CHAPTER

01

Indefinite Integral

Learning Part

Session 1

- Fundamental of Indefinite Integral

Session 2

- Methods of Integration

Session 3

- Some Special Integrals

Session 4

- Integration by Parts

Session 5

- Integration Using Partial Fractions

Session 6

- Indirect and Derived Substitutions

Session 7

- Euler's Substitution, Reduction Formula and Integration Using Differentiation

Practice Part

- JEE Type Examples
- Chapter Exercises

Arihant on Your Mobile !

Exercises with the symbol can be practised on your mobile. See inside cover page to activate for free.

It was in this aspect that the process of integration was treated by Leibnitz, the symbol of $\int$ being regarded as the initial letter of the word sum, in the same way as the symbol of differentiation d is the initial letter in the word difference.

Definition

If f and g are functions of x such that $g'(x) = f(x)$, then the function g is called a anti-derivative (or primitive function or simply integral) of f w.r.t. x. It is written symbolically, $\int f(x)\,dx = g(x)$, where, $\frac{d}{dx} g(x) = f(x)$

Remarks

1. In other words, $\int f(x)\,dx = g(x)$ iff $g'(x) = f(x)$

2. $\int f(x)\,dx = g(x) + c$, where c is constant,
$[\because (g(x)+C)' = g'(x) = f(x)]$ and C is called constant of integration.

Example 1 If $\frac{d}{dx}[x^{n+1}+c] = (n+1)\,x^n$, then find $\int x^n\,dx$.

Sol. As, $\frac{d}{dx}[x^{n+1}+C] = (n+1)\,x^n$

$\Rightarrow$ $(x^{n+1}+C)$ is anti-derivative or integral of $(n+1)\,x^n$.

$$\therefore \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Example 2 If $\frac{d}{dx}(\sin x + c) = \cos x$, then find $\int \cos x\,dx$.

Sol. As, $\frac{d}{dx}(\sin x + C) = \cos x$

$\Rightarrow$ $\sin x + C$ is anti-derivative or integral of $\cos x$.

$$\therefore \quad \int \cos x\,dx = (\sin x) + C$$

Session 1

Fundamental of Indefinite Integral

Fundamental of Indefinite Integral

Since, $\quad \frac{d}{dx}\{g(x)\} = f(x)$

$\Leftrightarrow \quad \int f(x)\,dx = g(x) + C$

Therefore, based upon this definition and various standard differentiation formulas, we obtain the following integration formulae

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \log|x| + C$, when $x \neq 0$

(iii) $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x + C$

(iv) $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x, a > 0, a \neq 1$

$\Rightarrow \int a^x\,dx = \frac{a^x}{\log_e a} + C$

(v) $\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x\,dx = -\cos x + C$

(vi) $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\,dx = \tan x + C$

(viii) $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x \Rightarrow \int \text{cosec}^2 x\,dx = -\cot x + C$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\Rightarrow \int \sec x \tan x\,dx = \sec x + C$

(x) $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$

$\Rightarrow \int \text{cosec}\, x \cot x\,dx = -\text{cosec}\, x + C$

(xi) $\frac{d}{dx}(\log|\sin x|) = \cot x$

$\Rightarrow \int \cot x\,dx = \log|\sin x| + C$

(xii) $\frac{d}{dx}(-\log|\cos x|) = \tan x$

$\Rightarrow \int \tan x \, dx = -\log|\cos x| + C$

(xiii) $\frac{d}{dx}(\log|\sec x + \tan x|) = \sec x$

$\Rightarrow \int \sec x \, dx = \log|\sec x + \tan x| + C$

(xiv) $\frac{d}{dx}(\log|\text{cosec}\, x - \cot x|) = \text{cosec}\, x$

$\Rightarrow \int \text{cosec}\, x \, dx = \log|\text{cosec}\, x - \cot x| + C$

(xv) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$

$\Rightarrow \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$

$\Rightarrow \int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2 + x^2}$

$\Rightarrow \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(xviii) $\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right) = \frac{-1}{a^2 + x^2}$

$\Rightarrow \int \frac{-1}{a^2 + x^2}\, dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(xix) $\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2 - a^2}}$

$\Rightarrow \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xx) $\frac{d}{dx}\left(\frac{1}{a}\text{cosec}^{-1}\frac{x}{a}\right) = \frac{-1}{x\sqrt{x^2 - a^2}}$

$\Rightarrow \int \frac{-dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$

Example 3 Evaluate

(i) $\int \frac{x^2 + 5x - 1}{\sqrt{x}}\, dx$ (ii) $\int (x^2 + 5)^3\, dx$

Sol. (i) $I = \int \frac{x^2 + 5x - 1}{\sqrt{x}}\, dx = \int \left(\frac{x^2}{x^{1/2}} + \frac{5x^1}{x^{1/2}} - \frac{1}{x^{1/2}}\right) dx$

$= \int (x^{3/2} + 5x^{1/2} - x^{-1/2})\, dx$

$\left[\text{using} \int x^n\, dx = \frac{x^{n+1}}{n+1} + C\right]$

$= \frac{x^{3/2+1}}{3/2+1} + \frac{5x^{1/2+1}}{1/2+1} - \frac{x^{-1/2+1}}{-1/2+1} + C$

$\Rightarrow \quad I = \frac{2}{5}x^{5/2} + \frac{2}{3}\cdot 5x^{3/2} - 2x^{1/2} + C$

(ii) $I = \int (x^2 + 5)^3\, dx$ [using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

$I = \int (x^6 + 15x^4 + 75x^2 + 125)\, dx$

$I = \frac{x^7}{7} + \frac{15x^5}{5} + \frac{75x^3}{3} + 125x + C$

$I = \frac{x^7}{7} + 3x^5 + 25x^3 + 125x + C$

Example 4 Evaluate

(i) $\int \tan^2 x\, dx$ (ii) $\int \frac{dx}{\sin^2 x \cos^2 x}$

(iii) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$ (iv) $\int \frac{\cos x - \cos 2x}{1 - \cos x}\, dx$

Sol. (i) $I = \int \tan^2 x\, dx \Rightarrow I = \int (\sec^2 x - 1)\, dx$

$I = \int \sec^2 x\, dx - \int 1\, dx$ [using $\int \sec^2 x dx = \tan x + C$]

$\Rightarrow I = \tan x - x + C$

(ii) $I = \int \frac{1}{\sin^2 x \cos^2 x}\, dx$

$I = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx$ [Using $\sin^2 x + \cos^2 x = 1$]

$I = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx$

$I = \int \sec^2 x\, dx + \int \text{cosec}^2 x\, dx$

$I = \tan x - \cot x + C$

(iii) $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$

$I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x}\, dx$

[using $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$]

$I = \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}\, dx$

$I = \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x}\, dx$

$$I = \int \frac{1}{\sin^2 x \cos^2 x} dx - \int 3\, dx$$

$$I = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx - 3x + C$$

$$I = \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx - 3x + C$$

$$I = \tan x - \cot x - 3x + C$$

(iv) $I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx$ [using $\cos 2x = 2\cos^2 x - 1$.]

$$I = \int \frac{\cos x - (2\cos^2 x - 1)}{1 - \cos x} dx$$

$$= \int \frac{-2\cos^2 x + \cos x + 1}{1 - \cos x} dx$$

$$\Rightarrow I = \int \frac{-(2\cos x + 1)(\cos x - 1)}{-(\cos x - 1)} dx$$

[as $-2\cos^2 x + \cos x - 1 = -(2\cos x + 1)\cdot(\cos x - 1)$]

$$\Rightarrow \quad I = \int (2\cos x + 1)\, dx$$

$$\therefore \quad I = 2\sin x + x + C$$

Remark

In rational algebraic functions if the degree of numerator is greater than or equal to degree of denominator, then always divide the numerator by denominator and use the result of integration.

Example 5 Evaluate

(i) $\int \frac{x^3}{x+2} dx$ (ii) $\int \frac{x^2}{x^2+5} dx$

Sol. (i) $I = \int \frac{x^3}{x+2} dx = \int \frac{x^3 + 8 - 8}{x+2} dx$

$$I = \int \left(\frac{(x^3 + 2^3)}{x+2} - \frac{8}{x+2} \right) dx$$

$$\Rightarrow \quad I = \int \left(\frac{(x+2)(x^2 - 2x + 4)}{x+2} - \frac{8}{x+2} \right) dx$$

$$I = \int \left(x^2 - 2x + 4 - \frac{8}{x+2} \right) dx$$

$$\therefore \quad I = \frac{x^3}{3} - x^2 + 4x - 8\log|x+2| + C$$

(ii) $I = \int \frac{x^2}{x^2+5} dx = \int \frac{x^2+5-5}{x^2+5} dx = \int \left(\frac{x^2+5}{x^2+5} - \frac{5}{x^2+5} \right) dx$

$$I = \int \left(1 - \frac{5}{x^2+5} \right) dx = x - 5\int \frac{dx}{x^2 + (\sqrt{5})^2}$$

$$I = x - \frac{5}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}} \right) + C$$

$$I = x - \sqrt{5} \tan^{-1}\left(\frac{x}{\sqrt{5}} \right) + C$$

Example 6 Evaluate

(i) $\int 5^{\log_e x} dx$ (ii) $\int 2^{\log_4 x} dx$

Sol. (i) $I = \int 5^{\log_e x} dx = \int x^{\log_e 5} dx$ [Using $a^{\log_e b} = b^{\log_e a}$]

$$= \frac{x^{\log_e 5 + 1}}{(\log_e 5 + 1)} + C$$

$$\therefore \quad \int 5^{\log_e x} dx = \frac{x^{\log_e 5 + 1}}{\log_e 5 + 1} + C$$

(ii) $I = \int 2^{\log_4 x} dx = \int 2^{\log_{2^2} x} dx = \int 2^{1/2 \log_2 x} dx$

$\left[\text{using } \log_{b^n} x = \frac{1}{n} \log_b x \right]$

$$= \int 2^{\log_2 \sqrt{x}} dx = \int \sqrt{x}\, dx$$ [using $a^{\log_a b} = b$]

$$= \frac{x^{3/2}}{3/2} + C$$

$$\therefore \quad \int 2^{\log_4 x} dx = \frac{2}{3} x^{3/2} + C$$

Example 7 Evaluate $\int \frac{(\sqrt{x}+1)(x^2 - \sqrt{x})}{x\sqrt{x} + x + \sqrt{x}} dx$.

Sol. Here, $I = \int \frac{(\sqrt{x}+1)\cdot\sqrt{x}(x^{3/2} - 1)}{\sqrt{x}(x + \sqrt{x} + 1)} dx$

$$\therefore \quad I = \int \frac{(\sqrt{x}+1)[(\sqrt{x})^3 - 1^3]}{(x + \sqrt{x} + 1)} dx$$

$$= \int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x + \sqrt{x} + 1)}{(x + \sqrt{x} + 1)} dx$$

[Using, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

$$= \int (x - 1)\, dx = \frac{x^2}{2} - x + C$$

Example 8 Evaluate

(i) $\int \frac{1+2x^2}{x^2(1+x^2)} dx$ (ii) $\int \frac{x^6 - 1}{(x^2+1)} dx$

Sol. (i) Here, $I = \int \frac{1+2x^2}{x^2(1+x^2)} dx = \int \frac{1 + x^2 + x^2}{x^2(1+x^2)} dx$

$$= \int \frac{1+x^2}{x^2(1+x^2)} dx + \int \frac{x^2}{x^2(1+x^2)} dx$$

$$= \int \frac{1}{x^2} dx + \int \frac{1}{1+x^2} dx = -\frac{1}{x} + \tan^{-1} x + C$$

(ii) Here, $I = \int \frac{x^6 - 1}{x^2+1} dx = \int \frac{x^6 + 1 - 2}{x^2+1} dx$

$$= \int \frac{(x^2)^3 + 1^3}{x^2+1} dx - \int \frac{2}{x^2+1} dx$$

$$I=\int\frac{(x^2+1)(x^4-x^2+1)}{(x^2+1)}\,dx-2\int\frac{dx}{x^2+1}$$

[Using, $a^3+b^3=(a+b)(a^2-ab+b^2)$]

$$=\int(x^4-x^2+1)\,dx-2\int\frac{1}{x^2+1}\,dx$$

$$=\frac{x^5}{5}-\frac{x^3}{3}+x-2\tan^{-1}x+C$$

| Example 9 Evaluate

(i) $\displaystyle\int\left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}+\frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right)dx$

(ii) $\displaystyle\int\left(\frac{x^{-6}-64}{4+2x^{-1}+x^{-2}}\cdot\frac{x^2}{4-4x^{-1}+x^{-2}}-\frac{4x^2(2x+1)}{1-2x}\right)dx$

Sol. (i) Here, $I=\displaystyle\int\left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}+\frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right)dx$

$$=\int\left(\frac{(1-x^{-2})+(x^{-2}-x)}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}\right)dx$$

$$=\int\left(\frac{1-x}{\sqrt{x}-\frac{1}{\sqrt{x}}}-\frac{2}{x^{3/2}}\right)dx=\int\left(\frac{1-x}{\frac{x-1}{\sqrt{x}}}-\frac{2}{x^{3/2}}\right)dx$$

$$=\int(-\sqrt{x}-2x^{-3/2})\,dx$$

$$=\left(-\frac{x^{3/2}}{3/2}-2\cdot\frac{x^{-1/2}}{-1/2}\right)+C=-\frac{2}{3}x^{3/2}+\frac{4}{\sqrt{x}}+C$$

(ii) Here

$$I=\int\left(\frac{x^{-6}-64}{4+2x^{-1}+x^{-2}}\cdot\frac{x^2}{4-4x^{-1}+x^{-2}}-\frac{4x^2(2x+1)}{1-2x}\right)dx$$

$$=\int\left(\frac{\frac{1-64x^6}{x^6}}{\frac{4x^2+2x+1}{x^2}}\cdot\frac{x^2}{\frac{4x^2-4x+1}{x^2}}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{1-(4x^2)^3}{x^6\cdot(4x^2+2x+1)}\cdot\frac{x^6}{(4x^2-4x+1)}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{(1-4x^2)(1+4x^2+16x^4)}{(4x^2+2x+1)(4x^2-4x+1)}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{(1-4x^2)\cdot(4x^2+2x+1)(4x^2-2x+1)}{(4x^2+2x+1)(2x-1)^2}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

[using, $16x^4+4x^2+1=16x^4+8x^2+1-4x^2$
$=(4x^2+1)^2-(2x)^2=(4x^2+1+2x)$]

$$=\int\left(\frac{(1-2x)(1+2x)(4x^2-2x+1)}{(1-2x)^2}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\left(\frac{(1+2x)\cdot(4x^2-2x+1)}{(1-2x)}-\frac{4x^2(2x+1)}{(1-2x)}\right)dx$$

$$=\int\frac{(2x+1)\{4x^2-2x+1-4x^2\}\,dx}{1-2x}$$

$$=\int\frac{(2x+1)(1-2x)}{(1-2x)}\,dx$$

$$=\int(2x+1)\,dx=x^2+x+C$$

| Example 10 Evaluate

(i) $\displaystyle\int\frac{1}{\sin(x-a)\cos(x-b)}\,dx$

(ii) $\displaystyle\int\frac{1}{\cos(x-a)\cos(x-b)}\,dx$

Sol. (i) $I=\displaystyle\int\frac{1}{\sin(x-a)\cos(x-b)}\,dx$

$$I=\frac{\cos(a-b)}{\cos(a-b)}\cdot\int\frac{dx}{\sin(x-a)\cos(x-b)}$$

$$=\frac{1}{\cos(a-b)}\cdot\int\frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\,dx$$

$$=\frac{1}{\cos(a-b)}\cdot\int\left\{\frac{\cos(x-b)\cdot\cos(x-a)}{\sin(x-a)\cos(x-b)}+\frac{\sin(x-b)\cdot\sin(x-a)}{\sin(x-a)\cos(x-b)}\right\}dx$$

$$=\frac{1}{\cos(a-b)}\int\{\cot(x-a)+\tan(x-b)\}\,dx$$

$$=\frac{1}{\cos(a-b)}\{\log|\sin(x-a)|-\log|\cos(x-b)|\}+C$$

$$=\frac{1}{\cos(a-b)}\log_e\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+C$$

(ii) $I=\displaystyle\int\frac{1}{\cos(x-a)\cos(x-b)}\,dx$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin\{(x-b)-(x-a)\}}{\cos(x-a)\cos(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\left\{\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)}-\frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right\}dx$$

$$=\frac{1}{\sin(a-b)}\int\{\tan(x-b)-\tan(x-a)\}\,dx$$

$$=\frac{1}{\sin(a-b)}[-\log|\cos(x-b)|+\log|\cos(x-a)|]+C$$

$$=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Example 11 Evaluate $\int \frac{\sin (x+a)}{\sin (x+b)} dx$.

Sol. Let $I = \int \frac{\sin (x + a)}{\sin (x + b)} dx$. Put $x + b = t \Rightarrow dx = dt$

$$\therefore \quad I = \int \frac{\sin (t - b + a)}{\sin t} dt$$

$$= \int \left\{ \frac{\sin t \cos (a - b)}{\sin t} + \frac{\cos t \sin (a - b)}{\sin t} \right\} dt$$

$$= \cos (a - b) \int 1 dt + \sin (a - b) \int \cot (t) dt$$

$$= t \cos (a - b) + \sin (a - b) \log | \sin t | + C$$

$$= (x + b) \cos (a - b) + \sin (a - b) \log | \sin (x + b)| + C$$

Example 12

(i) If $f'(x) = \frac{x}{2} + \frac{2}{x}$ and $f(1) = \frac{5}{4}$, then find $f(x)$.

(ii) The gradient of the curve is given by $\frac{dy}{dx} = 2x - \frac{3}{x^2}$.

The curve passes through (1, 2) find its equation.

Sol. (i) Given, $f'(x) = \frac{x}{2} + \frac{2}{x}$,

On integrating both sides w.r.t. x,

we get $\int f'(x) dx = \int \left(\frac{x}{2} + \frac{2}{x} \right) dx$

$$\Rightarrow \quad f(x) = \frac{1}{2} \cdot \frac{x^2}{2} + 2 \log |x| + c \quad \ldots \text{(i)}$$

Now, as $f(1) = \frac{5}{4}$ (called as initial value problem

i.e. when $x = 1, y = \frac{5}{4}$ or $f(1) = \frac{5}{4}$)

Putting, $x = 1$ in Eq. (i),

$$f(1) = \frac{1}{4} + 2 \log |1| + C, \text{ but } f(1) = \frac{5}{4}$$

$$\therefore \quad \frac{5}{4} = \frac{1}{4} + C \Rightarrow C = 1$$

$$\Rightarrow \quad f(x) = \frac{x^2}{4} + 2 \log |x| + 1$$

(ii) Given, $\frac{dy}{dx} = 2x - \frac{3}{x^2}$ or $dy = \left(2x - \frac{3}{x^2} \right) dx$,

On integrating both sides w.r.t. x, we get

$$\int dy = \int \left(2x - \frac{3}{x^2} \right) dx$$

$$\Rightarrow \quad y = \frac{2x^2}{2} + \frac{3}{x} + C$$

Since, curve passes through (1, 1).

$$\Rightarrow \quad 1 = 1 + 3 + C \Rightarrow C = -3$$

$$\therefore \quad f(x) = x^2 + \frac{3}{x} - 3$$

Important Points Related to Integration

1. $\int k f(x) dx = k \int f(x) dx$, where k is constant. i.e. the integral of the product of a constant and a function = the constant × integral of the function

2. $\int \{ f_1 (x) \pm f_2 (x) \pm \ldots \pm f_n (x) \} dx$
$$= \int f_1 (x) dx \pm \int f_2 (x) dx \pm \ldots \pm \int f_n (x) dx.$$
i.e. the integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the various functions.

3. **Geometrical interpretation of constant of integration** By adding C means the graph of function would shift in upward or downward direction along y-axis as C is +ve or − ve respectively.

e.g. $$y = \int x \, dx = \frac{x^2}{2} + C$$

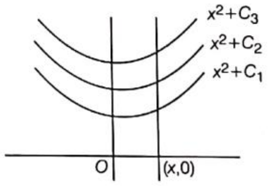

Figure. 1.1

$$\therefore \quad y = \int f(x) dx = F(x) + C$$

$$\Rightarrow \quad F'(x) = f(x); F'(x_1) = f(x_1)$$

Hence, $y = \int f(x) dx$ denotes a family of curves such that the slope of the tangent at $x = x_1$ on every member is same i.e. $F'(x_1) = f(x)$ [when x_1 lies in the domain of $f(x)$]

Hence, anti-derivative of a function is not unique. If $g_1(x)$ and $g_2(x)$ are two anti-derivatives of a function $f(x)$ on $[a, b]$, then they differ only by a constant.

i.e. $$g_1(x) - g_2(x) = C$$

Anti-derivative of a continuous function is differentiable.

4. If $f(x)$ is continuous, then
$$\int f(x) dx = F(x) + C$$
$$\Rightarrow \quad F'(x) = f(x)$$
$\Rightarrow$ always exists and is continuous.
$$\Rightarrow \quad F'(x)$$

5. If integral is discontinuous at $x = x_1$, then its anti-derivative at $x = x_1$ need not be discontinuous.

e.g. $\int x^{-1/3} dx$. Here, $x^{-1/3}$ is discontinuous at $x = 0$. But $\int x^{-1/3} dx = \frac{3}{2} x^{2/3} + C$ is continuous at $x = 0$.

6. Anti-derivative of a periodic function need not be a periodic function. e.g. $f(x) = \cos x + 1$ is periodic but $\int (\cos x + 1)\, dx = \sin x + x + C$ is a periodic.

Daily Life Applications

The Derivative			The Integral		
Function	**Its derivative function**	**In symbols**	**Function**	**It's Anti-derivative Function**	**In symbols**
Distance (s)	Velocity (v)	$v = \frac{ds}{dt}$	Velocity	Distance	$s = \int v(t)\, dt$
Velocity (v)	Acceleration (a)	$a = \frac{dv}{dt}$	Acceleration	Velocity	$v = \int a(t)\, dt$
Mass (μ)	Liner Density (ρ)	$\rho = \frac{d\mu}{dx}$	Linear Density	Mass	$\mu = \int \rho(x)\, dx$
Population (P)	Instantaneous growth	$\frac{dP}{dt}$	Instantaneous Growth	Population	$\rho = \int \left(\frac{dP}{dt}\right) dt$
Cost (C)	Marginal cost (MC)	$\text{MC} = \frac{dC}{dq}$	Marginal Cost	Cost	$C(q) = \int \left(\frac{dC}{dq}\right) dq$
Revenue (R)	Marginal Revenue (MR)	$\text{MR} = \frac{dR}{dq}$	Marginal Revenue	Revenue	$R(q) = \int \left(\frac{dR}{dq}\right) dq$

Here, q is quantity of products.

Exercise for Session 1

- Evaluate the following integration

1. $\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}$

2. $\int \frac{x^2 + 3}{x^6(x^2 + 1)} dx$

3. $\int \frac{(1+x)^2}{x(1+x^2)} dx$

4. $\int \frac{x^4}{1+x^2} dx$

5. $\int \frac{x^4 + x^2 + 1}{2(1+x^2)} dx$

6. $\int \frac{(x^2 + \sin^2 x)\sec^2 x}{(1+x^2)} dx$

7. $\int \frac{x^2}{(a+bx)^2} dx$

8. $\int 2^x \cdot e^x \cdot dx$

9. $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}} dx$

10. $\int (e^{a \log x} + e^{x \log a})\, dx$

11. $\int \frac{1 + \cos 4x}{\cot x - \tan x} dx$

12. $\int \tan x \tan 2x \tan 3x\, dx$

13. $\int \frac{\sin 4x}{\sin x} dx$

14. $\int \cos^3 x\, dx$

15. $\int \sin^3 x \cos^3 x\, dx$

Session 2

Methods of Integration

Methods of Integration

If the integral is not a derivative of a simple function, then the corresponding integrals cannot be found directly. In order to find the integral of complex problems.

e.g. $\int \frac{\sin x}{x}\, dx, \int \frac{\cos x}{x}\, dx, \int \frac{1}{\log x}\, dx$

Some Integrals which Cannot be Found

Any function continuous on an interval (a, b) has an anti-derivative in that interval. In other words, there exists a function $F(x)$ such that $F'(x) = f(x)$.

However, not every anti-derivative $F(x)$, even when it exists, is expressible in closed form in terms of elementary functions such as polynomials, trigonometric, logarithmic, exponential functions etc. Then, we say that such anti-derivatives or integrals "cannot be found".

Some typical examples are

(i) $\int \frac{\sin x}{x}\, dx$ (ii) $\int \frac{\cos x}{x}\, dx$

(iii) $\int \frac{1}{\log x}\, dx$ (iv) $\int \sqrt{1-k^2 \sin^2 x}\, dx$

(v) $\int \sqrt{\sin x}\, dx$ (vi) $\int \sin(x^2)\, dx$

(vii) $\int \cos(x^2)\, dx$ (viii) $\int x \tan x\, dx$

(ix) $\int e^{-x^2}\, dx$ (x) $\int e^{x^2}\, dx$

(xi) $\int \frac{x^2}{1+x^5}\, dx$ (xii) $\int \sqrt[3]{1+x^2}\, dx$

(xiii) $\int \sqrt{1+x^3}\, dx$ etc.

Integration by Substitution

(or by change of the independent variable)

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the form,

$$I = \int f(g(x)) \cdot g'(x)\, dx,$$

we substitute $g(x) = t$ and $g'(x)\, dx = dt$

The substitution reduces the integral to $\int f(t)\, dt$. After evaluating this integral we substitute back the value of t.

Example 13 Prove that

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C, n \neq 1.$$

Sol. Putting, $ax + b = t$, we get

$$adx = dt \text{ or } dx = \frac{1}{a} dt$$

$$\therefore \quad I = \int (ax+b)^n\, dx = \int t^n \cdot \frac{dt}{a} = \frac{1}{a} \cdot \frac{t^{n+1}}{n+1} + C$$

$$= \frac{1\,(ax+b)^{n+1} dx}{a(n+1)} + C.$$

Remarks

1. If $\int f(x)\, dx = g(x) + C$, then $\int f(ax+b)\, dx = \frac{1}{a} g(ax+b) + C$
2. If $\int \frac{1}{x}\, dx = \log|x| + C$, then $\int \frac{1}{ax+b}\, dx = \frac{1}{a} \log|ax+b| + C$

Thus, in any fundamental integral formulae given in article fundamental integration formulae if in place of x we have $(ax + b)$, then same formula is applicable but we must divide by coefficient of x or derivative of $(ax + b)$ i.e. a.

