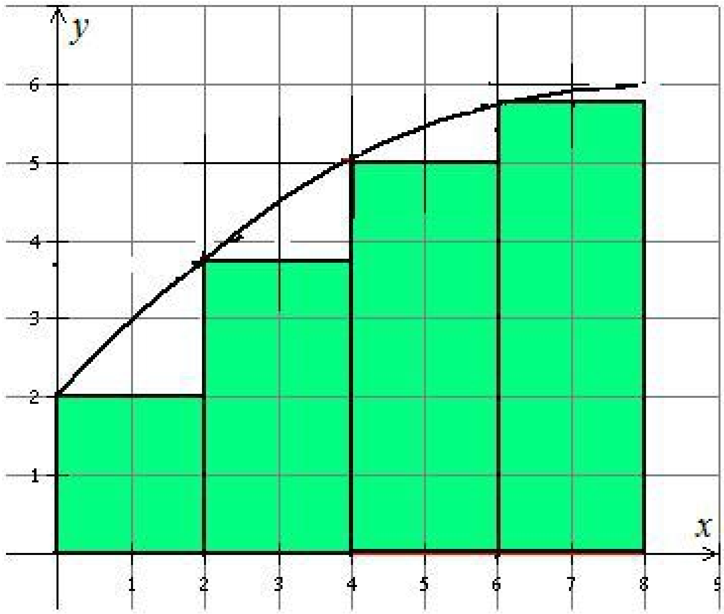
Exercise 4.1

Answer 1E.

(a)

Consider the graph of the function f for lower estimate as shown below:



Find a lower estimate by drawing four rectangles between $x=0$ and $x=8$. To make it a lower estimate, the heights of the rectangles are the values of f at the left endpoints of the subintervals.

Here $n=4, a=0, b=8.$

The width of each of the n strips is,

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{8-0}{4} \\ &= \frac{8}{4} \\ &= 2\end{aligned}$$

Lower estimate is,

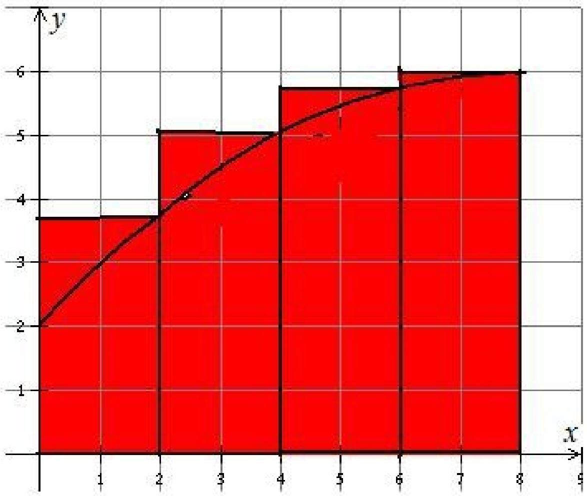
$$\begin{aligned} L_4 &= \left[f(x_0)+f(x_2)+f(x_4)+f(x_6)\right]\Delta x \\ &= \left[f(0)+f(2)+f(4)+f(6)\right]2 \\ &= (2+3.8+5+5.8)2 \\ &= 33.2 \end{aligned}$$

≈ 33

Therefore, lower estimate by drawing four rectangles between $x=0$ and $x=8$ is,

$\boxed{L_4 = 33}$.

Consider the graph of the function f for upper estimate as shown below:



Find an upper estimate by drawing four rectangles between $x=0$ and $x=8$. To make it an upper estimate, the heights of the rectangles are the values of f at the right endpoints of the subintervals.

Upper estimate is,

$$\begin{aligned} R_4 &= \left[f(x_2)+f(x_4)+f(x_6)+f(x_8)\right]\Delta x \\ &= \left[f(2)+f(4)+f(6)+f(8)\right]2 \\ &= (3.8+5+5.8+6)2 \\ &= 41.2 \end{aligned}$$

≈ 41

Therefore, upper estimate by drawing four rectangles between $x=0$ and $x=8$ is,

$\boxed{R_4 = 41}$.

Find an upper estimate by drawing four rectangles between $x=0$ and $x=8$. To make it an upper estimate, the heights of the rectangles are the values of f at the right endpoints of the subintervals.

Upper estimate is,

$$\begin{aligned} R_4 &= \left[f(x_2)+f(x_4)+f(x_6)+f(x_8)\right]\Delta x \\ &= \left[f(2)+f(4)+f(6)+f(8)\right]2 \\ &= (3.8+5+5.8+6)2 \\ &= 41.2 \end{aligned}$$

≈ 41

Therefore, upper estimate by drawing four rectangles between $x=0$ and $x=8$ is,

$\boxed{R_4 = 41}$.

Lower estimate is,

$$L_8 = \left[f(x_0)+f(x_1)+f(x_2)+...+f(x_6)+f(x_7)\right]\Delta x$$
$$= \left[f(0)+f(1)+f(2)+...+f(7)\right]1$$
$$\approx 2+3+3.8+4.5+5+5.2+5.7+5.9$$
$$= 35.2$$

Therefore, lower estimate by drawing eight rectangles between $x=0$ and $x=8$ is,

$\boxed{L_8 \approx 35.2}$.

Find an upper estimate by drawing eight rectangles between $x=0$ and $x=8$.

Upper estimate is,

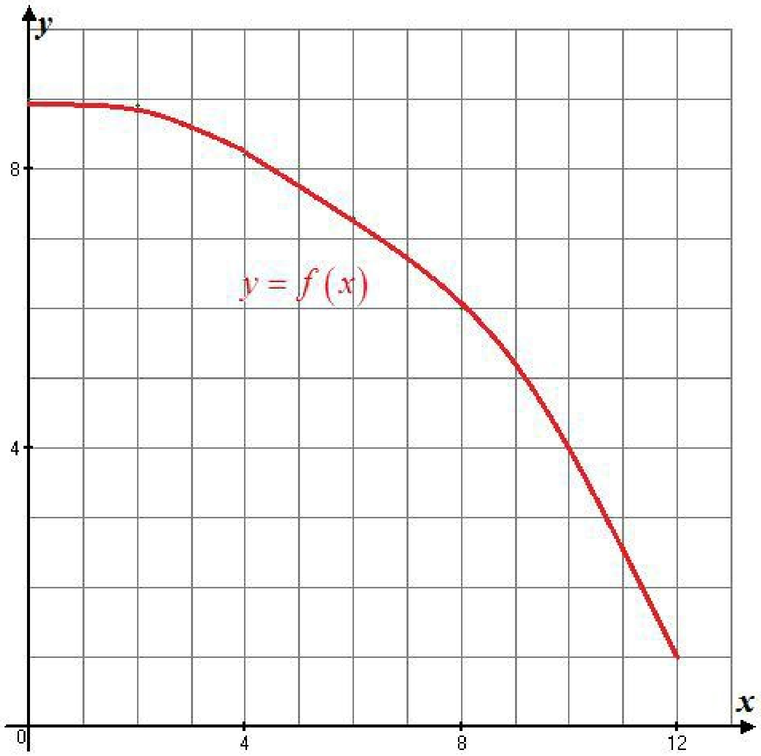
$$R_8 = \left[f(x_1)+f(x_2)+f(x_3)+...+f(x_7)+f(x_8)\right]\Delta x$$
$$= \left[f(1)+f(2)+f(3)+...+f(7)+f(8)\right]1$$
$$\approx 3+3.8+4.5+5+5.2+5.7+5.9+6$$
$$= 39.2$$

Therefore, upper estimate by drawing eight rectangles between $x=0$ and $x=8$ is,

$\boxed{R_8 \approx 39.2}$.

Answer 2E.

Consider the following graph:



Consider the interval $[a,b]=[0,12]$ and $n=6$

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{12-0}{6}$$
$$= 2$$

Since $(x_i = a + i\Delta x)$, thus the sample points are as follows:

$$x_0 = 0, x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 8, x_5 = 10, x_6 = 12$$

The corresponding function values from the above graph are as follows:

$$f(x_0)=9, f(x_1)=8.9, f(x_2)=8.2, f(x_3)=7.3, f(x_4)=6, f(x_5)=4, f(x_6)=1$$

(a) (i) Estimate the area using the left end points.

Here, $n = 6$.

Therefore, $L_6 = \sum_{i=0}^{5} f(x_i)\Delta x$

$= \Delta x\left[f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right]$

$= 2[9+8.9+8.2+7.3+6+4]$

$= 2[43.4]$

$= \boxed{86.8}$

(ii) Estimate the area using the right end points.

Here, $n = 6$.

Therefore, $R_6 = \sum_{i=1}^{6} f(x_i)\Delta x$

$= \Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)\right]$

$= 2[8.9+8.2+7.3+6+4+1]$

$= 2[35.4]$

$= \boxed{70.8}$

(iii) The midpoints are as follows:

$x_1^* = \frac{x_0+x_1}{2} = \frac{0+2}{2} = 1 \qquad x_4^* = \frac{x_3+x_4}{2} = \frac{6+8}{2} = 7$

$x_2^* = \frac{x_1+x_2}{2} = \frac{2+4}{2} = 3$ and $x_5^* = \frac{x_4+x_5}{2} = \frac{8+10}{2} = 9$

$x_3^* = \frac{x_2+x_3}{2} = \frac{4+6}{2} = 5 \qquad x_6^* = \frac{x_5+x_6}{2} = \frac{10+12}{2} = 11$

Now, estimate the area using these mid points.

Here, $n = 6$

Therefore, $M_6 = \sum_{i=1}^{6} f(x_i^*)\Delta x$

$= \Delta x\left[f(x_1^*)+f(x_2^*)+f(x_3^*)+f(x_4^*)+f(x_5^*)+f(x_6^*)\right]$

$= 2[9+8.7+7.9+6.7+5+3]$

$= 2[40.3]$

$= \boxed{80.6}$

(b) Consider part (a)- (i) and (ii). It is evident that $\boxed{L_6 > R_6}$

Thus, L_6 is an **Overestimate** of the true area.

(c) Consider part (a)- (i) and (ii). It is evident that $\boxed{L_6 > R_6}$

Thus R_6 is an **Underestimate** of the true area.

(d) In general, the Area for best estimation is the unique number that is smaller than all the upper sums and bigger than all the lower sums.

That is, $\boxed{70.8 < A < 80.6}$

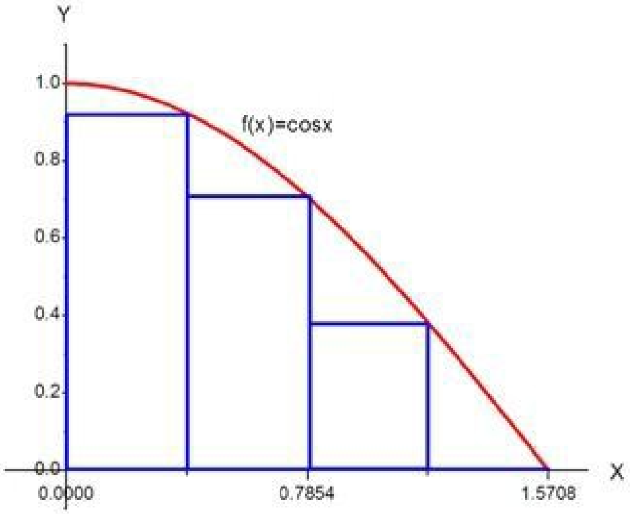
Consider part (a)-(i), (ii) and (iii) and obtain the following:

$\boxed{R_6 > M_6 > L_6}$

Thus, $\boxed{M_6}$ gives the **best estimate.**

Answer 3E.

f(x) = cos x , we graph this function in [0 , π/2] and use the right end points of the rectangles below the curve to estimate the area of the curve.



We divide the interval [0, π/2] in to 4 subintervals, then width of the subintervals is

$$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

And subintervals are

[0, π/8], [π/8, π/4], [π/4, 3π/8] and [3π/8, π/2]

Right end points are π/8, π/4, 3π/8 and π/2

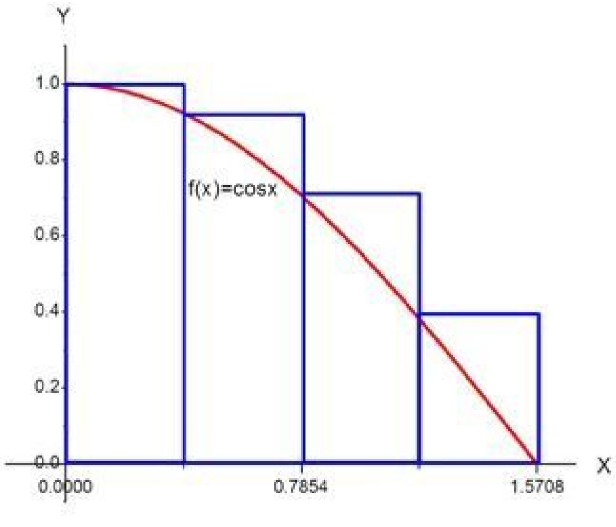
Then area

$$A \approx \Delta x[\cos(\pi/8) + \cos(\pi/4) + \cos(3\pi/8) + \cos(\pi/2)]$$

$= \frac{\pi}{8}\left[\cos(\pi/8)+\cos(\pi/4)+\cos(3\pi/8\right)+\cos\left(\pi/2\right)\right] \approx 0.7908$

This is an underestimate

(b)



We divide the interval [0, π/2] in to 4 subintervals, then width of the subintervals is

$$\Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$$

And subintervals are

We divide the interval [0, π/2] in to 4 subintervals, then width of the subintervals is

$$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

And subintervals are

[0, π/8], [π/8, π/4], [π/4, 3π/8] and [3π/8, π/2]

Left end points are 0, π/8, π/4, and 3π/8

Then area

$$A \approx \Delta x[\cos(0) + \cos(\pi/8) + \cos(\pi/4) + \cos(3\pi/8)]$$

$$= \frac{\pi}{8}\left[\cos(0) + \cos(\pi/8) + \cos(\pi/4) + \cos(3\pi/8)\right] \approx 1.1835$$

This is an overestimate

Answer 4E.

Consider the following function:

$f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$

(a)

The objective is to estimate the area under the graph, using four approximating rectangles and right end points and estimate the result is over estimate or under estimate.

Find the right end points and estimate it is an over estimate or it is an under estimate.

