.

Cor. If each constituent of one row, or column, is the same multiple of the corresponding constituent of another row, or column, the determinant vanishes.

495. *If each constituent in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants.*

Thus we have

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix};$$

for the expression on the left

$$= (a_1+\alpha_1)A_1 - (a_2+\alpha_2)A_2 + (a_3+\alpha_3)A_3$$
$$= (a_1A_1 - a_2A_2 + a_3A_3) + (\alpha_1A_1 - \alpha_2A_2 + \alpha_3A_3)\,;$$
which proves the proposition.

In like manner if each constituent in any one row, or column, consists of m terms, the determinant can be expressed as the sum of m other determinants.

Similarly, we may shew that

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}.$$

These results may easily be generalised; thus if the constituents of the three columns consist of m, n, p terms respectively, the determinant can be expressed as the sum of mnp determinants.

Example 1. Shew that $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$

The given determinant

$$= \begin{vmatrix} b & a & a \\ c & b & b \\ a & c & c \end{vmatrix} - \begin{vmatrix} b & b & a \\ c & c & b \\ a & a & c \end{vmatrix} + \begin{vmatrix} c & a & a \\ a & b & b \\ b & c & c \end{vmatrix} - \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix}.$$

Of these four determinants the first three vanish, Art. 493; thus the expression reduces to the last of the four determinants; hence its value

$$= -\{c(c^2 - ab) - b(ac - b^2) + a(a^2 - bc)\}$$
$$= 3abc - a^3 - b^3 - c^3.$$

Example 2. Find the value of $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$.

We have

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10+57 & 19 & 21 \\ 0+39 & 13 & 14 \\ 9+72 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 21 \\ 0 & 13 & 14 \\ 9 & 24 & 26 \end{vmatrix} + \begin{vmatrix} 57 & 19 & 21 \\ 39 & 13 & 14 \\ 72 & 24 & 26 \end{vmatrix}$$

$$= \begin{vmatrix} 10 & 19 & 21 \\ 0 & 13 & 14 \\ 9 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 19+2 \\ 0 & 13 & 13+1 \\ 9 & 24 & 24+2 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 2 \\ 0 & 13 & 1 \\ 9 & 24 & 2 \end{vmatrix}$$

$$= 10 \begin{vmatrix} 13 & 1 \\ 24 & 2 \end{vmatrix} + 9 \begin{vmatrix} 19 & 2 \\ 13 & 1 \end{vmatrix} = 20 - 63 = -43.$$

496. Consider the determinant

$$\begin{vmatrix} a_1 + pb_1 + qc_1 & b_1 & c_1 \\ a_2 + pb_2 + qc_2 & b_2 & c_2 \\ a_3 + pb_3 + qc_3 & b_3 & c_3 \end{vmatrix};$$

as in the last article we can shew that it is equal to

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qc_1 & b_1 & c_1 \\ qc_2 & b_2 & c_2 \\ qc_3 & b_3 & c_3 \end{vmatrix};$$

and the last two of these determinants vanish [Art. 494 Cor.]. Thus we see that the given determinant is equal to a new one whose first column is obtained by subtracting from the constituents of the first column of the original determinant equimultiples of the corresponding constituents of the other columns, while the second and third columns remain unaltered.

Conversely,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + pb_1 + qc_1 & b_1 & c_1 \\ a_2 + pb_2 + qc_2 & b_2 & c_2 \\ a_3 + pb_3 + qc_3 & b_3 & c_3 \end{vmatrix};$$

and what has been here proved with reference to the first column is equally true for any of the columns or rows; hence it appears that in reducing a determinant we may replace any one of the rows or columns by a new row or column formed in the following way:

Take the constituents of the row or column to be replaced, and increase or diminish them by any equimultiples of the corresponding constituents of one or more of the other rows or columns.