Here is the list of some of frequently used formulae

(i) $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$

(ii) $\int \frac{1}{ax+b}\, dx = \frac{1}{a} \log|ax+b| + C$

(iii) $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$

(iv) $\int a^{bx+c}\, dx = \frac{1}{b} \cdot \frac{a^{bx+c}}{\log a} + C$

(v) $\int \sin(ax+b)\, dx = -\frac{1}{a} \cos(ax+b) + C$

(vi) $\int \cos(ax+b)\, dx = \frac{1}{a} \sin(ax+b) + C$

(vii) $\int \sec^2(ax+b)\, dx = \frac{1}{a} \tan(ax+b) + C$

(viii) $\int \operatorname{cosec}^2 (ax+b)\, dx = -\frac{1}{a} \cot(ax+b) + C$

(ix) $\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + C$

(x) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$

(xi) $\int \tan(ax+b)\, dx = -\frac{1}{a}\log|\cos(ax+b)| + C$

(xii) $\int \cot(ax+b)\, dx = \frac{1}{a}\log|\sin(ax+b)| + C$

(xiii) $\int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)| + C$

(xiv) $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b)-\cot(ax+b)| + C$

Example 14 Evaluate

(i) $\int \frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}\,dx$ (ii) $\int \frac{8x+13}{\sqrt{4x+7}}\,dx.$

(iii) $\int (7x-2)\sqrt{3x+2}\,dx.$ (iv) $\int \frac{2+3x^2}{x^2(1+x^2)}\,dx.$

Sol. (i) Here, $I = \int \frac{dx}{\sqrt{3x+4}-\sqrt{3x+1}}$

$$= \int \frac{(\sqrt{3x+4}+\sqrt{3x+1})}{(\sqrt{3x+4}-\sqrt{3x+1})(\sqrt{3x+4}+\sqrt{3x+1})}\,dx$$

$$= \int \frac{(\sqrt{3x+4}+\sqrt{3x+1})}{(3x+4)-(3x+1)}\,dx$$

$$= \frac{1}{3}\int \sqrt{3x+4}\,dx + \frac{1}{3}\int \sqrt{3x+1}\,dx$$

$$= \frac{1}{3}\left\{\frac{(3x+4)^{3/2}}{3/2\times 3}\right\} + \frac{1}{3}\left\{\frac{(3x+1)^{3/2}}{3/2\times 3}\right\} + C$$

Using, $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)\,a} + C$

$$= \frac{2}{27}[(3x+4)^{3/2} + (3x+1)^{3/2}] + C$$

(ii) Here, $I = \int \frac{8x+13}{\sqrt{4x+7}}\,dx = \int \frac{8x+14-1}{\sqrt{4x+7}}\,dx$

$$= \int \frac{2(4x+7)-1}{\sqrt{4x+7}}\,dx$$

$$= 2\int \frac{4x+7}{\sqrt{4x+7}}\,dx - \int \frac{dx}{\sqrt{4x+7}}$$

$$= 2\int (\sqrt{4x+7})dx - \int (4x+7)^{-1/2}dx$$

$$= 2\left(\frac{(4x+7)^{3/2}}{3/2\times 4}\right) - \left(\frac{(4x+7)^{1/2}}{1/2\times 4}\right) + C$$

$$= \frac{1}{3}(4x+7)^{3/2} - \frac{1}{2}(4x+7)^{1/2} + C$$

(iii) Here, $I = \int (7x-2)\sqrt{3x+2}\,dx = 7\int \left(x-\frac{2}{7}\right)\sqrt{3x+2}\,dx$

$$= \frac{7}{3}\int \left(3x-\frac{6}{7}\right)\sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\int \left(3x+2-2-\frac{6}{7}\right)\sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\int (3x+2)\sqrt{3x+2}\,dx - \frac{20}{3}\int \sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\int (3x+2)^{3/2}\,dx - \frac{20}{3}\int \sqrt{3x+2}\,dx$$

$$= \frac{7}{3}\left(\frac{(3x+2)^{5/2}}{\frac{5}{2}\times 3}\right) - \frac{20}{3}\left(\frac{(3x+2)^{3/2}}{\frac{3}{2}\times 3}\right) + C$$

$$= \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2} + C$$

(iv) Here, $I = \int \frac{(2+3x^2)\,dx}{x^2(1+x^2)} = \int \frac{2+2x^2+x^2}{x^2(1+x^2)}\,dx$

$$= \int \left(\frac{2(1+x^2)}{x^2(1+x^2)} + \frac{x^2}{x^2(1+x^2)}\right)dx$$

$$= \int \left(\frac{2}{x^2} + \frac{1}{1+x^2}\right)dx = 2\int x^{-2}dx + \int \frac{1}{1+x^2}\,dx$$

$$= 2\cdot\frac{x^{-1}}{-1} + \tan^{-1}x + C = -\frac{2}{x} + \tan^{-1}x + C$$

Example 15 Evaluate

(i) $\int \frac{\sin(\log x)}{x}\,dx$ (ii) $\int \left(\frac{3\sin x + 4\cos x}{4\sin x - 3\cos x}\right)dx$

(iii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ (iv) $\int x\sin(4x^2+7)\,dx$

Sol. (i) $I = \int \frac{\sin(\log x)}{x}\,dx$

We know that, $\frac{d}{dx}(\log x) = \frac{1}{x}$

Thus, let $\log x = t$

$\Rightarrow \quad \frac{1}{x}\,dx = dt$ ...(i)

$\therefore \quad I = \int \sin(t)\,dt = -\cos(t) + C$

$= -\cos(\log x) + C$ [using Eq. (i)]

(ii) $I = \int \frac{3\sin x + 4\cos x}{4\sin x - 3\cos x}\,dx$

We know, $\frac{d}{dx}(4\sin x - 3\cos x) = (4\cos x + 3\sin x)$

Thus, let $4\sin x - 3\cos x = t$ ...(i)

$\Rightarrow \quad (4\cos x + 3\sin x)\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t} = \log|t| + C$

$= \log|4\sin x - 3\cos x| + C$ [using Eq. (i)]

(iii) $I = \int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ Let $m\tan^{-1}x = t$

$\Rightarrow \quad \frac{m}{1+x^2}\,dx = dt \Rightarrow \frac{1}{1+x^2}\,dx = \frac{1}{m}\,dt$

$\therefore \quad I = \int e^t \cdot \frac{dt}{m} \Rightarrow I = \frac{1}{m}\int e^t\,dt$

$I = \frac{1}{m}e^t + C = \frac{1}{m}e^{m\tan^{-1}x} + C$

(iv) $I = \int x\sin(4x^2+7)\,dx$

Let $4x^2 + 7 = t \Rightarrow 8x\,dx = dt \Rightarrow x\,dx = \frac{1}{8}\,dt$

$\therefore \quad I = \int \sin(t)\frac{dt}{8} = -\frac{1}{8}\cos(t) + C$

$= -\frac{1}{8}\cos(4x^2+7) + C$

Remarks

While solving product of trignometric, it is expedient to use the following trigonometric identities

1. $\sin mx \cos nx = \frac{1}{2}\{\sin(m-n)x + \sin(m+n)x\}$
2. $\cos mx \sin nx = \frac{1}{2}\{\sin(m+n)x - \sin(m-n)x\}$
3. $\sin mx \sin nx = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$
4. $\cos mx \cos nx = \frac{1}{2}\{\cos(m-n)x + \cos(m+n)x\}$

Example 16 Evaluate

(i) $\int \cos 4x \cos 7x\,dx$ (ii) $\int \cos x \cos 2x \cos 5x\,dx$

Sol. When calculating such integrals it is advisable to use the trigonometric product formulae.

(i) $\int \cos 4x \cos 7x\,dx$

Here, $\cos 4x \cos 7x = \frac{1}{2}(\cos 3x + \cos 11x)$,

[using $\cos mx \cdot \cos nx = \frac{1}{2}\{\cos(m-n)x + \cos(m+n)x\}$]

$\therefore \; I = \int \cos 4x \cos 7x\,dx = \frac{1}{2}\int(\cos 3x + \cos 11x)\,dx$

$= \frac{1}{2}\int \cos 3x\,dx + \frac{1}{2}\int \cos 11x\,dx$

$= \frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$

(ii) $\int \cos x \cos 2x \cos 5x\,dx$,

We have $(\cos x \cos 2x)\cos 5x = \frac{1}{2}(\cos x + \cos 3x)\cos 5x$,

[using $\cos mx.\cos nx = \frac{1}{2}\{\cos(m-n)x + \cos(m+n)x\}$]

$= \frac{1}{4}\{2\cos x\cos 5x + 2\cos 3x\cos 5x\}$

$= \frac{1}{4}\{(\cos 4x + \cos 6x) + (\cos 2x + \cos 8x)\}$

$\therefore \; \cos x\cos 2x\cos 5x = \frac{1}{4}\{\cos 2x + \cos 4x + \cos 6x + \cos 8x\}$

$\therefore \quad I = \int(\cos x\cos 2x\cos 5x)\,dx$

$= \frac{1}{4}\int(\cos 2x + \cos 4x + \cos 6x + \cos 8x)\,dx$

$= \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + \frac{\sin 8x}{32} + C$

Example 17 Evaluate

(i) $\int \sin x\cos x \cdot \cos 2x \cdot \cos 4x\,dx$

(ii) $\int \frac{1-\tan^2 x}{1+\tan^2 x}\,dx$ (iii) $\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$

(iv) $\int \frac{\cos 2x}{\cos^2 x\sin^2 x}\,dx$ (v) $\int \frac{\sec 2x - 1}{\sec 2x + 1}\,dx$

Sol. (i) Here, $I = \int \sin x\cos x \cdot \cos 2x\cos 4x\,dx$

$= \frac{1}{2}\int 2\sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x\,dx$

[using, $\sin 2x = 2\sin x \cdot \cos x$]

$= \frac{1}{2\times 2}\int 2\sin 2x \cdot \cos 2x \cdot \cos 4x \cdot dx$

$= \frac{1}{4}\int \sin 4x \cdot \cos 4x dx = \frac{1}{2\times 4}\int 2\sin 4x\cos 4x\,dx$

$= \frac{1}{8}\int \sin 8x\,dx = \frac{-\cos 8x}{64} + C$

(ii) Here, $I = \int \frac{1-\tan^2 x}{1+\tan^2 x}\,dx$

$\therefore \quad I = \int \cos 2x\,dx$ Using, $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$

$= \frac{\sin 2x}{2} + c$

(iii) Here, $I = \int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$,

$= \int \frac{1+\cos^2 x}{1+2\cos^2 x - 1}\,dx = \int \frac{1+\cos^2 x}{2\cos^2 x}\,dx$

[Using, $\cos 2x = 2\cos^2 x - 1$]

$= \frac{1}{2}\int(\sec^2 x + 1)\,dx = \frac{1}{2}(\tan x + x) + C$

(iv) Here, $I = \int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x}\,dx$

$I = \int \frac{(\cos^2 x - \sin^2 x)\,dx}{\cos^2 x \cdot \sin^2 x}$

[Using, $\cos 2x = \cos^2 x - \sin^2 x$]

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int (\text{cosec}^2 x - \sec^2 x)\, dx = -\cot x - \tan x + C$$

(v) Here, $I = \int \frac{\sec 2x - 1}{\sec 2x + 1} \cdot dx = \int \frac{1 - \cos 2x}{1 + \cos 2x}\, dx,$

We get, $I = \int \frac{2\sin^2 x}{2\cos^2 x}\, dx$

[using, $1 - \cos 2x = 2\sin^2 x$ and $1 + \cos 2x = 2\cos^2 x$]

$$= \int \tan^2 x \cdot dx$$

As, $\tan^2 x = \sec^2 x - 1$

$\therefore \quad I = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

| Example 18 Evaluate

$$I = \int \left(\frac{\cot^2 2x - 1}{2\cot 2x} - \cos 8x \cdot \cot 4x \right) dx.$$

Sol. Here, $I = \int \left(\frac{\cot^2 2x - 1}{2\cot 2x} - \cos 8x \cdot \cot 4x \right) dx$

$$I = \int (\cot 4x - \cos 8x \cdot \cot 4x)\, dx$$

$$\left[\text{using } \cot 2A = \frac{\cot^2 A - 1}{2\cot A} \right]$$

$$= \int \cot 4x (1 - \cos 8x)\, dx$$

[using $1 - \cos 2A = 2\sin^2 A$]

$$= \int \cot 4x \cdot 2\sin^2 (4x)\, dx$$

$$= \int \frac{\cos 4x}{\sin 4x} \cdot 2\sin^2 4x\, dx$$

$$= \int 2\sin 4x \cdot \cos 4x dx,$$

using $\sin 2A = 2\sin A \cos A$

$$= \int \sin 8x\, dx = \frac{-\cos 8x}{8} + C$$

Exercise for Session 2

- Solve the following integration

1. $\int \frac{dx}{1 + \sin x}$

2. $\int \frac{\cos x - \sin x}{\cos x + \sin x} \cdot (2 + 2\sin 2x)\, dx$

3. $\int (3\sin x \cos^2 x - \sin^3 x)\, dx$

4. $\int \cos x^\circ\, dx$

5. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} \cdot dx$, here $(\sin x + \cos x) > 0$

6. $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \cdot dx$

7. $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cdot \cos^2 x}\, dx$

8. $\int \sec^2 x \cdot \text{cosec}^2 x\, dx$

9. $\int \sqrt{1 - \sin 2x}\, dx$

10. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x}\, dx$

11. $\int \left(\sin^2 \left(\frac{9\pi}{8} + \frac{x}{4} \right) - \sin^2 \left(\frac{7\pi}{8} + \frac{x}{4} \right) \right) dx$

12. $\int \frac{\cos 4x + 1}{\cot x - \tan x}\, dx$

13. $\int \left(\sin \alpha \cdot \sin (x - \alpha) + \sin^2 \left(\frac{x}{2} - \alpha \right) \right) dx$

14. $\int \frac{\sin 2x + \sin 5x - \sin 3x}{\cos x + 1 - 2\sin^2 2x}\, dx$

15. $\int \frac{\cos^4 x + \sin^4 x}{\sqrt{1 + \cos 4x}}\, dx$, here $\cos 2x > 0$

Session 3

Some Special Integrals

Some Special Integrals

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$

(v) $\int \frac{dx}{\sqrt{a^2+x^2}} = \log|x+\sqrt{x^2+a^2}|+C$

(vi) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+C$

(vii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+C$

(viii) $\int \sqrt{a^2+x^2}\,dx$

$=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\log|x+\sqrt{a^2+x^2}|+C$

(ix) $\int \sqrt{x^2-a^2}\,dx$

$=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\log|x+\sqrt{x^2-a^2}|+C$

Some Important Substitutions

Expression	Substitution
a^2+x^2	$x=a\tan\theta$ or $a\cot\theta$
a^2-x^2	$x=a\sin\theta$ or $a\cos\theta$
x^2-a^2	$x=a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x=\alpha\cos^2\theta+\beta\sin^2\theta$

Application of these Formulae

The above standard integrals are very important. Given below are integrals which are applications of these.

Type I

(i) $\int \frac{dx}{ax^2+bx+c}$ (ii) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(iii) $\int \sqrt{ax^2+bx+c}\,dx$

If ax^2+bx+c can be factorised, then the integration is easily done by the method of partial fractions (explained later). If the denominator cannot be factorised, then express it as the sum or difference of two squares by the method of completing the square

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}\right)-\frac{b^2}{4a^2}\right\}$$

make the substitution $x+\frac{b}{2a}=t$.

| Example 19 Evaluate

(i) $\int \frac{1}{x^2-x+1}dx$ (ii) $\int \frac{1}{2x^2+x-1}dx$

(iii) $\int \frac{1}{\sqrt{x^2-2x+3}}dx$ (iv) $\int \sqrt{2x^2-3x+1}\,dx$

Sol. (i) $I=\int \frac{dx}{x^2-x+1}$,

completing x^2-x+1 into perfect square.

$$=\int \frac{dx}{x^2-x+1/4-1/4+1}=\int \frac{dx}{(x-1/2)^2+3/4}$$

$$\therefore I=\int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2}$$

$$=\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)+C$$

$$\therefore I=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

$$\left[\text{using} \int \frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right]$$

(ii) $I=\int \frac{1}{2x^2+x-1}dx=\frac{1}{2}\int \frac{1}{x^2+x/2-1/2}dx$

$$=\frac{1}{2}\int \frac{1}{x^2+x/2+1/16-1/16-1/2}dx$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2 - 9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2 - (3/4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right| + C$$

$$\left[\text{using } \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right| + C$$

$$\therefore \quad I = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + C$$

(iii) $I = \int \frac{dx}{\sqrt{x^2-2x+3}} = \int \frac{dx}{\sqrt{x^2-2x+1-1+3}}$

$$= \int \frac{dx}{\sqrt{(x-1)^2 + (\sqrt{2})^2}}$$

$$= \log|(x-1) + \sqrt{(x-1)^2 + (\sqrt{2})^2}|$$

$$\left[\text{using } \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x + \sqrt{x^2+a^2}| + C\right]$$

$$\therefore \quad I = \log|(x-1) + \sqrt{x^2-2x+3}| + C$$

(iv) $I = \int (\sqrt{2x^2-3x+1})\,dx = \sqrt{2}\int(\sqrt{x^2-3x/2+1/2})\,dx$

$$= \sqrt{2}\int(\sqrt{x^2-3x/2+9/16-9/16+1/2})\,dx$$

$$= \sqrt{2}\int(\sqrt{(x-3/4)^2-1/16})\,dx$$

$$= \sqrt{2}\left\{\begin{array}{l}\frac{1}{2}(x-3/4)\sqrt{(x-3/4)^2-1/16} \\ -\frac{1}{16\times 2}\log|(x-3/4)+\sqrt{(x-3/4)^2-1/16}|\end{array}\right\} + C$$

$$= \sqrt{2}\left\{\begin{array}{l}\frac{1}{8}(4x-3)\sqrt{x^2-3x/2+1/2} \\ -\frac{1}{32}\log|(x-3/4)+\sqrt{x^2-3x/2+1/2}|\end{array}\right\} + C$$

Example 20 Evaluate

(i) $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$ (ii) $\int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx$

(iii) $\int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$ (iv) $\int \sqrt{\frac{x}{a^3-x^3}}\,dx$

Sol. (i) Here, $I = \int \frac{dx}{\sqrt{1-e^{2x}}}$

Let, $1 - e^{2x} = t^2$

Then, $-2e^{2x}dx = 2t\,dt$

$$\Rightarrow \quad dx = -\frac{-t\,dt}{e^{2x}},$$

$$dx = \frac{t\,dt}{t^2-1} = \int \frac{1}{\sqrt{t^2}}\cdot\frac{t}{t^2-1}\,dt$$

$$I = \int \frac{dt}{t^2-1} = \frac{1}{2\cdot 1}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1}\right| + C$$

(ii) Here, $I = \int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx$

Let, $x^2 = t, 2x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{\sqrt{1-t-t^2}} = \int \frac{dt}{\sqrt{1+\frac{1}{4}-\frac{1}{4}-t-t^2}}$$

$$I = \int \frac{dt}{\sqrt{5/4-(t+1/2)^2}} = \int \frac{dt}{\sqrt{(\sqrt{5}/2)^2-(t+1/2)^2}}$$

$$= \sin^{-1}\left(\frac{t+1/2}{\sqrt{5}/2}\right) + C = \sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right) + C$$

(iii) Here, $I = \int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$. Let, $a^x = t$

$$\therefore \quad a^x \log a\,dx = dt, \quad a^x\,dx = \frac{dt}{\log a}$$

$$\therefore \quad I = \int \frac{1}{\sqrt{1-t^2}}\cdot\frac{dt}{\log a} = \frac{1}{\log a}\cdot\sin^{-1}(t) + C$$

$$I = \frac{1}{\log a}\sin^{-1}(a^x) + C$$

(iv) Here, $I = \int \sqrt{\frac{x}{a^3-x^3}}\,dx = \int \frac{x^{1/2}\,dx}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}$

Let, $x^{3/2} = t, \frac{3}{2}x^{1/2}dx = dt, x^{1/2}\,dx = \frac{2}{3}dt$

$$= \int \frac{\frac{2}{3}dt}{\sqrt{(a^{3/2})^2-(t)^2}}$$

$$= \frac{2}{3}\cdot\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

Example 21 Evaluate

(i) $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}\,dx$. (ii) $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$.

(iii) $\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}\,dx$.

Sol. (i) Let $I=\int \frac{\cos x\,dx}{\sqrt{\sin^2 x-2\sin x-3}}$

Put $\sin x=t \quad \therefore \cos x\,dx=dt$

$\Rightarrow I=\int \frac{dt}{\sqrt{t^2-2t-3}}=\int \frac{dt}{\sqrt{t^2-2t+1-1-3}}$

$I=\int \frac{dt}{\sqrt{(t-1)^2-(2)^2}}$

$=\log|(t-1)+\sqrt{(t-1)^2-(2)^2}|+C$

$\therefore I=\log|(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}|+C$

(ii) Let $I=\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

$I=\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\times\frac{\sin(x-\alpha)}{\sin(x-\alpha)}}\,dx$

$=\int \frac{\sin(x-\alpha)}{\sqrt{\sin^2 x-\sin^2\alpha}}\,dx$

$I=\int \frac{\sin x\cos\alpha-\cos x\sin\alpha}{\sqrt{\sin^2 x-\sin^2\alpha}}\,dx$

$=\cos\alpha\int \frac{\sin x\,dx}{\sqrt{\sin^2 x-\sin^2\alpha}}-\sin\alpha\int \frac{\cos x\,dx}{\sqrt{\sin^2 x-\sin^2\alpha}}$

$=\cos\alpha\int \frac{\sin x\,dx}{\sqrt{1-\cos^2 x-\sin^2\alpha}}$

$-\sin\alpha\int \frac{\cos x\,dx}{\sqrt{\sin^2 x-\sin^2\alpha}}$

$=\cos\alpha\int \frac{\sin x\,dx}{\sqrt{\cos^2\alpha-\cos^2 x}}-\sin\alpha\int \frac{\cos x\,dx}{\sqrt{\sin^2 x-\sin^2\alpha}}$

In the first part, put $\cos x=t$, so that $-\sin x\,dx=dt$ and in second part, put $\sin x=u$, so that $\cos x\,dx=du$

$\therefore I=-\cos\alpha\int \frac{dt}{\sqrt{\cos^2\alpha-t^2}}-\sin\alpha\int \frac{du}{\sqrt{u^2-\sin^2\alpha}}$

$=-\cos\alpha\cdot\sin^{-1}\left(\frac{t}{\cos\alpha}\right)-\sin\alpha\cdot\log$
$|u-\sqrt{(u^2-\sin^2\alpha)}|+C$

$=-\cos\alpha\cdot\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right)-\sin\alpha\cdot\log$
$|\sin x-\sqrt{\sin^2 x-\sin^2\alpha}|+C$

(iii) $I=\int \frac{2\sin 2x-\cos x}{6-\cos^2 x-4\sin x}\,dx$

$=\int \frac{(4\sin x-1)\cos x}{6-(1-\sin^2 x)-4\sin x}\,dx$

$=\int \frac{(4\sin x-1)\cos x}{\sin^2 x-4\sin x+5}\,dx$

Put $\sin x=t$, so that $\cos x\,dx=dt$

$\therefore \quad I=\int \frac{(4t-1)\,dt}{(t^2-4t+5)}$ ...(i)

Now, let $(4t-1)=\lambda(2t-4)+\mu$

Comparing coefficients of like powers of t, we get

$2\lambda=4, -4\lambda+\mu=-1$

$\Rightarrow \lambda=2, \mu=7$ ...(ii)

$\therefore I=\int \frac{2(2t-4)+7}{t^2-4t+5}dt$ [using Eqs. (i) and (ii)]

$=2\int \frac{2t-4}{t^2-4t+5}\,dt+7\int \frac{dt}{t^2-4t+5}$

$=2\log|t^2-4t+5|+7\int \frac{dt}{t^2-4t+4-4+5}$

$=2\log|t^2-4t+5|+7\int \frac{dt}{(t-2)^2+(1)^2}$

$=2\log|t^2-4t+5|+7\tan^{-1}(t-2)+C$

$=2\log|\sin^2 x-4\sin x+5|+7\tan^{-1}(\sin x-2)+C$

Type II

(i) $\int \frac{(px+q)\,dx}{ax^2+bx+c}$ (ii) $\int \frac{(px+q)}{\sqrt{ax^2+bx+c}}\,dx$

(iii) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

The linear factor $(px+q)$ is expressed in terms of the derivative of the quadratic factor ax^2+bx+c together with a constant as $px+q=\frac{\lambda d}{dx}(ax^2+bx+c)+\mu$

$\Rightarrow \quad px+q=\lambda(2ax+b)+\mu$

Here, we have to find λ and μ and replace $(px+q)$ by $\{\lambda(2ax+b)+\mu\}$ in (i), (ii) and (iii).

Example 22 Evaluate

(i) $\int \sqrt{\frac{a-x}{a+x}}\,dx$ (ii) $\int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

Sol. (i) Let $I=\int \sqrt{\frac{a-x}{a+x}}\,dx$

$I=\int \sqrt{\frac{a-x}{a+x}\times\frac{a-x}{a-x}}\,dx=\int \frac{a-x}{\sqrt{a^2-x^2}}\,dx$

$I=\int \frac{a}{\sqrt{a^2-x^2}}\,dx-\int \frac{x}{\sqrt{a^2-x^2}}\,dx$

$=a\cdot\sin^{-1}\left(\frac{x}{a}\right)+\int \frac{t\,dt}{t}$

Put $a^2-x^2=t^2 \Rightarrow -2x\,dx=2t\,dt \Rightarrow x\,dx=-t\,dt$

$$= a \cdot \sin^{-1}\left(\frac{x}{a}\right) + t + C$$

$$= a \cdot \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + C$$

(ii) Let $I = \int x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx$

Put $x^2 = t \Rightarrow 2x\, dx = dt$

$$\therefore \quad I = \int \sqrt{\frac{a^2 - t}{a^2 + t}} \cdot \frac{dt}{2} = \frac{1}{2}\int \sqrt{\frac{a^2 - t}{a^2 + t} \cdot \frac{a^2 - t}{a^2 - t}}\, dt$$

$$= \frac{1}{2}\int \frac{a^2 - t}{\sqrt{a^4 - t^2}}\, dt$$

$$= \frac{1}{2} a^2 \int \frac{dt}{\sqrt{(a^2)^2 - (t)^2}} - \frac{1}{2}\int \frac{t\, dt}{\sqrt{a^4 - t^2}}$$

$$= \frac{1}{2} a^2 \cdot \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{4}\int \frac{du}{\sqrt{u}},$$

where $a^4 - t^2 = u \Rightarrow -2t\, dt = du$

$$= \frac{1}{2} a^2 \cdot \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{4}\cdot\left(\frac{u^{1/2}}{1/2}\right) + C$$

[where $t = x^2$ and $u = a^4 - x^4$]

$$= \frac{1}{2} a^2 \cdot \sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\cdot\sqrt{a^4 - x^4} + C$$

Integrals of the Form $\int \frac{k(x)}{ax^2 + bx + c}\, dx$, where $k(x)$ is a Polynomial of Degree Greater than 2

To evaluate this type of integrals we divide the numerator by denominator and express the integral as

$Q(x) + \frac{R(x)}{ax^2 + bx + c}$, where $R(x)$ is a linear function of x.

| Example 23 Evaluate

(i) $\int x\sqrt{1 + x - x^2}\, dx$ (ii) $\int (x + 1)\sqrt{1 - x - x^2}\, dx$

Sol. (i) Let $I = \int x\sqrt{1 + x - x^2}\, dx$

Put $x = \lambda\left\{\frac{d}{dx}(1 + x - x^2)\right\} + \mu$

Then, comparing the coefficients of like powers of x, we get

$1 = -2\lambda$ and $\lambda + \mu = 0 \Rightarrow \lambda = -1/2, \mu = 1/2$

$\therefore \quad I = \int x\sqrt{1 + x - x^2}\, dx$

$$= \int\left\{-\frac{1}{2}(1 - 2x) + \frac{1}{2}\right\}\sqrt{1 + x - x^2}\, dx$$

$$= -\frac{1}{2}\int (1 - 2x)\sqrt{1 + x - x^2}\, dx + \frac{1}{2}\int \sqrt{1 + x - x^2}\, dx$$

$$= -\frac{1}{2}\int \sqrt{t}\, dt + \frac{1}{2}\int \sqrt{-(x^2 - x + \frac{1}{4} - \frac{1}{4} - 1)}\, dx$$

[where $t = 1 + x - x^2$]

$$= -\frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\, dx$$

$$= -\frac{1}{3} t^{3/2} + \frac{1}{2}$$

$$\left[\frac{(x - 1/2)}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2 \sin^{-1}\left(\frac{x - 1/2}{\sqrt{5}/2}\right)\right] + C$$

$$= -\frac{1}{3}(1 + x - x^2)^{3/2} + \frac{1}{2}\left[\frac{(x - 1/2)}{2}\sqrt{1 + x - x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x - 1}{\sqrt{5}}\right)\right] + C$$

(ii) Let $I = \int (x + 1)\sqrt{1 - x - x^2}\, dx$

Put, $(x + 1) = \lambda \cdot \left(\frac{d}{dx}(1 - x - x^2)\right) + \mu$

Then, $(x + 1) = \lambda(-1 - 2x) + \mu$ comparing the coefficients of like powers of x, we get $-2\lambda = 1$ and $\mu - \lambda = 1 \Rightarrow \lambda = -1/2$ and $\mu = 1/2$.

$$\therefore \quad (x + 1) = -\frac{1}{2}(-1 - 2x) + \frac{1}{2}$$

So, $\int (x + 1)\sqrt{1 - x - x^2}\, dx = \int\left\{-\frac{1}{2}(-1 - 2x) + \frac{1}{2}\right\}\sqrt{1 - x - x^2}\, dx$

$$= -\frac{1}{2}\int (-1 - 2x)\sqrt{1 - x - x^2}\, dx + \frac{1}{2}\int \sqrt{1 - x - x^2}\, dx$$

$$= -\frac{1}{2}\int (-1 - 2x)\sqrt{1 - x - x^2}\, dx + \frac{1}{2}\int \sqrt{\left\{1 - \left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right)\right\}}\, dx$$

$$= -\frac{1}{2}\sqrt{t}\, dt + \frac{1}{2}\int \sqrt{(\sqrt{5}/2)^2 - (x + 1/2)^2}\, dx$$

[where $t = 1 + x - x^2$]

$$= -\frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + \frac{1}{2}\left\{\frac{1}{2}\left(x + \frac{1}{2}\right)\sqrt{1 - x - x^2} + \frac{1}{2}\cdot\frac{5}{4}\sin^{-1}\left(\frac{x + 1/2}{\sqrt{5}/2}\right)\right\} + C$$

$$= -\frac{1}{3}(1 - x - x^2)^{3/2} + \frac{1}{8}(2x + 1)\sqrt{1 - x - x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x + 1}{\sqrt{5}}\right) + C$$

Example 24 Evaluate $\int \frac{x^2+x+3}{x^2-x-2}\,dx$.