Since, $b = 4,\ a = 0,$ and $n = 4$

$$\begin{aligned}\Delta x &= \frac{b-a}{n}\\ &= \frac{4-0}{4}\\ \Delta x &= 1\end{aligned}$$

Use the formula $x_i = a + i\Delta x$ to obtain the end points are as follows:

$$\begin{aligned}x_0 &= a + 0\cdot\Delta x\\ &= 0 + 0\cdot 0\\ &= 0\\ x_1 &= a + 1\cdot\Delta x\\ &= 0 + 1\cdot 1\\ &= 1\\ x_2 &= a + 2\cdot\Delta x\\ &= 0 + 2\cdot 1\\ &= 2\end{aligned}$$

Continuing the above,

$$\begin{aligned} x_3 &= a + 3\cdot\Delta x \\ &= 0 + 3\cdot 1 \\ &= 3 \\ x_4 &= a + 4\cdot\Delta x \\ &= 0 + 4\cdot 1 \\ &= 4 \end{aligned}$$

Consider the intervals are,

$$(0,1),(1,2),(2,3), \text{ and } (3,4)$$

The right end points are,

$$1, 2, 3, \text{and } 4$$

Let R_4 be the sum of the areas of these approximating rectangles, then

Find the areas of the sum of the approximating rectangles formula is,

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \end{aligned}$$

Since $n = 4,$ and write the right end points of the rectangle is,

$$R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$\begin{aligned} R_4 &= f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x \\ &= \Delta x\left(f(1) + f(2) + f(3) + f(4)\right) \end{aligned}$$

$$\begin{aligned} &= (1)\left[\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}\right] \\ &= 1 + 1.4142 + 1.732 + 2 \\ &= 6.1462 \end{aligned}$$

Therefore, the area of the right end points are, $\boxed{6.1462}$.

Find the actual integral value.

Let $f(x)=\sqrt{x}$ from $x=0$ to $x=4$

Now evaluate the integral.

$$\int_{x=0}^{4}\sqrt{x}dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 \quad \left(\text{Use } \int\sqrt{x}dx=\frac{2}{3}x^{\frac{3}{2}}\right)$$

$$=\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$=\frac{2}{3}\left[4^{\frac{3}{2}}-0\right]$$

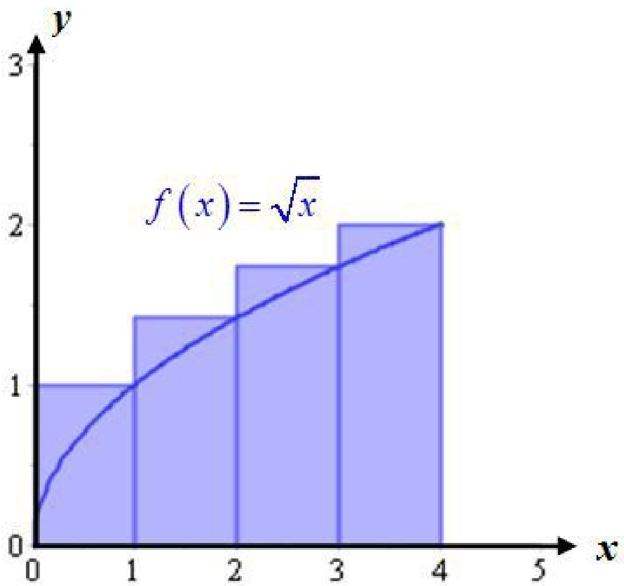
$$=\frac{2}{3}[8-0]$$

$$=\frac{16}{3}$$

$$=5.3333$$

By comparing both the right end points and the actual integral values, conclude that the estimation of the right end point is an overestimate the actual integral.

The sketch for the function $f(x)=\sqrt{x}$ using right end points is as below:



Clearly from the graph it is an overestimate.

(b)

Find the left end points and estimate it is an over estimate or it is an under estimate.

Find left end points, using the part (a).

The four rectangle points are,

$(0,1),(1,2),(2,3)$, and $(3,4)$

The left end points are,

1,2,3, and 4.

Let R_4 be the sum of the areas of these approximating rectangles, then

Find the areas of the sum of the approximating rectangles formula is,

$$R_n=f(x_1)\Delta x+f(x_2)\Delta x+...+f(x_n)\Delta x$$

Since $n=4,$ and write the left end points is,

Estimate the area using the left end points.

$$R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$R_4 = f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x$$
$$= \Delta x\left(f(0)+f(1)+f(2)+f(3)\right)$$
$$=(1)\left[\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}\right]$$
$$=0+1+1.4142+1.732$$
$$=4.1462$$

Therfore, the area of the left end points are, $\boxed{4.1462}$.

Find the actual integral value.

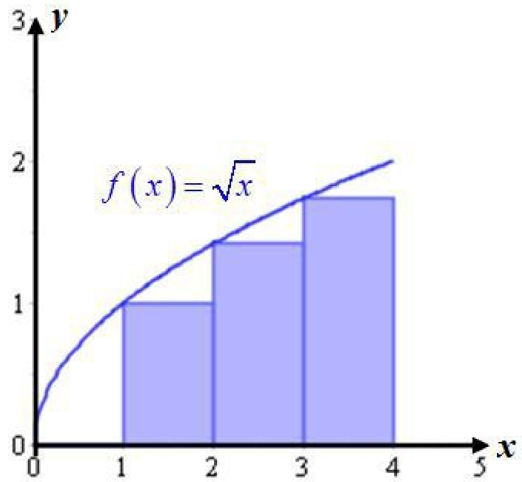
Let $f(x)=\sqrt{x}$ from $x=0$ to $x=4$

Now evaluate the integral.

$$\int_{x=0}^{4}\sqrt{x}dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 \quad \left(\text{Use } \int\sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}}\right)$$
$$=\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4$$
$$=\frac{2}{3}\left[4^{\frac{3}{2}}-0\right]$$
$$=\frac{2}{3}[8-0]$$
$$=\frac{16}{3}$$
$$=5.3333$$

By comparing both the left end points and the actual integral values, conclude that the estimation of the left end point is an underestimate the actual integral.

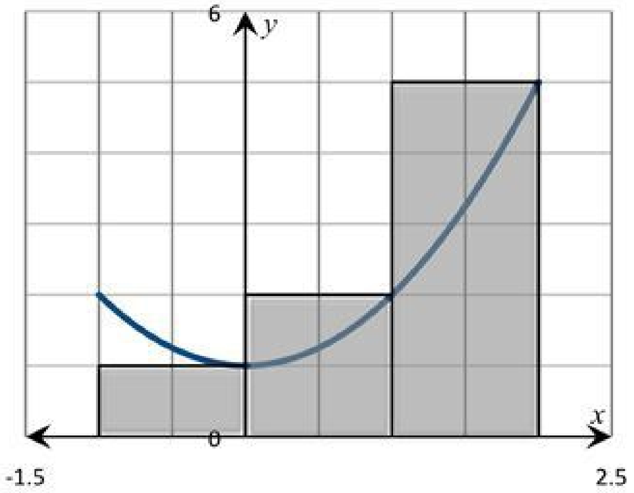
The sketch for the function $f(x)=\sqrt{x}$ using left end points is as below:



Clearly from the graph it is an underestimate.

Answer 5E.

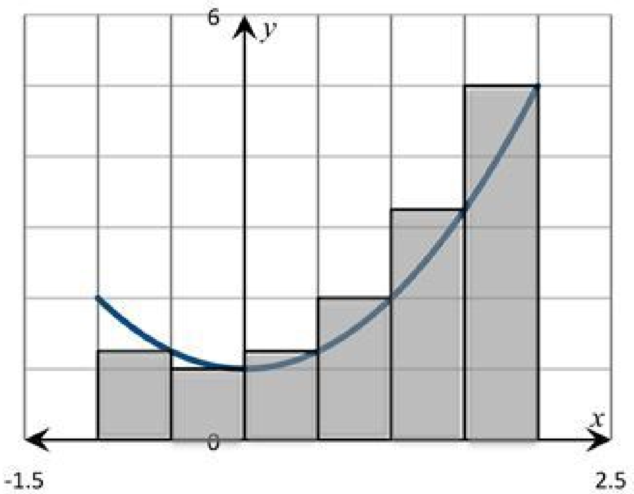
(a) Sketch the graph of $f(x)=1+x^2$ from $x=-1$ to $x=2$ with three approximating rectangles using right endpoints.



Each rectangle has width 1. The heights are 1, 2, and 5. If we let R_3 be the sum of the areas of these rectangles, we get

$$R_3 = 1\cdot 1 + 1\cdot 2 + 1\cdot 5 \\ = \boxed{8}$$

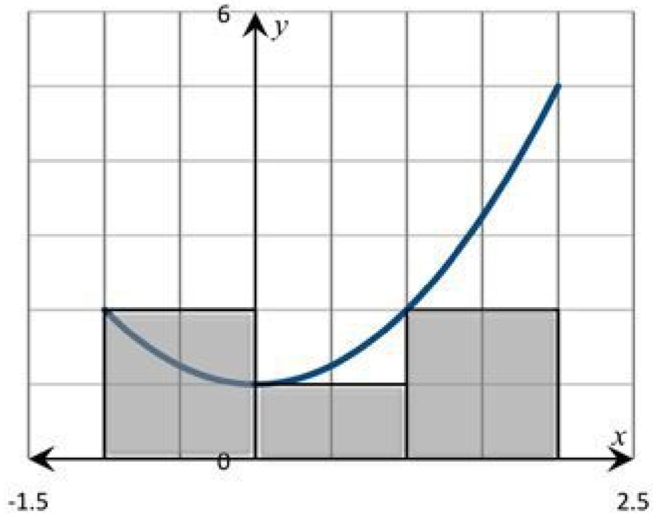
Sketch the graph of f with six approximating rectangles using right endpoints.



Each rectangle has width $\frac{1}{2}$. The heights are $\frac{5}{4}, 1, \frac{5}{4}, 2, \frac{13}{4}$ and 5, if we let R_6 be the sum of the areas of these rectangles, we get

$$R_6 = \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{13}{4} + \frac{1}{2}\cdot 5 \\ = \boxed{6.875}$$

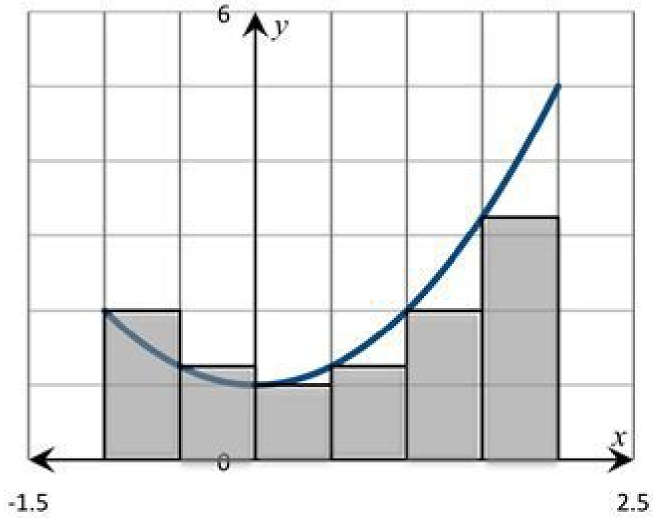
(b) Sketch the graph of f with three approximating rectangles using left endpoints.



Each rectangle has width 1. The heights are 2, 1, and 2. If we let L_3 be the sum of the areas of these rectangles, we get

$$\begin{aligned} L_3 &= 1\cdot 2 + 1\cdot 1 + 1\cdot 2 \\ &= \boxed{5} \end{aligned}$$

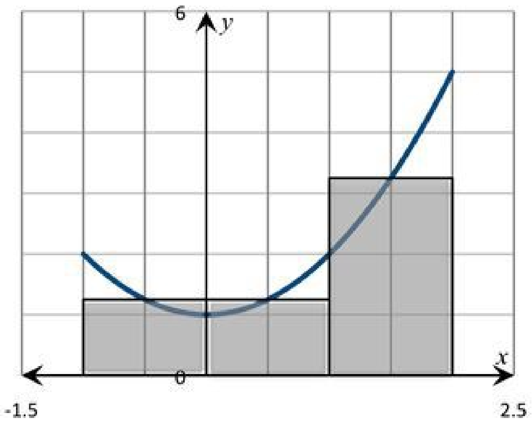
Sketch the graph of f with six approximating rectangles using left endpoints.



Each rectangle has width $\frac{1}{2}$. The heights are 2, $\frac{5}{4}$, 1, $\frac{5}{4}$, 2, $\frac{13}{4}$. If we let L_6 be the sum of the areas of these rectangles, we get

$$\begin{aligned} L_6 &= \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{5}{4} + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{13}{4} \\ &= \boxed{5.375} \end{aligned}$$

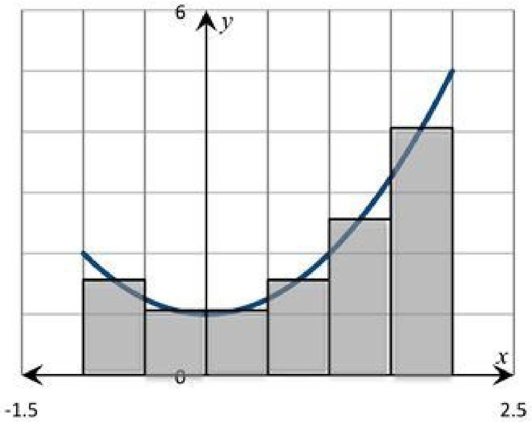
(c) Sketch the graph of f with three approximating rectangles using midpoints.



Each rectangle has width 1. The heights are $\frac{5}{4}$, $\frac{5}{4}$, and $\frac{13}{4}$. If we let M_3 be the sum of the areas of these rectangles, we get

$$M_3 = 1\cdot\frac{5}{4} + 1\cdot\frac{5}{4} + 1\cdot\frac{13}{4}$$
$$= \boxed{5.75}$$

Sketch the graph of *f* with six approximating rectangles using midpoints.



Each rectangle has width $\frac{1}{2}$. The heights are $\frac{25}{16}$, $\frac{17}{16}$, $\frac{17}{16}$, $\frac{25}{16}$, $\frac{41}{16}$, and $\frac{65}{16}$. If we let M_6 be the sum of the areas of these rectangles, we get

$$M_6 = \frac{1}{2}\cdot\frac{25}{16} + \frac{1}{2}\cdot\frac{17}{16} + \frac{1}{2}\cdot\frac{17}{16} + \frac{1}{2}\cdot\frac{25}{16} + \frac{1}{2}\cdot\frac{41}{16} + \frac{1}{2}\cdot\frac{65}{16}$$
$$= \boxed{5.9375}$$

(d) From the sketches, $\boxed{M_6}$ appears to be the best estimate.

