Chapter - 4

Quadratic Equations

Quadratic Equations

We come across quadratic equations in many real-life situations.

Quadratic equations are widely used in the field of communication

They are useful in describing the trajectory of a moving ball or a satellite.

They are used to determine the height of the thrown object.

Quadratic equations are commonly used to find the maximum and minimum values of something.

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$ is called the standard form of a quadratic equation.

$3x^2 + 2x + 5 = 0$	$a = 3, b = 2, c = 5$
$3x - 5x^2 + 12 = 0$	$a = -5, b = 3, c = 12$
$-12 + 3x^2$	$a = 3, b = 0$ and $c = -12$
$4x - 12 = 0$	$a = 0, b = 4$ and $c = -12$. This is not a quadratic equation as $a = 0$

Example: Check whether the following are quadratic equations

i) $(x + 1)^2 = 2(x - 3)$

$(x + 1)^2 = x^2 + 2x + 1 \because (a + b)^2 = a^2 + 2ab + b^2$

$x^2 + 2x + 1 = 2(x - 3) \Rightarrow x^2 + 2x + 1 = 2x - 6$

$x^2 + 2x + 1 - 2x + 6 = 0 \Rightarrow x^2 + 2x - 2x + 6 + 1 = 0$

$$x^2 + 7 = 0$$

The above equation is a quadratic equation, where the coefficient of x is zero, i.e. $b = 0$

ii) $x(x + 1)(x + 8) = (x + 2)(x - 2)$

LHS

$x(x + 1)(x + 8) = x(x^2 + 8x + x + 8)$

$= x(x^2 + 9x + 8) = x^3 + 9x^2 + 8x$

RHS

$(x + 2)(x - 2) = x^2 - 4 \because (a + b)(a - b) = a^2 - b^2$

Now, $x^3 + 9x^2 + 8x = x^2 - 4$

$x^3 + 9x^2 - x^2 + 8x + 4 = 0$

$x^3 + 8x^2 + 8x + 4 = 0$

It is not a quadratic equation as it is an equation of degree 3.

iii) $(x - 2)^2 + 1 = 2x - 3$

LHS

$(x - 2)^2 + 1 = x^2 - 2x + 4 + 1$

$\because (a - b)^2 = a^2 - 2ab + b^2$

$= x^2 - 2x + 5$

RHS

$2x - 3$

$x^2 - 2x + 5 = 2x - 3$
$x^2 - 2x - 2x + 5 + 3 = 0$
$x^2 - 4x + 8 = 0$

The above equation is quadratic as it is of the form,

$ax^2 + bx + c = 0$

Example: The product of two consecutive positive integers is 420. Form the equation satisfying this scenario.

Let the two consecutive positive integers be x and x + 1 Product of the two consecutive integers= $x(x + 1) = 420$

$\Rightarrow x^2 + x = 420$
$x^2 + x - 420 = 0$

$x^2 + x - 420 = 0$, is the required quadratic equation and the two integers satisfy this quadratic equation. Example: A train travels a distance of 480 km at a

uniform speed. If the speed had been 8 km/hr less, then it would have taken 4 hr more to cover the distance. We need to find the speed of the train. Form the equation

satisfying this scenario

Let the speed of the train be x km/hr

Distance travelled by train = 480 km

Time taken to cover the distance of 480 km = $\frac{480}{x}hr$

$$\because \text{Time} = \frac{Distance}{Speed}$$

If the speed was 8 km/hr less, i.e. $(x - 8)$km/hr, then the time taken for travelling 480 km = $\frac{480}{(x-8)}hr$

According to the question,

$\frac{480}{(x-8)} = 4 + \frac{480}{x} \Rightarrow - \frac{480}{(x-8)} = 4$

$$\frac{480x - 480(x-8)}{x(x-8)} = 4$$

$$\frac{120x - 120x + 960}{x(x-8)} = 1$$

$$960 = x(x-8) \Rightarrow x^2 - 8x - 960 = 0$$

$x^2 - 8x - 960 = 0$

$x^2 - 8x - 960 = 0$, is the required quadratic equation and the speed of the train satisfies the equation.