Sol. Let $I = \int \frac{x^2+x+3}{x^2-x-2}\,dx$

$\therefore \quad I = \int \left(1 + \frac{2x+5}{x^2-x-2}\right) dx$

$\Rightarrow \quad I = \int 1\,dx + \int \frac{2x+5}{x^2-x-2}\,dx$

$I = x + \int \frac{2x+5}{x^2-x-2}\,dx$

Put, $2x+5 = \lambda \left\{\frac{d}{dx}(x^2-x-2)\right\} + \mu$. Then, obtaining $2x+5 = \lambda\,(2x-1)+\mu$, comparing the coefficients of like terms. We get, $2 = 2\lambda$ and $5 = \mu - \lambda$

$\therefore \quad \lambda = 1$ and $\mu = 6$

$$\therefore \ I = x + \int \frac{\lambda\,(2x-1)+\mu}{x^2-x-2}\,dx$$

$$= x + \int \frac{2x-1}{x^2-x-2}\,dx + 6\int \frac{1}{x^2-x-2}\,dx$$

$$= x + \int \frac{1}{t}\,dt + 6\int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}-2}$$

[where $t = x^2-x-2$]

$$= x + \log|t| + 6\int \frac{dx}{(x-1/2)^2-(3/2)^2}\,dx$$

$$= x + \log|x^2-x-2| + 6\cdot\frac{1}{2(3/2)}\log\left|\frac{x-\frac{1}{2}-\frac{3}{2}}{x-\frac{1}{2}+\frac{3}{2}}\right| + C$$

$$= x + \log|x^2-x-2| + 2\cdot\log\left|\frac{x-2}{x+1}\right| + C$$

Integrals of the Type

1. $\int \frac{ax^2+bx+c}{(px^2+qx+r)}\,dx$

2. $\int \frac{(ax^2+bx+c)}{\sqrt{px^2+qx+r}}\,dx$

3. $\int (ax^2+bx+c)\sqrt{px^2+qx+r}\,dx$

In above cases; substitute

$$ax^2+bx+c = \lambda\,(px^2+qx+r) + \mu\left\{\frac{d}{dx}(px^2+qx+r)\right\} + \gamma.$$

Find λ, μ and γ. These integrations reduces to integration of three independent functions.

Example 25 Evaluate $\int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$.

Sol. Let $I = \int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$

Put $2x^2+5x+4 = \lambda\,(x^2+x+1) + \mu\left\{\frac{d}{dx}(x^2+x+1)\right\} + \gamma$

or $2x^2+5x+4 = \lambda\,(x^2+x+1) + \mu\,(2x+1) + \gamma$

Comparing the coefficients of like terms, we get

$2 = \lambda, \quad 5 = \lambda + 2\mu, \quad 4 = \lambda + \mu + \gamma$

$\therefore \quad \lambda = 2, \quad \mu = 3/2, \quad \gamma = 1/2$

Hence, the above integral reduces to

$$I = \int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$$

$$= \int \left(2\cdot\frac{(x^2+x+1)}{\sqrt{x^2+x+1}} + \frac{3}{2}\cdot\frac{(2x+1)}{\sqrt{x^2+x+1}} + \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+x+1}}\right) dx$$

$$= 2\int \sqrt{x^2+x+1}\,dx + \frac{3}{2}\int \frac{dt}{\sqrt{t}} + \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

[where $t = x^2+x+1$]

$$= 2\int \sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}\,dx + \frac{3}{2}\cdot\frac{t^{1/2}}{1/2} + \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}}$$

$$= 2\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{1}{2}\cdot\frac{3}{4}\cdot\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right]$$
$$+ 3\sqrt{x^2+x+1} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + C$$

$$\therefore \ I = \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right\}$$
$$+ 3\sqrt{x^2+x+1} + \frac{1}{2}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right\} + C$$

$$\Rightarrow I = \left(x+\frac{7}{2}\right)\sqrt{x^2+x+1} + \frac{5}{4}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right\} + C$$

Trigonometric Integrals

(a) Integrals of the Form

$$\int \frac{1}{a\cos^2 x + b\sin^2 x}\,dx, \int \frac{1}{a+b\sin^2 x}\,dx, \int \frac{1}{a+b\cos^2 x}\,dx,$$
$$\int \frac{1}{(a\sin x + b\cos x)^2}\,dx, \int \frac{1}{a+b\sin^2 x + c\cos^2 x}\,dx$$

To evaluate this type of integrals, divide numerator and denominator both by $\cos^2 x$, replace $\sec^2 x$, if any, in denominator by $(1+\tan^2 x)$ and put $\tan x = t$. So that $\sec^2 x\, dx = dt$.

| Example 26 Evaluate

(i) $\int \frac{1}{4\sin^2 x + 9\cos^2 x} dx$ (ii) $\int \frac{\sin x}{\sin 3x} dx$

Sol. (i) $I = \int \frac{dx}{4\sin^2 x + 9\cos^2 x}$

Here, dividing numerator and denominator by $\cos^2 x$. We get

$$I = \int \frac{\sec^2 x}{4\tan^2 x + 9} dx$$

Put $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{4t^2+9} = \frac{1}{4}\int \frac{dt}{t^2+(3/2)^2}$$

$$= \frac{1}{4}\cdot\frac{1}{3/2}\tan^{-1}\left(\frac{t}{3/2}\right) + C$$

$$I = \frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right) + C$$

(ii) Let $I = \int \frac{\sin x}{\sin 3x} dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x} dx$

$$I = \int \frac{dx}{3 - 4\sin^2 x}$$

Dividing numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x\, dx}{3\sec^2 x - 4\tan^2 x} = \int \frac{\sec^2 x\, dx}{3(1+\tan^2 x) - 4\tan^2 x}$$

$$I = \int \frac{\sec^2 x\, dx}{3 - \tan^2 x}$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{(\sqrt{3})^2 - (t)^2} = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C$$

$$I = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$$

(b) Integrals of the Form

$$\int \frac{1}{a\sin x + b\cos x} dx, \int \frac{1}{a + b\sin x} dx,$$

$$\int \frac{1}{a + b\cos x} dx, \int \frac{1}{a\sin x + b\cos x + c} dx$$

To evaluate this type of integrals we put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ and replace $\tan x/2 = t$, by performing these steps the integral reduces to the form $\int \frac{1}{at^2+bt+c} dt$ which can be evaluated by methods discussed earlier.

| Example 27 Evaluate

(i) $\int \frac{dx}{2+\sin x + \cos x}$ (ii) $\int \frac{dx}{\sqrt{3}\sin x + \cos x}$

Sol. For this type we use, $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$,

$$\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}.$$

(i) Let $I = \int \frac{dx}{2+\sin x+\cos x}$

$$= \int \frac{dx}{2 + \frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{2 + 2\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$I = \int \frac{\sec^2\frac{x}{2} dx}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 3}$$

Put $\tan\frac{x}{2} = t$

$$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2} dx = dt = \int \frac{2\,dt}{t^2+2t+3} = 2\int \frac{dt}{t^2+2t+1+2}$$

$$= 2\int \frac{dt}{(t+1)^2 + (\sqrt{2})^2}$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + C$$

$$I = \sqrt{2}\tan^{-1}\left(\frac{\tan x/2 + 1}{\sqrt{2}}\right) + C$$

(ii) Let $I = \int \frac{dx}{\sqrt{3}\sin x + \cos x}$

$$= \int \frac{dx}{\frac{\sqrt{3}\cdot 2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{\sec^2\frac{x}{2} dx}{2\sqrt{3}\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

$$\therefore \quad I=\int\frac{2dt}{-t^2+2\sqrt{3}t+1}=2\int\frac{dt}{-t^2+2\sqrt{3}t-3+3+1}$$

$$=2\int\frac{dt}{4-(t-\sqrt{3})^2}=2\int\frac{dt}{(2)^2-(t-\sqrt{3})^2}$$

$$=2\cdot\frac{1}{2(2)}\log\left|\frac{2+t-\sqrt{3}}{2-t+\sqrt{3}}\right|+C$$

$$\therefore \quad I=\frac{1}{2}\log\left|\frac{2-\sqrt{3}+\tan x/2}{2+\sqrt{3}-\tan x/2}\right|+C$$

(c) Alternative Method to Evaluate the Integrals of the Form

$$\int\frac{1}{a\sin x+b\cos x}\,dx$$

To evaluate this type of integrals we substitute $a=r\cos\theta$, $b=r\sin\theta$ and so

$$r=\sqrt{a^2+b^2},\quad \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \quad a\sin x+b\cos x=r\sin(x+\theta)$$

So, $$\int\frac{1}{a\sin x+b\cos x}\,dx=\frac{1}{r}\int\frac{1}{\sin(x+\theta)}\,dx$$

$$=\frac{1}{r}\int\operatorname{cosec}(x+\theta)\,dx=\frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right|+C$$

$$\therefore \quad \int\frac{1}{a\sin x+b\cos x}\,dx=\frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|+C$$

Example 28 Evaluate $\int\frac{1}{\sqrt{3}\sin x+\cos x}\,dx$.

Sol. Let $\sqrt{3}=r\sin\theta$ and $1=r\cos\theta$.

Then, $r=\sqrt{(\sqrt{3})^2+(1)^2}=2$ and $\tan\theta=\frac{\sqrt{3}}{1} \Rightarrow \theta=\frac{\pi}{3}$

$$\therefore \quad I=\int\frac{1}{\sqrt{3}\sin x+\cos x}\,dx$$

$$=\int\frac{1}{r\sin\theta\sin x+r\cos\theta\cos x}\,dx$$

$$=\frac{1}{r}\int\frac{dx}{\cos(x-\theta)}=\frac{1}{r}\int\sec(x-\theta)\,dx$$

$$=\frac{1}{r}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\theta}{2}\right)\right|+C$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{6}\right)\right|+C$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+C$$

(d) Integrals of the Form

$$\int\frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}\,dx,\quad \int\frac{p\cos x+q\sin x}{a\cos x+b\sin x}\,dx$$

Rule for (i) In this integral express numerator as,

λ (denominator) $+\mu$ (diffn. of denominator) $+\gamma$.

Find λ,μ and γ by comparing coefficients of $\sin x,\cos x$ and constant term and split the integral into sum of three integrals.

$$\lambda\int dx+\mu\int\frac{\text{d.c.of (denominator)}}{\text{denominator}}\,dx+\gamma\int\frac{dx}{a\sin x+b\cos x+c}$$

Rule for (ii) Express numerator as λ (denominator) $+\mu$ (differentiation of denominator) and find λ and μ as above.

Example 29 Evaluate $\int\frac{(2+3\cos x)}{\sin x+2\cos x+3}\,dx$.

Sol. Write the numerator $=\lambda$ (denominator) $+\mu$ (d.c. of denominator) $+\gamma$

$\Rightarrow 2+3\cos x=\lambda(\sin x+2\cos x+3)+\mu(\cos x-2\sin x)+\gamma$

Comparing the coefficients of $\sin x$, $\cos x$ and constant terms, we get

$$0=\lambda-2\mu,\ 3=2\lambda+\mu,\ 2=3\lambda+\gamma$$

$$\Rightarrow \quad \lambda=6/5,\ \mu=3/5,\ \gamma=-8/5$$

Hence, $$I=\frac{6}{5}\int 1\,dx+\frac{3}{5}\int\frac{\cos x-2\sin x}{\sin x+2\cos x+3}\,dx-\frac{8}{5}\int\frac{dx}{\sin x+2\cos x+3}$$

$$=\frac{6}{5}\cdot x+\frac{3}{5}\log|\sin x+2\cos x+3|-\frac{8}{5}I_3 \quad \text{...(i)}$$

Where, $$I_3=\int\frac{dx}{\sin x+2\cos x+3}$$

$$=\int\frac{dx}{\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2}+3}$$

$$=\int\frac{\sec^2\frac{x}{2}\,dx}{2\tan\frac{x}{2}+2-2\tan^2\frac{x}{2}+3+3\tan^2\frac{x}{2}}$$

$$=\int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+5},\quad \text{let } \tan\frac{x}{2}=t$$

$$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt = \int\frac{2dt}{t^2+2t+5} = 2\int\frac{dt}{(t+1)^2+2^2}$$

$$I_3 = 2\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$I = \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\tan^{-1}\left|\left(\frac{\tan\frac{x}{2}+1}{2}\right)\right| + C$$

Example 30 The value of $\int\{1+\tan x\cdot\tan(x+A)\}\,dx$ is equal to

(a) $\cot A\cdot\log\left|\frac{\sec x}{\sec(x+A)}\right|+C$

(b) $\tan A\cdot\log|\sec(x+A)|+C$

(c) $\cot A\cdot\log\left|\frac{\sec(x+A)}{\sec(x)}\right|+C$

(d) None of the above

Sol. Let $I = \int\{1+\tan x\cdot\tan(x+A)\}\,dx$

$$= \int\left\{1+\frac{\sin x\cdot\sin(x+A)}{\cos x\cdot\cos(x+A)}\right\}dx$$

$$= \int\frac{\cos x\cdot\cos(x+A)+\sin x\cdot\sin(x+A)}{\cos x\cdot\cos(x+A)}dx$$

$$= \int\frac{\cos(x+A-x)\,dx}{\cos x\cdot\cos(x+A)}$$

$$= \cos A\cdot\int\frac{dx}{\cos x\cdot\cos(x+A)}$$

Multiplying and dividing by $\sin A$, we get

$$= \cot A\cdot\int\frac{\sin A\,dx}{\cos x\cdot\cos(x+A)}$$

$$= \cot A\cdot\int\frac{\sin(x+A-x)\,dx}{\cos x\cdot\cos(x+A)}$$

$$= \cot A\cdot\int\left\{\frac{\sin(x+A)\cdot\cos x}{\cos x\cos(x+A)} - \frac{\cos(x+A)\cdot\sin x}{\cos x\cdot\cos(x+A)}\right\}dx$$

$$= \cot A\cdot\{\int\tan(x+A)dx - \int\tan x\,dx\}$$

$$= \cot A\cdot\{\log|\sec(x+A)| - \log|\sec x|\} + C$$

Hence, (c) is the correct answer.

Example 31 The value of $\int\frac{\sqrt{\cos 2x}}{\sin x}dx$, is equal to

(a) $\log|\cot x+\sqrt{\cot^2 x-1}|+\sqrt{2}\log|\cos x + \sqrt{\cos^2 x-1/2}|+C$

(b) $-\log|\cot x+\sqrt{\cot^2 x-1}|+\sqrt{2}\log|\cos x + \sqrt{\cos^2 x-1/2}|+C$

(c) $\log|\cot x+\sqrt{\cot^2 x-1}|+2\log|\cos x + \sqrt{\cos^2 x-1/2}|+C$

(d) $-\log|\cot x+\sqrt{\cot^2 x-1}|+2\log|\cos x + \sqrt{\cos^2 x-1/2}|+C$

Sol. Let $I = \int\frac{\sqrt{\cos 2x}}{\sin x}dx = \int\frac{\cos 2x}{\sin x\sqrt{\cos 2x}}dx$

$$= \int\frac{1-2\sin^2 x}{\sin x\sqrt{\cos 2x}}dx$$

$$= \int\frac{1}{\sin x\sqrt{\cos^2 x-\sin^2 x}}dx - 2\int\frac{\sin x}{\sqrt{2\cos^2 x-1}}dx$$

$$= \int\frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x-1}}dx - \frac{2}{\sqrt{2}}\int\frac{\sin x}{\sqrt{\cos^2 x-1/2}}dx$$

$$= -\int\frac{dt}{\sqrt{t^2-1}} - \sqrt{2}\int\frac{-ds}{\sqrt{s^2-(1/\sqrt{2})^2}}$$

[where $t = \cot x$ and $s = \cos x$]

$$= -\log|t+\sqrt{t^2-1}| + \sqrt{2}\log|s+\sqrt{s^2-1/2}|+C$$

$$= -\log|\cot x+\sqrt{\cot^2 x-1}|+\sqrt{2}\log|\cos x + \sqrt{\cos^2 x-1/2}|+C$$

Hence, (b) is the correct answer.

Exercise for Session 3

- **Evaluate the following integrals**

1. $\int \frac{x\,dx}{9-16x^4}$

2. $\int \frac{x^2}{9+16x^6}\,dx$

3. $\int \frac{x^3\,dx}{16x^8-25}$

4. $\int \sqrt{\frac{x}{a^3-x^3}}\,dx$

5. $\int \sqrt{\frac{x^4}{a^6+x^6}}\,dx$

6. $\int \frac{1}{4e^{-x}-9e^{x}}\,dx$

7. $\int \frac{2^x}{\sqrt{4^x-25}}\,dx$

8. $\int \frac{8x-11}{\sqrt{5+2x-x^2}}\,dx$

9. $\int \frac{x+2}{x^2+2x+2}\,dx$

10. $\int \frac{x-3}{3-2x-x^2}\,dx$

11. $\int \frac{3x-1}{4x^2-4x+17}\,dx$

12. $\int \frac{\sqrt{x}\,dx}{\sqrt{2x+3}}$

13. $\int \sqrt{\frac{a-x}{x-b}}\,dx$

14. $\int \sqrt{\frac{1-x}{1+x}}\,dx$

15. $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx$

16. $\int \frac{dx}{1+\sin x+\cos x}$

17. $\int \frac{dx}{\sin x+\sqrt{3}\cos x}$

18. $\int \frac{\cos^2 x\sin x}{\sin x-\cos x}\,dx$

19. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$

20. $\int \sqrt{\frac{\cos x-\cos^3 x}{(1-\cos^3 x)}}\,dx$

21. Evaluate $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

22. Evalute $\int (2x-4)\sqrt{4+3x-x^2}\,dx$.

23. $\int \frac{(2x^2+5x+9)\,dx}{(x+1)\sqrt{x^2+x+1}}$

24. The value of $\int \frac{dx}{\sec x+\operatorname{cosec} x}$, is equal to

(a) $\left\{(\sin x+\cos x)+\frac{1}{\sqrt{2}}\log\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x/2-1+\sqrt{2}}\right|\right\}+C$

(b) $2\left\{(\sin x+\cos x)+\frac{1}{\sqrt{2}}\log\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x/2-1+\sqrt{2}}\right|\right\}+C$

(c) $\frac{1}{2}\left\{(\sin x-\cos x)+\frac{1}{\sqrt{2}}\log\left|\frac{\tan x/2-1-\sqrt{2}}{\tan x-1+\sqrt{2}}\right|\right\}+C$

(d) None of these

Session 4

Integration by Parts

Integration by Parts

Theorem If u and v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

i.e. The integral of product of two functions = (first function) × (integral of second function) − integral of (differential of first function × integral of second function).

Proof For any two functions $f(x)$ and $g(x)$, we have

$$\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}$$

$$\therefore \int\left(f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}\right)dx=\int f(x)\cdot g(x)\,dx$$

$$\Rightarrow \int\left(f(x)\cdot\frac{d}{dx}\{g(x)\}\right)dx + \int\left(g(x)\cdot\frac{d}{dx}\{f(x)\}\right)dx$$

$$= \int f(x)\cdot g(x)dx$$

$$\Rightarrow \int\left(f(x)\cdot\frac{d}{dx}\{g(x)\}\right)dx$$

$$= \int f(x)\cdot g(x)\,dx - \int\left(g(x)\cdot\frac{d}{dx}\{f(x)\}\right)dx$$

Let $f(x) = u$ and $\frac{d}{dx}\{g(x)\} = v$

So that, $g(x) = \int v\,dx$

$$\therefore \int uv\,dx = u\cdot\int v\,dx - \int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}\cdot dx$$

Remarks

While applying the above rule, care has to be taken in the selection of first function (u) and selection of second function (v). Normally we use the following methods :

1. If in the product of the two functions, one of the functions is not directly integrable (e.g. $\log|x|$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, …, etc.) Then, we take it as the first function and the remaining function is taken as the second function. i.e. In the integration of $\int x\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.
2. If there is no other function, then unity is taken as the second function. e.g. In the integration of $\int \tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as first function and 1 as the second function.
3. If both of the function are directly integrable, then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.

Usually we use the following preference order for selecting the first function. (Inverse, Logarithmic, Algebraic, Trigonometric, Exponent).

In above stated order, the function on the left is always chosen as the first function. This rule is called as **ILATE**.

Example 32 Evaluate

(i) $\int \sin^{-1}x\,dx$ (ii) $\int \log_e|x|\,dx$

Sol. (i) $I = \int \sin^{-1}x\,dx = \int \underset{\text{I}}{\sin^{-1}x}\cdot \underset{\text{II}}{1}\,dx$

Here, we know by definition of integration by parts that order of preference is taken according to ILATE. So, '$\sin^{-1}x$' should be taken as first and '1' as the second function to apply by parts.

Applying integration by parts, we get

$$I = \sin^{-1}x\cdot(x) - \int\frac{1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$= x\cdot\sin^{-1}x + \frac{1}{2}\int\frac{dt}{t^{1/2}}$$

Let $1 - x^2 = t$

$$-2x\,dx = dt \Rightarrow x\,dx = -\frac{1}{2}dt$$

$$= x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{1/2}}{1/2} + C$$

$$I = x\sin^{-1}x + \sqrt{1-x^2} + C$$

$$\therefore \int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

(ii) $I = \int \log_e|x|\,dx = \int \underset{\text{I}}{\log_e|x|}\cdot \underset{\text{II}}{1}\,dx$

Applying integration by parts, we get

$$= \log|x|\cdot x - \int\frac{1}{x}\cdot x\,dx$$

$$= x\log|x| - \int 1\,dx$$

$$I = x\log|x| - x + C$$

Example 33 Evaluate

(i) $\int x\cos x\,dx$ (ii) $\int x^2\cos x\,dx$

Sol. (i) $\int x\cos x\,dx$, $I = \int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos x}\,dx$

Applying integration by parts,

$$I = x\left(\int \cos x\, dx\right) - \int \left\{\frac{d}{dx}(x)\right\}\left\{\int (\cos x)\, dx\right\} dx$$

$$I = x \sin x - \int 1 \cdot \sin x\, dx = x \sin x + \cos x + C$$

(ii) $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\cos x}\, dx$

Applying integration by parts,

$$I = x^2 \left(\int \cos x\, dx\right) - \int \left\{\frac{d}{dx}(x^2)\right\} \cdot \left\{\int \cos x\right\} dx$$

$$= x^2 \sin x - \int 2x \cdot (\sin x)\, dx$$

$$= x^2 \sin x - 2\int x (\sin x)\, dx$$

We again have to integrate $\int x \sin x\, dx$ using integration by parts,

$$= x^2 \cdot \sin x - 2 \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\sin x}\, dx$$

$$= x^2 \sin x - 2\left\{x\left(\int \sin x\, dx\right) - \int \left(\frac{dx}{dx}\right)\left(\int \sin x\, dx\right) dx\right\}$$

$$= x^2 \sin x - 2\{- x \cos x - \int 1 \cdot (- \cos x)\, dx\}$$

$$I = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Example 34 Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$.

Sol. Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

$$= \int \frac{\sin^{-1}\sqrt{x} - (\pi/2 - \sin^{-1}\sqrt{x})}{\pi/2}\, dx$$

$[\because \sin^{-1}\theta + \cos^{-1}\theta = \pi/2]$

$$\Rightarrow \quad I = \frac{2}{\pi} \int (2\sin^{-1}\sqrt{x} - \pi/2)\, dx$$

$$= \frac{4}{\pi} \int \sin^{-1}\sqrt{x}\, dx - \int 1\, dx$$

$$I = \frac{4}{\pi} \int \sin^{-1}\sqrt{x}\, dx - x + C \quad \text{...(i)}$$

Let $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\, d\theta = \sin 2\theta\, d\theta$

$$\therefore \quad \int \sin^{-1}\sqrt{x}\, dx = \int \underset{\text{I}}{\theta} \cdot \underset{\text{II}}{\sin 2\theta}\, d\theta$$

Applying integration by parts

$$\int \sin^{-1}\sqrt{x} dx = -\theta \cdot \frac{\cos 2\theta}{2} + \int \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \frac{-\theta}{2} \cdot \cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{-1 \cdot \theta}{2} \cdot (1 - 2\sin^2\theta) + \frac{1}{2} \cdot \sin\theta \cdot \sqrt{1 - \sin^2\theta}$$

$$= \frac{-1}{2}\sin^{-1}\sqrt{x}\,(1 - 2x) + \frac{1}{2} \cdot \sqrt{x}\,\sqrt{1 - x} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$I = \frac{4}{\pi}\left\{\frac{-1}{2}(\sin^{-1}\sqrt{x})(1 - 2x) + \frac{1}{2}\sqrt{x}\sqrt{1 - x}\right\} - x + C$$

$$= \frac{2}{\pi}\{\sqrt{x - x^2} - (1 - 2x)\sin^{-1}\sqrt{x}\} - x + C$$

Integral of Form $\int e^x \{f(x) + f'(x)\} dx$

Theorem Prove that

$$\int e^x \{f(x) + f'(x)\}\, dx = e^x f(x) + C$$

Proof We have, $\int e^x \{f(x) + f'(x)\}\, dx$

$$= \int \underset{\text{II}}{e^x} \cdot \underset{\text{I}}{f(x)}\, dx + \int e^x \cdot f'(x)\, dx$$

$$= f(x) \cdot e^x - \int f'(x) \cdot e^x dx + \int e^x \cdot f'(x)\, dx + C$$

$$= f(x) \cdot e^x + C$$

Thus, to evaluate the integrals of the type

$$\int e^x \{f(x) + f'(x)\}\, dx,$$

we first express the integral as the sum of two integrals $\int e^x f(x)\, dx$ and $\int e^x f'(x)\, dx$ and then integrate the integral involving $e^x f(x)$ as integral by parts taking e^x as second function.