Answer 6E.

(A)

We calculate the value of $f(x) = \frac{1}{(1+x^2)}$ in the interval [-2, 2] for different values of x

x	-2	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2
$f(x)$	0.2	0.31	0.5	0.8	1	0.8	0.5	0.31	0.2

With help of this data, we sketch the curve of $f(x)$ in figure 1

The figure is as follows:

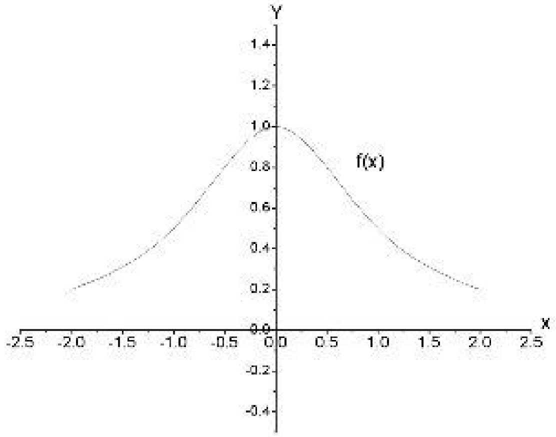


Fig.1

(B)

(1)

Now we divide the interval [-2, 2] in to four subintervals [-2, -1], [-1, 0], [0, 1], and [1, 2] and we sketch four rectangles whose base is same as width of the subinterval (= 1), and height is same as the right edge of the strip. [We already divide the interval [-2, 2] in to four strips for sketching approximating rectangle] [Figure 2].

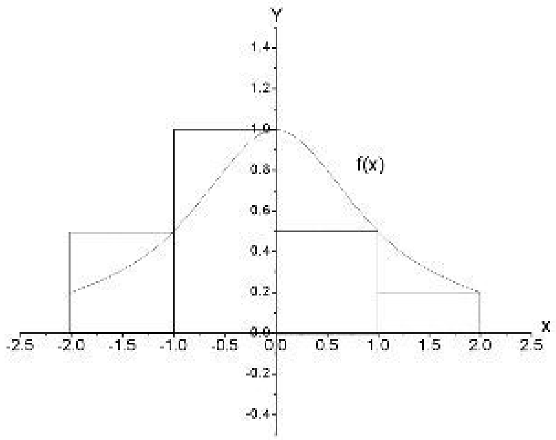


Fig.2

The sum of the area of these four approximating rectangles is

$$R_4 = 1.\left(\frac{1}{\left(1+(-1)^2\right)}\right)+1.\left(\frac{1}{\left(1+(0)^2\right)}\right)+1.\left(\frac{1}{\left(1+1^2\right)}\right)+1.\left(\frac{1}{\left(1-2^2\right)}\right)$$

Or $\quad R_4 = 1.\frac{1}{2}+1+1.\frac{1}{2}+1.\frac{1}{5}$

Or $\quad R_4 = 0.5+1+0.5+0.2 = 2.2$

So an estimate for the area under the graph of $f(x)$ is $\boxed{A \approx 2.2}$.

(2) Again we divide [-2, 2] in to subintervals [-2, -1], [-1, 0], [0, 1], and [1, 2]. The mid points of these intervals are

$$x_1^* = -1.5, \qquad x_2^* = -0.5, \qquad x_3^* = 0.5, \qquad x_4^* = 1.5$$

Now we sketch four approximating rectangles such that the midpoints of the upper edge of the rectangles are touching the graph of $f(x)$ (Figure 3).

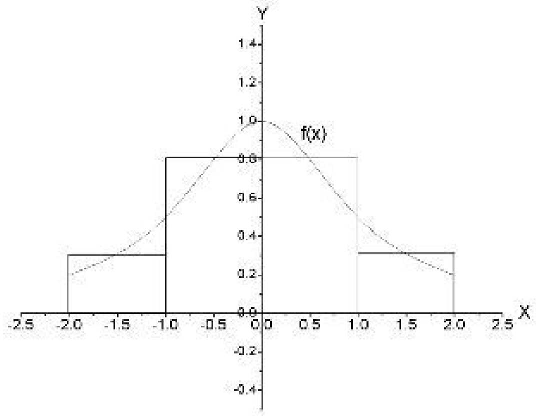


Fig.3

The sum of the area of the four approximating rectangles is

$$M_4 = \sum_{i=1}^{4} f\left(x_i^*\right).\Delta x,$$

where $\Delta x = 1$, the width of the subinterval or rectangles.

So $\quad M_4 = 1.f(-1.5) + 1.f(-0.5) + 1.f(0.5) + 1.f(1.5)$

Or $\quad M_4 \approx 0.31 + 0.8 + 0.8 + 0.31$

Or $\quad M_4 \approx 2.22$

So an estimate for the area under the graph of $f(x)$ is

$$\boxed{A \approx 2.22}$$

(C)

(1) For improving our estimation we divide the interval [-2, 2] to following 8 subintervals:
[-2, 1.5], [-1.5, -1], [-1, -0.5], [-0.5, 0], [0, 0.5], [0.5, 1.0], [1.0, 1.5], and [1.5, 2.0]
And we sketch 8 rectangles whose base is same as width of the subintervals (= 0.5) and height is same as the right edge of the strip [We already divide the interval [-2, 2] in the eight strips for sketching approximating rectangles] [Figure 4].

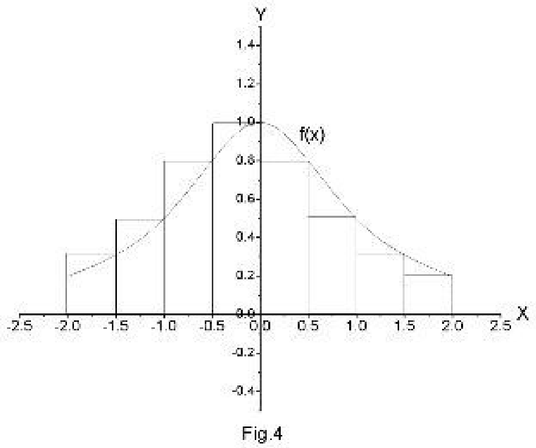


Fig.4

The sum of the areas of these eight rectangles is

$$R_8 = (0.5)\left(f(-1.5)\right) + (0.5).\left(f(-1)\right) + (0.5)f(-0.5) + (0.5).f(0) + (0.5).f(0.5) + (0.5).f(1.0) + (0.5).f(1.5) + 0.5.f(2.0)$$

Or

$$R_8 = (0.5).(0.31) + (0.5)(0.5) + (0.5).(0.8) + (0.5)(1.0) + (0.5).(0.8) + (0.5).(0.5) + (0.5.).(0.31) + (0.5).(0.2)$$

Or $\quad R_8 \approx 0.155 + 0.25 + 0.4 + 0.5 + 0.4 + 0.25 + 0.155 + 0.1$

Or $\quad R_8 \approx 2.21$

So an estimate for the area under the graph of $f(x)$ is

$$\boxed{A \approx 2.21}$$

(2) Again we divide the intervals [-2, 2] in to subintervals [-2, -1.5], [-1.5, -1], [-1, -0.5], [-0.5, 0], [0, 0.5], [0.5, 1], [1, 1.5], and [1.5, 2].
The midpoints of these intervals are

$x_1^* = -1.75,\quad x_2^* = -1.25,\quad x_3^* = -0.75,\quad x_4^* = -0.25,\quad x_5^* = 0.25,$

$x_7^* = 1.25,\quad$ and $\quad x_8^* = 1.75$

Now we sketch eight approximating rectangle such that the mid points of the upper edge (top) of the rectangles are touching the graph of $f(x)$ (Figure 5) is

$$M_8 = \sum_{i=1}^{8} f\left(x_i^*\right).\Delta x$$

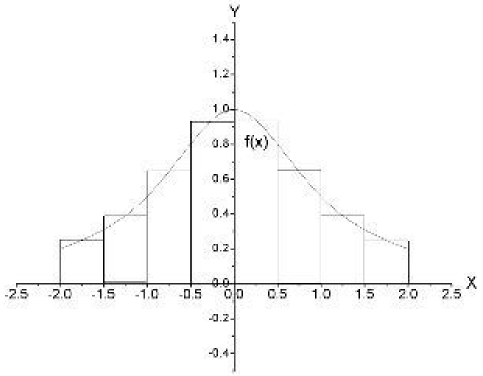


Fig.5

Here $\Delta x = 0.5$, the width of the subintervals or rectangles.
Or

$$M_4 = (0.5)f(-1.75)+(0.5)f(-1.25)+(0.5)f(-0.75)+(0.5)f(-0.25)+$$
$$(0.5)f(0.25)+(0.5)f(0.75)+(0.5)f(1.25)+(0.5)f(1.75)$$

Or

$$M_4 \approx (0.5)(0.246)+(0.5)(0.39)+(0.5)(0.64)+(0.5)(0.941)+$$
$$(0.5)(0.941)+(0.5)(0.64)+(0.5)(0.39)+(0.5)(0.246)$$

Or $\quad M_4 \approx 0.123+0.195+0.32+0.471+0.471+0.32+0.195+0.123$

Or $\quad M_4 \approx 2.218$

So an estimate for the area of under the graph of $f(x)$ is $\boxed{A \approx 2.218}$.

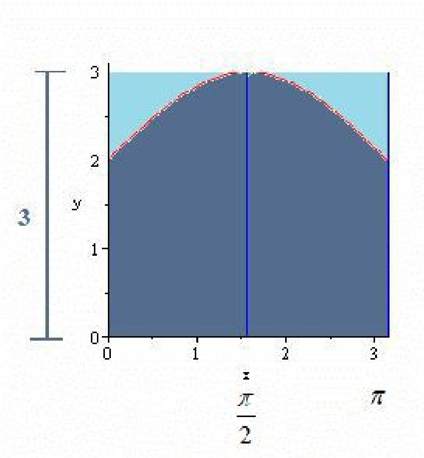
Answer 7E.

Given $f(x) = 2+\sin x, 0 \le x \le \pi$, $n = 2, 4$ and 8

Now $n = 2$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{2} = \frac{\pi}{2}$$

Upper Sum:
The graph is shown below.



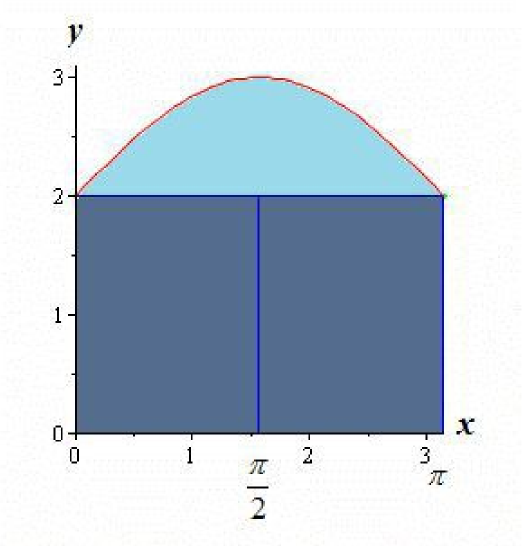
Now

$$f\left(\frac{\pi}{2}\right) = 2 + \sin\frac{\pi}{2}$$
$$= 2 + 1$$
$$= 3$$

The height of the function from the above graph is also 3.
Thus the upper sum value is

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \Delta x\left[f(x_1) + f(x_2)\right]$$
$$= \frac{\pi}{2}[3+3]$$
$$= \frac{\pi}{2}[6]$$
$$= 3\pi$$
$$= 9.42$$

Lower Sum:
The graph is shown below.



Now

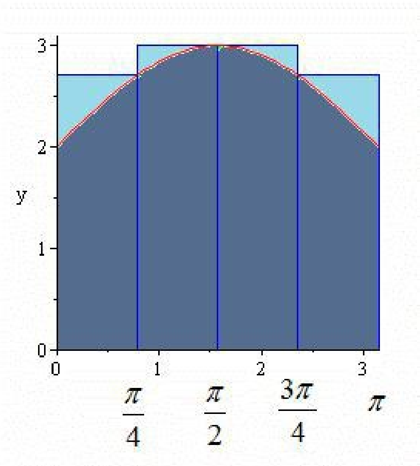
$$f(0) = 2 + \sin 0$$
$$= 2 + 0$$
$$= 2$$

The height of the function from the above graph is also 2.
Thus the lower sum value is

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \Delta x\left[f(x_1) + f(x_2)\right]$$

$$= \frac{\pi}{2}[2+2]$$
$$= \frac{\pi}{2}[4]$$
$$= 2\pi$$
$$= 6.28$$

Now $n = 4$
Upper Sum:
The graph is shown below.



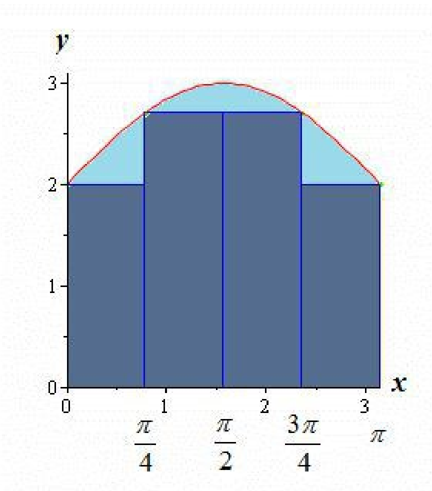
Now

$$f\left(\frac{\pi}{4}\right) = 2 + \sin\frac{\pi}{4}$$
$$= 2 + \frac{1}{\sqrt{2}}$$
$$= 2.7071$$
$$f\left(\frac{\pi}{2}\right) = 2 + \sin\frac{\pi}{2}$$
$$= 2 + 1$$
$$= 3$$

Thus the upper sum value is

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4)\right]$$
$$= \frac{\pi}{4}\left[2 + \frac{1}{\sqrt{2}} + 3 + 2 + \frac{1}{\sqrt{2}} + 3\right]$$
$$= \frac{\pi}{4}\left[10 + \sqrt{2}\right]$$
$$= 8.96$$

Lower Sum:
The graph is shown below.