Solution of Quadratic Equations by Factorisation

Solution of Quadratic Equation by Factorisation

A real number α is called a root of the quadratic equation $x^2 + bx + c = 0$, $a \neq 0$ if $a\alpha^2 + b\alpha + c = 0$

We say that $x = \alpha$ is a solution of the quadratic equation.

Example: $x^2 - 2x - 3 = 0$

If we put $x = -1$ in the LHS of the above equation we get,

$(-1)^2 - 2(-1) - 3$

$1 + 2 - 3 = 0$

Thus $x = -1$ is a solution of the equation $x^2 - 2x - 3 = 0$.

To find the roots of the quadratic equations we follow these steps.

Transpose all the terms of the equation to LHS to obtain quadratic equation of the form $ax^2 + bx + c = 0$

Factorise the quadratic expression into linear factors, equating each factor equal to zero.

Solve the resulting linear equation to get the roots of the quadratic equation.

Example: Find the roots of the equation $x^2 - 3x - 10 = 0$

(REFERENCE: NCERT)

The given equation is $x^2 - 3x - 10 = 0$.

Here $a = 1$, $b = -3$ and $c = -10$

1) Find the product of a and c .

Here, the product of a and c= -10 → (ac) is negative

2) Write the factors of this product (ac) such that the sum of the two factors is equal to b.

$\therefore ac = m \times n$ and $m + n = b$

Factors of $10 = 2\times5$

Let $m = -5$ and $n = 2$ → (ac=-10)

We write the given equation as,
$x^2 - 5x + 2x - 10 = 0$
$x(x - 5) + 2(x - 5) = 0$
$x - 5\ x + 2 = 0$

Equate each factor to zero to get the roots of the equation.

$x - 5 = 0$ and $x + 2 = 0$

$$x = 5, -2$$

Therefore, 5 and -2 are the roots of the equation x

$2 - 3x - 10 = 0$

Example: Solve the following quadratic equation by factorisation method.

i) $4\sqrt{3x^2} + 5x - 2\sqrt{3} = 0$

The given equation is $4\sqrt{3x^2} + 5x - 2\sqrt{3} = 0$

Here, $a = 4\sqrt{3}$, $b = 5$and$c = -2\sqrt{3}$

The product of a and $c = 4\sqrt{3} \times (-2\sqrt{3}) = -8 \times 3 = -24$

Factors of $24 = 3\times8$ and $8 + (-3) = 5$

The factorsof the equation are $8, -3$

So, the given equation can be written as,

$4\sqrt{3x^2} + (8-3)x - 2\sqrt{3} = 0 \Rightarrow 4\sqrt{3x^2} + 8x - 3x - 2\sqrt{3} = 0$

$4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2) = 0 \Rightarrow (4x - \sqrt{3})(\sqrt{3}x + 2) = 0$

Equating each factor to zero we get,

$(4x - \sqrt{3})=0$ and $(\sqrt{3}x + 2) = 0$

$x = \frac{\sqrt{3}}{4}$ and $x = \frac{-2}{\sqrt{3}}$

The roots of the equation $4\sqrt{3x^2} + 5x - 2\sqrt{3} = 0$ are $\frac{\sqrt{3}}{4}, \frac{-2}{\sqrt{3}}$

ii) $\frac{2}{x^2} - \frac{5}{x} + 2 = 0$

The given equation is $\frac{2}{x^2} - \frac{5}{x} + 2 = 0$

Multiplying the above equation by x^2 we get,

$$x^2\left(\frac{2}{x^2} - \frac{5}{2} + 2 = 0\right) \Rightarrow \frac{2x^2}{x^2} - \frac{5x^2}{x} + 2x^2 = 0.$$

