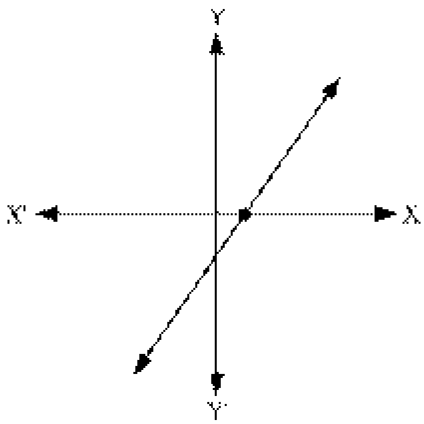
Polynomials

- **Graphic Interpretation of the number of Zeros of a Polynomial**

The zero of a polynomial, $y = p(x)$, (if it exists) is the *x*-coordinate of the point where the graph of $y = p(x)$ intersects the *x*-axis.

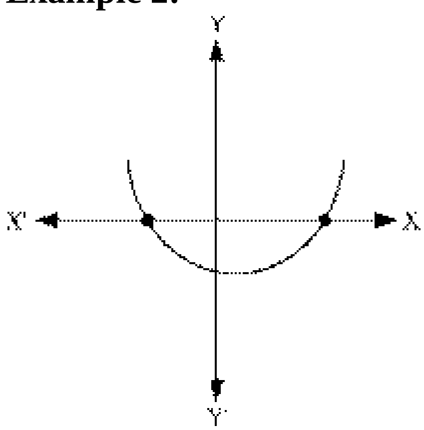
Example 1:



In the above graph, the graph intersects the *x*-axis at only one point.

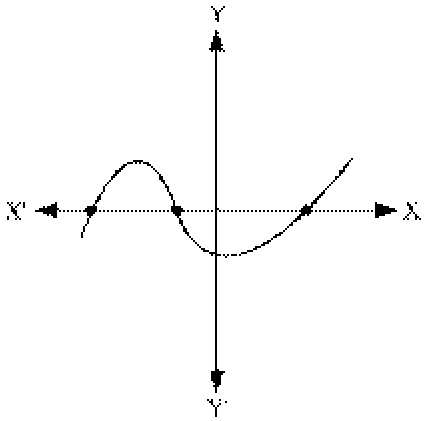
The number of zeroes of the corresponding polynomial is 1.

Example 2:



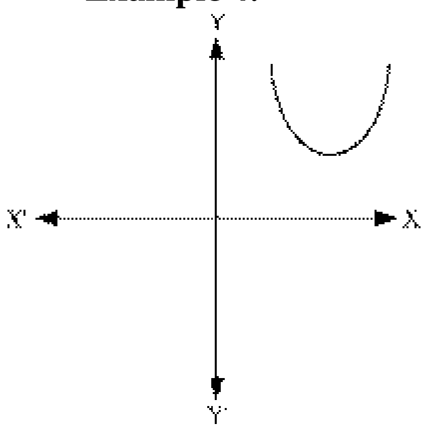
In the above graph, the graph intersects the *x*-axis at exactly two points.
The number of zeroes of the corresponding polynomial is 2.

Example 3:



In the above graph, the graph intersects the x-axis at three points.
The number of zeroes of the corresponding polynomial is 3.

Example 4:



In the above graph, the graph does not intersect the x-axis.
The corresponding polynomial has no zeroes.

- **Zeroes of a polynomial**

A real number 'k' is a zero of a polynomial $p(x)$, if $p(k) = 0$. In this case, 'k' is also called the root of the equation, $p(x) = 0$.

Note: A polynomial of degree n can have at most n zeroes.

Example:

1. $-\frac{9}{2}$ is the zero of the linear polynomial, $2x + 9$, because $2\times\left(-\frac{9}{2}\right)+9=-9+9=0$.
2. 2 and –3 are the zeroes of the quadratic polynomial, x^2+x-6.
$\left[\because 2^2+2-6=0,\ (-3)^2+(-3)-6=0\right]$

Example:

Find the zeroes of the polynomial, $p(x)=x^3-3x^2-6x+8$

Solution:

By trial, we obtain
p (1) = 1 – 3 – 6 + 8 = 0
(*x* – 1) is a factor *p* (*x*) [By factor theorem]

$$
\begin{aligned}
p(x) &= x^3-x^2-2x^2+2x-8x+8 \\
&= x^2(x-1)-2x(x-1)-8(x-1) \\
&= (x-1)(x^2-2x-8) \\
&= (x-1)\left[x^2-4x+2x-8\right] \\
&= (x-1)[x(x-4)+2(x-4) \\
&= (x-1)(x-4)(x+2)
\end{aligned}
$$

p (*x*) = 0, if *x* = 1, 4, or –2

Thus, the zeroes of *p*(*x*) are 1, 4, and –2.

- **Relationship between zeroes and Coefficients of a polynomial**
- **Linear Polynomial**

The zero of the linear polynomial, $ax+b$, is $\frac{-b}{a}=\frac{-(\text{Constant term})}{\text{Coefficient of } x}$

Example: 3*x* – 5

$3x-5=0 \Rightarrow x=\frac{5}{3}$

Zero of 3*x* – 5 is $\frac{5}{3}=\frac{-(-5)}{3}=\frac{-(\text{Constant term})}{\text{Coefficient of } x}$

- **Quadratic Polynomial**

If α and β are the zeroes of the quadratic polynomial, $p(x)=ax^2+bx+c$, then $(x-\alpha),(x-\beta)$ are the factors of *p*(*x*).