Now

$$f(0) = 2+\sin 0$$
$$= 2+0$$
$$= 2$$
$$f\left(\frac{\pi}{4}\right) = 2+\sin\frac{\pi}{4}$$
$$= 2+\frac{1}{\sqrt{2}}$$

Thus the upper sum value is

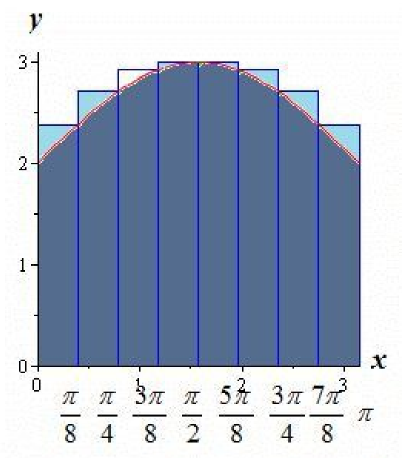
$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)\right]$$
$$= \frac{\pi}{4}\left[2+\left(2+\frac{1}{\sqrt{2}}\right)+\left(2+\frac{1}{\sqrt{2}}\right)+2\right]$$
$$= \frac{\pi}{4}\left[8+\sqrt{2}\right]$$
$$= 7.39$$

Now n= 8

Upper sum:

The graph is shown below.



Now

$$f\left(\frac{\pi}{8}\right) = 2+\sin\frac{\pi}{8}$$
$$= 2+\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$
$$= \frac{4+\sqrt{2-\sqrt{2}}}{2}$$
$$f\left(\frac{\pi}{4}\right) = 2+\sin\frac{\pi}{4}$$
$$= 2+\left(\frac{1}{\sqrt{2}}\right)$$
$$= 2+\frac{1}{\sqrt{2}}$$

Also

$$f\left(\frac{3\pi}{8}\right)=2+\sin\frac{3\pi}{8}$$
$$=2+\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)$$
$$=\frac{4+\sqrt{2+\sqrt{2}}}{2}$$

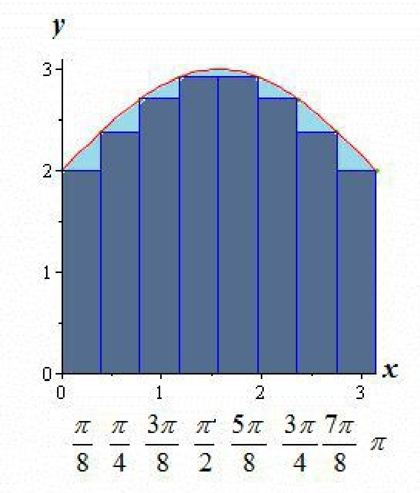
$$f\left(\frac{\pi}{2}\right)=2+\sin\frac{\pi}{2}$$
$$=2+(1)$$
$$=3$$

Thus the upper sum value is

$$=\sum_{i=1}^{n} f(x_i)\Delta x$$

$$=\Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)+f(x_8)\right]$$
$$=\frac{\pi}{8}\left[\frac{4+\sqrt{2-\sqrt{2}}}{2}+\left(2+\frac{1}{\sqrt{2}}\right)+\frac{4+\sqrt{2+\sqrt{2}}}{2}+3+3+\frac{4+\sqrt{2+\sqrt{2}}}{2}+\left(2+\frac{1}{\sqrt{2}}\right)+\frac{4+\sqrt{2-\sqrt{2}}}{2}\right]$$
$$=8\left[18+\sqrt{2}+\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}\right]$$
$$=8.65$$

Lower Sum
The graph is shown below.



Now

$$f(0)=2+\sin 0$$
$$=2+0$$
$$=2$$

$$f\left(\frac{\pi}{8}\right)=2+\sin\frac{\pi}{8}$$
$$=2+\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$
$$=\frac{4+\sqrt{2-\sqrt{2}}}{2}$$

Also

$$f\left(\frac{\pi}{4}\right)=2+\sin\frac{\pi}{4}$$
$$=2+\left(\frac{1}{\sqrt{2}}\right)$$
$$=2+\frac{1}{\sqrt{2}}$$

$$f\left(\frac{3\pi}{8}\right)=2+\sin\frac{3\pi}{8}$$
$$=2+\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)$$
$$=\frac{4+\sqrt{2+\sqrt{2}}}{2}$$

Thus the lower sum value is

$$=\sum_{i=1}^{n} f(x_i)\Delta x$$

$$=\Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)+f(x_8)\right]$$
$$=\frac{\pi}{8}\left[2+\frac{4+\sqrt{2-\sqrt{2}}}{2}+\left(2+\frac{1}{\sqrt{2}}\right)+\frac{4+\sqrt{2+\sqrt{2}}}{2}+\frac{4+\sqrt{2+\sqrt{2}}}{2}+\left(2+\frac{1}{\sqrt{2}}\right)+\frac{4+\sqrt{2-\sqrt{2}}}{2}+2\right]$$
$$=\frac{\pi}{8}\left[16+\sqrt{2}+\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}\right]$$
$$=7.86$$

Answer 8E.

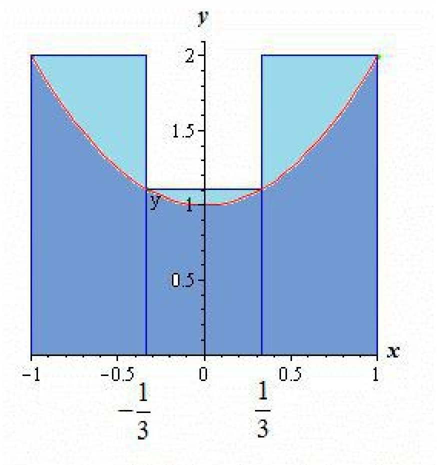
Given $f(x)=1+x^2, -1\le x\le 1$, $n=3,4$

(a) Now $n=3$

Upper Sum:

$$\Delta x=\frac{1-(-1)}{3}$$
$$=\frac{2}{3}$$

The graph is shown below.



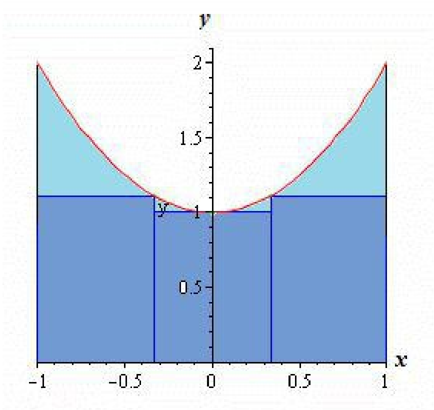
The values of

$$x_0 = -1$$
$$f(x_0) = 1 + (-1)^2$$
$$= 2$$
$$x_1 = a + \Delta x$$
$$= -1 + \frac{2}{3}$$
$$= \frac{-1}{3}$$
$$f(x_1) = f\left(\frac{-1}{3}\right)$$
$$= 1 + \left(\frac{-1}{3}\right)^2$$
$$= \frac{10}{9}$$
$$f(1) = 1 + (1)^2$$
$$= 2$$

Thus Upper sum is

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{2}{3}\left[2 + \frac{10}{9} + 2\right]$$
$$= \frac{2}{3}\left[\frac{36+10}{9}\right]$$
$$= \frac{2}{3}\left[\frac{46}{9}\right]$$
$$= \frac{92}{27}$$

Lower Sum:
The graph is shown below.



Now from the graph we have
Lower Sum

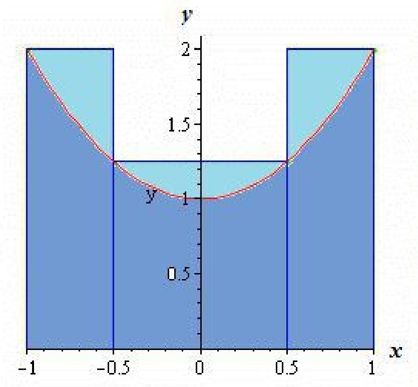
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x$$
$$= \frac{2}{3}\left[\frac{10}{9}+1+\frac{10}{9}\right]$$
$$= \frac{2}{3}\left[\frac{20+9}{9}\right]$$
$$= \frac{2}{3}\left[\frac{29}{9}\right]$$
$$= \frac{58}{27}$$
$$= 2.148$$

Therefore Riemann lower sum $\boxed{2.148}$

(b) Now when $n=4$

$$\Delta x = \frac{1-(-1)}{4}$$
$$= \frac{2}{4}$$
$$= \frac{1}{2}$$

The graph is shown below.



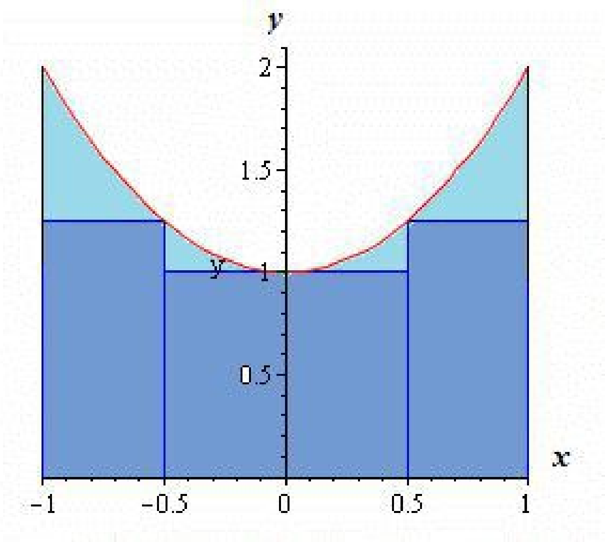
From the graph we have

$$f(-1) = 1+(-1)^2$$
$$= 2$$
$$f(-0.5) = 1+\left(\frac{1}{4}\right)$$
$$= \frac{5}{4}$$
$$f(0.5) = 1+\left(\frac{1}{4}\right)$$
$$= \frac{5}{4}$$

Thus Upper sum is

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{1}{2}\left[\frac{5}{4}+2+\frac{5}{4}+2\right]$$
$$= \frac{1}{2}\left[\frac{5}{2}+4\right]$$
$$= \frac{1}{2}\left[\frac{13}{2}\right]$$
$$= \frac{13}{4}$$
$$= 3.25$$

Lower Sum:
The graph is shown below.



From the graph shown below we have
Lower Sum:

$$= \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$\begin{aligned}
&= \frac{1}{2}\left[1+\frac{5}{4}+1+\frac{5}{4}\right] \\
&= \frac{1}{2}\left[\frac{5}{2}+2\right] \\
&= \frac{1}{2}\left[\frac{9}{2}\right] \\
&= \frac{9}{4} \\
&= 2.25
\end{aligned}$$

Therefore Riemann lower sum is $\boxed{2.25}$

Answer 9E.

f (x) = x 4 on [0 , 1].

f is an increasing function on [0,1]. consider 10 subintervals on [0 ,1] using n = 10, we get

Δx = 0.1 and f (x) = f (0 + kΔ) where k varies from 1 to 10.

then $\int_0^1 x^4 \, dx = \sum_{k=1}^{10} \Delta x * f(0 + k\Delta x) = 0.1 * \sum_{k=1}^{10} (k * \Delta x)^4 =$

$$0.1 * \left(0.1^4\right) * \sum_{k=1}^{10} k^4 = (0.1)^5 \left\{ \frac{6n^5 + 15n^4 + 10n^3 - n}{30} \right\}_{n=1}^{10}$$

$$= (0.1)^5 \left\{ \frac{6 * 10^5 + 15 * 10^4 + 10 * 10^3 - 10}{30} \right\} = 0.25333$$

----

putting n = 30 , the length of the subinterval Δx becomes 0.033 and the above integrand

becomes $= (0.033)^5 \left\{ \frac{6 * 30^5 + 15 * 30^4 + 10 * 30^3 - 30}{30} \right\} = 0.2064$

----

putting n = 50 , Δx becomes 1/50 = 0.02 , so, the integrand becomes

$$= (0.02)^5 \left\{ \frac{6 * 50^5 + 15 * 50^4 + 10 * 50^3 - 50}{30} \right\}$$

= 0.21013

putting n = 100 , Δx = 0.01 , the integrand becomes 0.20503.

continuing in the same way the integrand becomes 0.2.

Answer 10E.

f(x) = cos x in [0 , π/2].

putting n = 10, consider a uniform partition on the interval to give the length of the subinterval Δx = π/20 = 0.157142 , the integration of the given function on the given interval is

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = \sum_{k=1}^{10} \Delta x * f(0+\Delta x) = \Delta x * \sum_{k=1}^{10} \cos(k\Delta x)$$

using the computer programming to find this summation , we get

0.157142*5.850257 = 0.9193261

----

putting n = 30, Δx becomes 0.05238095 and the above integrand becomes

$$= \Delta x * \sum_{k=1}^{30} \cos(k\Delta x)$$ = 18.58622 *0.05238095 =0.973563.

----

putting n = 50 , Δx becomes 0.031428571 and the integrand becomes

$$= \Delta x * \sum_{k=1}^{50} \cos(k\Delta x)$$ = 31.31524 *0.031428571 = 0.984193.