$$2 - 5x + 2x^2 = 0 \Rightarrow 2x^2 - 5x + 2 = 0$$

Here, $a = 2, b = -5$ and $c = 2$

The product of a and $c = 2 \times 2 = 4$

The factors of $4 = 4 \times 1$ and $4 + 1 = 5$

$2x^2 - (4 + 1)x + 2 = 0 \Rightarrow 2x^2 - 4x - 1x + 2 = 0$

$2x(x - 2) - (x - 2) = 0$

$(2x - 1)(x - 2) = 0$

Equating each factor to zero we get,

$(2x - 1) = 0$ and $(x - 2) = 0$

$x = \frac{1}{2}$ and $x = 2$

The roots of equation $2x^2 - 5x + 2 = 0$ are $\frac{1}{2}$ and 2

Example: The altitude of a right-angled triangle is 7 cm less than its base. If the hypotenuse is 13 cm long, then find the other two sides.

(REFERENCE: NCERT)

Let the length of the base be x cm, then altitude $= x - 7$ cm

Hypotenuse = 13 cm

We know, $H^2 = P^2 + B^2$

$132 = (x - 7)^2 + x^2 \Rightarrow 169 = x^2 - 14x + 49 + x^2$

$x^2 - 14x + 49 + x^2 = 169 \Rightarrow 2x2 - 14x + 49 - 169 = 0$

$2x2 - 14x - 120 = 0$

Dividing the above equation by 2 we get,

$x^2 - 7x - 60 = 0$

Here, $a = 1, b = -7$ and $c = -60$

The product of a and c $= 1 \times (-60) = -60$

The factors of $60 = 5 \times 12$ and $-12 + 5 = 7$

The given equation can be written as,

$x^2 - 12x + 5x - 60 = 0$

$x(x - 12) + 5(x - 12) = 0 \Rightarrow (x + 5)(x - 12) = 0$

Equating each factor to zero we get,

$(x + 5) = 0$ and $(x - 12) = 0 \Rightarrow x = -5$ and $x = 12$

The length of the base cannot be negative.

Therefore, Base $= 12$ cm

Altitude $= x - 7$ cm $= 12 - 7 = 5$ cm, Hypotenuse $= 13$ cm

Solution of Quadratic Equations by Completing the Square

Solution of the Quadratic Equations by Completing the Square

If we have to find the solution of a quadratic equation by completing the square, we follow the steps given below.

We first write the given equation in standard form,

$ax^2 + bx + c = 0, a \neq 0$

The coefficient of x^2 should be 1. If it is not 1 then divide the whole equation by the coefficient of x^2, that is a.

$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = 0$

Move $\frac{c}{a}$ to the RHS.

$x^2 + \frac{bx}{a} = -\frac{c}{a}$

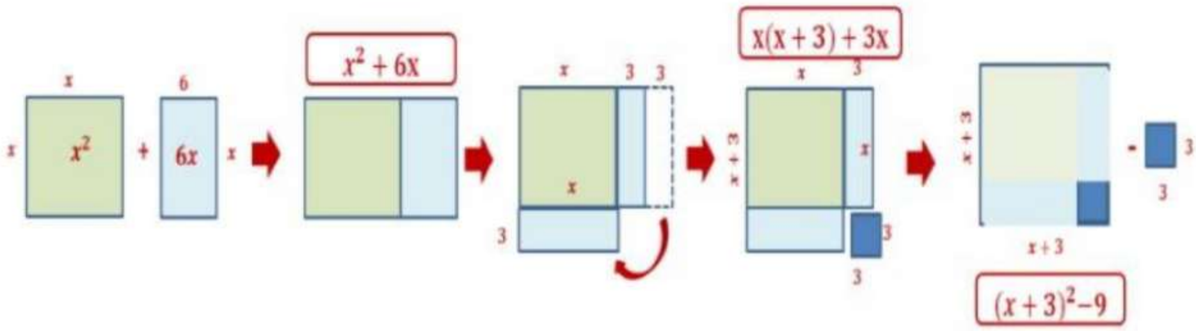
Add $(\frac{b}{2a})^2$ to both sides.