After a little practice it will be found that determinants may often be quickly simplified by replacing two or more rows or columns simultaneously: for example, it is easy to see that

$$\begin{vmatrix} a_1 + pb_1 & b_1 - qc_1 & c_1 \\ a_2 + pb_2 & b_2 - qc_2 & c_2 \\ a_3 + pb_3 & b_3 - qc_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

but in any modification of the rule as above enunciated, care must be taken to leave one row or column unaltered.

Thus, if on the left-hand side of the last identity the constituents of the third column were replaced by $c_1 + ra_1$, $c_2 + ra_2$, $c_3 + ra_3$ respectively, we should have the former value increased by

$$\begin{vmatrix} a_1 + pb_1 & b_1 - qc_1 & ra_1 \\ a_2 + pb_2 & b_2 - qc_2 & ra_2 \\ a_3 + pb_3 & b_3 - qc_3 & ra_3 \end{vmatrix},$$

and of the four determinants into which this may be resolved there is one which does not vanish, namely

$$\begin{vmatrix} pb_1 & -qc_1 & ra_1 \\ pb_2 & -qc_2 & ra_2 \\ pb_3 & -qc_3 & ra_3 \end{vmatrix}.$$

Example 1. Find the value of $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$.

The given determinant

$$= \begin{vmatrix} 3 & 26 & -4 \\ -6 & 31 & -4 \\ 9 & 54 & -8 \end{vmatrix} = -3\times 4 \times \begin{vmatrix} 1 & 26 & 1 \\ -2 & 31 & 1 \\ 3 & 54 & 2 \end{vmatrix} = -12 \times \begin{vmatrix} 1 & 26 & 1 \\ -3 & 5 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= -12 \begin{vmatrix} 1 & 1 & 26 \\ 0 & -3 & 5 \\ 0 & 1 & 2 \end{vmatrix} = -12 \begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix} = 132.$$

[*Explanation.* In the first step of the reduction keep the second column unaltered; for the first new column diminish each constituent of the first column by the corresponding constituent of the second; for the third new column diminish each constituent of the third column by the corresponding constituent of the second. In the second step take out the factors 3 and -4. In the third step keep the first row unaltered; for the second new row diminish the constituents of the second by the corresponding ones of the first; for the third new row diminish the constituents of the third by twice the corresponding constituents of the first. The remaining steps will be easily seen.]

Example 2. Shew that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

The given determinant

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)\times \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\times \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)\times \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} = (a+b+c)^3.$$

[*Explanation.* In the first new determinant the first row is the sum of the constituents of the three rows of the original determinant, the second and third rows being unaltered. In the third of the new determinants the first column remains unaltered, while the second and third columns are obtained by subtracting the constituents of the first column from those of the second and third respectively. The remaining transformations are sufficiently obvious.]

497. Before shewing how to express the product of two determinants as a determinant, we shall investigate the value of

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}.$$

From Art. 495, we know that the above determinant can be expressed as the sum of 27 determinants, of which it will be sufficient to give the following specimens:

$$\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 \end{vmatrix}, \quad \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix}, \quad \begin{vmatrix} a_1\alpha_1 & c_1\gamma_2 & b_1\beta_3 \\ a_2\alpha_1 & c_2\gamma_2 & b_2\beta_3 \\ a_3\alpha_1 & c_3\gamma_2 & b_3\beta_3 \end{vmatrix},$$

these are respectively equal to

$$\alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix}, \quad \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \alpha_1\beta_3\gamma_2 \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix};$$

the first of which vanishes; similarly it will be found that 21 out of the 27 determinants vanish. The six determinants that remain are equal to

$$(\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1) \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

that is,

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

hence the given determinant can be expressed as the product of two other determinants.