Remark

The above theorem is also true, if we have e^{kx} in place of e^x

i.e. $\quad \int e^{kx} \{f(kx) + f'(kx)\}\, dx = e^{kx} f(kx) + C$

General Concept

$$\int e^{g(x)} \{f(x) g'(x) + f'(x)\}\, dx$$

Proof $I = \int \underbrace{e^{g(x)}}_{\text{II}} \underbrace{f(x)}_{\text{I}} \underbrace{g'(x)}_{\text{II}}\, dx + \underbrace{\int e^{g(x)} f'(x)\, dx}_{\text{as it is}}$

Using, $\int e^{g(x)} \cdot g'(x)\, dx = e^{g(x)}$, we get

$$= f(x) \cdot e^{g(x)} - \int f'(x) \cdot e^{g(x)} dx + \int e^{g(x)} \cdot f'(x)\, dx$$

$$= f(x) \cdot e^{g(x)} + C$$

e.g. $\quad = \int e^{(x \sin x + \cos x)} \left(\frac{x^2 \cos^2 x - (x \sin x + \cos x)}{x^2}\right) dx$

$$\Rightarrow \quad \int e^{(x \sin x + \cos x)} \left(\cos^2 x - \frac{x \sin x + \cos x}{x^2}\right) dx$$

$$\Rightarrow \int e^{(x\sin x+\cos x)}\left(x\cos x\left(\frac{\cos x}{x}\right)+\left(\frac{\cos x}{x}\right)'\right)dx$$

$$\Rightarrow e^{(x\sin x+\cos x)}\cdot\frac{\cos x}{x}+C$$

e.g. $=\int e^{\tan x}(\sin x-\sec x)\,dx$

$$=\int e^{\tan x}\sin x\,dx-\int e^{\tan x}\sec x\,dx$$

$$\Rightarrow -e^{\tan x}\cdot\cos x+\int e^{\tan x}\sec^2 x\cos x dx-\int e^{\tan x}\sec x dx$$

$$\Rightarrow -e^{\tan x}\cdot\cos x$$

| Example 35 Evaluate

(i) $\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)dx$ (ii) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$

Sol. (i) $I=\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)dx$

$$I=\int e^x\left\{\frac{1}{\cos^2 x}+\frac{\sin x\cos x}{\cos^2 x}\right\}dx$$

$$I=\int e^x\{\tan x+\sec^2 x\}\,dx$$

$$I=\int \underset{\text{II}}{e^x}\cdot\underset{\text{I}}{\tan x}\,dx+\int e^x(\sec^2 x)\,dx$$

$$I=\tan x\cdot e^x-\int\sec^2 x\cdot e^x\,dx+\int e^x\cdot\sec^2 x\,dx+C$$

$$I=e^x\tan x+C$$

(ii) $I=\int e^{2x}\left\{\frac{1+\sin 2x}{1+\cos 2x}\right\}dx$

$$=\int e^{2x}\left\{\frac{1+2\sin x\cos x}{2\cos^2 x}\right\}dx$$

$$=\int e^{2x}\left\{\frac{1}{2\cos^2 x}+\frac{2\sin x\cos x}{2\cos^2 x}\right\}dx$$

$$=\int e^{2x}\left\{\frac{1}{2}\sec^2 x+\tan x\right\}dx$$

$$=\int \underset{\text{II}}{e^{2x}}\cdot\underset{\text{I}}{\tan x}\,dx+\frac{1}{2}\int e^{2x}\cdot\sec^2 x\,dx$$

$$=\tan x\cdot\frac{e^{2x}}{2}-\int\sec^2 x\cdot\frac{e^{2x}}{2}\,dx+\frac{1}{2}\int e^{2x}\cdot\sec^2 x\,dx$$

$$I=\frac{1}{2}e^{2x}\cdot\tan x+C$$

| Example 36 Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$.

Sol. $I=\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx=\int e^x\frac{(1-2x+x^2)}{(1+x^2)^2}dx$

$$=\int e^x\left\{\frac{1+x^2}{(1+x^2)^2}-\frac{2x}{(1+x^2)^2}\right\}dx$$

$$=\int e^x\left\{\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right\}dx\quad\left[\text{as }\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=-\frac{2x}{(1+x^2)^2}\right]$$

$$=\frac{e^x}{1+x^2}+C$$

$$\therefore\quad I=\frac{e^x}{1+x^2}+C$$

Integrals of the Form $\int e^{ax}\sin bx\,dx$, $\int e^{ax}\cos bx\,dx$

Let $I=\int e^{ax}(\sin bx)\,dx$

Then, $I=\int \underset{\text{I}}{\sin bx}\cdot\underset{\text{II}}{e^{ax}}\,dx$

$$=\sin bx\cdot\left(\frac{e^{ax}}{a}\right)-\int b\cos bx\cdot\frac{e^{ax}}{a}\,dx$$

$$=\frac{1}{a}\sin bx\cdot e^{ax}-\frac{b}{a}\left\{\cos bx\cdot\frac{e^{ax}}{a}-\int(-b\sin bx)\cdot\frac{e^{ax}}{a}\,dx\right\}$$

$$=\frac{1}{a}\sin bx\cdot e^{ax}-\frac{b}{a^2}\cos bx\cdot e^{ax}-\frac{b^2}{a^2}\int\sin bx\cdot e^{ax}\,dx$$

$$I=\frac{1}{a}\sin bx\cdot e^{ax}-\frac{b}{a^2}\cos bx\cdot e^{ax}-\frac{b^2}{a^2}I$$

$$\therefore\quad I+\frac{b^2}{a^2}I=\frac{1\cdot e^{ax}}{a^2}\cdot(a\sin bx-b\cos bx)$$

$$\Rightarrow\quad I\left(\frac{a^2+b^2}{a^2}\right)=\frac{e^{ax}}{a^2}(a\sin bx-b\cos bx)$$

or $I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

Thus, $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

Similarly, $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$

Aliter Use Euler's equation

Let $P=\int e^{ax}\cos bx\,dx$ and $Q=\int e^{ax}\sin bx\,dx$

Hence, $P+iQ=\int e^{ax}\cdot e^{ibx}\,dx=\int e^{(a+ib)x}\,dx$

$$P+iQ=\frac{1}{a+ib}e^{(a+ib)x}=\frac{a-ib}{a^2+b^2}e^{ax}(\cos bx+i\sin bx)$$

$$= \frac{(ae^{ax}\cos bx + be^{ax}\sin bx) - i(ae^{ax}\sin bx - be^{ax}\cos bx)}{a^2+b^2}$$

$$\therefore \quad P = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$Q = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

Example 37 Evaluate

(i) $\int e^x \cos^2 x\, dx$

(ii) $\int \sin(\log x)\, dx$

Sol. (i) $I = \int e^x \cdot \cos^2 x\, dx = \int e^x \cdot \left\{\frac{1+\cos 2x}{2}\right\} dx$

$$I = \frac{1}{2}\int e^x\, dx + \frac{1}{2}\int \cos 2x \cdot e^x\, dx$$

$$I = \frac{1}{2}e^x + \frac{1}{2}I_1 \quad \text{...(i)}$$

where, $I_1 = \int \cos 2x \cdot e^x\, dx$

$$I_1 = \int \underset{\text{I}}{\cos 2x} \cdot \underset{\text{II}}{e^x}\, dx = \cos 2x \cdot e^x - \int -2\sin 2x \cdot e^x\, dx$$

$$= e^x \cdot \cos 2x + 2\int \underset{\text{I}}{\sin 2x} \cdot \underset{\text{II}}{e^x}\, dx$$

$$= e^x \cdot \cos 2x + 2\left\{\sin 2x \cdot e^x - \int 2\cos 2x \cdot e^x\, dx\right\}$$

$$= e^x \cdot \cos 2x + 2\sin 2x \cdot e^x - 4I_1$$

$$\therefore \quad I_1 = \frac{1}{5}\{e^x \cos 2x + 2\sin 2x\, e^x\} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$I = \frac{1}{2}e^x + \frac{1}{2}\cdot\frac{1}{5}\{e^x \cos 2x + 2\sin 2x \cdot e^x\}$$

$$I = \frac{1}{2}e^x + \frac{1}{10}e^x\{\cos 2x + 2\sin 2x\} + C$$

(ii) $I = \int \sin(\log x)\, dx$

Let $\log x = t$

$\Rightarrow \quad x = e^t$ or $dx = e^t\, dt$

$$\therefore \quad I = \int \underset{\text{I}}{(\sin t)} \cdot \underset{\text{II}}{e^t}\, dt = \sin t \cdot e^t - \int \underset{\text{I}}{\cos t} \cdot \underset{\text{II}}{e^t}\, dt$$

$$I = \sin t \cdot e^t - \left\{\cos t \cdot e^t - \int -\sin t \cdot e^t\, dt\right\}$$

$$I = e^t \cdot \sin t - e^t \cdot \cos t - I$$

$$\therefore \quad I = \frac{1}{2}e^t(\sin t - \cos t) + C$$

$$I = \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$$

Example 38 Evaluate $\int \frac{x^2 dx}{(x\sin x + \cos x)^2}$.

Sol. Let $I = \int \frac{x^2}{(x\sin x + \cos x)^2}\, dx$

Multiplying and dividing it by $(x\cos x)$, we get

$$I = \int \underset{\text{I}}{(x\sec x)} \cdot \underset{\text{II}}{\frac{(x\cos x)}{(x\sin x + \cos x)^2}}\, dx$$

$$I = x\sec x \cdot \int \frac{x\cos x}{(x\sin x + \cos x)^2}\, dx - \int \left\{\frac{d}{dx}(x\sec x)\right\}\left\{\int \frac{x\cos x}{(x\sin x + \cos x)^2}\, dx\right\} dx$$

$$= x\sec x \cdot \frac{-1}{(x\sin x + \cos x)} - \int (x\sec x \cdot \tan x + \sec x) \cdot \frac{-1}{(x\sin x + \cos x)}\, dx$$

$$= \frac{-x\sec x}{(x\sin x + \cos x)} + \int \frac{(x\sin x + \cos x)}{\cos^2 x \cdot (x\sin x + \cos x)}\, dx$$

$$= \frac{-x\sec x}{(x\sin x + \cos x)} + \int \sec^2 x\, dx$$

$$I = \frac{-x\sec x}{(x\sin x + \cos x)} + \tan x + C$$

Example 39 The value of

$\int \sqrt{\frac{3-x}{3+x}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx$, is equal to

(a) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2} \cdot \cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(b) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\, \sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(c) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}\, \sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(d) None of the above

Sol. Here, $I = \int \sqrt{\frac{3-x}{3+x}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right) dx$

Put $\quad x = 3\cos 2\theta$

$\Rightarrow \quad dx = -6\sin 2\theta\, d\theta$

$$\therefore \quad I = \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}} \cdot \sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)\, d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \cdot \sin^{-1}(\sin\theta) \cdot (-6\sin 2\theta)\, d\theta$$

$$= -6\int \theta \cdot (2\sin^2\theta)\, d\theta = -6\int \theta(1-\cos 2\theta)\, d\theta$$

$$= -6\left\{\frac{\theta^2}{2} - \int \theta \cos 2\theta \, d\theta\right\}$$

$$= -6\left\{\frac{\theta^2}{2} - \left(\theta \frac{\sin 2\theta}{2} - \int 1 \cdot \left(\frac{\sin 2\theta}{2}\right) d\theta\right)\right\}$$

$$= -3\theta^2 + 6\left\{\frac{\theta \sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + C$$

$$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2} \cdot \cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$$

Hence, (a) is the correct answer.

Example 40 The value of $\int \frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2} dx$, is equal to

(a) $\frac{\sin x}{2+\cos x} + C$ (b) $\frac{\cos x}{2+\cos x} + C$

(c) $\frac{-\sin x}{2+\sin x} + C$ (d) $\frac{\cos x}{2+\sin x} + C$

Sol. Let $I = \int \frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2} dx = \int \frac{2\cos x + 1}{(\cos x + 2)^2} dx$

$$= \int \frac{\cos x\,(\cos x + 2) + \sin^2 x}{(2+\cos x)^2} dx$$

$$= \int \frac{\cos x}{2+\cos x} dx + \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

$$= \int \underset{\text{II}}{\cos x} \cdot \underset{\text{I}}{\frac{1}{(2+\cos x)}} dx + \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

Applying integration by parts to first by taking $\cos x$ as second function, keeping $\int \frac{\sin^2 x}{(2+\cos x)^2} dx$ as it is.

$$\therefore \quad I = \frac{1}{2+\cos x} \cdot (\sin x) - \int \sin x \cdot \frac{\sin x}{(2+\cos x)^2} dx + \int \frac{\sin^2 x}{(2+\cos x)^2} dx$$

$$\therefore \quad I = \frac{\sin x}{2+\cos x} + C$$

Hence, (a) is the correct answer.

Example 41 The value of $\int \log(\sqrt{1-x} + \sqrt{1+x})\, dx$, is equal to

(a) $x \log(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2}x - \frac{1}{2}\sin^{-1}(x) + C$

(b) $x \log(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2}x + \frac{1}{2}\sin^{-1}(x) + C$

(c) $x \log(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2}x + \frac{1}{2}\sin^{-1}(x) + C$

(d) None of the above

Sol. Here, we have only one function. This can be solved easily by applying integration by parts taking unity as second function.

If we take $u = \log(\sqrt{1-x} + \sqrt{1+x})$ as the first function and $v = 1$ as the second function.

Then,

$$I = \int 1 \cdot \log(\sqrt{1-x} + \sqrt{1+x})\, dx$$

$$= \{\log(\sqrt{1-x} + \sqrt{1+x})\} \cdot x - \int \frac{1}{\sqrt{1-x} + \sqrt{1+x}} \left(-\frac{1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right) \cdot x\, dx$$

$$= x \log(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2}\int \frac{\sqrt{1-x} - \sqrt{1+x}}{\sqrt{1-x} + \sqrt{1+x}} \cdot \frac{1}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= x \log(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2}\int \frac{(1-x)+(1+x)-2\sqrt{1-x^2}}{(1-x)-(1+x)} \cdot \frac{1}{\sqrt{1-x^2}} \cdot 2x\, dx$$

$$= x \log(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2}\int \frac{\sqrt{1-x^2} - 1}{\sqrt{1-x^2}} dx$$

$$= x \log(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2}\int 1\, dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= x \log(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2}x + \frac{1}{2}\sin^{-1} x + C$$

Hence, (c) is the correct answer.

Example 42 The value of $\int e^x \left(\frac{x^4+2}{(1+x^2)^{5/2}}\right) dx$, is equal to

(a) $\frac{e^x\,(x+1)}{(1+x^2)^{3/2}}$ (b) $\frac{e^x\,(1-x+x^2)}{(1+x^2)^{3/2}}$

(c) $\frac{e^x\,(1+x)}{(1+x^2)^{3/2}}$ (d) None of these

Sol. Let $I = \int e^x \left(\frac{x^4+2}{(1+x^2)^{5/2}}\right) dx$

$$= \int e^x \left(\frac{1}{(1+x^2)^{1/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right) dx$$

$$= \int e^x \left(\frac{1}{(1+x^2)^{1/2}} - \frac{x}{(1+x^2)^{3/2}} + \frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right) dx$$

$$= \frac{e^x}{(1+x^2)^{1/2}} + \frac{xe^x}{(1+x^2)^{3/2}} + C = \frac{e^x\,\{1+x^2+x\}}{(1+x^2)^{3/2}} + C$$

Hence, (d) is the correct answer.

Example 43 If $\int (\sin 3\theta + \sin\theta)\, e^{\sin\theta} \cos\theta\, d\theta$
$= (A\sin^3\theta + B\cos^2\theta + C\sin\theta + D\cos\theta + E)\, e^{\sin\theta} + F,$
then

(a) $A = -4, B = 12$ (b) $A = -4, B = -12$
(c) $A = 4, B = 12$ (d) $A = 4, B = -12$

Sol. Let $I = \int (\sin 3\theta + \sin\theta)\, e^{\sin\theta} \cdot \cos\theta\, d\theta$

$$= \int (3\sin\theta - 4\sin^3\theta) \cdot e^{\sin\theta} \cdot \cos\theta\, d\theta;$$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$$= \int (3t - 4t^3)\, e^t\, dt \quad \text{...(i)}$$

As, $I = (A\sin^3\theta + B\cos^2\theta + C\sin\theta + D\cos\theta + E) e^{\sin\theta} + F$

$$= (A\sin^3\theta - B\sin^2\theta + C\sin\theta + D\cos\theta + B + E) e^{\sin\theta} + F$$

When, $\sin\theta = t$

$$I = (At^3 - Bt^2 + Ct + B + E)\, e^t + F$$

as by Eq. (i) $D = 0$ ...(ii)

From Eqs. (i) and (ii),

$$\int (3t - 4t^3)\, e^t\, dt = \underbrace{(At^3 - Bt^2 + Ct + B + E)}_{f(t)}\, e^t + F,$$

differentiating both sides

$$(3t - 4t^3)\, e^t = (At^3 - Bt^2 + Ct + B + E) e^t + (3At^2 - 2Bt + C) e^t$$

$\Rightarrow \quad A = -4$ and $3A = B \Rightarrow B = -12$

Hence, (b) is the correct answer.

Example 44 The value of

$\int e^{(x\sin x + \cos x)} \cdot \left(\dfrac{x^4 \cos^3 x - x\sin x + \cos x}{x^2 \cos^2 x} \right) dx$, is equal to

(a) $e^{(x\sin x + \cos x)} \cdot \left(x + \dfrac{1}{x\cos x} \right) + C$

(b) $e^{(x\sin x + \cos x)} \cdot \left(x\cos x + \dfrac{1}{x} \right) + C$

(c) $e^{(x\sin x + \cos x)} \cdot \left(x - \dfrac{1}{x\cos x} \right) + C$

(d) None of the above

Sol. Let $I = \int e^{(x\sin x + \cos x)} \cdot \left(\dfrac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x} \right) dx$

$$= \int (x \cdot e^{(x\sin x + \cos x)} \cdot x\cos x)\, dx - \int e^{(x\sin x + \cos x)} \cdot \left(\frac{x\sin x - \cos x}{(x\cos x)^2} \right) dx$$

Applying integration by parts

$$= \{x \cdot e^{(x\sin x + \cos x)} - \int e^{(x\sin x + \cos x)}\, dx\} - \left\{ e^{(x\sin x + \cos x)} \cdot \frac{1}{x\cos x} - \int e^{(x\sin x + \cos x)}\, dx \right\}$$

$$= e^{(x\sin x + \cos x)} \left(x - \frac{1}{x\cos x} \right) + C$$

Hence, (c) is the correct answer.

Example 45 Evaluate $\int \sin^{-1} \left\{ \dfrac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right\} dx.$

[IIT JEE 2001]

Sol. Here, $I = \int \sin^{-1} \left(\dfrac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right) dx$

$$= \int \sin^{-1} \left(\frac{2x+2}{\sqrt{(2x+2)^2 + 3^2}} \right) dx$$

Put $2x + 2 = 3\tan\theta \Rightarrow 2\, dx = 3\sec^2\theta\, d\theta$

$$= \int \sin^{-1} \left(\frac{3\tan\theta}{3\sec\theta} \right) \frac{3}{2} \sec^2\theta\, d\theta = \frac{3}{2} \int \theta \sec^2\theta\, d\theta$$

$$= \frac{3}{2} \{\theta\tan\theta - \int \tan\theta\, d\theta\}$$

$$= \frac{3}{2} \{\theta\tan\theta - \log|\sec\theta|\} + C$$

$$\Rightarrow I = \frac{3}{2} \left[\frac{2x+2}{3} \tan^{-1}\left(\frac{2x+2}{3} \right) - \log\left(\sqrt{1 + \left(\frac{2x+2}{3} \right)^2} \right) \right] + C$$

$$= \frac{3}{2} \left\{ \frac{2}{3}(x+1) \tan^{-1}\left(\frac{2}{3}(x+1) \right) - \log\sqrt{4x^2 + 8x + 13} \right\} + C$$

$$\Rightarrow I = (x+1)\tan^{-1}\left(\frac{2}{3}(x+1) \right) - \frac{3}{4} \log(4x^2 + 8x + 13) + C$$

Example 46 Evaluate $\int \dfrac{x^2 (x\sec^2 x + \tan x)\, dx}{(x\tan x + 1)^2}.$

Sol. Here, $I = \int x^2 \cdot \dfrac{x\sec^2 x + \tan x}{(x\tan x + 1)^2}\, dx$

$$= x^2 \left(-\frac{1}{(x\tan x + 1)} \right) - \int 2x \cdot \left(-\frac{1}{(x\tan x + 1)} \right) dx \quad \text{...(i)}$$

$$\left[\text{using} \int \frac{x\sec^2 x + \tan x}{(x\tan x + 1)^2}\, dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{(x\tan x + 1)} \right]$$

$$\Rightarrow I = -\left(\frac{x^2}{x\tan x + 1} \right) + \int \frac{2x(\cos x)}{x\sin x + \cos x}\, dx$$

[put, $x\sin x + \cos x = u$
$\Rightarrow (x\cos x + \sin x - \cos x)\, dx = du$]

$$\Rightarrow \quad I = -\frac{x^2}{(x\tan x + 1)} + 2\int \frac{du}{u}$$

$$= -\frac{x^2}{x\tan x + 1} + 2\log|u| + C$$

$$= -\frac{x^2}{x\tan x + 1} + 2\log|x\sin x + \cos x| + C$$

Exercise for Session 4

1. $\int x^2 e^x dx$

2. $\int x^2 \sin x \, dx$

3. $\int \log x \cdot dx$

4. $\int (\log x)^2 dx$

5. $\int (\tan^{-1} x) \, dx$

6. $\int (\sec^{-1} x) \, dx$

7. $\int x \tan^{-1} x \, dx$

8. $\int \frac{\log x}{x^2} dx$

9. $\int \frac{x - \sin x}{1 - \cos x} dx$

10. $\int \log (1 + x^2) \, dx$

11. $\int e^x (\tan x + \log \sec x) \, dx$

12. $\int e^x \left\{ \frac{1 + \sin x \cos x}{\cos^2 x} \right\} dx$

13. $\int \left(\log (\log x) + \frac{1}{(\log x)^2} \right) dx$

14. $\int e^{2x} \cdot \left\{ \frac{1 + \sin 2x}{1 + \cos 2x} \right\} dx$

15. $\int e^x \frac{(1 - x)^2}{(1 + x^2)^2} dx$

16. $\int \frac{e^x \cdot (2 - x^2)}{(1 - x)\sqrt{1 - x^2}} dx$

17. $\int e^{ax} \cdot \cos (bx + c) \, dx$

18. $\int \sec^3 x \, dx$

19. $\int \sin \sqrt{x} \, dx$

20. $\int (\sin^{-1} x)^2 dx$

21. $\int \cot^{-1} (1 - x + x^2) \, dx$

22. $\int \sin^{-1} \sqrt{\frac{x}{a + x}} \, dx$

23. $\int \frac{\sqrt{x^2 + 1} \{\log (x^2 + 1) - 2 \log x\}}{x^4} dx$

24. $\int \frac{\cos^2 x + \sin 2x}{(2 \cos x - \sin x)^2} dx$

25. $\int e^{\sin x} \left(\frac{x \cos^2 x - \sin x}{\cos^2 x} \right) dx$

Session 5

Integration Using Partial Fraction

This section deals with the integration of general algebraic rational functions, of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials. We already have seen some examples of this form. For example, we know how to integrate functions of the form $\frac{1}{Q(x)}$ or $\frac{L(x)}{Q(x)}$ or $\frac{P(x)}{Q(x)}$ where $L(x)$ is a linear factor, $Q(x)$ is a quadratic factor and $P(x)$ is a polynomial of degree $n \geq 2$. We intend to generalise that previous discussion in this section.

We are assuming the scanario where $g(x)$ (the denominator) is decomposible into linear or quadratic factors. These are the only cases relevant to us right now. Any linear or quadratic factor in $g(x)$ might also occur repeatedly.

Thus, $g(x)$ could be of the following general forms.

- $g(x) = L_1(x)L_2(x)\ldots L_n(x)$ (n linear factors)
- $g(x) = L_1(x)\ldots L_r^k(x)\ldots L_n(x)$ $\begin{pmatrix} n \text{ linear factors; the } r\text{th} \\ \text{factor is repeated } k \text{ times} \end{pmatrix}$
- $g(x) = L_1^{k_1}(x)L_2^{k_2}(x)\ldots L_n^{k_n}(x)$ $\begin{pmatrix} n \text{ linear factors, the } i\text{th} \\ \text{factor is repeated } k_i \text{ times} \end{pmatrix}$
- $g(x) = L_1(x)L_2(x)\ldots L_n(x)Q(x)Q_2(x)\ldots Q_m(x)$ $\begin{pmatrix} n \text{ linear factors and} \\ m \text{ quadratic factors} \end{pmatrix}$
- $g(x) = \ldots Q_r^k(x)\ldots$ $\begin{pmatrix} \text{a particular quadratic factor} \\ \text{repeats more than once} \end{pmatrix}$
- A combination of any of the above

Suppose that the degree of $g(x)$ is n and that of $f(x)$ is m. If $m \geq n$, we can always divide $f(x)$ by $g(x)$ to obtain a quotient $q(x)$ and a remainder $r(x)$ whose degree would be less than n.

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \qquad \ldots\text{(i)}$$

If $m < n$, $\frac{f(x)}{g(x)}$ is termed a proper rational function.

The partial fraction expansion technique says that a proper rational function can be expressed as a sum of simpler rational functions each possessing one of the factors of $g(x)$. The simpler rational functions are called partial fractions.

From now on, we consider only proper rational functions. If $\frac{f(x)}{g(x)}$ is not proper, we make it proper $\left(\frac{r(x)}{g(x)}\right)$ by the procedure described in (1) above. Let us consider a few examples.

Let $g(x)$ be a product of non-repeated, linear factors :

$$g(x) = L_1(x)L_2(x)\ldots L_n(x)$$

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1(x)} + \frac{A_2}{L_2(x)} + \ldots + \frac{A_n}{L_n(x)}$$

where the $A_i's$ are all constants that need to be determined.

Suppose $f(x) = x + 1$ and $g(x) = (x-1)(x-2)(x-3)$. Let us write down the partial fraction expansion of $\frac{f(x)}{g(x)}$:

$$\frac{f(x)}{g(x)} = \frac{x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

We need to determine A, B and C. Cross multiplying in the expression above, we obtain :

$$(x+1) = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

A, B, C can now be determined by comparing coefficients on both sides. More simply since this relation that we have obtained should held true for all x,we substitute those values of x that would straight way give us the required values of A, B and C. These values are obviously the roots of $g(x)$.

$x = 1 \Rightarrow 2 = A(-1)(-2) + B(0) + C(0)$

$\Rightarrow A = 1$

$x = 2 \Rightarrow 3 = A(0) + B(1)(-1) + C(0)$

$\Rightarrow B = -3$

$x = 3 \Rightarrow 4 = A(0) + B(0) + C(2)(1)$

$\Rightarrow C = 2$

Thus, $A = 1$, $B = -3$ and $C = 2$.

We can therefore write $\frac{f(x)}{g(x)}$ as a sum of partial fractions.

$$\frac{f(x)}{g(x)} = \frac{1}{x-1} - \frac{3}{x-2} + \frac{2}{x-3}$$

Integrating $\frac{f(x)}{g(x)}$ is now a simple matter of integrating the partial fractions. This was our sole motive in writing such an expansion, so that integration could be carried out easily. In the example above :

$$\int \frac{f(x)}{g(x)} dx = \ln(x-1) - 3\ln(x-2) + 2\ln(x-3) + C$$

Now, suppose that $g(x)$ contains all linear factors, but a particular factor, say $L_1(x)$, is repeated k times.

Thus, $\quad g(x) = L_1^k(x) L_2(x) \dots L_n(x)$

$\frac{f(x)}{g(x)}$ can now be expanded into partial fractions as follows

$$\frac{f(x)}{g(x)} = \underbrace{\frac{A_1}{L_1(x)} + \frac{A_2}{L_1^2(x)} + \frac{A_3}{L_1^3(x)} + \dots \frac{A_k}{L_1^k(x)}}_{k \text{ partial fractions corresponding to } L_1(x)} + \frac{B_2}{L_2(x)} + \dots + \frac{B_n}{L_n(x)}$$

This means that we will have k terms corresponding to $L_1(x)$. The rest of the linear factors will have single corresponding terms in the expansion. Here are some examples.

$\Rightarrow \quad \frac{1}{(x-1)^2(x-2)}$

can be expanded as $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$\Rightarrow \quad \frac{1}{(x-1)^3(x-2)(x-3)}$

can be expanded as

$\Rightarrow \quad \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-2)} + \frac{C}{(x-3)}$

$\Rightarrow \quad \frac{1}{(x-1)^2(x+5)^3}$

can be expanded as

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+5)} + \frac{D}{(x+5)^2} + \frac{E}{(x+5)^3}$$

| Example 47 $\frac{2x+1}{(x+1)(x-2)}$ into partial fractions

Sol. Here, $\frac{2x+1}{(x+1)(x-2)}$ has $Q(x) = (x+1)(x-2)$ i.e linear and non-repeated roots.

$\therefore \quad \frac{2x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$\Rightarrow \quad (2x+1) = A(x-2) + B(x+1)$

On putting, $x = 2$ we get

$$5 = A(0) + B(3) \Rightarrow B = \frac{5}{3}$$

Again, let $x = -1$

$\Rightarrow \quad 2(-1) + 1 = A(-1-2) + B(0)$

$\therefore \quad A = \frac{1}{3}$.