----

putting n = 100, Δx becomes 0.0157142 and the integrand is $= \Delta x * \sum_{k=1}^{100} \cos(k\Delta x)$

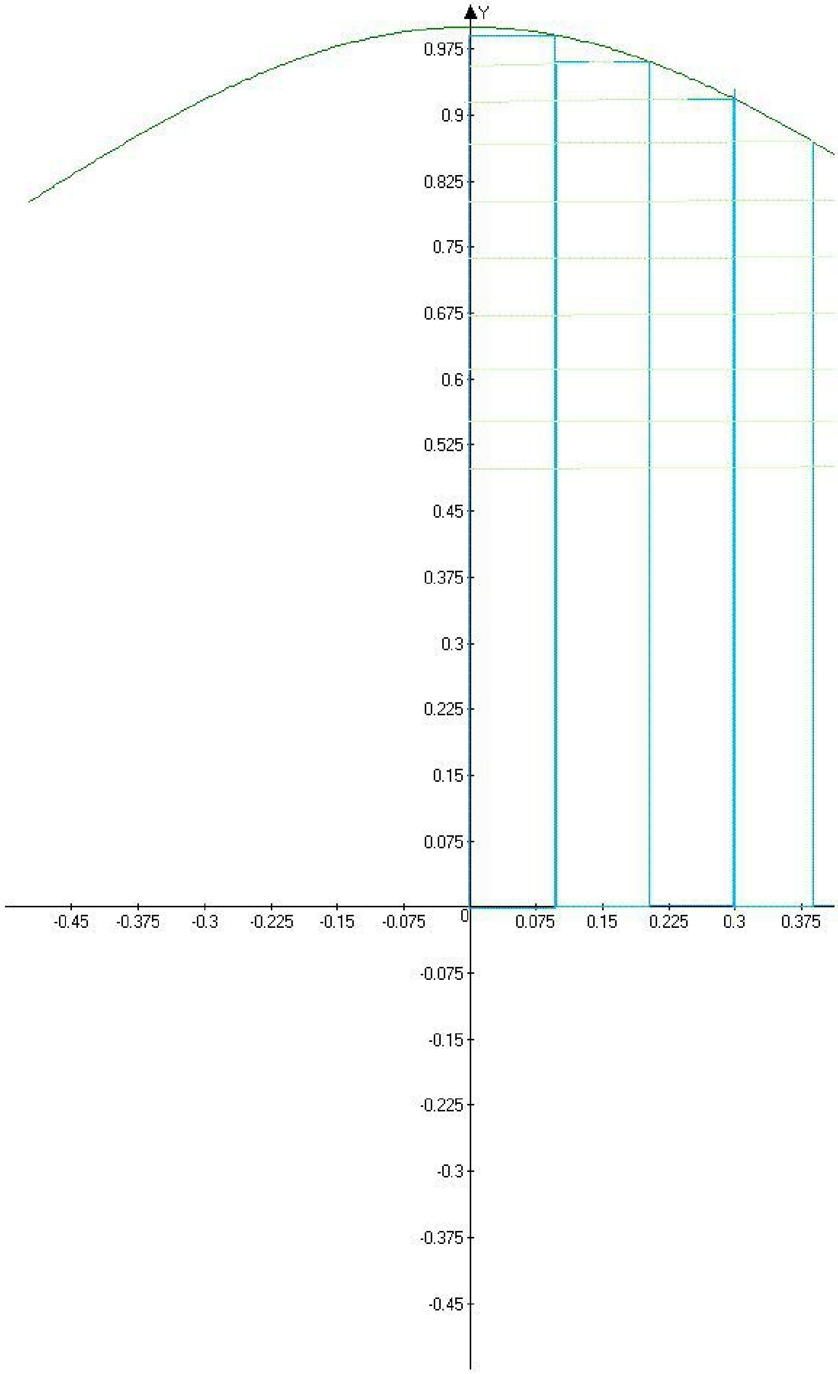
= 63.13508*0.0157142 = 0.992117

note that as n becomes larger , the integrand reaches 1.

Answer 11E.

$f\left(x\right) = \frac{1}{1+x^2}$ on [0 ,1] , the integrand to be evaluated using n rectangles below the curve whose left end points touch the curve .

putting n = 10 ,



from this , the integrand is 0.1{ 0.99182 + 0.95512+0.92134+ 0.87 + 0. 8 + 0.71+ 0.67 + 0.62 + 0.54 +0.48} = 0.675828

similarly , when n = 30 , the integrand becomes 0.721312

when n = 50 , it is 0.745535

when n = 100 , it is 0.761215.

Answer 12E.

Given

f(x)=x/(x+2) ; 1≤ x ≤4

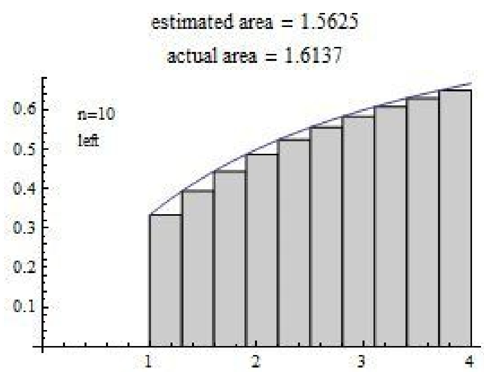
a)

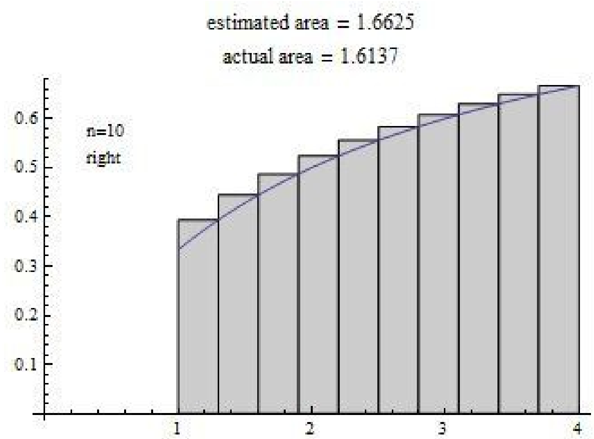
Using the commands

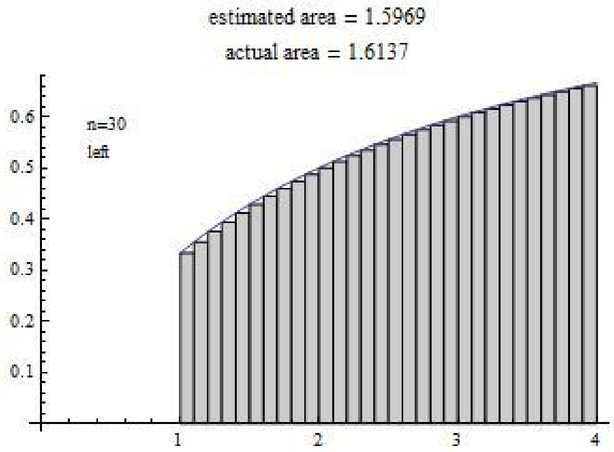
n	Left	Right
10	1.5625	1.6625
30	1.5969	1.6302
50	1.6037	1.6237

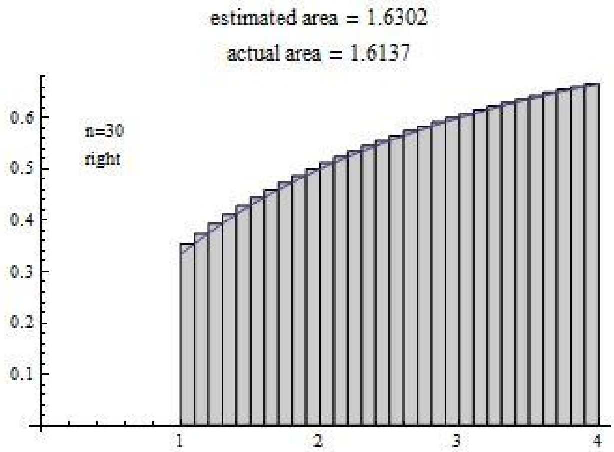
b)

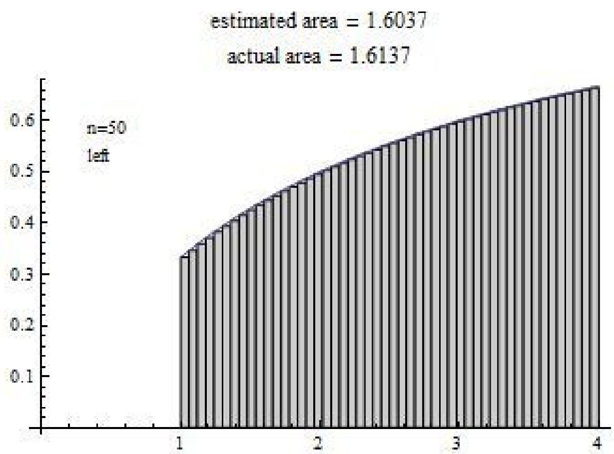
Using CAS

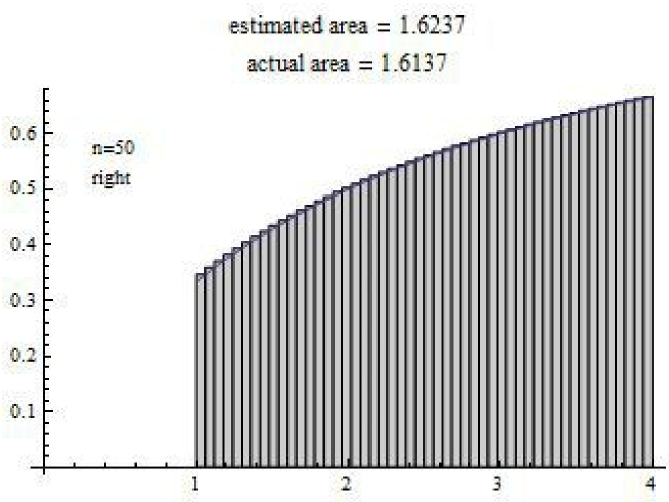





estimated area = 1.5969
actual area = 1.6137
n=30
left
0.6
0.5
0.4
0.3
0.2
0.1
1
2
3
4


estimated area = 1.6302
actual area = 1.6137
n=30
right
0.6
0.5
0.4
0.3
0.2
0.1
1
2
3
4


estimated area = 1.6037
actual area = 1.6137
n=50
left
0.6
0.5
0.4
0.3
0.2
0.1
1
2
3
4



c)

Since f(x) is an increasing function on [1,4] all of the left sum are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with n=50 is about 1.6037 and the right sum with n=50 is about 1.6237

So we conclude that

$1.603 < L50 \leq \text{actual area} \leq R50 < 1.624$

$1.603 < \text{actual area} < 1.624$

The actual area is between 1.603 and 1.624

Answer 13E.

Consider the data for the speed of a runner in half-second intervals.

$t(\text{s})$	$v(\text{ft/s})$
0	0
0.5	6.2
1.0	10.8
1.5	14.9
2.0	18.1
2.5	19.4
3.0	20.2

Take the subintervals, $[0,0.5],[0.5,1.0].[1.0,1.5],[1.5,2.0],[2.0,2.5],[2.5,3.0].$, each of length $0.5.$

During first 0.5 sec, the velocity does not change much, so estimate the distance travelled during that time.

Take the velocity during that time interval to be the initial velocity (0.5 ft/s).

The distance travelled during first 0.5 seconds:

$$0.5\times 0.5 = 0.25 \text{ ft.}$$

Similarly during the second time interval, the velocity is 6.2 ft/s and the distance travelled from $t = 0.5\text{s to } t = 1.0 \text{ s}$ is $6.2\times 0.5 = 3.1 \text{ ft.}$

The lower estimate is obtained by considering the left end points of the intervals.

That is $0,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5.$

The velocities corresponding to the left end points are:

$$0,\ 6.2,\ 10.8,\ 14.9,\ 18.1,\ 19.4.$$

The lower estimate of the distance travelled in 3 seconds:

$$\begin{aligned}&0\times 0.5+6.2\times 0.5+10.8\times 0.5+14.9\times 0.5+18.1\times 0.5+19.4\times 0.5\\&=0+3.1+5.4+7.45+9.05+9.7\\&=34.7\end{aligned}$$

Therefore, the lower estimate of the data is $\boxed{34.7 \text{ ft}}$.

The lower estimate is obtained by considering the right end points of the intervals.

That is $0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0.$

The velocities corresponding to the right end points are:

$$6.2,\ 10.8,\ 14.9,\ 18.1,\ 19.4,\ 20.2.$$

The upper estimate of the distance travelled in 3 seconds:

$$\begin{aligned}&6.2\times 0.5+10.8\times 0.5+14.9\times 0.5+18.1\times 0.5+19.4\times 0.5+20.2\times 0.5\\&=3.1+5.4+7.45+9.05+9.7+10.1\\&=44.8\end{aligned}$$

Therefore, the upper estimate of the data is $\boxed{44.8 \text{ ft}}$.

Answer 14E.

Then given data is

$t(S)$	0	12	24	26	48	60
$V(ft/S)$	30	28	25	22	24	27

(A)

Here five time intervals are given as [0, 12], [12, 24], [24, 36], [36, 48] and [48, 60]

Left end points of the interval are 0, 12, 24, 36 and 48

And width of the intervals is = 12

So the distance traveled by the motorcycle during this time period (using the velocity at the beginning of the time intervals) can be estimated as

[We used distance = velocity × time]

$$d = 12\times 30 + 12\times 28 + 12\times 25 + 12\times 22 + 12\times 24$$

Or $\boxed{d = 1548 \text{ feet}}$

[Velocities at left end points of time interval are 30, 28, 25, 22, and 24]

(B) Now the right end points of the intervals are 12, 24, 36, 48 and 60

And the velocities at these points are 28, 25, 22, 24, and 27

So w can estimate the distance traveled in the time period [0, 60] as

$$d = 12\times 28 + 12\times 25 + 12\times 22 + 12\times 24 + 12\times 27$$

Or $\boxed{d = 1512 \text{ feet}}$

(C) The estimates are neither lower nor upper estimates because the velocity function V is not monotonic means it is neither increasing nor decreasing.

Answer 15E.

Then given data is

$t(h)$	0	2	4	6	8	10
$(L/h)\ r(t)$	8.7	7.6	6.8	6.2	5.7	5.3

Here $r(t)$ is rate of leaking out the oil from a tank in liters/hour.

During first 2 hours the rate of leaking is 7.6 L/h.

So amount of oil leaked out during first two hours = time × rate = $2\times 7.6 = 15.2$ L.

Now we have 5 time intervals as [0, 2], [2, 4], [4, 6], [6, 8], and [8, 10].

So width of intervals is 2.

So left end points of the intervals are 0, 2, 4, 6, and 8.

And the rates of leaking at these points are 8.7, 7.6, 6.8, 6.2, and 5.7.

So we can estimate the amount of oil as

Amount of oil $= 2\times 8.7 + 2\times 7.6 + 2\times 6.8 + 2\times 6.2 + 2\times 5.7 \boxed{= 70 \text{ liters}}$

For confirming that this estimation is upper or lower, we sketch the curve of $r(t)$ and approximating rectangles such as left upper vertexes of the rectangles touch the graph of $r(t)$ [figure 1].

From figure 1 we see that the arc of region under the graph of $r(t)$ is less than the sum of areas of these given rectangles.

So our estimation = 70 L is upper estimation.