498. *The product of two determinants is a determinant.*

Consider the two linear equations

$$\left.\begin{aligned} a_1X_1 + b_1X_2 = 0 \\ a_2X_1 + b_2X_2 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where
$$\left.\begin{array}{l} X_1 = a_1 x_1 + a_2 x_2 \\ X_2 = \beta_1 x_1 + \beta_2 x_2 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Substituting for X_1 and X_2 in (1), we have
$$\left.\begin{array}{l} (a_1\alpha_1 + b_1\beta_1)\, x_1 + (a_1\alpha_2 + b_1\beta_2)\, x_2 = 0 \\ (a_2\alpha_1 + b_2\beta_1)\, x_1 + (a_2\alpha_2 + b_2\beta_2)\, x_2 = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(3).$$

In order that equations (3) may simultaneously hold for values of x_1 and x_2 other than zero, we must have
$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots(4).$$

But equations (3) will hold if equations (1) hold, and this will be the case either if
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

or if
$$X_1 = 0 \text{ and } X_2 = 0;$$

which last condition requires that
$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Hence if equations (5) and (6) hold, equation (4) must also hold; and therefore the determinant in (4) must contain as factors the determinants in (5) and (6); and a consideration of the dimensions of the determinants shews that the remaining factor of (4) must be numerical; hence
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix},$$
the numerical factor, by comparing the coefficients of $a_1 b_2 \alpha_1 \beta_2$ on the two sides of the equations, being seen to be unity.

Cor.
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2 + b_1^2 & a_1 a_2 + b_1 b_2 \\ a_1 a_2 + b_1 b_2 & a_2^2 + b_2^2 \end{vmatrix}.$$

The above method of proof is perfectly general, and holds whatever be the order of the determinants.

Since the value of a determinant is not altered when we write the rows as columns, and the columns as rows, the product of two determinants may be expressed as a determinant in several ways; but these will all give the same result on expansion.

Example. Shew that $\begin{vmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2,$

the capital letters denoting the minors of the corresponding small letters in the determinant on the right.

Let D, D' denote the determinants on the right and left-hand sides respectively; then

$$DD' = \begin{vmatrix} a_1A_1 - b_1B_1 + c_1C_1 & a_2A_1 - b_2B_1 + c_2C_1 & a_3A_1 - b_3B_1 + c_3C_1 \\ -a_1A_2 + b_1B_2 - c_1C_2 & -a_2A_2 + b_2B_2 - c_2C_2 & -a_3A_2 + b_3B_2 - c_3C_2 \\ a_1A_3 - b_1B_3 + c_1C_3 & a_2A_3 - b_2B_3 + c_2C_3 & a_3A_3 - b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}; \quad \text{[Art. 493.]}$$

thus $DD' = D^3$, and therefore $D' = D^2$.

EXAMPLES. XXXIII. a.

Calculate the values of the determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 35 & 37 & 34 \\ 23 & 26 & 25 \end{vmatrix}$. 2. $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$. 3. $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$.

4. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. 5. $\begin{vmatrix} 1 & z & -y \\ -z & 1 & x \\ y & -x & 1 \end{vmatrix}$. 6. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$.

7. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$. 8. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$.

If ω is one of the imaginary cube roots of unity, find the value of

9. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$. 10. $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$.

11. Eliminate l, m, n from the equations

$$al + cm + bn = 0, \quad cl + bm + an = 0. \quad bl + am + cn = 0.$$

and express the result in the simplest form.

12. Without expanding the determinants, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}.$$

13. Solve the equations:

(1) $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0.$ (2) $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0.$

Prove the following identities:

14. $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

15. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$$

16. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$$

17. $$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy).$$

18. $$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b).$$

19. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

20. Express as a determinant $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2.$

21. Find the condition that the equation $lx+my+nz=0$ may be satisfied by the three sets of values (a_1, b_1, c_1) (a_2, b_2, c_2) (a_3, b_3, c_3); and shew that it is the same as the condition that the three equations

$$a_1x+b_1y+c_1z=0, \quad a_2x+b_2y+c_2z=0, \quad a_3x+b_3y+c_3z=0$$

may be simultaneously satisfied by l, m, n.

22. Find the value of

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}.$$

23. Prove that $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & \alpha+i\beta \end{vmatrix}$,

where $i=\sqrt{-1}$, can be written in the form

$$\begin{vmatrix} A-iB & C-iD \\ -C-iD & A+iB \end{vmatrix};$$

hence deduce the following theorem, due to Euler:

The product of two sums each of four squares can be expressed as the sum of four squares.