$\therefore \quad \frac{2x+1}{(x+1)(x-2)} = \frac{1/3}{x+1} + \frac{5/3}{x-2}$

| Example 48 Resolve $\frac{1}{(x-1)(x+2)(2x+3)}$ into partial fractions.

Sol. Let $\frac{1}{(x-1)(x+2)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{2x+3}$,

where A, B, C are constants.

$$1 = A(x+2)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+2) \quad \dots\text{(i)}$$

For finding A, let $x - 1 = 0$ or $x = 1$ in Eq. (i), we get

$$1 = A(1+2)(2+3) + B(0) + C(0)$$

$\therefore \quad A = \frac{1}{15}$

Similarly, for getting B, let $x + 2 = 0$ or $x = -2$ in Eq. (i), we get

$$1 = A(0) + B(-2-1)(-4+3) + C(0)$$

$\Rightarrow \quad B = \frac{1}{3}$

For getting C, let $2x + 3 = 0$ or $x = -3/2$ in Eq. (i), we get

$$1 = A(0) + B(0) + C\left(-\frac{3}{2} - 1\right)\left(-\frac{3}{2} + 2\right)$$

$\Rightarrow \quad C = -\frac{4}{5}$

Hence, $\frac{1}{(x-1)(x+2)(2x+3)} = \frac{1}{15(x-1)} + \frac{1}{3(x+2)} - \frac{4}{5(2x+3)}$

| Example 49 Resolve $\frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)}$ into partial fractions.

Sol. This is not a proper fraction. Hence, by division process it is to be expressed as the sum of an integral polynomial and a fraction.

Now, $3x^3 + 2x^2 + x + 1 = 3x(x^2 + 3x + 2)$
$- 7(x^2 + 3x + 2) + (16x + 15)$

So, the given polynomial

$$\frac{3x^3 + 2x^2 + x + 1}{(x+1)(x+2)} = (3x - 7) + \frac{(16x+15)}{(x+1)(x+2)} \quad \dots\text{(i)}$$

Now, the second term is proper fraction hence it can be expressed as a sum of partial fractions.

$$\frac{16x + 15}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

To find A put $x+1=0$, *ie*, $x=-1$ in the fraction except in the factor $(x+1)$.

$$\therefore \quad \frac{16(-1)+15}{(-1+2)}=A \Rightarrow A=-1 \quad \text{...(ii)}$$

To find B put $x+2=0$, *ie*, $x=-2$ in the fraction except in the factor $(x+2)$.

$$\therefore \quad \frac{16(-2)+15}{(-2+1)}=B \Rightarrow B=17 \quad \text{...(iii)}$$

$\Rightarrow$ The given expression $=(3x-7)-\frac{1}{x+1}+\frac{17}{x+2}$

[using Eqs. (i), (ii) and (iii)]

Case II When the denominator $g(x)$ is expressible as the product of the linear factors such that some of them are repeating. (Linear and Repeated)

Let $Q(x)=(x-a)^k(x-a_1)(x-a_2)\dots(x-a_r)$. Then, we assume that

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+\dots+\frac{B_r}{(x-a_r)}$$

| Example 50 Expression $\frac{x+5}{(x-2)^2}$ has repeated (twice) linear factors in denominator, so find partial fractions.

Sol. Let $\frac{x+5}{(x-2)^2}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}$

$\therefore \quad (x+5)=A(x-2)+B$

Comparing the like terms, $A=1, -2A+B=5$ or $B=7$

$$\therefore \quad \frac{x+5}{(x-2)^2}=\frac{1}{(x-2)}+\frac{7}{(x-2)^2}$$

| Example 51 Resolve $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions.

Sol. Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$

$$=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}+\frac{D}{(x+2)}$$

$\therefore$ $3x-2=A(x-1)(x+1)(x+2)+B(x+1)(x+2)+C(x-1)^2(x+2)+D(x-1)^2(x+1)$

Putting $(x-1)=0$, we get $B=1/6$,

Putting $(x+1)=0$, we get $C=-5/4$

Putting $(x+2)=0$, we get $D=8/9$

Now, equating the coefficient of x^3 on both the sides, we get

$$0=A+C+D \Rightarrow A=\frac{5}{4}-\frac{8}{9}=\frac{13}{36}$$

$$\therefore \quad \frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{13}{36(x-1)}+\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{8}{9(x+2)}$$

Case III When some of the factors in denominator are quadratic but non-repeating. Corresponding to each quadratic factor ax^2+bx+c, we assume the partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in numerator of both the sides.

| Example 52 Resolve $\frac{2x+7}{(x+1)(x^2+4)}$ into partial fractions.

Sol. Let $\frac{2x+7}{(x+1)(x^2+4)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}$

$\therefore \quad 2x+7=A(x^2+4)+(Bx+C)(x+1)$

Put $x=-1$

$\therefore \quad 5=5A$ or $A=1$

Comparing the terms, $0=A+B \Rightarrow B=-1$

$7=4A+C \Rightarrow C=3$

$$\therefore \quad \frac{2x+7}{(x+1)(x^2+4)}=\frac{1}{x+1}+\frac{(-x+3)}{x^2+4}$$

Aliter To obtain values of A, B and C from

$2x+7=A(x^2+4)+(Bx+C)(x+1)$

i.e., $2x+7=(A+B)x^2+(B+C)x+4A+C$

Equating the coefficients of identical powers of x, we get

$A+B=0, B+C=2$ and $4A+C=7$.

Solving, we get $A=1, B=-1, C=3$

| Example 53 Find the partial fraction $\frac{2x+1}{(3x+2)(4x^2+5x+6)}$.

Sol. Let $\frac{2x+1}{(3x+2)(4x^2+5x+6)}=\frac{A}{(3x+2)}+\frac{Bx+C}{(4x^2+5x+6)}$,

then $2x+1=A(4x^2+5x+6)+(Bx+C)(3x+2)$

where A, B, C are constants.

For A, let $3x+2=0$,

i.e., $x=-2/3$

$$2\left(-\frac{2}{3}\right)+1=A\left\{4\cdot\frac{4}{9}-\frac{10}{3}+6\right\}+\left\{B\left(-\frac{2}{3}\right)+C\right\}(0)$$

$$-\frac{1}{3}=A\left(\frac{40}{9}\right) \Rightarrow A=-\frac{3}{40}$$

Comparing coefficients of x^2 and constant term on both the sides for B and C, we get

$$4A+3B=0,$$

$$\therefore \quad B=-\frac{4}{3}A \Rightarrow B=\frac{1}{10} \text{ and } 6A+2C=1,$$

$$\therefore \quad C=\frac{1-6A}{2} \Rightarrow C=\frac{29}{40}$$

$$\therefore \frac{2x+1}{(3x+2)(4x^2+5x+6)}=\frac{-3}{40(3x+2)}+\frac{\left(x+\frac{29}{4}\right)}{10(4x^2+5x+6)}$$

Case IV When some of the factors of the denominator are quadratic and repeating. For every quadratic repeating factor of the type $(ax^2+bx+c)^k$, we assume :

$$\frac{A_1x+A_2}{ax^2+bx+c}+\frac{A_3x+A_4}{(ax^2+bx+c)^2}+\ldots+\frac{A_{2k-1}x+A_{2k}}{(ax^2+bx+c)^k}$$

| Example 54 Resolve $\dfrac{2x^4+2x^2+x+1}{x(x^2+1)^2}$ into partial fractions.

Sol. Let $\dfrac{2x^4+2x^2+x+1}{x(x^2+1)^2}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

or $2x^4+2x^2+x+1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x$

Comparing coefficients of x^4, x^3, x^2, x and constant term

$\therefore \quad A+B=2, C=0, 2A+D+B=2, E=1, A=1$

$\therefore$ we get $A=1, B=1, C=0, D=-1, E=1$

Hence, the partial fraction,

$$\frac{2x^4+2x^2+x+1}{x(x^2+1)^2}=\frac{1}{x}+\frac{x}{1+x^2}+\frac{1-x}{(1+x^2)^2}$$

| Example 55 Evaluate the following integrals:

(i) $\displaystyle\int\frac{(1-x^2)\,dx}{x(1-2x)}$ (ii) $\displaystyle\int\frac{3x-1}{(x-2)^2}dx$

(iii) $\displaystyle\int\frac{x^2+x+1}{x^2(x+2)}dx$ (iv) $\displaystyle\int\frac{8dx}{(x+2)(x^2+4)}$

Sol. (i) Let, $\dfrac{1-x^2}{x(1-2x)}=\dfrac{1}{2}+\dfrac{A}{x}+\dfrac{B}{(1-2x)}$

$$\Rightarrow \quad (1-x^2)=\frac{1}{2}x(1-2x)+A(1-2x)+B(x)$$

On putting $x=0$ and $x=\frac{1}{2}$, we get

$$1=A \text{ and } 1-\frac{1}{4}=B.\frac{1}{2}\Rightarrow A=1, B=\frac{3}{2}$$

$$\therefore \int\frac{1-x^2}{x(1-2x)}dx=\int\left(\frac{1}{2}+\frac{1}{x}+\frac{3}{2(1-2x)}\right)dx$$

$$=\frac{1}{2}x+\log|x|+\frac{3}{2}\frac{\log|1-2x|}{-2}+C$$

$$=\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+C.$$

(ii) Let, $\dfrac{3x-1}{(x-2)^2}=\dfrac{A}{(x-2)}+\dfrac{B}{(x-2)^2}$

$$\Rightarrow \quad 3x-1=A(x-2)+B, \quad \ldots(i)$$

On putting $x=2$ in Eq. (i), we get $B=5$.

On equating coefficients of x on both sides of (i), we get $A=3$,

$$\therefore \int\frac{3x-1}{(x-2)^2}dx=\int\left(\frac{3}{(x-2)}+\frac{5}{(x-2)^2}\right)dx$$

$$=3\log|x-2|-\frac{5}{x-2}+C$$

(iii) Let, $\dfrac{x^2+x+1}{x^2(x+2)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+2}$

$$\Rightarrow \quad x^2+x+1=Ax(x+2)+B(x+2)+C(x^2) \quad \ldots(i)$$

On putting, $x=-2$ and $x=0$ in Eq. (i), we get

$$C=3/4 \text{ and } B=1/2$$

On equation Coefficient of x^2 on both sides of (i), we get $1=A+C\Rightarrow A=1/4$.

$$\therefore \int\frac{x^2+x+1}{x^2(x+2)}dx=\int\left(\frac{1/4}{x}+\frac{1/2}{x^2}+\frac{3/4}{x+2}\right)dx$$

$$=\frac{1}{4}\log|x|-\frac{1}{2x}+\frac{3}{4}\log|x+2|+C$$

(iv) Let, $\dfrac{8}{(x+2)(x^2+4)}=\dfrac{A}{x+2}+\dfrac{Bx+C}{x^2+4}$

$$\Rightarrow \quad 8=A(x^2+4)+(Bx+C)(x+2) \quad \ldots(i)$$

On putting $x=-2$ in Eq. (i), we get $A=1$.

On equating coefficient of x^2 on both sides we get,

$$0=A+B\Rightarrow B=-1$$

On equating constant term on both sides, we get,

$$8=4A+2C\Rightarrow C=2$$

$$\therefore \int\frac{8}{(x+2)(x^2+4)}dx=\int\left(\frac{1}{x+2}+\frac{(-x+2)}{x^2+4}\right)dx$$

$$=\int\frac{1}{x+2}dx-\frac{1}{2}\int\frac{2x\,dx}{x^2+4}+2\int\frac{dx}{x^2+4}$$

$$=\log|x+2|-\frac{1}{2}\log|x^2+4|+2.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C.$$

| Example 56 Evaluate $\displaystyle\int\frac{1}{\sin x-\sin 2x}dx$.

Sol. Let $I=\displaystyle\int\frac{1}{\sin x-\sin 2x}dx=\int\frac{1}{\sin x-2\sin x\cos x}dx$

$$=\int\frac{1}{\sin x(1-2\cos x)}dx=\int\frac{\sin x}{\sin^2 x(1-2\cos x)}dx$$

$$= \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)}\, dx$$

(put $\cos x = t \Rightarrow -\sin x\, dx = dt$)

$$\therefore \quad I = \int \frac{-dt}{(1-t^2)(1-2t)} = \int \frac{-1}{(1-t)(1+t)(1-2t)}\, dt \quad ...(i)$$

Here, in Eq. (i) we have linear and non-repeated factors thus we use partial fractions for;

$$\frac{-1}{(1-t)(1+t)(1-2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1-2t)}$$

or $-1 = A(1+t)(1-2t) + B(1-t)(1-2t) + C(1-t)(1+t)$ ...(ii)

Putting $(t+1) = 0$ or $t = -1$, we get

$$-1 = B(2)(1+2) \Rightarrow B = -\frac{1}{6}$$

Putting $(1-t) = 0$ or $t = 1$, we get

$$-1 = A(2)(-1) \Rightarrow A = \frac{1}{2}$$

Putting $(1-2t) = 0$ or $t = 1/2$, we get

$$-1 = C\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right) \Rightarrow C = -\frac{4}{3}$$

$$\therefore \quad \frac{-1}{(1-t)(1+t)(1-2t)} = \frac{1}{2(1-t)} - \frac{1}{6(1+t)} - \frac{4}{3(1-2t)}$$

So, Eq. (i) reduces to

$$I = \frac{1}{2}\int \frac{1}{1-t}\, dt - \frac{1}{6}\int \frac{1}{1+t}\, dt - \frac{4}{3}\int \frac{1}{1-2t}\, dt$$

$$= -\frac{1}{2}\log|1-t| - \frac{1}{6}\log|1+t| - \frac{4}{3} \times -\frac{1}{2}\log|1-2t| + C$$

$$= -\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + C$$

| Example 57 Evaluate $\int \frac{(1-x\sin x)\, dx}{x(1-x^3e^{3\cos x})}$.

Sol. Here, $I = \int \frac{(1-x\sin x)\, dx}{x(1-(xe^{\cos x})^3)}$

Put $xe^{\cos x} = t$

$\Rightarrow (xe^{\cos x}\cdot(-\sin x) + e^{\cos x})\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t(1-t^3)} = \int \frac{dt}{t(1-t)(1+t+t^2)}$$

$$= \int \left(\frac{A}{t} + \frac{B}{1-t} + \frac{Ct+D}{1+t+t^2}\right) dt$$

Comparing coefficients, we get

$$A = 1, B = \frac{1}{3}, C = -\frac{2}{3}, D = -\frac{1}{3}$$

$$\therefore \quad I = \int \frac{dt}{t} + \frac{1}{3}\int \frac{dt}{1-t} + \int \frac{\left(-\frac{2}{3}t - \frac{1}{3}\right)}{1+t+t^2}\, dt$$

$$= \log|t| - \frac{1}{3}\log|1-t| - \frac{1}{3}\log|1+t+t^2|$$

[where, $t = xe^{\cos x}$]

| Example 58 Evaluate $\int \sin 4x \cdot e^{\tan^2 x}\, dx$.

Sol. The given integral could be written as,

$$I = \int 4\sin x\cdot\cos x\cdot\cos 2x\cdot e^{\tan^2 x} dx$$

$$= 4\int \tan x\cdot\cos^2 x(\cos^2 x - \sin^2 x)\cdot e^{\tan^2 x} dx$$

$$= 4\int \tan x\cdot\cos^4 x(1-\tan^2 x)\cdot e^{\tan^2 x}\, dx$$

$$= 4\int \frac{\tan x}{(\sec^2 x)^2}\cdot(1-\tan^2 x)\cdot e^{\tan^2 x}\, dx$$

Put $\tan^2 x = t$

$\Rightarrow 2\tan x\cdot\sec^2 x\, dx = dt$

$$\Rightarrow \quad I = 4\int \frac{(1-t)e^t}{(1+t)^3}\cdot\frac{dt}{2} = 2\int \frac{(1-t)e^t}{(1+t)^3}\, dt$$

$$= 2\left\{\int \frac{2e^t - (1+t)e^t}{(1+t)^3} dt\right\}$$

$$= 2\left\{\int e^t \frac{2}{(1+t)^3} - \frac{1}{(1+t)^2}\right\} dt$$

$$= -2\frac{1}{(1+t)^2}e^t + C$$

[using $\int e^x(f(x)) + f'(x))dx = e^x.f(x) + c$]

$$= -\frac{2e^{\tan^2 x}}{(1+\tan^2 x)^2} + C$$

$$I = -2\cos^4 x\cdot e^{\tan^2 x} + C$$

| Example 59 Solve $\int \frac{1+x\cos x}{x(1-x^2e^{2\sin x})}\, dx$.

Sol. Let $I = \int \frac{1+x\cos x}{x(1-x^2e^2\sin x)}\, dx$

Put $(x\, e^{\sin x}) = t$

Differentiating both the sides, we get

$$(x\, e^{\sin x}\cdot\cos x + e^{\sin x})\, dx = dt$$

$$\Rightarrow \quad e^{\sin x}(x\cos x + 1)\, dx = dt$$

$$\Rightarrow \quad I = \int \frac{dt}{t(1-t^2)}$$

$$= \int \frac{dt}{t(1-t)(1+t)} \quad \text{[using partial fraction]}$$

$$= \int \left\{\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)}\right\} dt$$

$$= \log|t| - \frac{1}{2}\log|1-t| - \frac{1}{2}\log|1+t| + C$$

$$= \log|x\, e^{\sin x}| - \frac{1}{2}\log|1-x^2e^{2\sin x}| + C$$

| Example 60 Evaluate;

$$\int \frac{1}{x}\{\log e^{ex} . \log e^{e^2 x} . \log e^{e^3 x}\} dx$$

Sol. We have,

$$I = \int \frac{1}{x} \{\log e^{ex} . \log e^{e^2 x} . \log_e e^3 x\} dx$$

$$= \int \frac{1}{x} \left\{ \frac{1}{\log e^{ex}} \cdot \frac{1}{\log e^{e^2 x}} \cdot \frac{1}{\log e^{e^3 x}} \right\} dx$$

$$= \int \frac{dx}{x\{\log e^e + \log e^x\}\{\log e^{e^2} + \log e^{x^2}\}\{\log e^{e^3} + \log x^2\}}$$

$$= \int \frac{dx}{x\{1 + \log e^x\}\{2 + \log e^x\}\{3 + \log e^x\}}$$

Put $\log e^x = t \Rightarrow \frac{1}{x} dx = dt$

$$I = \int \frac{dt}{(1+t)(2+t)(3+t)} = \int \left(\frac{1}{2} \cdot \frac{1}{(1+t)} - \frac{1}{(2+t)} + \frac{1}{(3+t)} \right) dt$$

[using partial fraction]

$$= \frac{1}{2} \log |1 + t| - \log |2 + t| + \log |3 + t| + C.$$

$$= \frac{1}{2} \log |1 + \log e^x| - \log |2 + \log e^x| + \log |3 + \log e^x| + C.$$

Exercise for Session 5

- **Evaluate the following Integrals :**

1. $\int \frac{x^2}{(x-1)(x-2)(x-3)} dx$
2. $\int \frac{dx}{1+x^3}$
3. $\int \frac{dx}{x(x^n+1)}$
4. $\int \frac{2x}{(x^2+1)(x^2+3)}$
5. $\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$
6. $\int \frac{dx}{\sin x (3 + 2\cos x)} dx$
7. $\int \frac{\sec x}{1+\operatorname{cosec} x} dx$
8. $\int \frac{\tan x + \tan^3 x}{1+\tan^3 x} dx$
9. $\int \frac{dx}{x\,|6(\log x)^2 + 7\log x + 2|}$
10. $\int \frac{\tan^{-1}}{x^2} . dx$

Session 6

Indirect and Derived Substitutions

Indirect and Derived Substitutions

(i) Indirect Substitution

If the integral is of the form $f(x) \cdot g(x)$, where $g(x)$ is a function of the integral of $f(x)$, then put integral of $f(x) = t$.

Example 61 The value of $\int \frac{d(x^2+1)}{\sqrt{x^2+2}}$, is

(a) $2\sqrt{x^2+2}+C$ (b) $\sqrt{x^2+2}+C$

(c) $x\sqrt{x^2+2}+C$ (d) None of these

Sol. Here, $I = \int \frac{d(x^2+1)}{\sqrt{x^2+2}}$

We know, $d(x^2+1) = 2x\,dx$

$\therefore \quad I = \int \frac{2x\,dx}{\sqrt{x^2+2}}$

Put, $\quad x^2+2 = t^2$

$\therefore \quad 2x\,dx = 2t\,dt \Rightarrow I = \int \frac{2t\,dt}{t} = 2t + C$

$\Rightarrow \quad I = 2\sqrt{x^2+2}+C$

Hence, (a) is the correct answer.

Example 62 If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}\,dx = a\log\left(\frac{x^k}{1+x^k}\right)+C$, then a and k are

(a) $2/5, 5/2$ (b) $1/5, 2/5$

(c) $5/2, 1/2$ (d) $2/5, 1/2$

Sol. Here, $I = \int \frac{(\sqrt{x})^5}{(\sqrt{x})^7+x^6}\,dx = \int \frac{dx}{(\sqrt{x})^2+(\sqrt{x})^7}$

$$= \int \frac{dx}{x^{7/2}\left\{1+\frac{1}{x^{5/2}}\right\}},$$

Put $\frac{1}{x^{5/2}} = y \Rightarrow -\frac{5}{2x^{7/2}}\,dx = dy$

$$I = -\frac{2}{5}\int \frac{dy}{1+y}$$

$$= -\frac{2}{5}\log|1+y| + C = \frac{2}{5}\log\left|\frac{1}{1+y}\right| + C$$

$$= \frac{2}{5}\log\left|\frac{x^{5/2}}{x^{5/2}+1}\right| + C \quad \text{...(i)}$$

where, $I = a\log\left(\frac{x^k}{1+x^k}\right)+C$ (given) ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$$a\log\left(\frac{x^k}{1+x^k}\right)+C = \frac{2}{5}\log\left(\frac{x^{5/2}}{1+x^{5/2}}\right)+C$$

$\Rightarrow \quad a = 2/5$ and $k = 5/2$

Hence, (a) is the correct answer.

Example 63 Evaluate $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}\,dx$.

Sol. Here, $I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2}\,dx = \int \frac{x^4(5+4x)}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}\,dx$

$$= \int \frac{5/x^6+4/x^5}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}\,dx$$

Put $\quad 1+\frac{1}{x^4}+\frac{1}{x^5} = t$

$\Rightarrow \quad \left(-\frac{4}{x^5}-\frac{5}{x^6}\right)dx = dt$

$$I = \int -\frac{dt}{t^2} = \frac{1}{t}+C = \frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}}+C$$

$$= \frac{x^5}{x^5+x+1}+C$$

Example 64 For any natural number m, evaluate $\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx,\ x>0$

[IIT JEE 2002]

Sol. Here, $I = \int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx$

$$= \int (x^{3m}+x^{2m}+x^m)\frac{(2x^{3m}+3x^{2m}+6x^m)^{1/m}}{x}\,dx$$

$$= \int (x^{3m-1}+x^{2m-1}+x^{m-1})(2x^{3m}+3x^{2m}+6x^m)^{1/m}\,dx \quad \text{...(i)}$$

Put $\qquad 2x^{3m}+3x^{2m}+6x^m=t$

$\Rightarrow \qquad 6m\,(x^{3m-1}+x^{2m-1}+x^{m-1})\,dx=dt$

$\therefore$ Eq. (i) becomes,

$$I=\int t^{1/m}\,\frac{dt}{6m}=\frac{1}{6m}\cdot\frac{t^{(1/m)+1}}{(1/m)+1}+C$$

$$I=\frac{1}{6(m+1)}\,(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}+C$$

| Example 65 $\displaystyle\int\frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$ is equal to

(a) $\frac{1}{2}\ln(1+\sqrt{1+x^2})+C$ (b) $2\sqrt{1+\sqrt{1+x^2}}+C$

(c) $2(1+\sqrt{1+x^2})+C$ (d) None of these

Sol. $\displaystyle\int\frac{x\,dx}{\sqrt{(1+x^2)}\,\sqrt{1+\sqrt{1+x^2}}}$

Put $\qquad 1+\sqrt{1+x^2}=t^2 \Rightarrow \dfrac{2x}{2\sqrt{1+x^2}}\,dx=2t\,dt$

$\therefore \qquad \dfrac{x\,dx}{\sqrt{1+x^2}}=2t\,dt$

$\therefore \qquad I=\displaystyle\int\frac{2t\,dt}{t}=2t+C=2\sqrt{1+\sqrt{1+x^2}}+C$

Hence, (b) is the correct answer.

| Example 66 $\displaystyle\int\frac{(2x+1)}{(x^2+4x+1)^{3/2}}\,dx$

(a) $\dfrac{x^3}{(x^2+4x+1)^{1/2}}+C$ (b) $\dfrac{x}{(x^2+4x+1)^{1/2}}+C$

(c) $\dfrac{x^2}{(x^2+4x+1)^{1/2}}+C$ (d) $\dfrac{1}{(x^2+4x+1)^{1/2}}+C$

Sol. $\displaystyle\int\frac{2x+1}{(x^2+4x+1)^{3/2}}\,dx=\int\frac{2x+1}{x^3\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}\,dx$

$$=\int\frac{2x^{-2}+x^{-3}}{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}\,dx$$

Now, put $\dfrac{1}{x^2}+\dfrac{4}{x}+1=t^2 \Rightarrow \left(-\dfrac{2}{x^3}-\dfrac{4}{x^2}\right)dx=2t\,dt$

$\therefore \qquad I=\displaystyle\int\frac{-t\,dt}{t^3}=\frac{1}{t}+C$

$$=\frac{x}{\sqrt{x^2+4x+1}}+C$$

Hence, (b) is the correct answer.

| Example 67 The evaluation of

$\displaystyle\int\frac{p\,x^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}\,dx$ is

(a) $-\dfrac{x^p}{x^{p+q}+1}+C$ (b) $\dfrac{x^q}{x^{p+q}+1}+C$

(c) $-\dfrac{x^q}{x^{p+q}+1}+C$ (d) $\dfrac{x^p}{x^{p+q}+1}+C$

Sol. Here, $I=\displaystyle\int\frac{px^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}$

$$\int\frac{px^{p+2q-1}-qx^{q-1}}{(x^{p+q}+1)^2}\,dx=\int\frac{px^{p-1}-qx^{-q-1}}{(x^p+x^{-q})^2}\,dx$$

Taking x^q as x^{2q} common from denominator and take it in numerator.