The figure is as follows:

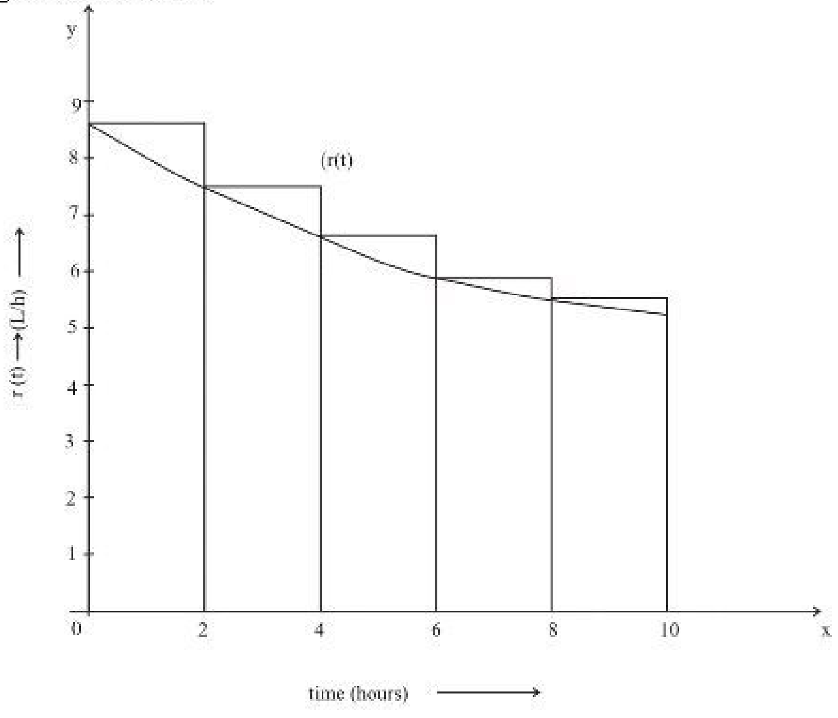


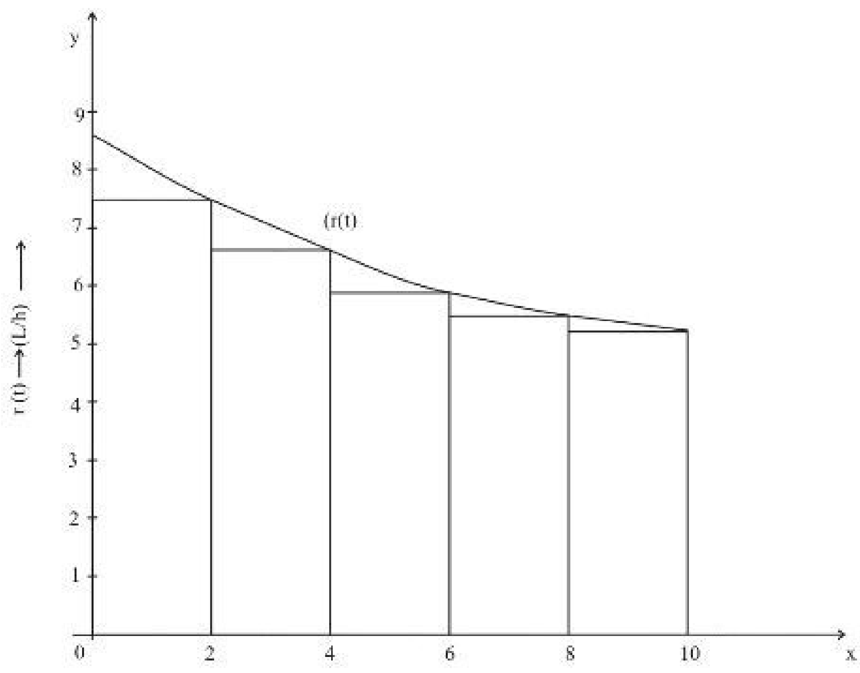
Figure 1

Now the right end point of the intervals are 2, 4, 6, 8 and 10
And rates of leaking at these points are 7.6, 6.8, 6.2, 5.7, 5.3
So we can estimate the amount of oil as

Amount of oil $= 2\times 7.6 + 2\times 6.8 + 2\times 6.2 + 2\times 5.7 + 2\times 5.3$

Or Amount of oil $= 63.2$ liters

If we sketch the graph of $r(t)$ and the approximating rectangles such that the upper right vertex of the rectangles are touching the graph of $r(t)$ in figure 2, then we see that this estimation of the amount of oil is lower estimate $\boxed{= 63.2L}$



Answer 16E.

The given date is

$T(s)$	0	10	15	20	32	59	62	125
$v(ft/s)$	0	185	319	447	742	1325	1445	4151

Here time intervals are not equally spaced. So we have to calculate the width of each time interval separately.

$\Delta t_1 = 10 - 0 = 10$

$\Delta t_2 = 15 - 10 = 5$

$\Delta t_3 = 20 - 15 = 5$

$\Delta t_4 = 32 - 20 = 12$

$\Delta t_5 = 59 - 32 = 27$

$\Delta t_6 = 62 - 59 = 3$

$\Delta t_7 = 125 - 62 = 63$

Now we have the time sub-intervals [0, 10], [10, 15], [15, 20], [20, 32], [32, 59] [59, 62], and [125, 62].
And left end points of intervals are 0, 10, 15, 20, 32, 59, and 12.
And velocities at these points are 0, 185, 319, 447, 742, 1325, and 1445 ft/s.

Now we sketch the curve of $V(t)$ with the help of given data and sketch the approximating rectangles whose width is equal to the width of sub-intervals. The left upper vertex of the rectangles touches the graph of $V(t)$, so the area of these rectangles $= \sum_{i=1}^{7} t_i \left(V(t_i)\right)$ (figure 1)

The graph is as follows:

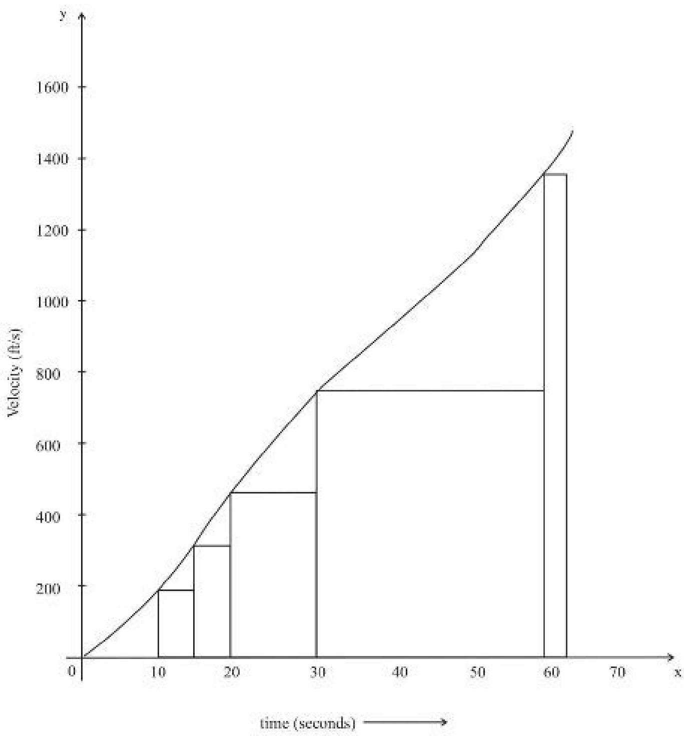


Figure 1

Now using the formula

$$\text{Distance} = \text{Time} \times \text{Velocity} = \Delta t_i \times V(t_i)$$

We obtain a lower estimate for the distance by adding the areas of all rectangles in the interval [0, 62].

$$h(\text{height}) = V(t_1)\Delta t_1 + V(t_2)\Delta t_2 + V(t_3)\Delta t_3 + V(t_4)\Delta t_4 + V(t_5)\Delta t_5 + V(t_6)\Delta t_6$$

$$= 0 + 185\times 5 + 319\times 5 + 447\times 12 + 742\times 27 + 1325\times 3$$

$\boxed{h = 31893}$ feet

Similarly we sketch the curve of $f(t)$ and rectangles whose width is equal to the width of sub-intervals and right upper vertexes of the rectangles touch the graph in (figure 2).

We obtain an upper estimate for the distance by adding the areas of these rectangles.

So $\quad h(\text{height}) = 185\times10+319\times5+447\times5+742\times12+1325\times27+1445\times3$

$$\boxed{h = 54694 \text{ feet}}$$

Now we can approximate the height by taking the average of these two estimations, so height of space shuttle from the earth. After 62 seconds is

$$\boxed{h \approx 43293.5 \text{ feet}}$$

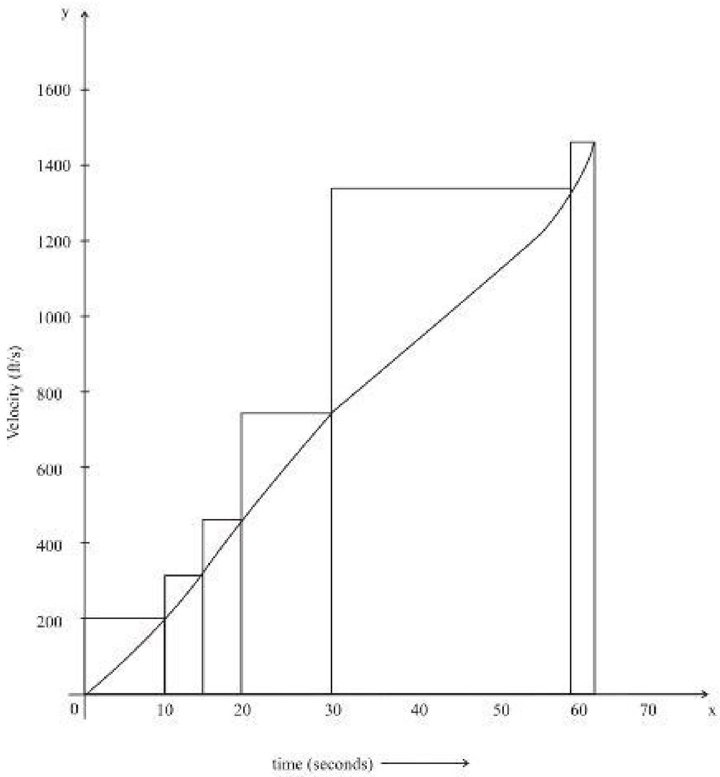
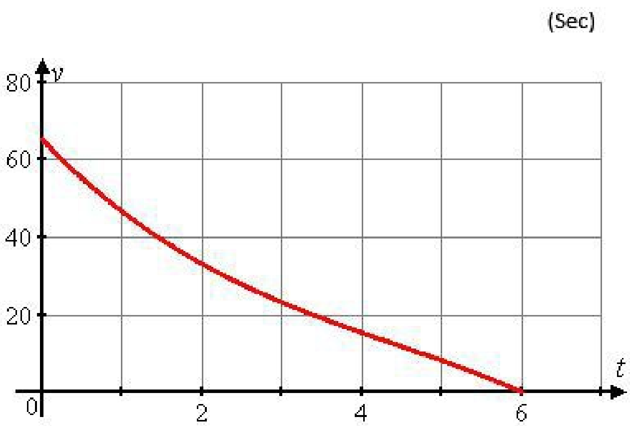


Figure 2

Answer 17E.

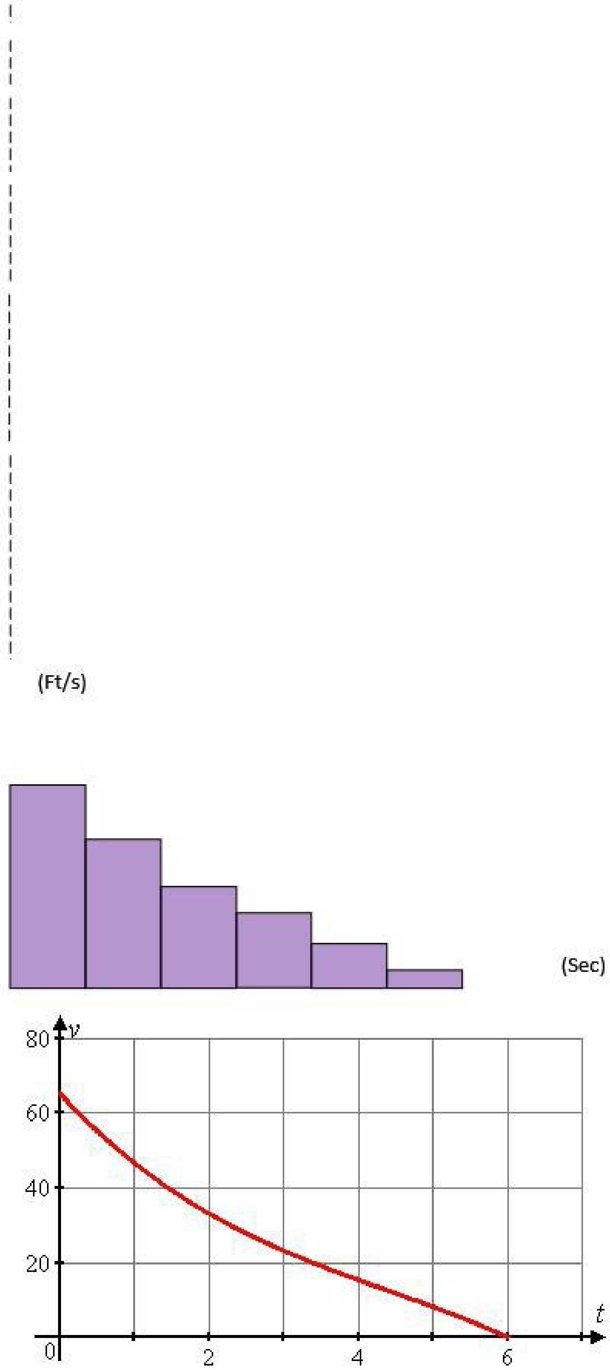
Evaluate the distance travelled by the car using the following graph:

(Ft/s)



The more rectangles used, better is the estimation of the distance traveled.

In this case, use midpoints to estimate the graph using six rectangles as shown below:



Find the area of the each rectangle using the formula $r_n = wh$, (width multiplied by height).

From the figure, it is observed that the all the rectangle have same width 1 unit.

$$r_1 = 1(55)$$
$$r_2 = 1(40)$$
$$r_3 = 1(25)$$
$$r_4 = 1(20)$$
$$r_5 = 1(10)$$
$$r_6 = 1(5)$$

Find the sum of the areas of the rectangles.

$$\begin{aligned} r_{\text{total}} &= 55+40+25+20+10+5 \\ &= 155 \end{aligned}$$

Hence, the area covered by the estimated rectangles is 155 square units.

This follows that, the distance travelled by the car is **155 km**.

Answer 18E.

Here for an increasing function, using left end points gives us an under-estimate and using right endpoints results in an over-estimate.
Here we will calculate M_6 to get an estimate.

$$\Delta t = \frac{30-0}{6}$$
$$\Delta t = 5\text{s}$$
$$\Delta t = \frac{5}{3600} h$$
$$\Delta t = \frac{1}{720} h$$

$$M_6 = \frac{1}{720}\left[v(2.5)+v(7.5)+v(12.5)+v(17.5)+v(22.5)+v(27.5)\right]$$
$$= \frac{1}{720}\left[31.25+66+88+103.5+113.75+119.25\right]$$
$$= \frac{1}{720}(521.75)$$
$$\approx 0.725 \text{ km}$$

Hence, the distance traveled by the car is ≈ 0.725 km.