Prove the following identities:

24. $$\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix}$$
$$= -(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

25. $$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ -bc+ca+ab & bc-ca+ab & bc+ca-ab \\ (a+b)(a+c) & (b+c)(b+a) & (c+a)(c+b) \end{vmatrix}$$
$$=3(b-c)(c-a)(a-b)(a+b+c)(bc+ca+ab).$$

26. $$\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$
$$=2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y).$$

27. Find in the form of a determinant the condition that the expression

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta$$

may be the product of two factors of the first degree in α, β, γ.

28. Solve the equation:

$$\begin{vmatrix} u+a^2x & w'+abx & v'+acx \\ w'+abx & v+b^2x & u'+bcx \\ v'+acx & u'+bcx & w+c^2x \end{vmatrix} = 0,$$

expressing the result by means of determinants.

499. The properties of determinants may be usefully employed in solving simultaneous linear equations.

Let the equations be

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0;$$

multiply them by A_1, $-A_2$, A_3 respectively and add the results, A_1, A_2, A_3 being minors of a_1, a_2, a_3 in the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The coefficients of y and z vanish in virtue of the relations proved in Art. 493, and we obtain

$$(a_1A_1 - a_2A_2 + a_3A_3)\,x + (d_1A_1 - d_2A_2 + d_3A_3) = 0.$$

Similarly we may shew that

$$(b_1B_1 - b_2B_2 + b_3B_3)\,y + (d_1B_1 - d_2B_2 + d_3B_3) = 0,$$

and

$$(c_1C_1 - c_2C_2 + c_3C_3)\,z + (d_1C_1 - d_2C_2 + d_3C_3) = 0.$$

Now $\quad a_1A_1 - a_2A_2 + a_3A_3 = -(b_1B_1 - b_2B_2 + b_3B_3)$

$$= c_1C_1 - c_2C_2 + c_3C_3 = D;$$

hence the solution may be written

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} d_1 & a_1 & c_1 \\ d_2 & a_2 & c_2 \\ d_3 & a_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or more symmetrically

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

500. Suppose we have the system of four homogeneous linear equations :

$$a_1x + b_1y + c_1z + d_1u = 0,$$
$$a_2x + b_2y + c_2z + d_2u = 0,$$
$$a_3x + b_3y + c_3z + d_3u = 0,$$
$$a_4x + b_4y + c_4z + d_4u = 0.$$

From the last three of these, we have as in the preceding article

$$\frac{x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}} = \frac{-u}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}.$$

Substituting in the first equation, the eliminant is

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

This may be more concisely written in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0;$$

the expression on the left being a determinant of the fourth order.

Also we see that the coefficients of a_1, b_1, c_1, d_1 taken with their proper signs are the *minors* obtained by omitting the row and column which respectively contain these constituents.

501. More generally, if we have n homogeneous linear equations

$$a_1x_1 + b_1x_2 + c_1x_3 + \ldots\ldots + k_1x_n = 0,$$
$$a_2x_1 + b_2x_2 + c_2x_3 + \ldots\ldots + k_2x_n = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_nx_1 + b_nx_2 + c_nx_3 + \ldots\ldots + k_nx_n = 0,$$

involving n unknown quantities x_1, x_2, x_3, ... x_n, these quantities can be eliminated and the result expressed in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \ldots\ldots\ldots & k_1 \\ a_2 & b_2 & c_2 \ldots\ldots\ldots & k_2 \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ a_n & b_n & c_n \ldots\ldots\ldots & k_n \end{vmatrix} = 0.$$

The left-hand member of this equation is a determinant which consists of n rows and n columns, and is called a determinant of the n^{th} order.

The discussion of this more general form of determinant is beyond the scope of the present work; it will be sufficient here to remark that the properties which have been established in the case of determinants of the second and third orders are quite general, and are capable of being extended to determinants of any order.