$x^2 + \frac{bx}{a} + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$

$(x + \frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$

The complete square is, $(x + \frac{2}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$

Let’s learn to complete the square with the help of a diagram



Example: Find the roots of the following quadratic equations by the method of completing the square:

$2x^2 - 7x + 3 = 0$

The given quadratic equation is $2x^2 - 7x + 3 = 0$

The coefficient of x^2 is not 1, so we divide the whole equation by 2.

$$x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

Now move $\frac{3}{2}$ to RHS

$$x^2 - \frac{7}{2}x = -\frac{3}{2}$$

Adding $(\frac{7}{4})^2$ to both sides we get,

$$x^2 - \frac{7}{2}x + (\frac{7}{4})^2 = -\frac{3}{2} + (\frac{7}{4})^2$$

$$(x - \frac{7}{4})^2 = -\frac{3}{2} + \frac{49}{16}$$

$$(x - \frac{7}{4})^2 = \frac{-24 + 49}{16}$$

$$(x - \frac{7}{4})^2 = \frac{25}{16}$$

Taking square root of both sides we get,

$$x - \frac{7}{4} = \pm\frac{5}{4}$$

$x - \frac{7}{4} = +\frac{5}{4} \Rightarrow x = \frac{7+5}{4} = \frac{12}{4} = 3$

$x - \frac{7}{4} = -\frac{5}{4} \Rightarrow x = \frac{7-5}{4} = \frac{2}{4} = \frac{1}{2}$

The roots of the equation are 3 and $\frac{1}{2}$

i) $4x^2 + 4\sqrt{3x} + 3 = 0$

Dividing the whole equation by 4, so that the coefficient of x^2 is 1.

$$\frac{4}{4}x^2 + \frac{4\sqrt{3}}{4}x + \frac{3}{4} = 0 \Rightarrow x^2 + \sqrt{3x} + \frac{3}{4} = 0$$

Shifting $\frac{3}{4}$ to RHS

$$x^2 + \sqrt{3x} = -\frac{3}{4}$$

Adding $(\frac{\sqrt{3}}{2})^2$ to both sides we get,

$$x^2 + \sqrt{3x} + (\frac{\sqrt{3}}{2})^2 = -\frac{3}{4} + (\frac{\sqrt{3}}{2})^2 \Rightarrow (x + \frac{\sqrt{3}}{2})^2 = -\frac{3}{4} + \frac{3}{4}$$

$$(x + \frac{\sqrt{3}}{2})^2 = 0$$

Taking the square root of both sides

$$x + \frac{\sqrt{3}}{2} = 0 \Rightarrow x = -\frac{\sqrt{3}}{2}$$

The roots of the given equation are $-\frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{2}$

Example: Solve the quadratic equation

$x^2 - (\sqrt{5} + 1)x + \sqrt{5} = 0$ by completing the square method.

The given quadratic equation is, $x^2 - (\sqrt{5} + 1)x = -\sqrt{5} = 0$

Shifting $\sqrt{5}$ to RHS we get,

$x^2 - (\sqrt{5} + 1)x = -\sqrt{5}$

Adding $(\frac{\sqrt{5}+1}{2})^2$ to both sides we get,

$$x^2 - (\sqrt{5}+1)x + (\frac{\sqrt{5}+1}{2})^2 = -\sqrt{5} + (\frac{\sqrt{5}+1}{2})^2$$

$$[x - \frac{\sqrt{5}+1}{2}]^2 = -\sqrt{5} + \frac{5 + 2\sqrt{5} + 1}{4}$$

$$[x - \frac{\sqrt{5}+1}{2}]^2 = \frac{5 + 2\sqrt{5} + 1 - 4\sqrt{5}}{4} \Rightarrow \frac{(\sqrt{5})^2 + 2\sqrt{5} + 1 - 4\sqrt{5}}{4}$$

$$[x - \frac{\sqrt{5}+1}{2}]^2 = \frac{(\sqrt{5})^2 - 2\sqrt{5} + 1}{4}$$

$$[x - \frac{(\sqrt{5}+1)}{2}]^2 = (\frac{\sqrt{5}-1}{2})^2 \Rightarrow x - \frac{(\sqrt{5}+1)}{2} = \pm(\frac{\sqrt{5}-1}{2})$$