Definition: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x\right]$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

Given $f(x) = \frac{2x}{x^2+1}$, $1 \le x \le 3$

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

Answer 19E.

Definition: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x\right]$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

Given $f(x) = \frac{2x}{x^2+1}$, $1 \le x \le 3$

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

The values of

$$x_0 = a = 1,\quad x_1 = a + \Delta x = 1 + \frac{2}{n},$$

$$x_2 = a + 2\Delta x = 1 + 2\cdot\frac{2}{n}$$

$$\Rightarrow x_2 = 1 + \frac{4}{n}\ ,\ldots\ \textit{and so on}$$

$$\therefore\ x_i = a + i\Delta x = 1 + i.\frac{2}{n} = 1 + \frac{2i}{n}$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n}$$

$$= \sum_{i=1}^{n} \frac{2\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{2i}{n}\right)^2 + 1}.\frac{2}{n}$$

Further,

$$R_n = \frac{4}{n}\sum_{i=1}^{n} \frac{\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{4i^2}{n^2} + \frac{4i}{n}\right) + 1}$$

$$= \frac{4}{n}\sum_{i=1}^{n} \frac{\left(1 + \frac{2i}{n}\right)}{\left(2 + \frac{4i^2}{n^2} + \frac{4i}{n}\right)}$$

$$= \frac{4}{n}\sum_{i=1}^{n} \frac{\left(1 + \frac{2i}{n}\right)}{2\left(1 + \frac{2i^2}{n^2} + \frac{2i}{n}\right)}$$

$$\Rightarrow R_n = \frac{2}{n}\sum_{i=1}^{n} \frac{\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{2i^2}{n^2} + \frac{2i}{n}\right)}$$

Thus $\lim_{n\to\infty} R_n = \sum_{i=1}^{n} \frac{2\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{2i}{n}\right)^2 + 1}.\frac{2}{n}$ (OR) $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{2i^2}{n^2} + \frac{2i}{n}\right)}$

Answer 20E.

Definition: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x\right]$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

Given $f(x) = x^2 + \sqrt{1+2x}$ $,4 \le x \le 7$

$$\Delta x = \frac{7-4}{n} = \frac{3}{n}$$

The values of

$$x_0 = a = 4, \quad x_1 = a + \Delta x = 4 + \frac{3}{n},$$

$$x_2 = a + 2\Delta x = 4 + 2 \cdot \frac{3}{n}$$

$$\Rightarrow x_2 = 4 + \frac{6}{n}, \ldots \textit{and so on}$$

$$\therefore x_i = a + i\Delta x = 4 + i\frac{3}{n} = 4 + \frac{3i}{n}$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \sum_{i=1}^{n} f\left(4 + \frac{3i}{n}\right)\frac{3}{n}$$

$$= \sum_{i=1}^{n}\left[\left(4 + \frac{3i}{n}\right)^2 + \sqrt{1 + 2\left(4 + \frac{3i}{n}\right)}\right]\frac{3}{n}$$

$$\boxed{\therefore \lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(4 + \frac{3i}{n}\right)^2 + \sqrt{1 + 2\left(4 + \frac{3i}{n}\right)}\right]}$$

Answer 21E.

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Consider the function,

$f(x) = \sqrt{\sin x}, 0 \le x \le \pi$.

The object is to find an expression for the area under the graph of f as a limit.

Since the lower limit of the function is $a = 0$, then the width of a subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$$

So, $x_1 = \frac{\pi}{n}, x_2 = \frac{2\pi}{n}, x_3 = \frac{3\pi}{n}, x_i = \frac{i\pi}{n}$, and $x_n = \frac{n\pi}{n}$.

Since the sum of the areas of the approximating rectangles is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right]\Delta x$$

$$= \left[\sqrt{\sin\left(\frac{\pi}{n}\right)} + \sqrt{\sin\left(\frac{2\pi}{n}\right)} + \sqrt{\sin\left(\frac{3\pi}{n}\right)} + \cdots + \sqrt{\sin\left(\frac{n\pi}{n}\right)}\right]\frac{\pi}{n}$$

$$= \sum_{i=1}^{n}\sqrt{\sin\left(\frac{\pi i}{n}\right)}\frac{\pi}{n}$$

Hence, the area under the graph of the function f is

$$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\sin\left(\frac{\pi i}{n}\right)}\frac{\pi}{n}$$

$$= \boxed{\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n}\sqrt{\sin\left(\frac{\pi i}{n}\right)}}.$$

Answer 22E.

Given

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$$

Taking $a = 5, b = 7$

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{7-5}{n}$$
$$= \frac{2}{n}$$

And

$$x_i = a + i\Delta x$$
$$= 5 + \frac{2i}{n}$$

Hence $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$ represents the function $f(x) = x^{10}$

So the area under the curve $f(x) = x^{10}$ between the points $x = 5$ and $x = 7$

Answer 23E.

Given area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n}.\tan\frac{i\pi}{4n}$

We expand the sigma notation.

$$A = \lim_{n\to\infty}\left[\frac{\pi}{4n}\tan\frac{\pi}{4n} + \frac{\pi}{4n}\tan\frac{2\pi}{4n} + \frac{\pi}{4n}\tan\frac{3\pi}{4n} + --+\frac{\pi}{4n}\tan\frac{\pi}{4}\right]$$

Or we can write

$$A = \lim_{n\to\infty}\left[\frac{\pi}{4n}\tan\left(0+\frac{\pi}{4n}\right) + \frac{\pi}{4n}\tan\left(0+\frac{2\pi}{4n}\right) + \frac{\pi}{4n}\tan\left(0+\frac{3\pi}{4n}\right) + --+\frac{\pi}{4n}\tan\left(0+\frac{n\pi}{4n}\right)\right]$$

We can compare this by the formula of area as

$$A = \lim_{n\to\infty}\left[\Delta x.f(x_1) + \Delta x.f(x_2) + --+\Delta x.f(x_n)\right]$$

Or

$$A = \lim_{n\to\infty}\left[\Delta x.f(x_0+\Delta x) + \Delta x.f(x_0+2\Delta x) + \Delta x.f(x_0+3\Delta x) + --+\Delta x.f(x_0+n\Delta x)\right]$$

So we have $\Delta x = \frac{\pi}{4n}$ and $x_0 = 0$ so $x_n = \frac{\pi}{4}$

And the function is $\tan x$.

So given limit is equal to the area of the region under the graph of the function $\boxed{f(x) = \tan x}$ in the interval $\left[0, \frac{\pi}{4}\right]$.

Answer 24E.

Definition2: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x\right]$$

(a)

Given function $y = x^3$, $[0,1]$

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-0}{n}$$
$$= \frac{1}{n}$$
$$x_i = a + i\Delta x$$
$$= 0 + i.\frac{1}{n}$$
$$= \frac{i}{n}$$

Substituting these values in the above definition

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x\right]$$
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\frac{1}{n}$$

Therefore $\boxed{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n}(i)^3\frac{1}{n^4}}$

(b)

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{n(n+1)}{2}\right]^2\frac{1}{n^4}$$
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n^2}\right)}{2}\right]^2$$
$$A = \lim_{n\to\infty} R_n = \frac{1}{4}\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\right]^2$$
$$A = \lim_{n\to\infty} R_n = \frac{1}{4}\left[\left(1+\frac{1}{0}\right)\right]^2$$

Therefore $\boxed{A = \lim_{n\to\infty} R_n = \frac{1}{4}}$

Answer 25E.

(a) $\boxed{L_n \le A \le R_n}$

(b)

Upper Riemann sum

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \Delta x\left[f(x_1)+f(x_2)+f(x_3)+...+f(x_{n-1})+f(x_n)\right]$$

Lower Riemann sum

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$
$$= \Delta x\left[f(x_0)+f(x_1)+f(x_2)+...+f(x_{n-1})\right]$$

Now $R_n - L_n = \left[f(x_n)-f(x_0)\right]\Delta x$

$$\boxed{\therefore R_n - L_n = \left(\frac{b-a}{n}\right)\left[f(b)-f(a)\right]}$$

(c) We have to prove $R_n - A < \frac{b-a}{n}\left[f(b)-f(a)\right]$

Consider $L_n \le A \le R_n$

$$\Rightarrow L_n - R_n \le A - R_n \le R_n - R_n$$
$$\Rightarrow L_n - R_n \le A - R_n \le 0$$
$$\Rightarrow L_n - R_n \le A - R_n$$
$$\Rightarrow R_n - L_n \ge R_n - A \quad \text{[Multiplying with } -1 \text{ reverses the inequality]}$$
$$\Rightarrow R_n - A \le R_n - L_n \quad [b \ge a \Rightarrow a \le b]$$
$$\Rightarrow R_n - A < \frac{b-a}{n}\left[f(b)-f(a)\right] \quad \text{[by part (b)]}$$

$$\boxed{\therefore R_n - A < \frac{b-a}{n}\left[f(b)-f(a)\right]}$$

Answer 26E.

Given $y = \sin x$, 0 to $\frac{\pi}{2}$

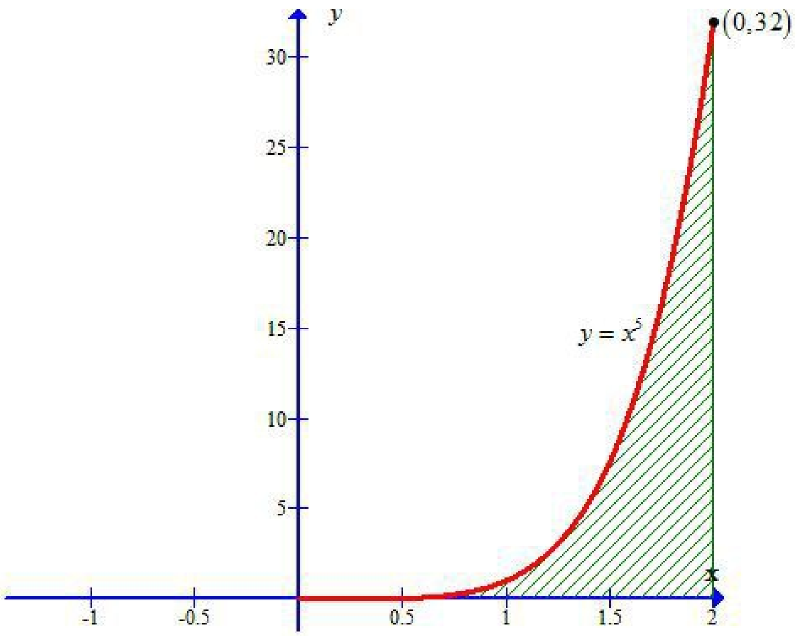
Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n, *and* R_n be the approximations to A with n subintervals using left and right end points the we have $R_n - A < \frac{b-a}{n}\left[f(b)-f(a)\right]$

$$R_n - A < 0.0001$$
$$\Rightarrow R_n - A < \frac{b-a}{n}\left[f(b)-f(a)\right]$$
$$\Rightarrow \frac{b-a}{n}\left[f(b)-f(a)\right] = 0.0001$$
$$\Rightarrow \frac{\frac{\pi}{2}-0}{n}\left[f\left(\frac{\pi}{2}\right)-f(0)\right] = 0.0001$$
$$\Rightarrow \frac{\pi}{2n}\left[\sin\left(\frac{\pi}{2}\right)-\sin(0)\right] = 0.0001$$
$$\Rightarrow \frac{\pi}{2n}\left[1-0\right] = 0.0001$$
$$\Rightarrow \frac{\pi}{2n} = 0.0001$$
$$\Rightarrow n = \frac{\pi}{2}\times 10^4$$

$\boxed{n = 15700}$ (approximately)

Answer 27E.

Firstly, graph the function $y=x^5$ and area covered in the interval $[0,2]$.



(a) Divide the curve $y=x^5$ from 0 to 2 into *n* subintervals.

Then $\Delta x = 2/n$ and $x_i = 2i/n$.

$$A=\lim_{n\to\infty}\left[\frac{2f(2/n)}{n}+\frac{2f(4/n)}{n}+\cdots+\frac{2f(2)}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2\cdot 2^5}{n^6}+\frac{2\cdot 4^5}{n^6}+\cdots+\frac{2\cdot(2n)^5}{n^6}\right]$$

$$=\lim_{n\to\infty}\frac{2}{n^6}\sum_{i=1}^{n}(2i)^5$$

$$=\lim_{n\to\infty}\frac{64}{n^6}\sum_{i=1}^{n}i^5$$

(a) Type the following commands in Maple then press ENTER to find the sum of the series $\sum_{i=1}^{n} i^5$.

sum$(i^5, i=1..n)$;

$$\frac{1}{6}(n+1)^6-\frac{1}{2}(n+1)^5+\frac{5}{12}(n+1)^4-\frac{1}{12}(n+1)^2$$

$$simplify\left(\frac{1}{6}(n+1)^6-\frac{1}{2}(n+1)^5+\frac{5}{12}(n+1)^4-\frac{1}{12}(n+1)^2\right);$$

$$\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$$

Observe the second output above (blue font), this follows that

$$\sum_{i=1}^{n} i^5 = 1^5+2^5+3^5+\cdots+n^5$$

$$=\boxed{\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}}.$$

(b) Use the formula to evaluate the limit.

$$A=\lim_{n\to\infty}\left[\frac{64}{n^6}\cdot\left(\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}\right)\right]$$
$$=\frac{64}{12}\lim_{n\to\infty}\left(\frac{2n^6+6n^5+5n^4-n^2}{n^6}\right)$$
$$=\frac{16}{3}\lim_{n\to\infty}\left(\frac{2n^6}{n^6}+\frac{6n^5}{n^6}+\frac{5n^4}{n^6}-\frac{n^2}{n^6}\right)$$
$$=\frac{16}{3}\lim_{\frac{1}{n}\to 0}\left(2+\frac{6}{n}+\frac{5}{n^2}-\frac{1}{n^4}\right)$$
$$=\frac{16}{3}\left(2+6(0)+5(0)^2-1(0)^4\right)$$
$$=\boxed{\frac{32}{3}}$$

Verify the result using CAS.

Type the following commands in Maple then press ENTER to find the limit of the series $\lim_{n\to\infty}\left[\frac{64}{n^6}\cdot\left(\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}\right)\right]$.

$$limit\left(\frac{64}{n^6}\cdot\left(\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2\right), n=infinity\right);$$

$$\frac{32}{3}$$

Hence verified.