For example, the above determinant of the n^{th} order is equal to

$$a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1 + \ldots + (-1)^{n-1} k_1K_1,$$

or

$$a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4 + \ldots + (-1)^{n-1} a_nA_n,$$

according as we develop it from the first row or the first column. Here the capital letters stand for the minors of the constituents denoted by the corresponding small letters, and are themselves determinants of the $(n-1)^{\text{th}}$ order. Each of these may be expressed as the sum of a number of determinants of the $(n-2)^{\text{th}}$ order; and so on; and thus the expanded form of the determinant may be obtained.

Although we may always develop a determinant by means of the process described above, it is not always the simplest method, especially when our object is not so much to find the value of the whole determinant, as to find the signs of its several elements.

502. The expanded form of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1\,;$$

and it appears that each element is the product of three factors, one taken from each row, and one from each column; also the signs of half the terms are + and of the other half −. The signs of the several elements may be obtained as follows. The first element $a_1b_2c_3$, in which the suffixes follow the arithmetical order, is positive; we shall call this the leading element; every other element may be obtained from it by suitably interchanging the suffixes. The sign + or − is to be prefixed to any element according as it can be deduced from the leading element by an even or odd number of permutations of two suffixes; for instance, the element $a_3b_2c_1$ is obtained by interchanging the suffixes 1 and 3, therefore its sign is negative; the element $a_3b_1c_2$ is obtained by first interchanging the suffixes 1 and 3, and then the suffixes 1 and 2, hence its sign is positive.

503. The determinant whose leading element is $a_1b_2c_3d_4 \ldots$ may thus be expressed by the notation

$$\Sigma \pm a_1b_2c_3d_4 \ldots\ldots,$$

the $\Sigma \pm$ placed before the leading element indicating the aggregate of all the elements which can be obtained from it by suitable interchanges of suffixes and adjustment of signs.

Sometimes the determinant is still more simply expressed by enclosing the leading element within brackets; thus $(a_1b_2c_3d_4 \ldots)$ is used as an abbreviation of $\Sigma \pm a_1b_2c_3d_4 \ldots.$

Example. In the determinant $(a_1b_2c_3d_4e_5)$ what sign is to be prefixed to the element $a_4b_3c_1d_5e_2$?

From the leading element by permuting the suffixes of a and d we get $a_4b_2c_3d_1e_5$; from this by permuting the suffixes of b and c we have $a_4b_3c_2d_1e_5$; by permuting the suffixes of c and d we have $a_4b_3c_1d_2e_5$; finally by permuting the suffixes of d and e we obtain the required element $a_4b_3c_1d_5e_2$; and since we have made four permutations the sign of the element is positive.

504. If in Art. 501, each of the constituents $b_1, c_1, \ldots k_1$ is equal to zero the determinant reduces to a_1A_1; in other words it is equal to the product of a_1 and a determinant of the $(n-1)^{\text{th}}$ order, and we easily infer the following general theorem.

If each of the constituents of the first row or column of a determinant is zero except the first, and if this constituent is equal to m, *the determinant is equal to* m *times that determinant of lower order which is obtained by omitting the first column and first row.*

Also since by suitable interchange of rows and columns any constituent can be brought into the first place, it follows that if *any* row or column has all its constituents except one equal to zero, the determinant can immediately be expressed as a determinant of lower order.

This is sometimes useful in the reduction and simplification of determinants.

Example. Find the value of

$$\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix}.$$

Diminish each constituent of the first column by twice the corresponding constituent in the second column, and each constituent of the fourth column by three times the corresponding constituent in the second column, and we obtain

$$\begin{vmatrix} 8 & 11 & 20 & 5 \\ 0 & 3 & 0 & 0 \\ 15 & -2 & 36 & 9 \\ 7 & 16 & 17 & 4 \end{vmatrix},$$

and since the second row has three zero constituents this determinant

$$=3\begin{vmatrix} 8 & 20 & 5 \\ 15 & 36 & 9 \\ 7 & 17 & 4 \end{vmatrix}=3\begin{vmatrix} 8 & 20 & 5 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix}=3\begin{vmatrix} 0 & 1 & 0 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix}=-3\begin{vmatrix} 8 & 5 \\ 7 & 4 \end{vmatrix}=9.$$