$p(x)=ax^2+bx+c=k\left[x^2-(\alpha+\beta)x+\alpha\beta\right]$, where $k \neq 0$ is constant.

Sum of zeroes = $\alpha+\beta=\frac{-b}{a}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of zeroes = $\alpha\beta=\frac{c}{a}=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Example:

Find the zeroes of the quadratic polynomial, $2x^2+17x-9$, and verify the relationship between the zeroes and the coefficients.

Solution:

$$\begin{aligned} p(x) &= 2x^2+17x-9 \\ &= 2x^2+18x-x-9 \\ &= 2x(x+9)-1(x+9) \\ &= (x+9)(2x-1) \end{aligned}$$

The zeroes of $p(x)$ are given by,

$$p(x)=0$$
$$\Rightarrow (x+9)(2x-1)=0$$
$$\Rightarrow 2x-1=0 \text{ or } x+9=0$$
$$\Rightarrow x=\frac{1}{2} \text{ or } x=-9$$

Zeroes of $p(x)$ are $\alpha=\frac{1}{2}$ and $\beta = -9$

Sum of zeroes $=\alpha+\beta=\frac{1}{2}-9=\frac{-17}{2}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of zeroes $=\alpha\beta=\frac{1}{2}\times -9=\frac{-9}{2}=-\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

- **Formation of Polynomial using the Sum and Product of Zeroes**

Example:

Find a quadratic polynomial, the sum and the product of whose zeroes are $\frac{-14}{3}$ and $\frac{-5}{3}$.

Solution:

Given that,

$\alpha+\beta=\frac{-14}{3}$, $\alpha\beta=\frac{-5}{3}$

The required polynomial is given by,

$$p(x)=k\left[x^2-(\alpha+\beta)x+\alpha\beta\right]$$

$$=k\left[x^2-\left(\frac{-14}{3}\right)x+\left(\frac{-5}{3}\right)\right]=k\left[x^2+\frac{14}{3}x-\frac{5}{3}\right]$$

For $k = 3$,

$$p(x)=3\left[x^2+\frac{14}{3}x-\frac{5}{3}\right]=3x^2+14x-5$$

One of the quadratic polynomials, which fit the given condition, is $3x^2+14x-5$.

- **Cubic polynomial**

If α,β,γ are the zeroes of the cubic polynomial, $f(x)=ax^3+bx^2+cx+d,$ then $(x-\alpha),(x-\beta),(x-\gamma)$ are the factors of $f(x)$.

$f(x)=ax^3+bx^2+cx+d=k\left[x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma\right]$, where k is a non-zero constant

$$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\alpha\beta\gamma=\frac{-d}{a}=-\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

- Division of polynomial by polynomial of degree more than 1 can be done as follows:

Example:

Divide $x^4 - x^3 + 3x^2 - x + 3$ by $x^2 - x + 1$.

Solution:

It is given that,

Dividend = $x^4 – x^3 + 3x^2 – x + 3$, Divisor = $x^2 – x + 1$

$$
\begin{array}{r|l}
 & x^2+2 \\
\hline
x^2-x+1 & x^4-x^3+3x^2-x+3 \\
 & x^4-x^3+x^2 \\
 & -\quad+\quad- \\
\hline
 & \qquad\quad 2x^2-x+3 \\
 & \qquad\quad 2x^2-2x+2 \\
 & \qquad\quad -\quad+\quad- \\
\hline
 & \qquad\qquad\quad x+1 \\
\hline
\end{array}
$$

- **Division Algorithm of Polynomials states that:**

Dividend = Divisor × Quotient + Remainder
i.e., $p(x) = g(x) \times q(x) + r(x)$
Here, degree of $r(x)$ < degree of $g(x)$ and degree of $q(x)$ = degree of $p(x)$ – degree of $g(x)$
Example:
By applying division algorithm, find the quotient and remainder when $p(x) = x^4 + 3x^3 + 2x^2 + 5x – \frac{5}{2}$ is divided by $g(x) = x^3 + 2x – 1$.
Solution:
$p(x) = x^4 + 3x^3 + 2x^2 + 5x – \frac{5}{2}$, $g(x) = x^3 + 2x – 1$
deg $p(x) = 4$, deg $g(x) = 3$

Degree of quotient $q(x) = 4 – 3 = 1$, and deg of remainder $r(x) <$ deg $g(x) = 3$
Let $q(x) = ax + b$, $r(x) = cx^2 + dx + e$

By division algorithm,

$p(x) = g(x) \times q(x) + r(x)$
$\Rightarrow x^4 + 3x^3 + 2x^2 + 5x – \frac{5}{2} = (ax + b)(x^3 + 2x – 1) + (cx^2 + dx + e)$
$= ax^4 + 2ax^2 – ax + bx^3 + 2bx – b + cx^2 + dx + e$
$= ax^4 + bx^3 + (2a + c)x^2 + (–a + 2b + d)x + (–b + e)$

Equating the coefficients of respective powers, we obtain
$a = 1, b = 3$
$2a + c = 2$
$\Rightarrow 2 + c = 2$
$\Rightarrow c = 0$

$-a + 2b + d = 5$
$\Rightarrow -1 + 6 + d = 5$
$\Rightarrow d = 0$
$-b + e = -\frac{5}{2}$

$\Rightarrow e = 3 = \frac{1}{2}$

Quotient, $q\,(x) = x + 3$

Remainder, $r\,(x) = \frac{1}{2}$