Answer 28E.

(a) y = x 4 + 5x2 + x from 2 to 7.

consider n equal subintervals on [2 , 7], then $\Delta x=\frac{|7-2|}{n}=\frac{5}{n}$

considering the left end points of the rectangles which touch the curve from below and we write the integral of the curve as a limit of the sum of areas of rectangles :

$$\int_2^7\left(x^4+5x^2+x\right)\mathrm{d}x=\lim_{\Delta x\to 0}\sum_{k=1}^{n}\Delta x*f(2+\Delta x)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{5}{n}\left\{38+\frac{145}{n}+\frac{725}{n^2}+\frac{1000}{n^3}+\frac{625}{n^4}\right\}$$

$$=\lim_{n\to\infty}\left(\frac{190}{n}\sum_{k=1}^{n}1+\frac{725}{n^2}\sum_{k=1}^{n}k+\frac{3625}{n^3}\sum_{k=1}^{n}k^2+\frac{5000}{n^4}\sum_{k=1}^{n}k^3+\frac{3125}{n^5}\sum_{k=1}^{n}k^4\right)$$

(b) the sum in the above result is

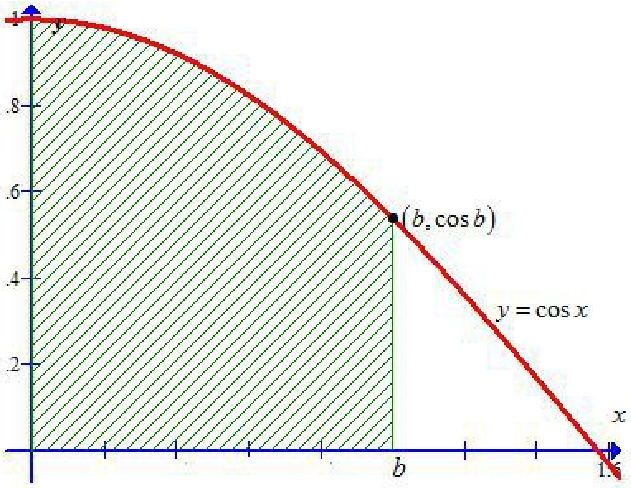
$$=\lim_{n\to\infty}\left\{\frac{190}{n}n+\frac{725}{n^2}*\frac{n(n+1)}{2}+\frac{3625}{n^3}*\frac{n(n+1)(2n+1)}{6}+\frac{5000}{n^4}*\frac{n^2(n+1)^2}{4}+\frac{3}{}\right.$$

(c) exact area of the given curve is $\int_2^7\left(x^4+5x^2+x\right)\mathrm{d}x=\frac{x^5}{5}+5\frac{x^3}{3}+\frac{x^2}{2}\Big\}_0^7=$

3483.5

Answer 29E.

Firstly, graph the function $y = \cos x$ and area covered in the interval $[0,b]$.



(a) Divide the curve $y = \cos x$; $x = 0$ to b into n subintervals.

Then $\Delta x = b/n$ and $x_i = bi/n$.

$$A = \lim_{n\to\infty}\left[\frac{bf(1b/n)}{n} + \frac{bf(2b/n)}{n} + \text{L} + \frac{bf(b)}{n}\right]$$

$$= \lim_{n\to\infty}\frac{b}{n}\sum_{i=1}^{n}\cos\frac{bi}{n}$$

Type the following commands in Maple then press ENTER to find the sum of the series $\sum_{i=1}^{n}\cos\frac{bi}{n}$.

$$sum\left(\cos\left(\frac{b\cdot i}{n}\right), i = 1..n\right);$$

$$-\frac{1}{2}\cos\left(\frac{b(n+1)}{n}\right) - \frac{1}{2}\frac{\sin\left(\frac{b}{n}\right)\sin\left(\frac{b(n+1)}{n}\right)}{\cos\left(\frac{b}{n}\right)-1} + \frac{1}{2}\cos\left(\frac{b}{n}\right) + \frac{1}{2}\frac{\sin\left(\frac{b}{n}\right)^2}{\cos\left(\frac{b}{n}\right)-1}$$

$$simplify\left(-\frac{1}{2}\cos\left(\frac{b(n+1)}{n}\right) - \frac{1}{2}\frac{\sin\left(\frac{b}{n}\right)\sin\left(\frac{b(n+1)}{n}\right)}{\cos\left(\frac{b}{n}\right)-1} + \frac{1}{2}\cos\left(\frac{b}{n}\right) + \frac{1}{2}\frac{\sin\left(\frac{b}{n}\right)^2}{\cos\left(\frac{b}{n}\right)-1}\right);$$

$$-\frac{1}{2}\frac{1}{\cos\left(\frac{b}{n}\right)-1}\left(\cos\left(\frac{b(n+1)}{n}\right)\cos\left(\frac{b}{n}\right) + \sin\left(\frac{b}{n}\right)\sin\left(\frac{b(n+1)}{n}\right) - \cos\left(\frac{b(n+1)}{n}\right) + \cos\left(\frac{b}{n}\right) - 1\right)$$

Observe the second output above (blue font), this follows that

$$\sum_{i=1}^{n}\cos\frac{bi}{n}=\frac{-1}{2\left(\cos\left(\frac{b}{n}\right)-1\right)}\left[\begin{array}{l}\cos\left(\frac{b(n+1)}{n}\right)\cos\left(\frac{b}{n}\right)+\sin\left(\frac{b}{n}\right)\sin\left(\frac{b(n+1)}{n}\right)\\-\cos\left(\frac{b(n+1)}{n}\right)+\cos\left(\frac{b}{n}\right)-1\end{array}\right]$$

$$=\frac{-1}{2\left(\cos\left(\frac{b}{n}\right)-1\right)}\left[\cos\left(\frac{b(n+1)}{n}-\frac{b}{n}\right)-\cos\left(\frac{b(n+1)}{n}\right)+\cos\left(\frac{b}{n}\right)-1\right]$$

$$=\frac{1-\cos b+\cos\left(b+\frac{b}{n}\right)-\cos\left(\frac{b}{n}\right)}{2\left(\cos\left(\frac{b}{n}\right)-1\right)}$$

$$=\frac{-\cos b+\cos b\cos\left(\frac{b}{n}\right)-\sin b\sin\left(\frac{b}{n}\right)+1-\cos\left(\frac{b}{n}\right)}{2\left(\cos\left(\frac{b}{n}\right)-1\right)}$$

Continue the above step.

$$=\frac{\cos b}{2}+\frac{\sin b\sin\left(\frac{b}{n}\right)}{4\sin^2\left(\frac{b}{2n}\right)}-\frac{1}{2}$$

$$=\frac{\cos b}{2}+\frac{2\sin b\sin\left(\frac{b}{2n}\right)\cos\left(\frac{b}{2n}\right)}{4\sin^2\left(\frac{b}{2n}\right)}-\frac{1}{2}$$

$$=\frac{\cos b}{2}+\frac{\sin b\cos\left(\frac{b}{2n}\right)}{2\sin\left(\frac{b}{2n}\right)}-\frac{1}{2}$$

$$=\frac{1}{2}\left(\cos b+\sin b\cot\left(\frac{b}{2n}\right)-1\right)$$

Hence $\sum_{i=1}^{n}\cos\frac{bi}{n}=\boxed{\frac{1}{2}\left(\cos b+\sin b\cot\left(\frac{b}{2n}\right)-1\right)}$

(b) Use the formula to evaluate the limit.

$$A = \lim_{n\to\infty}\left(\frac{b}{n}\sum_{i=1}^{n}\cos\frac{bi}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{b}{n}\cdot\frac{1}{2}\left(\cos b + \sin b\cot\left(\frac{b}{2n}\right) - 1\right)\right)$$

$$= \frac{b}{2}\lim_{\frac{1}{n}\to 0}\left(\frac{1}{n}\cos b + \sin b\frac{1}{n}\cot\left(\frac{b}{2n}\right) - \frac{1}{n}\right)$$

$$= \frac{b}{2}\lim_{\frac{1}{n}\to 0}\left((0)\cos b + \sin b\cdot\lim_{\frac{1}{n}\to 0}\left(\frac{1}{n}\cot\left(\frac{b}{2n}\right)\right) - (0)\right)$$

$$= \sin b\cdot\lim_{\frac{1}{n}\to 0}\left(\frac{b}{2n}\cot\left(\frac{b}{2n}\right)\right)$$

$$= \sin b\cdot\frac{\lim_{\frac{1}{n}\to 0}\cos\left(\frac{b}{2n}\right)}{\lim_{\frac{1}{n}\to 0}\frac{\sin\left(\frac{b}{2n}\right)}{\frac{b}{2n}}}$$

$$= \sin b\cdot\frac{\lim_{\frac{b}{2n}\to 0}\cos\left(\frac{b}{2n}\right)}{\lim_{\frac{b}{2n}\to 0}\frac{\sin\left(\frac{b}{2n}\right)}{\frac{b}{2n}}}$$

$$= \sin b\cdot\frac{\cos(0)}{1}$$

$$= \sin b$$

Hence $A = \boxed{\sin b}$.

Verify the result using CAS.

Type the following commands in Maple then press ENTER to find the limit of the series

$\lim_{n\to\infty}\left(\frac{b}{n}\cdot\frac{1}{2}\left(\cos b + \sin b\cot\left(\frac{b}{2n}\right) - 1\right)\right)$.

$$limit\left(\frac{b}{n}\cdot\frac{1}{2}\cdot\left(\cos(b) + \sin(b)\cdot\cot\left(\frac{b}{2\cdot n}\right) - 1\right), n = infinity\right)$$

$$\sin(b)$$

Hence verified.

(c) Evaluate the area when $b = \pi/2$

Substitute $\pi/2$ for b in $A = \sin b$.

$$A = \sin b$$

$$= \sin\frac{\pi}{2}$$

$$= 1$$

Hence the area of the required region is **1 square unit.**

Answer 30E.

(A)

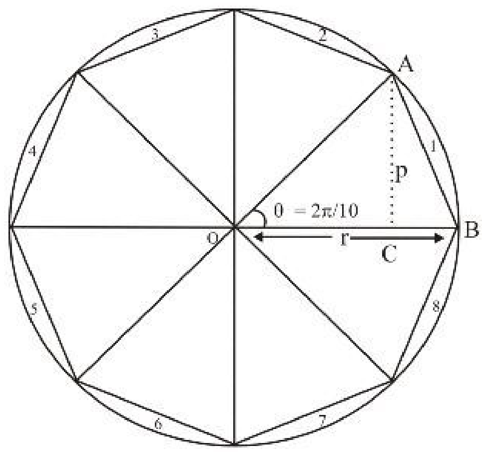


Figure 1

In (figure 1) a polygon with 8 equal sides has been inscribed in a circle of radius r.
Let the centre of the circle be at O.
Since all sides of the polygon are equal so the angles at the centre subtended by these sides are also equal. Let this angle be θ.
And the area of polygon $= 8 \times$ Area of one of the triangles made by the side of the polygon and the radii joining it.
Now we consider a triangle AOB

Area of a triangle AOB $= \frac{1}{2} \times \text{base} \times \text{height}(\text{as perpendicular})$

So we draw the perpendicular AC from A on |OB| in the triangle AOB triangle AOC.

$$|AC| = |OA| \sin\theta$$

Or $|AC| = r\sin\theta$ $\qquad [|OA| = \text{radius } r]$

Then the area of the triangle AOB is

$$= \frac{1}{2} \times |OB| \times |AC|$$

$$= \frac{1}{2} r.r \sin\theta \qquad [|OB| = \text{radius } r]$$

$$\boxed{= \frac{1}{2} r^2 \sin\theta}$$

Then the area of the polygon with 8 equal sides $(let\ A_8)$

$$A_8 = 8 \times \text{area of triangle AOB}$$

Or $A_8 = 8 \times \frac{1}{2} r^2 \sin\theta = \frac{1}{2} \times 8 \times r^2 \sin(\theta)$

$$= \frac{1}{2} \times \begin{pmatrix}\text{number of} \\ \text{sides}\end{pmatrix} \times \begin{pmatrix}\text{Square of radius} \\ \text{of circle}\end{pmatrix} \times \sin\begin{pmatrix}\text{angle subtended by} \\ \text{side on the centre}\end{pmatrix}$$

Now if number of sides of the polygon is n.

And $\theta = \frac{2\pi}{n}$

Then the area of the polygon is $(let\ A_n)$

$$A_n = \frac{n}{2}.r^2 \sin\left(\frac{2\pi}{n}\right)$$

Or $\boxed{A_n = \frac{1}{2} n.r^2 \sin\left(\frac{2\pi}{n}\right)}$ $\qquad$ Proved

(B) We have $A_n = \frac{1}{2} n.r^2 \sin\left(\frac{2\pi}{n}\right)$

Then taking limit as $n \to \infty$

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[\frac{1}{2} n.r^2 \sin\left(\frac{2\pi}{n}\right)\right]$$

Since $\frac{1}{2}$ and r are constant and $\lim_{n\to\infty} Cf\left(x_n\right) = C \lim_{n\to\infty} f\left(x\right)$ --- (1)

Here C is any constant, and we have

$$\lim_{n\to\infty} A_n = \frac{r^2}{2}\lim_{n\to\infty}\left(n \sin\left(\frac{2\pi}{n}\right)\right)$$

Divide and multiply by 2π, we have

$$\lim_{n\to\infty} A_n = \frac{r^2}{2}\lim_{n\to\infty}\left[\frac{n}{2\pi}.2\pi.\sin\left(\frac{2\pi}{n}\right)\right]$$

Or $$\lim_{n\to\infty} A_n = \frac{r^2}{2}\lim_{n\to\infty}\left[2\pi\frac{\sin\left(2\pi/n\right)}{\left(2\pi/n\right)}\right]$$

Let $\frac{2\pi}{n} = \phi$ so if $n \to \infty$ then $\phi \to 0$

So $$\lim_{n\to\infty} A_n = \frac{r^2}{2}\lim_{\phi\to 0} 2\pi.\frac{\sin\phi}{\phi}$$

Or $$\lim_{n\to\infty} A_n = \pi r^2 \lim_{\phi\to 0}\frac{\sin\phi}{\phi}$$ Since $\lim_{\theta\to 0} = \frac{\sin\theta}{\theta} = 1$

So we have

$$\boxed{\lim_{n\to\infty} A_n = \pi r^2}$$ Proved