505. The following examples shew artifices which are occasionally useful.

Example 1. Prove that

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}=(a+b+c+d)(a-b+c-d)(a-b-c+d)(a+b-c-d).$$

By adding together all the rows we see that $a+b+c+d$ is a factor of the determinant; by adding together the first and third rows and subtracting from the result the sum of the second and fourth rows we see that $a-b+c-d$ is also a factor; similarly it can be shewn that $a-b-c+d$ and $a+b-c-d$ are factors; the remaining factor is numerical, and, from a comparison of the terms involving a^4 on each side, is easily seen to be unity; hence we have the required result.

Example 2. Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}=(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

The given determinant vanishes when $b=a$, for then the first and second columns are identical; hence $a-b$ is a factor of the determinant [Art. 514]. Similarly each of the expressions $a-c$, $a-d$, $b-c$, $b-d$, $c-d$ is a factor of the determinant; the determinant being of six dimensions, the remaining factor must be numerical; and, from a comparison of the terms involving bc^2d^3 on each side, it is easily seen to be unity; hence we obtain the required result.

EXAMPLES. XXXIII. b.

Calculate the values of the determinants:

1. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$.

2. $\begin{vmatrix} 7 & 13 & 10 & 6 \\ 5 & 9 & 7 & 4 \\ 8 & 12 & 11 & 7 \\ 4 & 10 & 6 & 3 \end{vmatrix}$.

3. $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$.

4. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & b+c & a & a \\ 1 & b & c+a & b \\ 1 & c & c & a+b \end{vmatrix}$.

5. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$.

6. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$.

7. $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$.

8. $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix}$.

9. $\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$.

10. If ω is one of the imaginary cube roots of unity, shew that the square of

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix};$$

hence shew that the value of the determinant on the left is $3\sqrt{-3}$.

11. If
$$(f^2-bc)x+(ch-fg)y+(bg-hf)z=0,$$
$$(ch-fg)x+(g^2-ca)y+(af-gh)z=0,$$
$$(bg-hf)x+(af-gh)y+(h^2-ab)z=0,$$
shew that
$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Solve the equations:

12. $$\begin{aligned} x+y+z&=1,\\ ax+by+cz&=k,\\ a^2x+b^2y+c^2z&=k^2. \end{aligned}$$

13. $$\begin{aligned} ax+by+cz&=k,\\ a^2x+b^2y+c^2z&=k^2,\\ a^3x+b^3y+c^3z&=k^3. \end{aligned}$$

14. $$\begin{aligned} x+y+z+u&=1,\\ ax+by+cz+du&=k,\\ a^2x+b^2y+c^2z+d^2u&=k^2,\\ a^3x+b^3y+c^3z+d^3u&=k^3. \end{aligned}$$

15. Prove that

$$\begin{vmatrix} b+c-a-d & bc-ad & bc(a+d)-ad(b+c) \\ c+a-b-d & ca-bd & ca(b+d)-bd(c+a) \\ a+b-c-d & ab-cd & ab(c+d)-cd(a+b) \end{vmatrix}$$
$$= -2(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

16. Prove that

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$
$$=(b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$$

17. Shew that

$$\begin{vmatrix} a & b & c & d & e & f \\ f & a & b & c & d & e \\ e & f & a & b & c & d \\ d & e & f & a & b & c \\ c & d & e & f & a & b \\ b & c & d & e & f & a \end{vmatrix} = \begin{vmatrix} A & B & C \\ C & A & B \\ B & C & A \end{vmatrix},$$

where
$$\begin{aligned} A&=a^2-d^2+2ce-2bf,\\ B&=e^2-b^2+2ac-2df,\\ C&=c^2-f^2+2ae-2bd. \end{aligned}$$

18. If a determinant is of the n^{th} order, and if the constituents of its first, second, third, ...n^{th} rows are the first n figurate numbers of the first, second, third, ...n^{th} orders, shew that its value is unity.