Put $\quad x^p+x^{-q}=t \Rightarrow (px^{p-1}-qx^{-q-1})\,dx=dt$

$\therefore \qquad I=\displaystyle\int\frac{dt}{t^2}=-\frac{1}{t}+C=-\left(\frac{x^q}{x^{p+q}+1}\right)+C$

Hence, (c) is the correct answer.

| Example 68 $\displaystyle\int\frac{x^2(1-\ln x)}{\ln^4 x-x^4}\,dx$ is equal to

(a) $\dfrac{1}{2}\ln\left(\dfrac{x}{\ln x}\right)-\dfrac{1}{4}\ln(\ln^2 x-x^2)+C$

(b) $\dfrac{1}{4}\ln\left(\dfrac{\ln x-x}{\ln x+x}\right)-\dfrac{1}{2}\tan^{-1}\left(\dfrac{\ln x}{x}\right)+C$

(c) $\dfrac{1}{4}\ln\left(\dfrac{\ln x+x}{\ln x-x}\right)+\dfrac{1}{2}\tan^{-1}\left(\dfrac{\ln x}{x}\right)+C$

(d) $\dfrac{1}{4}\left(\ln\left(\dfrac{\ln x-x}{\ln x+x}\right)+\tan^{-1}\left(\dfrac{\ln x}{x}\right)\right)+C$

Sol. Here, $I=\displaystyle\int\frac{x^2(1-\ln x)}{\ln x^4 x-x^4}$

$$I=\int\frac{x^2(1-\ln x)}{x^4\left(\left(\frac{\ln x}{x}\right)^4-1\right)}\,dx=\int\frac{1-\ln x}{x^2\left(\left(\frac{\ln x}{x}\right)^4-1\right)}\,dx$$

Put $\dfrac{\ln x}{x}=t \Rightarrow \dfrac{1-\ln x}{x^2}=dt$

$$I=\int\frac{dt}{(t^4-1)}=\int\frac{dt}{(t^2+1)(t^2-1)}$$

$$=\frac{1}{2}\int\frac{(t^2+1)-(t^2-1)}{(t^2+1)(t^2-1)}\,dt$$

$$I=\frac{1}{2}\left(\int\frac{dt}{t^2-1}-\int\frac{dt}{t^2+1}\right)=\frac{1}{2}\left(\frac{1}{2}\ln\frac{t-1}{t+1}-\tan^{-1}t\right)$$

$$= \frac{1}{4} \ln \left(\frac{\ln x - x}{\ln x + x} \right) - \frac{1}{2} \tan^{-1} \left(\frac{\ln x}{x} \right) + C$$

Hence, (b) is the correct answer.

| Example 69 $\int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx$ is equal to **[IIT JEE 2006]**

(a) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$ (b) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$

(c) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$ (d) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

Sol. Let $I = \int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}} dx$

$$= \int \frac{x^2 - 1}{x^5} \cdot \frac{dx}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}$$

$$= \frac{1}{4} \int \frac{\frac{4}{x^3} - \frac{4}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx = \frac{1}{4} \cdot 2 \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}$$

$$\left(\because \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C \right)$$

$$= \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$$

Hence, (d) is the correct answer.

| Example 70 Let $f(x) = \frac{x}{(1 + x^n)^{1/n}}$ for $n \ge 2$ and $g(x) = \underbrace{fofo \dots of}_{n \text{ times}}(x)$, then $\int x^{n-2} g(x) dx$ equals to **[IIT JEE 2007]**

(a) $\frac{1}{n(n-1)} (1 + nx^n)^{1 - \frac{1}{n}} + C$ (b) $\frac{1}{n-1} (1 + nx^n)^{1 - \frac{1}{n}} + C$

(c) $\frac{1}{n(n+1)} (1 + nx^n)^{1 + \frac{1}{n}} + C$ (d) $\frac{1}{n+1} (1 + nx^n)^{1 + \frac{1}{n}} + C$

Sol. $\because$ $f(x) = \frac{x}{(1 + x^n)^{1/n}}$

$\therefore$ $f(f(x)) = \frac{y}{(1 + x^n)^{1/n}}$,

Where, $y = \frac{x}{(1 + x^n)^{1/n}} = \frac{x}{(1 + 2x^n)^{1/n}}$

Similarly, $f(f(f(x))) = \frac{x}{(1 + 3x^n)^{1/n}}$

and $\underbrace{(fofofo \dots of)}_{n \text{ times}}(x) = g(x) = \frac{x}{(1 + nx^n)^{1/n}}$

$$\therefore \int x^{n-2} g(x) dx = \int \frac{x^{n-1}}{(1 + nx^n)^{1/n}} dx$$

$$= \frac{1}{n^2} \int n^2 \cdot x^{n-1} \cdot (1 + nx^n)^{-1/n} dx$$

$$= \frac{1}{n^2} \cdot \frac{(1 + nx^n)^{1 - \frac{1}{n}}}{1 - \frac{1}{n}} + C$$

$$= \frac{(1 + nx^n)^{1 - \frac{1}{n}}}{n(n-1)} + C$$

Hence, (a) is the correct answer.

Derived Substitutions

Some times it is useful to write the integral as a sum of two related integrals which can be evaluated by making suitable substitutions.

Examples of such integrals are

Type I

(a) Algebraic Twins

$$\int \frac{2x^2}{x^4 + 1} dx = \int \frac{x^2 + 1}{x^4 + 1} dx + \int \frac{x^2 - 1}{x^4 + 1} dx$$

$$\int \frac{2}{x^4 + 1} dx = \int \frac{x^2 + 1}{x^4 + 1} dx - \int \frac{x^2 - 1}{x^4 + 1} dx$$

$$\int \frac{2x^2}{x^4 + 1 + kx^2} dx, \int \frac{2}{(x^4 + 1 + kx^2)} dx$$

(b) Trigonometric Twins

$$\int \sqrt{\tan x}\, dx, \int \sqrt{\cot x}\, dx,$$

$$\int \frac{1}{(\sin^4 x + \cos^4 x)} dx, \int \frac{1}{\sin^6 x + \cos^6 x} dx,$$

$$\int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos x} dx$$

Method of evaluating these integral are illustrated by mean of the following examples :

Integral of the Form

1. $\int f\left(x + \frac{1}{x} \right) \left(1 - \frac{1}{x^2} \right) dx$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2} \right) dx = dt$

2. $\int f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)dx$.

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx=dt$

3. $\int \frac{x^2+1}{x^4+kx^2+1}dx$.

Divide numerator and denominator by x^2.

4. $\int \frac{x^2-1}{x^4+kx^2+1}dx$

Divide numerator and denominator by x^2.

| Example 71 Evaluate $\int \frac{5}{1+x^4}dx$.

Sol. Let $I=\int \frac{5}{1+x^4}dx=\frac{5}{2}\int \frac{2}{1+x^4}dx$

$$=\frac{5}{2}\left[\int \frac{1+x^2}{1+x^4}dx+\int \frac{1-x^2}{1+x^4}dx\right]$$

$$=\frac{5}{2}\left[\int \frac{x^2+1}{x^4+1}dx-\int \frac{x^2-1}{x^4+1}dx\right]$$

Remark

Here, dividing Numerator and Denominator by x^2 and converting Denominator into perfect square so as to get differential in Numerator

i.e. $I=\frac{5}{2}\left[\int \frac{1+1/x^2}{x^2+1/x^2}dx-\int \frac{1-1/x^2}{x^2+1/x^2}dx\right]$

$$=\frac{5}{2}\left[\int \frac{1+1/x^2}{(x-1/x)^2+2}dx-\int \frac{1-1/x^2}{(x+1/x)^2-2}dx\right]$$

$$=\frac{5}{2}\left[\int \frac{dt}{t^2+(\sqrt{2})^2}-\int \frac{du}{u^2-(\sqrt{2})^2}\right],$$

[where $t=x-\frac{1}{x}$ and $u=x+\frac{1}{x}$]

$$=\frac{5}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|\right]+C$$

$$\therefore \quad I=\frac{5}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right]+C$$

| Example 72 Evaluate $\int \frac{1}{x^4+5x^2+1}dx$.

Sol. Let $I=\frac{1}{2}\int \frac{2}{x^4+5x^2+1}dx$

$$\Rightarrow I=\frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}dx+\frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}dx$$

$$=\frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}dx-\frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}dx$$

[(dividing Numerator and Denominator by x^2)]

$$=\frac{1}{2}\int \frac{(1+1/x^2)}{(x-1/x)^2+7}dx-\frac{1}{2}\int \frac{(1-1/x^2)}{(x+1/x)^2+3}dx$$

$$=\frac{1}{2}\int \frac{dt}{t^2+(\sqrt{7})^2}-\frac{1}{2}\int \frac{du}{u^2+(\sqrt{3})^2}$$

where $t=x-\frac{1}{x}$ and $u=x+\frac{1}{x}$

$$\therefore \quad I=\frac{1}{2}\cdot\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right)-\frac{1}{2}\cdot\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right)+C$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right]+C$$

| Example 73 Evaluate $\int \sqrt{\tan x}\,dx$.

Sol. Here $I=\int \sqrt{\tan x}\,dx$

Put $\tan x=t^2$

$\Rightarrow \sec^2 x\,dx=2t\,dt \quad \Rightarrow dx=\frac{2t\,dt}{1+t^4}$

$$\therefore \quad I=\int t\cdot\frac{2t}{1+t^4}dt=\int \frac{2t^2}{t^4+1}dt$$

$$=\int \frac{t^2+1}{t^4+1}dt+\int \frac{t^2-1}{t^4+1}dt$$

$$=\int \frac{1+1/t^2}{t^2+1/t^2}dt+\int \frac{1-1/t^2}{t^2+1/t^2}dt$$

$$=\int \frac{1+1/t^2}{(t-1/t)^2+2}dt+\int \frac{1-1/t^2}{(t+1/t)^2-2}dt$$

$$I=\int \frac{ds}{s^2+(\sqrt{2})^2}+\int \frac{dr}{r^2-(\sqrt{2})^2},\ \left[s=t-\frac{1}{t}\text{ and } r=t+\frac{1}{t}\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{s}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{r-\sqrt{2}}{r+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right)+\frac{1}{2}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|\right]+C$$

[(where $t=\sqrt{\tan x}$)]

| Example 74 Evaluate $\int \frac{4}{\sin^4 x+\cos^4 x}dx$.

Sol. Let $I=4\int \frac{dx}{\sin^4 x+\cos^4 x}$

Dividing numerator and denominator by $\cos^4 x$, we get

$$I = 4\int \frac{\sec^4 x}{\tan^4 x + 1}\,dx$$

$$I = 4\int \frac{\sec^2 x\,(1+\tan^2 x)}{1+\tan^4 x}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I = 4\int \frac{t^2+1}{t^4+1}\,dt$$

$$= 4\int \frac{1+1/t^2}{t^2+1/t^2}\,dt = 4\int \frac{1+1/t^2}{(1-1/t)^2+2}\,dt$$

Again, put $z = t - \frac{1}{t}$

$$\therefore \quad I = 4\int \frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C$$

$$I = 2\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1/\tan x}{\sqrt{2}}\right) + C$$

| Example 75 The value of $\int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$, is equal to

(a) $\frac{1}{c}\sin^{-1}\left(ax\frac{b}{x}\right)+k$ (b) $c\sin^{-1}\left(ax+\frac{b}{x}\right)+k$

(c) $\sin^{-1}\left[\frac{ax+b/x}{c}\right]+k$ (d) None of these

Sol. Here, $I = \int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}} = \int \frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$

Put $ax + \frac{b}{x} = t$

$$\therefore \quad \left(a - \frac{b}{x^2}\right)dx = dt$$

$$I = \int \frac{dt}{\sqrt{c^2-t^2}} = \sin^{-1}\left(\frac{t}{c}\right) + k$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{ax+b/x}{c}\right) + k$$

Hence (c) is the correct answer.

| Example 76 $\int \frac{x^x\,(x^{2x}+1)(\ln x + 1)}{x^{4x}+1}\,dx$

Sol. $I = \int \frac{x^x\,(x^{2x}+1)(\ln x+1)}{x^{4x}+1}\,dx$

Put $x^x = y \Rightarrow x^x(\ln x + 1)\,dx = dy$

$$I = \int \frac{y^2+1}{y^4+1}\,dy = \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\,dy = \int \frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+2}\,dy$$

Put, $y - \frac{1}{y} = t \Rightarrow \left(1+\frac{1}{y^2}\right)dy = dt$

$$I = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y-\frac{1}{y}}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^x-\frac{1}{x^x}}{\sqrt{2}}\right) + C$$

| Example 77 Evaluate

$$\int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}.$$

Sol. Here $I = \int \frac{(x^2-1)}{(x^4+x^3+1)\tan^{-1}}\left(x+\frac{1}{x}\right)$

The given integral can be written as

$$I = \int \frac{(1-1/x^2)\,dx}{(x^2+3+1/x^2)\tan^{-1}\left(x+\frac{1}{x}\right)}$$

(dividing numerator and denominator by x^2)

$$I = \int \frac{(1-1/x^2)\,dx}{\{(x+1/x)^2+1\}\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Put, $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t^2+1)\cdot\tan^{-1}(t)} \quad \text{...(i)}$$

Now, make one more substitution

$\tan^{-1} t = u$. Then, $\frac{dt}{t^2+1} = du$

$\therefore$ Eq. (i) becomes, $I = \int \frac{du}{u} = \log|u| + C$

$\Rightarrow I = \log|\tan^{-1} t| + C = \log|\tan^{-1}(x+1/x)| + C$

| Example 78 $\int \frac{(x^{-7/6}-x^{5/6})\,dx}{x^{1/3}(x^2+x+1)^{1/2}-x^{1/2}(x^2+x+1)^{1/3}}$

Sol. $I = \int \frac{x^{7/6}(x^{-7/6}-x^{5/6})\,dx}{x^{7/6}\cdot x^{1/3}(x^2+x+1)^{1/2}-x^{1/2}\cdot x^{7/6}(x^2+x+1)^{1/3}}$

$$= \int \frac{(1-x^2)\,dx}{x^{3/2}(x^2+x+1)^{1/2}-x^{5/3}(x^2+x+1)^{1/3}}$$

$$= \int \frac{-\left(1-\frac{1}{x^2}\right) dx}{\left(x+\frac{1}{x}+1\right)^{1/2} - \left(x+\frac{1}{x}+1\right)^{1/3}} \quad \left[\begin{array}{l}\text{Putting } x+\frac{1}{x}=t \\ \Rightarrow \left(1-\frac{1}{x^2}\right) dx = dt\end{array}\right]$$

$$= -\int \frac{dt}{(t+1)^{1/2} - (t+1)^{1/3}}$$

Substitute, $(t+1) = u^6$

$$= -\int \frac{6u^5\, du}{u^3 - u^2} = -6\int \frac{u^3}{u-1} du,$$

Put $u - 1 = z$

$$dz = -6\int \frac{(z+1)^3}{z} dz$$

$$= -6\int \frac{z^3 + 3z^2 + 3z + 1}{z} dz$$

$$= -6\int \left(z^2 + 3z + 3 + \frac{1}{z}\right) dz$$

$$= -6\left\{\frac{z^3}{3} + \frac{3z^2}{2} + 3z + \log|z|\right\} + C$$

where, $z = \left(x + \frac{1}{x} + 1\right)^{1/6} - 1$

Example 79 The value of $\int \{\{[x]\}\} dx$, where $\{.\}$ and $[.]$ denotes fractional part of x and greatest integer function, is equal to

(a) 0 (b) 1 (c) 2 (d) −1

Sol. Let $I = \int \{\{[x]\}\} dx$

where, $[x] =$ Integer and we know $\{n\} = 0; n \in$ Integer.

$\therefore \quad I = \int 0\, dx = 0$

Hence, (a) is the correct answer.

Type II

Integration of Some Special Irrational Algebraic Functions

In this case we shall discuss four integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$, where P and Q are polynomial functions of x and $\phi(x)$ is polynomial in x.

(a) Integrals of the Form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$, where P and Q are both linear of x

To evaluate this type of integrals we put $Q = t^2$, i.e. to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{cx+d}} dx$, put $cx + d = t^2$.

The following examples illustrate the procedure :

Example 80 Evaluate $\int \frac{1}{(x+1)\sqrt{x-2}} dx$.

Sol. Let $I = \int \frac{1}{(x+1)\sqrt{x-2}} dx$

Here, P and Q both are linear, so we put $Q = t^2$

i.e. $x - 2 = t^2$

So that $dx = 2t\, dt$

$$\therefore \quad I = \int \frac{1}{(t^2+2+1)\sqrt{t^2}} \cdot 2t\, dt = 2\int \frac{dt}{t^2+3}$$

$$= 2 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C$$

$$\therefore \quad I = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{3}}\right) + C$$

(b) Integrals of the Form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$, where P is a quadratic expression and Q is a linear expression

To evaluate this type of integrals we put $Q = t^2$ i.e. to evaluate the integrals of the form

$$\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}} dx$$

put $px + q = t^2$.

Example 81 Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$.

Sol. Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$

Put $x + 1 = t^2 \Rightarrow dx = 2t\, dt$

$$\therefore \quad I = \int \frac{(t^2-1)+2}{\{(t^2-1)^2 + 3(t^2-1) + 3\}\sqrt{t^2}} \cdot (2t)\, dt$$

$$= 2\int \frac{t^2+1}{t^4+t^2+1} dt = 2\int \frac{1+1/t^2}{t^2+1+1/t^2} dt$$

$$= 2\int \frac{1+1/t^2}{(t-1/t)^2 + (\sqrt{3})^2} dt = 2\int \frac{du}{u^2+(\sqrt{3})^2}$$

$$\left[\text{where } u = t - \frac{1}{t}\right]$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$\therefore \quad I = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right) + C$$

(c) Integral of the Form $\int \frac{1}{(ax+b)} \cdot \frac{dx}{\sqrt{px^2+qx+r}}$**, where in** $\int \frac{dx}{P\sqrt{Q}}$ **P is linear and Q is a quadratic put,** $ax+b=\frac{1}{t}$.

Example 82 Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$.

Sol. Let $I=\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Put $x-1=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt$

$$\therefore \quad I=\int \frac{-1/t^2\, dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}}$$

$$=-\int \frac{dt}{\sqrt{3t^2+3t+1}}=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}}$$

$$=-\frac{1}{\sqrt{3}}\log|(t+1/2)+\sqrt{(t+1/2)^2+1/12}|+C$$

$$=-\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+C$$

(d) Integrals of the Form $\int \frac{dx}{P\sqrt{Q}}$**, where P and Q both are pure quadratic expression in x, i.e.** $P=ax^2+b$ **and** $Q=cx^2+d$**, i.e.** $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$.

To evaluate this type of integrals of the form we put $x=\frac{1}{t}$

Example 83 Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Sol. Let $\quad I=\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $\quad x=\frac{1}{t}$, so that $dx=-\frac{1}{t^2}dt$

$$\therefore \quad I=\int \frac{-1/t^2\, dt}{(1+1/t^2)\sqrt{1-1/t^2}}=-\int \frac{t\, dt}{(t^2+1)\sqrt{t^2-1}}$$

Again, $t^2=u \Rightarrow 2t\, dt=du$.

$=-\frac{1}{2}\int \frac{du}{(u+1)\sqrt{u-1}}$ which reduces to the form $\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear so that we put $u-1=z^2$ so that $du=2z\, dz$

$$\therefore \quad I=-\frac{1}{2}\int \frac{2z\, dz}{(z^2+1+1)\sqrt{z^2}}=-\int \frac{dz}{(z^2+2)}$$

$$I=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$I=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+C$$

Aliter Put $x=\cos\theta$, $dx=-\sin\theta\, d\theta$

$$\therefore \quad I=-\int \frac{\sin\theta\, d\theta}{(1+\cos^2\theta)\sin\theta}=-\int \frac{d\theta}{1+\cos^2\theta}$$

$$=-\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta+1}=-\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta+2}$$

Put $\tan\theta=t. \Rightarrow \sec^2\theta\, d\theta=dt$

$$\therefore \quad I=-\int \frac{dt}{t^2+2}=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right)+C$$

$$=-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}\,x}\right)+C \qquad \left[\begin{array}{l}\text{where, } \cos\theta=x \\ \sin\theta=\sqrt{1-x^2} \\ \therefore \tan\theta=\frac{\sqrt{1-x^2}}{x}\end{array}\right]$$

Example 84 Evaluate

$$I=\int \frac{(x-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)}dx.$$

Sol. Here, $I=\int \frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^2(x+1)^2}dx$

$$=\int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{x^2\left(x^2+2x-1+\frac{2}{x}+\frac{1}{x^2}\right)}}{\frac{x^2(x^2+2x+1)}{x^2}}dx$$

$$=\int \frac{\left(1-\frac{1}{x^2}\right)\sqrt{\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-1}}{\left(x+\frac{1}{x}+2\right)}dx$$

Put $x+\frac{1}{x}=t$, i.e. $\left(1-\frac{1}{x^2}\right)dx=dt$

$$=\int \frac{\sqrt{(t^2-2)+2t-1}}{(t+2)}dt=\int \frac{\sqrt{t^2+2t-3}}{(t+2)}dt$$

$$= \int \frac{t^2+2t-3}{(t+2)\sqrt{t^2+2t-3}}\,dt$$

$$= \int \frac{t(t+2)}{(t+2)\sqrt{t^2+2t-3}}\,dt - 3\int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$$

$$I = I_1 - 3I_2 \quad \text{...(i)}$$

Where, $I_1 = \int \frac{t\,dt}{\sqrt{t^2+2t-3}}$ and $I_2 = \int \frac{dt}{(t+2)\sqrt{t^2+2t-3}}$

$$\therefore \quad I_1 = \int \frac{t\,dt}{\sqrt{(t+1)^2-4}}$$

Put, $t+1 = z = \int \frac{(z-1)\,dz}{\sqrt{z^2-2^2}}$

$$= \int \frac{z\,dz}{\sqrt{z^2-2^2}} - \int \frac{dz}{\sqrt{z^2-2^2}}$$

$$= \sqrt{z^2-2^2} - \log|z+\sqrt{z^2-4}|$$

$$= \sqrt{t^2+2t-3} - \log|(t+1)+\sqrt{t^2+2t+3}| \quad \text{...(ii)}$$

Also, $I_2 = \int \frac{dy}{y^2\cdot\frac{1}{y}\sqrt{\left(\frac{1}{y}-2\right)^2+2\left(\frac{1}{y}-2\right)-3}}$

Put $t+2 = \frac{1}{y} = \int \frac{dy}{\sqrt{1-2y-3y^2}} = \frac{1}{\sqrt{3}}\int \frac{dy}{\sqrt{\left(\frac{2}{3}\right)^2-\left(y+\frac{1}{3}\right)^2}}$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{y+\frac{1}{3}}{\frac{2}{3}}\right) = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{5+t}{2+t}\right) \quad \text{...(iii)}$$

$$\therefore \quad I = \sqrt{t^2+2t-3} - \log(t+1+\sqrt{t^2+2t-3}) - \sqrt{3}\sin^{-1}\left(\frac{t+5}{t+2}\right)$$

where, $t = x + \frac{1}{x}$

(e) Integrals of the Form $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$, **where $r \geq 2$ and $r \in I$**

Here, we substitute, $x-k = \frac{1}{t}$

| Example 85 Evaluate $\int \frac{dx}{(x-3)^3\sqrt{x^2-6x+10}}$.

Sol. Substitute $(x-3) = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$

We get, $\int \frac{dx}{(x-3)^3\sqrt{x^2-6x+10}}$

$$= \int \frac{-1/t^2\,dt}{1/t^3\sqrt{(1/t+3)^2-6(1/t+3)+10}}$$

$$= -\int \frac{t^2\,dt}{\sqrt{1+t^2}} = \int \frac{dt}{\sqrt{1+t^2}} - \int \sqrt{1+t^2}\,dt$$

$$= \log|t+\sqrt{1+t^2}| - \frac{t}{2}\sqrt{1+t^2} - \frac{1}{2}\log|t+\sqrt{1+t^2}| + C$$

$$= \frac{1}{2}\log|t+\sqrt{1+t^2}| - \frac{t}{2}\sqrt{1+t^2} + C$$

$$= \frac{1}{2}\left[\log\left|\frac{1+\sqrt{x^2-6x+10}}{|x-3|}\right| - \frac{\sqrt{x^2-6x+10}}{|x-3|^2}\right] + C$$

(f) Integrals of the Form $\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}}\,dx$

Here, we write

$$ax^2+bx+c = A_1(dx+e)(2fx+g) + B_1(dx+e) + C_1$$

Where A_1, B_1 and C_1 are constants which can be obtained by comparing the coefficients of like terms on both the sides.

| Example 86 Evaluate $\int \frac{2x^2+5x+9}{(x+1)\sqrt{x^2+x+1}}\,dx$.

Sol. Let $2x^2+5x+9 = A(x+1)(2x+1) + B(x+1) + C$

or $2x^2+5x+9 = x^2(2A) + x(3A+B) + (A+B+C)$

$\Rightarrow \quad A = 1, B = 2, C = 6$

Thus, $\int \frac{2x^2+5x+9}{(x+1)\sqrt{x^2+x+1}}\,dx$

$$= \int \frac{(x+1)(2x+1)}{(x+1)\sqrt{x^2+x+1}}\,dx + 2\int \frac{x+1}{(x+1)\sqrt{x^2+x+1}}\,dx + 6\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + 2\int \frac{dx}{\sqrt{x^2+x+1}} + 6\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$= \int \frac{du}{\sqrt{u}} + 2\int \frac{dx}{\sqrt{(x+1/2)^2+(3/4)}} + 6\int \frac{-dt}{\sqrt{t^2-t+1}}$$

[where $u = x^2+x+1$ and $\frac{1}{t} = x+1$]

$$= 2\sqrt{x^2+x+1} + 2\cdot 1\log|(x+1/2)+\sqrt{x^2+x+1}| - 6\int \frac{dt}{\sqrt{(t-1/2)^2+3/4}}$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| - 6\log\left|\left(t-\frac{1}{2}\right)+\sqrt{t^2-t+1}\right| + C$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| - 6\log\left|\frac{1-x+\sqrt{x^2+x+1}}{2(x+1)}\right| + C$$

Type III

Integration of Type $\int (\sin^m x \cdot \cos^n x)\, dx$

(i) Where m, n belongs to natural number.
(ii) If one of them is odd, then substitute for term of even power.
(iii) If both are odd, substitute either of them.
(iv) If both are even, use trigonometric identities only.
(v) If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitute $\cot x = p$ or $\tan x = p$ which so ever is found suitable.

Example 87 Evaluate $\int \sin^3 x \cdot \cos^5 x\, dx$.

Sol. $I = \int \sin^3 x \cdot \cos^5 x\, dx$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$I = -\int (1-t^2)\cdot t^5 dt$$

$$I = \int t^7\, dt - \int t^5\, dt = \frac{t^8}{8} - \frac{t^6}{6} + C$$

$$I = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$$

Aliter $I = \int R^3 (1-R^2)^2\, dR,$

if $\sin x = R,\ \cos x\, dx = dR$

$$I = \int R^3\, dR - \int 2R^5\, dR + \int R^7 dR$$

$$I = \frac{\sin^4 x}{4} - \frac{2\sin^6 x}{6} + \frac{\sin^8 x}{8} + C$$

Remark

This problem can also be handled by successive reduction or by trigonometrical identities. Answers will be in different form but identical with modified constant of integration.

Example 88 Evaluate $\int \sin^{-11/3} x \cdot \cos^{-1/3} x\, dx$.

Sol. Here, $\int \sin^{-11/3} x \cdot \cos^{-1/3} x\, dx$ i.e. $\left(\frac{-\frac{11}{3} - \frac{1}{3} - 2}{2} = -3\right)$

$$\therefore\quad I = \int \frac{\cos^{-1/3} x}{\sin^{-1/3} x \cdot \sin^4 x}\, dx = \int (\cot^{-1/3} x)(\text{cosec}^2 x)^2 dx$$

$$I = \int (\cot^{-1/3} x)(1+\cot^2 x)\,\text{cosec}^2 x\, dx$$

$$= -\int t^{-1/3}(1+t^2)\, dt = -\int (t^{-1/3} + t^{5/3})\, dt$$

[Put $\cot x = t, \Rightarrow -\text{cosec}^2 x\, dx = dt$]

$$= -\left\{\frac{3}{2} t^{2/3} + \frac{3}{8} t^{8/3}\right\} + C$$

$$= -\left\{\frac{3}{2}(\cot^{2/3} x) + \frac{3}{8}(\cot^{8/3} x)\right\} + C$$

Example 89 Evaluate $\int \frac{dx}{2\sin x + \sec x}$.

Sol. Let $I = \int \frac{dx}{2\sin x + \sec x} = \int \frac{\cos x\, dx}{\sin 2x + 1} = \frac{1}{2}\int \frac{2\cos x\, dx}{1+\sin 2x}$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{(\sin^2 + \cos^2 x + 2\sin x \cos x)}\, dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{(\sin x + \cos x)^2}\, dx + \frac{1}{2}\int \frac{(\cos x - \sin x)}{(\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2}\int \frac{dx}{\sin x + \cos x} + \frac{1}{2}\int \frac{dv}{v^2},\ \text{where, } v = \sin x + \cos x$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} - \frac{1}{2v} + C$$

$$= \frac{1}{2\sqrt{2}}\int \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} - \frac{1}{2(\sin x + \cos x)} + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\text{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right| - \frac{1}{2(\sin x + \cos x)} + C$$

Type IV

Integrals of the Form $\int x^m (a + bx^n)^P dx$

Case I If $P \in N$. We expand using binomial and integrate.

Case II If $P \in I^-$ (*ie*, negative integer), write $x = t^k$, where k is the LCM of m and n.

Case III If $\frac{m+1}{n}$ is an integer and $P \leftrightarrow$ fraction, put $(a + b\, x^n) = t^k$, where k is denominator of the fraction P.

Case IV If $\left(\frac{m+1}{n} + P\right)$ is an integer and $P \in$ fraction.

We put $(a + b\, x^n) = t^k x^n$, where k is denominator of the fraction P.

Example 90 Evaluate $\int x^{1/3} (2 + x^{1/2})^2\, dx$.