Taking +ve sign first

$$x - \frac{(\sqrt{5}+1)}{2} = +(\frac{\sqrt{5}-1}{2}) \Rightarrow x = (\frac{\sqrt{5}-1}{2}) + \frac{(\sqrt{5}+1)}{2}$$

$$x = (\frac{\sqrt{5}-1+\sqrt{5}+1}{2}) \Rightarrow (\frac{2\sqrt{5}}{2}) = \sqrt{5}$$

Taking –ve sign

$$x - \frac{(\sqrt{5}+1)}{2} = -(\frac{\sqrt{5}-1}{2}) \Rightarrow x = \frac{-\sqrt{5}+1}{2} + \frac{(\sqrt{5})+1}{2}$$

$$x = \frac{-\sqrt{5}+1\sqrt{5}+1}{2} \Rightarrow (\frac{2}{2}) = 1$$

The roots of the given equation are $\sqrt{5}$ and 1

Nature of Roots

Nature of Roots

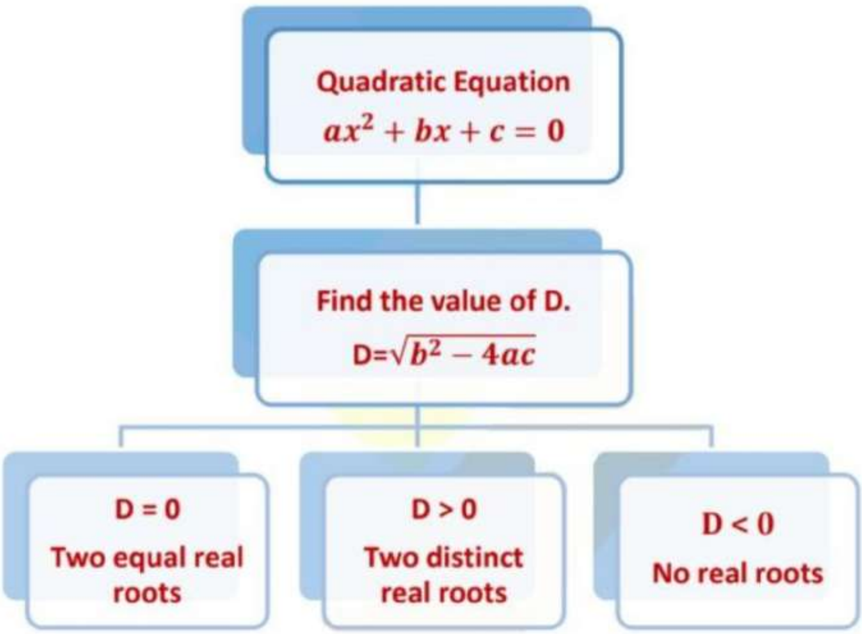
The roots of the quadratic equation $ax^2 + bx + c = 0$ are given

by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

Where $D = \sqrt{b^2 - 4ac}$ is called the discriminant.

This formula is known as the Quadratic Formula.

The nature of the roots depends upon the value of Discriminant, D.



Example: Find the roots of the equation,

$\sqrt{5x+7} = (2x - 7) = 0$

The given equation is $\sqrt{5x+7} = (2x - 7) = 0$

Squaring both sides of the equation we get,

$(\sqrt{9x+9})^2 = (2x - 7)^2$

$9x + 9 = 4x^2 - 28x + 49$

$4^2 - 28x + 49 - 9x - 9 = 0$

$4x^2 - 37x + 40 = 0$

Here, $a = 4$, $b = -37$ and $c = 40$

Substituting the value of a, b and c in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

x = $\frac{-(-37) \pm \sqrt{37^2 - 4X4X40}}{2X4} \Rightarrow \frac{37 \pm \sqrt{1369 - 640}}{8}$

$\Rightarrow \frac{37 \pm \sqrt{729}}{8} \Rightarrow \frac{37 \pm 27}{8}$

Taking +ve sign first,

x = $\frac{37 + 27}{8} \Rightarrow \frac{64}{8} = 8$

Taking -ve we get,

x = $\frac{37 - 27}{8} \Rightarrow \frac{10}{8} = \frac{5}{4}$

The roots of the given equation are 8 and $\frac{5}{4}$.