Sol. $I = \int x^{1/3}(2 + x^{1/2})^2\, dx$

Since, P is natural number.

$$\therefore\quad I = \int x^{1/3}(4 + x + 4x^{1/2})\, dx$$

$$= \int (4x^{1/3} + x^{4/3} + 4x^{5/6})\, dx$$

$$= \frac{4x^{4/3}}{4/3} + \frac{x^{7/3}}{7/3} + \frac{4x^{11/6}}{11/6} + C$$

$$= 3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + C$$

Example 91 Evaluate $\int x^{-2/3}(1+x^{2/3})^{-1}\,dx$.

Sol. If we substitute $x = t^3$ (as we know $P \in$ negative integer)

$\therefore$ Let $x = t^k$, where k is the LCM of m and n.

$\therefore \quad x = t^3 \Rightarrow dx = 3t^2\,dt$

or $\quad I = \int \frac{3t^2}{t^2(1+t^2)}\,dt = 3\int \frac{dt}{t^2+1} = 3\tan^{-1}(t) + C$

$\Rightarrow \quad I = 3\tan^{-1}(x^{1/3}) + C$

Example 92 Evaluate $\int x^{-2/3}(1+x^{1/3})^{1/2}\,dx$.

Sol. If we substitute $1 + x^{1/3} = t^2$, then $\frac{1}{3x^{2/3}}\,dx = 2t\,dt$

$\therefore \quad I = \int \frac{t \cdot 6t\,dt}{1} = 6\int t^2\,dt = 2t^3 + C$

or $\quad I = 2(1 + x^{1/3})^{3/2} + C$

Example 93 Evaluate $\int \sqrt{x}\,(1+x^{1/3})^4\,dx$.

Sol. Here, $m = \frac{1}{2}$ and $n = \frac{1}{3}$

Put $x = t^6 \Rightarrow dx = 6t^5\,dt$

$\Rightarrow \quad I = \int t^3(1+t^2)^4\, 6t^5\,dt$

$\Rightarrow \quad I = 6\int t^8(1 + 4t^2 + 6t^4 + 4t^6 + t^8)\,dt$

$= 6\int (t^8 + 4t^{10} + 6t^{12} + 4t^{14} + t^{16})\,dt$

$= 6\left\{\frac{t^9}{9} + \frac{4t^{11}}{11} + \frac{6t^{13}}{13} + \frac{4t^{15}}{15} + \frac{t^{17}}{17}\right\} + C$

$I = 6\left\{x^{2/3} + \frac{4}{11}x^{11/6} + \frac{6}{13}x^{13/6} + \frac{4}{15}x^{5/2} + \frac{1}{17}x^{17/6}\right\} + C$

Example 94 Evaluate $\int x^5(1+x^3)^{2/3}\,dx$.

Sol. Here, $\int x^5(1+x^3)^{2/3}\,dx$ have $m = 5, n = 3$ and $p = \frac{2}{3}$

$\therefore \quad \frac{m+1}{n} = \frac{6}{3} = 2$ [an integer]

So, we substitute $1 + x^3 = t^2$ and $3x^2\,dx = 2t\,dt$

$\therefore \int x^5(1+x^3)^{2/3}dx = \int x^3(1+x^3)^{2/3}x^2dx$

$= \int (t^2-1)(t^2)^{2/3}\frac{2}{3}t\,dt$

$= \frac{2}{3}\int (t^2-1)\,t^{7/3}dt = \frac{2}{3}\int (t^{13/3} - t^{7/3})\,dt$

$= \frac{2}{3}\left\{\frac{3}{16}t^{16/3} - \frac{3}{10}t^{10/3}\right\} + C$

$= \frac{1}{8}(1+x^3)^{8/3} - \frac{1}{5}(1+x^3)^{5/3} + C$

Example 95 Evaluate $\int x^{-11}(1+x^4)^{-1/2}\,dx$.

Sol. Here, $\left(\frac{m+1}{n} + p\right) = \left[\frac{-11+1}{4} - \frac{1}{2}\right] = -3$

If we substitute $(1 + x^4) = t^2x^4$,

then $1 + \frac{1}{x^4} = t^2$ and $\frac{-4}{x^5}\,dx = 2t\,dt$

$\therefore \quad I = \int \frac{dx}{x^{11}(1+x^4)^{1/2}} = \int \frac{dx}{x^{11}\cdot x^2(1+1/x^4)^{1/2}}$

$= \int \frac{dx}{x^{13}(1+1/x^4)^{1/2}} = -\frac{1}{4}\int \frac{2t\,dt}{x^8\,t}$

$= -\frac{1}{2}\int (t^2-1)^2\,dt = -\frac{1}{2}\int (t^4 - 2t^2 + 1)\,dt$

$= -\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C$

Where $t = \sqrt{1 + \frac{1}{x^4}}$

Example 96 Evaluate $\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx$.

Sol. Let $\quad I = \int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx$

Put $x^{1/12} = t, \Rightarrow x = t^{12}$ and $dx = 12t^{11}\,dt$

$\therefore \quad I = \int \frac{1}{t^4 + t^3}\cdot 12\,t^{11}dt = 12\int \frac{t^8}{t+1}\,dt$

Again put $(t + 1) = y$

$\therefore\ dt = dy = 12\int \frac{(y-1)^8}{y}\,dy$

$= 12\int \frac{(y^8 - 8y^7 + 28y^6 - 56y^5 + 70y^4 - 56y^3 + 28y^2 - 8y + 1)}{y}dy$

[using binomial]

$= 12\int (y^7 - 8y^6 + 28y^5 - 56y^4 + 70y^3 - 56y^2 + 28y - 8 + 1/y)dy$

$= 12\left(\begin{array}{l}\frac{y^8}{8} - \frac{8y^7}{7} + \frac{28y^6}{6} - \frac{56y^5}{5} + \frac{70y^4}{4} \\ -\frac{56y^3}{3} + \frac{28y^2}{2} - 8y + \log|y|\end{array}\right) + C_1$

Where $y = x^{1/12} + 1$

Exercise for Session 6

- Evaluate the following integrals

1. $\int \frac{x^4 - 1}{x^2(x^4 + x^2 + 1)^{1/2}} dx$

2. $\int \frac{(x+2)\,dx}{(x^2 + 3x + 3)\sqrt{x+1}}$

3. $\int \frac{dx}{(x+1)^{1/3} + (x+1)^{1/2}}$

4. $\int \frac{dx}{(x+a)^{8/7}(x-b)^{6/7}}$

5. $\int \frac{\sec x \, . dx}{\sqrt{\sin(x + 2A) + \sin A}}$

6. The value of $\int [\{x\}]\,dx$; (where [.] and {.} denotes greatest integer and fractional part of x) is equal to

(a) 0
(b) 1
(c) 2
(d) – 1

7. If $\int f(x) \cos x \, dx = \frac{1}{2} f^2(x) + C$, then $f(x)$ can be

(a) x
(b) 1
(c) $\cos x$
(d) $\sin x$

8. The value of, $\int \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$ is

(a) $\frac{1}{40} \log \left| \frac{5 + 4(\sin x - \cos x)}{5 - 4(\sin x - \cos x)} \right| + C$
(b) $\log \left| \frac{5 + 4(\sin x - \cos x)}{5 - 4(\sin x - \cos x)} \right| + C$
(c) $\frac{1}{10} \log \left| \frac{5 + 4(\sin x + \cos x)}{5 - 4(\sin x + \cos x)} \right| + C$
(d) None of these

9. The value of $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x} dx$, is

(a) $\frac{\sin 2x}{2} + \frac{\cos 3x}{3} + C$
(b) $\sin x - \cos x + C$
(c) $\frac{\sin 2x}{2} - \frac{\cos 3x}{3} + C$
(d) None of these

10. The value of $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$, is

(a) $\sin x + \sin 2x + C$
(b) $\sin x - \frac{\sin 2x}{2} + C$
(c) $-\sin x - \frac{\sin 2x}{2} + C$
(d) None of these

Session 7

Euler's Substitution, Reduction Formula and Integration Using Diffrentiation

Euler's Substitution, Reduction and Integration Using Diffrentiation

Integration Using Euler's Substitutions

Integrals of the form $\int f(x), \sqrt{ax^2+bx+c}\, dx$ are calculated with the aid of one of the three Euler's substitutions

(i) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$, if $a > 0$.

(ii) $\sqrt{ax^2+bx+c} = tx + \sqrt{c}$, if $c > 0$.

(iii) $\sqrt{ax^2+bx+c} = (x-\alpha)\,t$, if $ax^2+bx+c = a(x-\alpha)(x-\beta)$, i.e. If α is real root of (ax^2+bx+c).

Remark

The Euler's substitutions often lead to rather cumbersome calculations, therefore they should be applied only when it is difficult to find another method for calculating a given integral.

Example 97 Evaluate $I = \int \dfrac{x\,dx}{(\sqrt{7x-10-x^2})^3}$.

Sol. In this case $a < 0$ and $c < 0$. Therefore, neither (I) nor (II) Euler's Substitution is applicable. But the quadratic $7x - 10 - x^2$ has real roots $\alpha = 2, \beta = 5$.

$\therefore$ We use the substitution (III)

i.e. $\sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)\,t$

Where $(5-x) = (x-2)\,t^2$

or $5 + 2t^2 = x(1+t^2)$

$$\therefore \quad x = \frac{5+2t^2}{1+t^2}$$

$$(x-2)\,t = \left(\frac{5+2t^2}{1+t^2} - 2\right)t = \frac{3t}{1+t^2}$$

$$\therefore \quad dx = \frac{-6t}{(1+t^2)^2}\,dt$$

Hence, $I = \displaystyle\int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3} = \int \frac{\left(\frac{5+2t^2}{1+t^2}\right)\cdot\frac{-6t}{(1+t^2)^2}\,dt}{\left(\frac{3t}{1+t^2}\right)^3}$

$$= \frac{-6}{27}\int \frac{5+2t^2}{t^2}\,dt$$

$$= \frac{-2}{9}\int\left(\frac{5}{t^2}+2\right)dt = \frac{-2}{9}\left[\frac{-5}{t}+2t\right]+C$$

$$\therefore \quad \int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3} = \frac{-2}{9}\left(\frac{-5}{t}+2t\right)+C,$$

where, $t = \dfrac{\sqrt{7x-10-x^2}}{x-2}$

Example 98 Evaluate $\int \dfrac{dx}{x+\sqrt{x^2-x+1}}$.

Sol. Since, here $c = 1$, we can apply the second Euler's Substitution.

$$\sqrt{x^2-x+1} = tx - 1$$

Therefore, $(2t-1)\,x = (t^2-1)\,x^2 \Rightarrow x = \dfrac{2t-1}{t^2-1}$

$\therefore \quad dx = -\dfrac{2(t^2-t+1)\,dt}{(t^2-1)^2}$ and $x + \sqrt{x^2-x+1} = \dfrac{t}{t-1}$

$$\therefore \quad I = \int \frac{dx}{x+\sqrt{x^2-x+1}} = \int \frac{-2t^2+2t-2}{t(t-1)(t+1)^2}\,dt$$

Using partial fractions, we have

$$\frac{-2t^2+2t-2}{t(t-1)(t+1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t+1)} + \frac{D}{(t+1)^2}$$

or $(-2t^2+2t-2) = A(t-1)(t+1)^2 + Bt(t+1)^2 + C(t-1)(t+1)\,t + Dt$

we get $A = 2,\ B = -1/2,\ C = -3/2,\ D = -3$

Hence, $I = 2\displaystyle\int\frac{dt}{t} - \frac{1}{2}\int\frac{dt}{t-1} - \frac{3}{2}\int\frac{dt}{(t+1)} - 3\int\frac{dt}{(t+1)^2}$

$$= 2\log_e|t| - \frac{1}{2}\log_e|t-1| - \frac{3}{2}\log_e|t+1| + \frac{3}{(t+1)} + C$$

$$\left[\text{where } t = \frac{\sqrt{x^2-x+1}+1}{x}\right]$$

Introduction of Reduction Formulae (a recursive relation) Over Indefinite Integrals

Reduction formulae makes it possible to reduce an integral depending on the index $n > 0$, called the order of the integral, to an integral of the same type with smaller index. (i.e. To reduce the integrals into similar integrals of order less than or greater than given integral).

Application of reduction formula is given with the help of some examples.

Reduction Formula for $\int \sin^n x\, dx$

Let $I_n = \int \sin^n x\, dx = \int \underset{\text{I}}{\sin^{n-1} x} \underset{\text{II}}{\sin x}\, dx$

$$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int (\sin^{n-2} x - \sin^n x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$\Rightarrow I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

Thus, $\displaystyle\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx$

Reduction Formula for $\int \cos^n x\, dx$

Let $I_n = \int \cos^n x\, dx = \int \underset{\text{I}}{\cos^{n-1} x} \underset{\text{II}}{\cos x}\, dx$

$$= \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore\ n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

or $\displaystyle\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$

Reduction Formula for $\int \tan^n x\, dx$

Let $I_n = \int \tan^n x\, dx$

$$\Rightarrow I_n = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x - I_{n-2} = \int t^{n-2}\, dt - I_{n-2}$$

where, $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$\therefore \quad I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\Rightarrow \quad \int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$$

Reduction Formula for $\int \operatorname{cosec}^n x\, dx$

Let $I_n = \int \operatorname{cosec}^n x\, dx = \int \underset{\text{I}}{\operatorname{cosec}^{n-2} x} \underset{\text{II}}{\operatorname{cosec}^2 x}\, dx$

$$= \operatorname{cosec}^{n-2} x (-\cot x) - \int (n-2) \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int (\operatorname{cosec}^n x - \operatorname{cosec}^{n-2} x)\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore \quad (n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$$

or $\displaystyle I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

$$\therefore \int \operatorname{cosec}^n x\, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x\, dx$$

Reduction Formula for $\int \sec^n x\, dx$

Let $I_n = \int \sec^n x\, dx = \int \underset{\text{I}}{\sec^{n-2} x} \underset{\text{II}}{\sec^2 x}\, dx$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \cdot \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

or $\displaystyle I_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} I_{n-2}$

$$\therefore \int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x\, dx$$

Reduction Formula for $\int \cot^n x\, dx$

Let $I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx = \int \cot^{n-2} x\, dx$$

$$= \int t^{n-2}\, dt - I_{n-2},$$ where $t = \cot x$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore \quad \int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$$

Reduction Formula for $\int \sin^m x \cos^n x \, dx$

Let $A = \sin^{m-1} x \cos^{n+1} x$

$$\therefore \quad \frac{dA}{dx} = (m-1)\sin^{m-2} x \cos^{n+2} x - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x (1 - \sin^2 x) - (n+1)\sin^m x \cos^n x$$

$$= (m-1)\sin^{m-2} x \cos^n x - (m-1+n+1)\sin^m x \cos^n x$$

$$\Rightarrow \quad \frac{dA}{dx} = (m-1)\sin^{m-2} x \cos^n x - (m+n)\sin^m x \cos^n x$$

Integrating with respect to x on both the sides, we get

$$A = (m-1)\int \sin^{m-2} x \cos^n x \, dx - (m+n)\int \sin^m x \cos^n x \, dx$$

$$\Rightarrow \quad (m+n)\int \sin^m x \cos^n x \, dx = (m-1)\int \sin^{m-2} x \cos^n x \, dx - P$$

$$\Rightarrow \quad \int \sin^m x \cos^n x \, dx = \frac{(m-1)}{(m+n)}\int \sin^{m-2} x \cos^n x \, dx - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$$

or $$I_{m,n} = \frac{(m-1)}{(m+n)} I_{m-2,n} - \frac{\sin^{m-1} x \cos^{n+1} x}{(m+n)}$$

Remarks

Similarly, we can show

1. $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x \, dx$
2. $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2} x \cos^n x \, dx$
3. $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2} x \, dx$
4. $\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x \, dx$
5. $\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x \, dx$

Reduction Formula for $\int \cos^m x \sin nx \, dx$

Let $I_{m,n} = \int \underset{\text{I}}{\cos^m x} \underset{\text{II}}{\sin nx} \, dx$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx \, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \{\sin nx \cos x - \sin(n-1)x\} \, dx$$

[using $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$
$\Rightarrow \sin x \cos nx = \sin nx \cos x - \sin(n-1)x$]

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx \, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x \, dx$$

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad \frac{m+n}{n} I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

or $$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Remark

Similarly, we can show

1. $\int \cos^m x \cos nx \, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \cos(n-1)x \, dx$
2. $\int \sin^m x \sin nx \, dx = \frac{n\sin^m x \cos nx}{m^2 - n^2} - \frac{m\sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2}\int \sin^{m-2} x \sin nx \, dx$
3. $\int \sin^m x \cos nx \, dx = \frac{n\sin^m x \sin nx}{m^2 - n^2} - \frac{m\sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2}\int \sin^{m-2} x \cos nx \, dx$

| Example 99 Evaluate $I_n = \int \frac{dx}{(x^2 + a^2)^n}$.

Sol. Here, $I_n = \int \frac{dx}{(x^2 + a^2)^n} = \int \frac{1}{(x^2 + a^2)^n} \cdot 1 \, dx$

Applying Integration by parts, we get

$$= \frac{1}{(x^2 + a^2)^n} \cdot x - \int \frac{(2x)}{(x^2 + a^2)^{n+1}} \cdot (-n) \cdot (x) \, dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n\int \frac{x^2}{(x^2 + a^2)^{n+1}} \, dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n\int \frac{x^2 + a^2 - a^2}{(x^2 + a^2)^{n+1}} \, dx$$

$$\therefore \quad I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{1}{(x^2+a^2)^n}\,dx - 2a^2 n\int \frac{dx}{(x^2+a^2)^{n+1}}$$

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n\,I_n - 2n\,a^2\,I_{n+1}$$

$$\therefore \quad 2n\,a^2\,I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1)\,I_n$$

or $$I_{n+1} = \frac{1}{2n\,a^2}\cdot\frac{x}{(x^2+a^2)^n} + \frac{(2n-1)}{2n}\cdot\frac{1}{a^2}\,I_n$$

Remark

Above obtained formula reduces the calculations of the integral I_{n+1} to the calculations of the integral I_n and consequently, allows us to calculate completely an integral with natural index, as

$$I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$\therefore$ From above formula

Let $n=1$

$$I_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot I_1$$

$$= \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot\frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^3}\cdot\tan^{-1}\left(\frac{x}{a}\right) + C$$

Let $n=2$

$$I_3 = \int \frac{dx}{(x^2+a^2)^3} = \frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}\,I_2$$

$$= \frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4}\cdot\frac{x}{x^2+a^2} + \frac{3}{8a^5}\tan^{-1}\left(\frac{x}{a}\right) + C$$

...and so on.

| Example 100 Derive reduction formula for $I_{(n,m)} = \int \frac{\sin^n x}{\cos^m x}\,dx$.

Sol. Using Integration by parts for $I_{(n,m)}$, we get

$$I_{(n,m)} = \int \underset{\text{I}}{\sin^{n-1} x}\;\underset{\text{II}}{\frac{\sin x}{\cos^m x}}\,dx$$

$$= \sin^{n-1} x\cdot\frac{(\cos x)^{-m+1}}{(m-1)} - \int (n-1)\sin^{n-2} x\cdot\cos x\cdot\frac{(\cos x)^{-m+1}}{(m-1)}\,dx$$

$$= \frac{1}{m-1}\cdot\frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)}\cdot\int\frac{\sin^{n-2} x}{\cos^{m-2} x}\,dx$$

$$I_{n,m} = \frac{1}{(m-1)}\cdot\frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)}\cdot I_{(n-2,\,m-2)}$$

is required reduction formula.

Integration Using Differentiation

In $\int \frac{dx}{(a+b\cos x)^2}$, $\int \frac{dx}{(a+b\sin x)^2}$, $\int \frac{dx}{(\sin x + a\sec x)^2}$, $\int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx, \ldots$ we follow the following method.

1. Let $A = \frac{\sin x}{a+b\cos x}$ or $A = \frac{\cos x}{a+b\sin x}$ according to the integral to evaluated is of the form
$$\int \frac{dx}{(a+b\cos x)^2} \text{ or } \int \frac{dx}{(a+b\sin x)^2}$$
2. Find $\frac{dA}{dx}$ and express it in terms of $\frac{1}{a+b\cos x}$ or $\frac{1}{a+b\sin x}$ as the case may be.
3. Integrate both the sides of the expression obtained in step 2 to obtain the value of the required integral.

| Example 101 Evaluate $\int \frac{dx}{(5+4\cos x)^2}$.

Sol. Here, $A = \frac{\sin x}{5+4\cos x}$, then

$$\frac{dA}{dx} = \frac{(5+4\cos x)(\cos x) - \sin x\,(-4\sin x)}{(5+4\cos x)^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{5\cos x + 4}{(5+4\cos x)^2} = \frac{\frac{5}{4}(4\cos x + 5) + 4 - \frac{25}{4}}{(5+4\cos x)^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{5}{4}\cdot\frac{1}{(5+4\cos x)} - \frac{9}{4}\cdot\frac{1}{(5+4\cos x)^2}$$

Integrating both the sides w.r.t. 'x', we get

$$A = \frac{5}{4}\int\frac{dx}{5+4\cos x} - \frac{9}{4}\int\frac{dx}{(5+4\cos x)^2}$$

$$\Rightarrow \quad \frac{9}{4}\int\frac{dx}{(5+4\cos x)^2} = \frac{5}{4}\int\frac{dx}{5+4\cos x} - A$$

$$= \frac{5}{4}\int\frac{dx}{5+4\frac{(1-\tan^2 x/2)}{(1+\tan^2 x/2)}} - \frac{\sin x}{(5+4\cos x)}$$

$$\Rightarrow \int\frac{dx}{(5+4\cos x)^2} = \frac{5}{9}\int\frac{1+\tan^2 x/2}{9+\tan^2 x/2}\,dx - \frac{4}{9}\cdot\frac{\sin x}{5+4\cos x}$$

$$\Rightarrow \int\frac{dx}{(5+4\cos x)^2} = \frac{5}{9}\int\frac{2\,dt}{9+t^2} - \frac{4}{9}\cdot\frac{\sin x}{5+4\cos x}$$

(where $\tan x/2 = t$)

$$\Rightarrow \int \frac{dx}{(5+4\cos x)^2} = \frac{10}{9}\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - \frac{4}{9}\cdot\frac{\sin x}{5+4\cos x}$$

$$\Rightarrow \int \frac{dx}{(5+4\cos x)^2} = \frac{10}{27}\tan^{-1}\left(\frac{\tan x/2}{3}\right) - \frac{4}{9}\left(\frac{\sin x}{5+4\cos x}\right) + C$$

| Example 102 Evaluate $\int \frac{dx}{(16+9\sin x)^2}$.

Sol. Let $A = \frac{\cos x}{16+9\sin x}$ ...(i)

$$\Rightarrow \frac{dA}{dx} = \frac{(16+9\sin x)(-\sin x) - \cos x\,(9\cos x)}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-16\sin x - 9}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-\frac{16}{9}(9\sin x+16) + \frac{256}{9} - 9}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{dA}{dx} = -\frac{16}{9}\cdot\frac{1}{(16+9\sin x)} + \frac{175}{9(16+9\sin x)^2} \quad \text{...(ii)}$$

Integrating both the sides of Eq. (ii) w.r.t. 'x', we get

$$A = -\frac{16}{9}\int\frac{dx}{16+9\sin x} + \frac{175}{9}\int\frac{dx}{(16+9\sin x)^2}$$

$$\Rightarrow \frac{175}{9}\int\frac{dx}{(16+9\sin x)^2} = A + \frac{16}{9}\int\frac{(1+\tan^2 x/2)\,dx}{16+16\tan^2 x/2+18\tan x/2}$$

$$\Rightarrow \frac{175}{9}\int\frac{dx}{(16+9\sin x)^2} = A + \frac{16}{9}\int\frac{2\,dt}{16t^2+18t+16}$$

[where $\tan x/2 = t$]

$$\Rightarrow \frac{175}{9}\int\frac{dx}{(16+9\sin x)^2} = A + \frac{2}{9}\int\frac{dt}{t^2+\frac{9}{8}t+1}$$

$$= A + \frac{2}{9}\int\frac{dt}{\left(t+\frac{9}{16}\right)^2 + \left(\frac{\sqrt{175}}{16}\right)^2}$$

$$= A + \frac{2}{9}\times\frac{16}{\sqrt{175}}\tan^{-1}\left(\frac{16t+9}{\sqrt{175}}\right)$$

$$\Rightarrow \int\frac{dx}{(16+9\sin x)^2} = \frac{9}{175}\cdot\frac{\cos x}{(16+9\sin x)} + \frac{2}{(175)^{3/2}}\tan^{-1}\left(\frac{16\tan x/2+9}{\sqrt{175}}\right) + C$$

| Example 103 Evaluate $\int\frac{dx}{(\sin x + a\sec x)^2}$ when $|a| > 1/2$.

Sol. Here, $I = \int\frac{dx}{(\sin x + a\sec x)^2}$ or $I = \int\frac{\cos^2 x\,dx}{(\sin x\cos x + a)^2}$

$$= \int\frac{\cos^2 x\,dx}{a^2 + 2a\sin x\cos x + \sin^2 x\cos^2 x}$$

$$= \int\frac{\cos^2 x\,dx}{a^2 + a\sin 2x + \frac{1}{4}\sin^2 2x}$$

$$= \int\frac{4\cos^2 x\,dx}{(4a^2 + 4a\sin 2x + \sin^2 2x)} = 2\int\frac{(1+\cos 2x)\,dx}{(2a+\sin 2x)^2}$$

$$= 2\int\frac{dx}{(2a+\sin 2x)^2} + 2\int\frac{\cos 2x\,dx}{(2a+\sin 2x)^2}$$

$$\Rightarrow I = 2I_1 + \int\frac{dt}{t^2} \quad [\text{where } (2a+\sin 2x)=t,\ (2\cos 2x)\,dx = dt]$$

$$\Rightarrow I = 2I_1 - \frac{1}{(2a+\sin 2x)} + C$$

where $I_1 = \int\frac{dx}{(2a+\sin 2x)^2}$ ...(i)

Put $A = \frac{\cos 2x}{2a+\sin 2x}$

$$\Rightarrow \frac{dA}{dx} = \frac{(2a+\sin 2x)(-2\sin 2x) - \cos 2x\,(2\cos 2x)}{(2a+\sin 2x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-4a\sin 2x - 2}{(2a+\sin 2x)^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-4a(\sin 2x + 2a) - 2 + 8a^2}{(2a+\sin 2x)^2}$$

$$\Rightarrow \frac{dA}{dx} = -\frac{4a}{(2a+\sin 2x)} + \frac{(8a^2-2)}{(2a+\sin 2x)^2}$$

Integrating both the sides w.r.t. 'x', we get

$$\Rightarrow A = -4a\int\frac{dx}{(2a+\sin 2x)} + (8a^2-2)\,I_1$$

$$\Rightarrow (8a^2-2)\,I_1 = A + 4a\int\frac{\sec^2 x\,dx}{2a + 2\tan x + 2a\tan^2 x}$$

$$= A + \frac{4a}{2a}\int\frac{dt}{t^2+\frac{t}{a}+1}$$

$$= A + 2\int\frac{dt}{\left(t+\frac{1}{2a}\right)^2 + \left(1-\frac{1}{4a^2}\right)}$$

$$= A + 2\frac{(2a)}{\sqrt{4a^2-1}}\tan^{-1}\left(\frac{(2at+1)}{\sqrt{4a^2-1}}\right)$$

$$\Rightarrow (8a^2-2)\,I_1 = \frac{\cos 2x}{2a+\sin 2x} + \frac{4a}{\sqrt{4a^2-1}}\tan^{-1}\left(\frac{(2a\tan x+1)}{\sqrt{4a^2-1}}\right) \quad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$I = \frac{1}{(4a^2-1)}\cdot\frac{\cos 2x}{(2a+\sin 2x)} + \frac{4a}{(4a^2-1)^{3/2}}\tan^{-1}\left(\frac{2a\tan x+1}{\sqrt{4a^2-1}}\right) - \frac{1}{(2a+\sin 2x)} + C$$

JEE Type Solved Examples : Single Option Correct Type Questions

● **Ex. 1** *The value of* $\int \frac{dx}{\cos^6 x + \sin^6 x}$, *is equal to*

(a) $\tan^{-1}(2\cot 2x) + C$ (b) $\tan^{-1}(\cot 2x) + C$

(c) $\tan^{-1}\left(\frac{1}{2}\cot 2x\right) + C$ (d) $\tan^{-1}(-2\cot 2x) + C$

Sol. Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x\, dx}{1 + \tan^6 x}$

$$= \int \frac{(1+\tan^2 x)^2 \cdot \sec^2 x}{1 + \tan^6 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt = \int \frac{(1+t^2)^2}{1+t^6} dt$

$$= \int \frac{(1+t^2)^2}{(1+t^2)(1-t^2+t^4)} dt$$

$$= \int \frac{1+t^2}{1-t^2+t^4} dt = \int \frac{(1+1/t^2)\, dt}{(1/t^2 - 1 + t^2)} = \int \frac{\left(1+\frac{1}{t^2}\right) dt}{(t-1/t)^2+1},$$

Put $t - \frac{1}{t} = z$

$$\therefore \left(1+\frac{1}{t^2}\right) dt = dz = \int \frac{dz}{z^2+1} = \tan^{-1}(z) + C$$

$$= \tan^{-1}\left(\frac{t^2-1}{t}\right) + C = \tan^{-1}\left(\frac{\tan^2 x - 1}{\tan x}\right) + C$$

$$= \tan^{-1}(-2\cot 2x) + C$$

Hence, (d) is the correct answer.