Example: Find the numerical difference of the roots of the equation $x^2 - 7x - 30 = 0$

The given quadratic equation is $x^2 - 7x - 30 = 0$

Here a = 1, b = −7 and c = −30

Substituting the value of a, b and c in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{7^2 - 4X1X(-30)}}{2X1} \Rightarrow \frac{7 \pm \sqrt{49 + 120}}{2}$$

$$\Rightarrow \frac{7 \pm \sqrt{169}}{2} \Rightarrow \frac{7 \pm 13}{2}$$

Taking +ve sign first,

$$x = \frac{7 + 13}{2} \Rightarrow \frac{20}{2} = 10$$

Taking -ve we get,

$$x = \frac{7-13}{2} \Rightarrow \frac{-6}{2} = -3$$

The two roots are 10 and -3

The difference of the roots= $10 - (-3) = 10 + 3 = 13$

Example: Find the discriminant of the quadratic equation $x^2 - 4x - 5 = 0$

The given quadratic equation is $x^2 - 4x - 5 = 0$.

On comparing with $ax^2 + bx + c = 0$ we get,

$a = 1, b = -4$, and $c = -5$

Discriminant, D $= \sqrt{b^2 - 4ac}$

$$D = \sqrt{(-4)^2 - 4X1X(-5)} = \sqrt{16+20} = \sqrt{36}$$

$$D = \pm 6$$

Example: Find the value of p, so that the quadratic equation $px(x-2) + 9 = 0$ has equal roots.

The given quadratic equation is $px(x-2) + 9 = 0$

$px^2 - 2px + 9 = 0$

Now comparing with $ax^2 + bx + c = 0$ we get,

$a = p, b = -2p$ and $c = 9$

Discriminant, D $= \sqrt{b^2 - 4ac}$

D $= \sqrt{(-2p)^2 - 4XpX9} = \sqrt{4p^2 - 36p}$

The given quadratic equation will have equal roots if $D = 0$

$$D = \sqrt{4p^2 - 36p} = 0$$

$$4p^2 - 36p = 0 \Rightarrow 4p(p-9) = 0$$

$p = 0$ and $p - 9 = 0 \Rightarrow p = 9$
$p = 0$ and $p = 9$

The value of p cannot be zero as the coefficient of x, $(-2p)$ will

become zero.

Therefore, we take the value of $p = 9$.

Example: If $x = -1$ is a root of the quadratic equations $2x^2 + px + 5 = 0$ and the quadratic equation

$p(x^2 + x) + k = 0$ has equal roots, then find the value of k.

The given quadratic equation is $2x^2 + px + 5 = 0$. If $x = -1$ is

the root of the equation then,

$2(-1)^2 + p(-1) + 5 = 0$

$2 - p + 5 = 0 \Rightarrow -p = -7$

$p = 7$

Putting the value of p in the equation $p(x^2 + x) + k = 0$,

$7(x^2 + x) + k = 0 \Rightarrow 7x^2 + 7x + k = 0$

Now comparing with $ax^2 + bx + c = 0$ we get,

$a = 7, b = 7$ and $c = k$

Discriminant, $D = \sqrt{b^2 - 4ac}$

$D = \sqrt{(7)^2 - 4X7Xk} = \sqrt{49 - 28k}$

The given quadratic equation will have equal roots if $D = 0$

$D = \sqrt{49 - 28k} = 0$

$\sqrt{49 - 28k} = 0 \Rightarrow 49 - 28k = 0$

$28k = 49$

$$k = \frac{49}{28} = \frac{7}{4}$$

Therefore, the value of k is $\frac{7}{4}$.