● **Ex. 2** $\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[(\sec^{-1}\sqrt{1+x^2})^2 + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right] dx,$ *$(x > 0)$ is equal to*

(a) $e^{\tan^{-1}x} \cdot \tan^{-1} x + C$

(b) $\frac{e^{\tan^{-1}x} \cdot (\tan^{-1}x)^2}{2} + C$

(c) $e^{\tan^{-1}x} \cdot (\sec^{-1}(\sqrt{1+x^2}))^2 + C$

(d) $e^{\tan^{-1}x} \cdot (\text{cosec}^{-1}(\sqrt{1+x^2}))^2 + C$

Sol. Note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$,

For $x > 0$

$$\Rightarrow \quad I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\{(\tan^{-1}x)^2 + 2\tan^{-1}x\}\, dx,$$

Put $\tan^{-1}x = t$

$$= \int e^t(t^2 + 2t)\, dt = e^t \cdot t^2 = e^{\tan^{-1}x}(\tan^{-1}x)^2 + C$$

Hence, (c) is the correct answer.

● **Ex. 3** *Let* $I = \int \frac{e^x}{e^{4x}+e^{2x}+1} dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1} dx$. *Then, for an arbitrary constant c, the value of $J - I$ equals to* **[IIT JEE 2008]**

(a) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right) + C$ (b) $\frac{1}{2}\log\left(\frac{e^{2x}+e^x+1}{e^{2x}-e^x+1}\right) + C$

(c) $\frac{1}{2}\log\left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right) + C$ (d) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right) + C$

Sol. $J = \int \frac{e^{3x}}{1+e^{2x}+e^{4x}} dx$

$$J - I = \int \frac{(e^{3x}-e^x)}{1+e^{2x}+e^{4x}} dx = \int \frac{(u^2-1)}{1+u^2+u^4} du \qquad (u = e^x)$$

$$= \int \frac{\left(1-\frac{1}{u^2}\right) du}{1+\frac{1}{u^2}+u^2} = \int \frac{\left(1-\frac{1}{u^2}\right) du}{\left(u+\frac{1}{u}\right)^2 - 1} = \int \frac{dt}{t^2-1} \qquad \left(t = u + \frac{1}{u}\right)$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\log\left|\frac{u^2-u+1}{u^2+u+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right| + C$$

Hence, (c) is the correct answer.

● **Ex. 4** *Integral of* $\sqrt{1+2\cot x(\cot x + \text{cosec}\, x)}$ *w.r.t. x, is*

(a) $2\ln\cos\frac{x}{2} + C$ (b) $2\ln\sin\frac{x}{2} + C$

(c) $\frac{1}{2}\ln\cos\frac{x}{2} + C$ (d) $\ln\sin x - \ln(\text{cosec}\, x - \cot x) + C$

Sol. $I = \int \sqrt{1 + 2\,\text{cosec}\, x \cot x + 2\cot^2 x}\, dx$

$$= \int \sqrt{\text{cosec}^2 x + 2\,\text{cosec}\, x \cot x + \cot^2 x}\, dx$$

$$= \int (\text{cosec}\, x + \cot x)\, dx$$

$$= \int \frac{1+\cos x}{\sin x} dx = \int \cot\left(\frac{x}{2}\right) dx = 2\log\left|\sin\frac{x}{2}\right| + C$$

Hence, (b) is the correct answer.

● **Ex. 5** *If* $I_n = \int \cot^n x\, dx$, *then* $I_0 + I_1 + 2(I_2 + I_3 + \ldots + I_8) + I_9 + I_{10}$ *equals to (where $u = \cot x$)*

(a) $u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}$ (b) $-\left(u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}\right)$

(c) $-\left(u + \frac{u^2}{2!} + \ldots + \frac{u^9}{9!}\right)$ (d) $\frac{u}{2} + \frac{2u^2}{3} + \ldots + \frac{9u^9}{10}$

Sol. $I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cdot (\text{cosec}^2 x - 1)\, dx$

$$I_n = -\frac{u^{n-1}}{n-1} - I_{n-2} \text{ or } I_n + I_{n-2} = -\frac{u^{n-1}}{n-1} \quad [\text{put } n = 2, 3, 4, \ldots, 10]$$

$$I_2 + I_0 = -\frac{u}{1}$$

$$I_3 + I_1 = -\frac{u^2}{2}$$

$$I_4 + I_2 = -\frac{u^3}{3}$$

....

....

$$I_{10} + I_9 = -\frac{u^9}{9}$$

Adding, $I_0 + I_1 + 2(I_2 + I_3 + \ldots + I_8) + I_9 + I_{10}$

$$= -\left(u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}\right)$$

Hence, (b) is the correct answer.

- **Ex. 6** *Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ has the value equal to*

(a) $\sqrt{2} - 1$ (b) $\frac{\sqrt{2}+1}{\sqrt{2}}$

(c) $2 - \sqrt{2}$ (d) $\sqrt{2} + 1$

Sol.

$$f(x) = y = x + \sin x$$

$$\Rightarrow \frac{dy}{dx} = 1 + \cos x$$

$$g'(y) = \frac{dx}{dy} = \frac{1}{1 + \cos x}$$

where $y = \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x + \sin x \Rightarrow x = \frac{\pi}{4}$

$$\therefore g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right) = \frac{1}{1 + (1/\sqrt{2})}$$

$$= \frac{\sqrt{2}}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$$

Hence, (c) is the correct answer.

- **Ex. 7** *The value of $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$, is*

(a) $2\sin^{-1}\sqrt{\frac{x-a}{b-a}} + C$ (b) $2\sin^{-1}\sqrt{\frac{x-b}{b-a}} + C$

(c) $\sin^{-1}\sqrt{\frac{x-a}{b-a}} + C$ (d) None of these

Sol. Let $x = a\cos^2\theta + b\sin^2\theta$ in the given integral.

So that, $dx = a(2\cos\theta)(-\sin\theta) + b(2\sin\theta)(\cos\theta)\, d\theta$

$dx = 2(b-a)\sin\theta\cos\theta\, d\theta$

$$\therefore I = \int \frac{2(b-a)\sin\theta\cos\theta\, d\theta}{\sqrt{(a\cos^2\theta + b\sin^2\theta - a)(b - a\cos^2\theta - b\sin^2\theta)}}$$

$$= 2(b-a)\int \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{(b\sin^2\theta - a\sin^2\theta)(b\cos^2\theta - a\cos^2\theta)}}$$

$$= 2(b-a)\int \frac{\sin\theta\cos\theta\, d\theta}{(b-a)\sin\theta\cos\theta} = 2\int 1\, d\theta$$

$$= 2\theta + C = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}} + C$$

Hence, (a) is the correct answer.

- **Ex. 8** *The value of $\int \frac{(x-1)}{(x+1)\sqrt{x^3 + x^2 + x}}\, dx$, is*

(a) $2\tan^{-1}\sqrt{\frac{x+1}{x}} + C$ (b) $\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + C$

(c) $2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + C$ (d) None of these

Sol. Let $I = \int \frac{(x-1)}{(x+1)\sqrt{x^3+x^2+x}}\, dx$

$$= \int \frac{(x^2-1)}{(x+1)^2\sqrt{x^3+x^2+x}}\, dx$$

$$= \int \frac{x^2(1 - 1/x^2)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}\, dx$$

$$= \int \frac{x^2(1-1/x^2)}{x(x+2+1/x)\cdot x\sqrt{x+1+1/x}}\, dx$$

Put $x + \frac{1}{x} = t$,

$\Rightarrow (1 - 1/x^2)\, dx = dt$

$$= \int \frac{dt}{(t+2)\sqrt{t+1}}, \text{ which reduces to } \int \frac{dx}{P\sqrt{Q}}.$$

Let $t + 1 = z^2$

$$\therefore dt = 2z dz = \int \frac{2z\, dz}{(z^2+1)\sqrt{z^2}}$$

$$= 2\int \frac{dz}{z^2+1} = 2\tan^{-1}(z) + C$$

$$= 2\tan^{-1}(\sqrt{t+1}) + C = 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + C$$

Hence, (c) is the correct answer.

- **Ex. 9** *The value of $\int \frac{(1+x^2)\, dx}{(1-x^2)\sqrt{1+x^2+x^4}}$, is*

(a) $-\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^4+x^2+1} - \sqrt{3}x}{\sqrt{x^4+x^2+1} + \sqrt{3}x}\right| + C$

(b) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^4+x^2+1} + \sqrt{2}x}{\sqrt{x^4+x^2+1} - \sqrt{2}x}\right| + C$

(c) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^4-x^2+1} - \sqrt{3}x}{\sqrt{x^4+x^2+1} + \sqrt{3}x}\right| + C$

(d) None of the above

Sol. Let $I = \int \frac{(1+x^2)\,dx}{(1-x^2)\sqrt{1+x^2+x^4}}$

$$= \int \frac{x^2\left(1+\frac{1}{x^2}\right)dx}{x\left(\frac{1}{x}-x\right)x\sqrt{\frac{1}{x^2}+1+x^2}}$$

$$= -\int \frac{(1+1/x^2)\,dx}{(1-1/x)\sqrt{(x-1/x)^2+3}}$$

Put $x-\frac{1}{x}=t=\left(1+\frac{1}{x^2}\right)dx=dt \Rightarrow -\int \frac{dt}{t\sqrt{t^2+3}}$

Again, put $t^2+3=s^2$

$$\Rightarrow \quad 2t\,dt = 2s\,ds = -\int \frac{s\,ds}{s(s^2-3)} = -\int \frac{ds}{s^2-(\sqrt{3})^2}$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{s-\sqrt{3}}{s+\sqrt{3}}\right|+C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right|+C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{(x-1/x)^2+3}-\sqrt{3}}{\sqrt{(x-1/x)^2+3}+\sqrt{3}}\right|+C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right|+C$$

$$= -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^4+x^2+1}-\sqrt{3}x}{\sqrt{x^4+x^2+1}+\sqrt{3}x}\right|+C$$

Hence (a) is the correct answer.

● ***Ex. 10*** *The value of* $I = \int \frac{dx}{(a+bx^2)\sqrt{b-ax^2}}$, *is*

(a) $\frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C$

(b) $\frac{1}{\sqrt{(a^2+b^2)}}\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C$

(c) $\frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a}\right)+C$

(d) None of the above

Sol. Substituting $ax^2 = b\sin^2\theta$

$$\Rightarrow \quad dx = \sqrt{\frac{b}{a}}\cos\theta\,d\theta$$

$$\therefore \quad I = \int \frac{\sqrt{\frac{b}{a}}\cos\theta\,d\theta}{\left(a+\frac{b^2}{a}\sin^2\theta\right)\sqrt{b-b\sin^2\theta}}$$

$$= \sqrt{a}\int \frac{\cos\theta\,d\theta}{(a^2+b^2\sin^2\theta)\cdot\cos\theta}$$

$$= \sqrt{a}\int \frac{d\theta}{a^2+b^2\sin^2\theta},$$

dividing numerator and denominator by $\cos^2\theta$, we get

$$= \sqrt{a}\int \frac{\sec^2\theta\,d\theta}{a^2\sec^2\theta+b^2\tan^2\theta}, \text{ put } \tan\theta = t$$

$$= \sqrt{a}\int \frac{dt}{a^2(1+t^2)+b^2t^2} = \sqrt{a}\int \frac{dt}{(a^2+b^2)t^2+a^2}$$

$$= \frac{\sqrt{a}}{a^2+b^2}\int \frac{dt}{t^2+\frac{a^2}{a^2+b^2}}$$

$$= \frac{\sqrt{a}}{a^2+b^2}\cdot\left(\frac{\sqrt{a^2+b^2}}{a}\right)\tan^{-1}\left(\frac{t\sqrt{a^2+b^2}}{a}\right)+C$$

$$= \frac{1}{\sqrt{a(a^2+b^2)}}\cdot\tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right)+C \quad \left[\because t=\tan\theta=\frac{x}{\sqrt{b-ax^2}}\right]$$

Hence (a) is the correct answer.

● ***Ex. 11*** *The value of* $I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$

$$= -\frac{1}{2}\left\{\log\left(z+\frac{3}{2}+\sqrt{z^2+3z+3}\right)\right\}+\frac{1}{2}\left|\log s-\frac{1}{2}+\sqrt{s^2-s+1}\right|+C$$

and $s-z=\frac{k}{x}$, *then value of* k, *is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Here, $I = \int \frac{dx}{2x\sqrt{1-x}\sqrt{(2-x)+\sqrt{1-x}}}$,

put $(1-x)=t^2 - dx = 2t\,dt$

$$= -\int \frac{2t\,dt}{2(1-t^2)\cdot t\sqrt{1+t^2+t}}$$

$$= -\int \frac{dt}{(1-t^2)\sqrt{t^2+t+1}}$$

$$= \int \frac{dt}{(t-1)(t+1)\sqrt{t^2+t+1}} = \frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\cdot\frac{dt}{\sqrt{t^2+t+1}}$$

$$\because \frac{1}{(t-1)(t+1)} = \frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)$$

$$= \frac{1}{2}\int \frac{1}{(t-1)\sqrt{t^2+t+1}}dt - \frac{1}{2}\int \frac{1}{(t+1)\sqrt{t^2+t+1}}dt$$

Let $I = \frac{1}{2}I_1 - \frac{1}{2}I_2$ ...(i)

where, $I_1 = \int \frac{dt}{(t-1)\sqrt{t^2+t+1}}$

and $I_2 = \int \frac{dt}{(t+1)\sqrt{t^2+t+1}}$

For I_1, put $(t-1)=\frac{1}{z} \Rightarrow dt = -\frac{1}{z^2}dz$

$$I_1 = \int \frac{-1/z^2\,dz}{\frac{1}{z}\sqrt{\left(1+\frac{1}{z}\right)^2+\left(1+\frac{1}{z}\right)+1}} = -\int \frac{dz}{\sqrt{\left(z+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\log\left|\left(z+\frac{3}{2}\right)+\sqrt{z^2+3z+3}\right| \quad \text{...(ii)}$$

For I_2, put $(t+1) = \frac{1}{S}$

$$\Rightarrow \quad dt = -\frac{1}{s^2}ds$$

$$I_2 = -\int \frac{ds}{\sqrt{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$= -\log\left|\left(s-\frac{1}{2}\right)+\sqrt{s^2-s+1}\right| \quad \text{...(iii)}$$

$$\therefore \quad I = -\frac{1}{2}\left\{\log\left(z+\frac{3}{2}+\sqrt{z^2+3z+3}\right)\right\} + \frac{1}{2}\log\left|\left(s-\frac{1}{2}\right)+\sqrt{s^2-s+1}\right| + C$$

where, $z = \frac{1}{\sqrt{1-x}-1}$ and $s = \frac{1}{\sqrt{1-x}+1}$

$$\therefore \quad s - z = \frac{1}{\sqrt{1-x}+1} - \frac{1}{\sqrt{1-x}-1} = \frac{2}{x} \Rightarrow k = 2$$

Hence (b) is the correct answer.

• Ex. 12 *If* $\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{ka^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\} + C$. *Then the value of k, is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Here, we know

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad \text{...(i)}$$

Also, $\int \frac{1}{x^2+a^2}\cdot 1\,dx = \frac{1}{x^2+a^2}\,x - \int \frac{-2x}{(x^2+a^2)^2}\,x\,dx$

$$= \frac{x}{x^2+a^2} + 2\int \frac{x^2+a^2-a^2}{(x^2+a^2)^2}\,dx$$

I II

$$\int \frac{dx}{x^2+a^2} = \frac{x}{x^2+a^2} + 2\int \frac{dx}{x^2+a^2} - 2a^2\int \frac{dx}{(x^2+a^2)^2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{a}\tan^{-1}\frac{x}{a} = \frac{x}{x^2+a^2} + 2\frac{1}{a}\tan^{-1}\frac{x}{a} - 2a^2\int \frac{dx}{(x^2+a^2)^2}$$

$$\Rightarrow \quad 2a^2\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{x^2+a^2} + \frac{1}{a}\tan^{-1}\frac{x}{a}$$

or $\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\} + C$

$$= \frac{1}{ka^2}\left\{\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right\} + C$$

$$\therefore \quad k = 2$$

Hence, (b) is the correct answer.

• Ex. 13 *If* $\int \frac{dx}{(x^2+a^2)^3} = \frac{x}{4a^2(x^2+a^2)} + \frac{m}{na^2}\left\{\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right\} + C$. *Then* $|m-n|$ *is equal to*

(a) 4 (b) 3 (c) 2 (d) 1

Sol. Let $I = \int \frac{dx}{(x^2+a^2)^3}$ ...(i)

and $I_1 = \int \frac{1}{(x^2+a^2)^2}\,dx$ ...(ii)

$$= \int \frac{1}{(x^2+a^2)^2}\cdot 1\,dx = \frac{1}{(x^2+a^2)^2}\cdot x - \int \frac{-2(2x)}{(x^2+a^2)^3}\,x\,dx$$

I II

$$= \frac{x}{(x^2+a^2)^2} + 4\int \frac{x^2+a^2-a^2}{(x^2+a^2)^3}\,dx$$

$$= \frac{x}{(x^2+a^2)^2} + 4\int \frac{1}{(x^2+a^2)^2}\,dx - 4a^2\int \frac{dx}{(x^2+a^2)^3}$$

$$\Rightarrow \quad I_1 = \frac{x}{(x^2+a^2)^2} + 4I_1 - 4a^2\cdot I \quad \text{[using Eqs. (i) and (ii)]}$$

$$\Rightarrow 4a^2 I = \frac{x}{(x^2+a^2)^2} + 3I_1$$

$$\Rightarrow \quad I = \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2}I_1 \quad \text{...(iii)}$$

[using previous example,

$$I_1 = \int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + C\Big]$$

$$\Rightarrow I = \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2}\left\{\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right\} + C \quad \text{...(iv)}$$

$m = 3$ and $n = 4$

$|m-n| = |3-4| = |-1| = 1$

Hence, (d) is the correct answer.

• Ex. 14 *If* $y(x-y)^2 = x$, *then* $\int \frac{dx}{(x-3y)} = \frac{m}{n}\ln[(x-y)^2-1]$. *Then* $(m+2n)$ *is equal to*

(a) 1 (b) 3 (c) 5 (d) 7

Sol. Let $P = \int \frac{dx}{(x-3y)} = \frac{1}{2}\ln[(x-y)^2-1]$

$$\therefore \quad \frac{dP}{dx} = \frac{1}{x-3y} = \frac{(x-y)\left\{1-\frac{dy}{dx}\right\}}{\{(x-y)^2-1\}} \quad \text{...(i)}$$

Given, $y(x-y)^2 = x$, on differentiating both the sides, we get

$$\frac{dy}{dx} = \frac{1-2y(x-y)}{(x-y)(x-3y)} \quad \text{...(ii)}$$

$$\therefore \quad \frac{dP}{dx} = \frac{(x-y)\left\{1-\frac{1-2y(x-y)}{(x-y)(x-3y)}\right\}}{\{(x-y)^2-1\}}$$

$$= \frac{(x-y)(x-3y)-1+2y(x-y)}{(x-3y)\{(x-y)^2-1\}} = \frac{(x-y)^2-1}{(x-3y)\{(x-y)^2-1\}}$$

$$\therefore \frac{dP}{dx} = \frac{1}{(x-3y)} \qquad \text{...(iii)}$$

which is true as given

$$\therefore \int \frac{dx}{(x-3y)} = \frac{1}{2}\log\{(x-y)^2-1\},$$

$$\therefore m = 1, n = 2$$

$$\Rightarrow m + 2n = 5$$

Hence, (c) is the correct answer.

• Ex. 15 *If* $\int (x+\sqrt{1+x^2})^n\, dx$.

$$= \frac{1}{a(n+1)}\{x+\sqrt{1+x^2}\}^{n+1} + \frac{1}{-b(n-1)}\{x+\sqrt{1+x^2}\}^{n-1} + C$$

Then $(a+b)$ *is equal to*

(a) 2 (b) 3 (c) 4 (d) 5

Sol. Let $I = \int (x+\sqrt{1+x^2})^n\, dx$

Put $x+\sqrt{1+x^2} = t$ ...(i)

$$\Rightarrow \left(1 + \frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)dx = dt$$

$$\Rightarrow \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)dx = dt \qquad \text{...(ii)}$$

We know, $t = x+\sqrt{1+x^2} = x+\sqrt{1+x^2}\times\dfrac{x-\sqrt{1+x^2}}{x-\sqrt{1+x^2}}$

$$t = \frac{-1}{x-\sqrt{1+x^2}} \Rightarrow t = x+\sqrt{1+x^2}$$

$$-\frac{1}{t} = x-\sqrt{1+x^2}$$

Subtracting, we get

$$2\sqrt{1+x^2} = t + \frac{1}{t} \quad \text{or} \quad \frac{1}{\sqrt{1+x^2}} = \frac{2t}{t^2+1} \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$dx = \frac{t^2+1}{2t^2}\, dt$$

$$\therefore I = \int t^n \cdot \frac{t^2+1}{(2t^2)}\, dt = \frac{1}{2}\int (t^n + t^{n-2})\, dt$$

$$= \frac{1}{2}\left[\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1}\right] + C$$

$$\Rightarrow I = \frac{1}{2(n+1)}[x+\sqrt{(1+x^2)}]^{n+1}$$

$$+ \frac{1}{2(n-1)}(x+\sqrt{(1+x^2)})^{n-1} + C \text{ ...(iv)}$$

Then comparing the values of a and b by Eq. (iv) $a = 2, b = 2$

$$\therefore (a+b) = (2+2) = 4$$

Hence, (c) is the correct answer.

• Ex. 16 *If* $\int \dfrac{f(x)}{x^3-1}\, dx$, *where* $f(x)$ *is a polynomial of degree 2 in x such that* $f(0) = f(1) = 3f(2) = -3$ *and*

$$\int \frac{f(x)}{x^3-1}\, dx = -\log|x-1| + \log|x^2+x+1| + \frac{m}{\sqrt{n}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C. \text{ Then } (2m+n) \text{ is}$$

(a) 3 (b) 5 (c) 7 (d) 9

Sol. Let $f(x) = ax^2 + bx + c$

Given, $f(0) = f(1) = 3f(2) = -3$

$$\therefore f(0) = c = -3$$

$$f(1) = a + b + c = -3$$

$$3f(2) = 3(4a + 2b + c) = -3$$

On solving, we get $a = 1, b = -1, c = -3$

$$\therefore f(x) = x^2 - x - 3$$

$$\Rightarrow I = \int \frac{f(x)}{x^3-1}\, dx = \int \frac{x^2-x-3}{(x-1)(x^2+x+1)}\, dx$$

Using partial fractions, we get

$$\frac{(x^2-x-3)}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

we get, $A = -1$, $B = 2$, $C = 2$

$$\therefore I = \int -\frac{1}{x-1}\, dx + \int \frac{(2x+2)}{(x^2+x+1)}\, dx$$

$$= -\log|x-1| + \int \frac{(2x+1)\, dx}{x^2+x+1} + \int \frac{1\, dx}{x^2+x+1}$$

$$= -\log|x-1| + \log|x^2+x+1| + \int \frac{dx}{(x+1/2)^2 + (\sqrt{3}/2)^2}$$

$$= -\log|x-1| + \log|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$\therefore$ On comparing $m = 2, n = 3 \Rightarrow 2m + n = 7$.

Hence, (c) is the correct answer.

• Ex. 17 *The value of* $\int \dfrac{(1+x)}{x(1+xe^x)^2}\, dx$, *is equal to*

(a) $\log\left|\dfrac{x}{1+xe^x}\right| + \dfrac{1}{(1+xe^x)} + C$

(b) $\log\left|\dfrac{xe^x}{1+xe^x}\right| + \dfrac{1}{1+xe^x} + C$

(c) $\log\left|\dfrac{xe^x}{1+e^x}\right| + \dfrac{1}{1+xe^x} + C$

(d) None of the above

Sol. Let $I = \int \dfrac{(1+x)}{x(1+xe^x)^2}\, dx = \int \dfrac{(1+x)e^x}{(xe^x)(1+xe^x)^2}\, dx$,

put $1 + xe^x = t$

$\therefore (1+x)e^x\, dx = dt = \int \dfrac{dt}{(t-1)\cdot t^2}$, applying partial fraction,

we get $\dfrac{1}{(t-1)t^2} = \dfrac{A}{t-1} + \dfrac{B}{t} + \dfrac{C}{t^2}$

$\Rightarrow \quad 1 = A(t^2) + Bt(t-1) + C(t-1)$

For $\quad t = 1 \Rightarrow A = 1$

For $\quad t = 0 \Rightarrow C = -1$ and $B = -1$

$$\therefore \quad I = \int \left\{ \frac{1}{t-1} - \frac{1}{t} - \frac{1}{t^2} \right\} dt = \log|t-1| - \log|t| + \frac{1}{t} + C$$

$$= \log|xe^x| - \log|1 + xe^x| + \frac{1}{1 + xe^x} + C$$

$$= \log\left|\frac{xe^x}{1 + xe^x}\right| + \frac{1}{1 + xe^x} + C$$

Hence, (b) is the correct answer.

• Ex. 18 *The value of* $\int \frac{dx}{x + \sqrt{a^2 - x^2}}$, *is equal to*

(a) $\frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$

(b) $\frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$

(c) $\frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) - \log|x + \sqrt{a^2 - x^2}| + C_1$

(d) $\frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$

Sol. Let $I = \int \frac{dx}{x + \sqrt{a^2 - x^2}}$, Put $x = a\sin\theta$

$$\therefore \quad dx = a\cos\theta\, d\theta = \int \frac{a\cos\theta\, d\theta}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} = \frac{1}{2}\int \frac{\cos\theta + \sin\theta + \cos\theta - \sin\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \frac{1}{2}\int 1\, d\theta + \frac{1}{2}\int \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} d\theta$$

$$= \frac{1}{2}\cdot\theta + \frac{1}{2}\log(\sin\theta + \cos\theta) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log\left|\frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}}\right| + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\{\log|x + \sqrt{a^2 - x^2}|\} - \frac{1}{2}\log a + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\log|x + \sqrt{a^2 - x^2}| + C_1$$

$$\left[\text{where } C_1 = C - \frac{1}{2}\log a\right]$$

Hence, (a) is the correct answer.

• Ex. 19 *The value of* $\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}} dx$, *is equal to*

(a) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right) + C$ (b) $\sqrt{2}\sec^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right) + C$

(c) $\frac{1}{\sqrt{2}}\text{cosec}^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right) + C$ (d) $\sqrt{2}\,\text{cosec}^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right) + C$

Sol. Let $I = \int \frac{(x^2 - 1)}{(x^2 + 1)\sqrt{x^4 + 1}} dx = \int \frac{x^2(1 - 1/x^2)\, dx}{x^2\left(x + \frac{1}{x}\right)\sqrt{x^2 + \frac{1}{x^2}}}$

$$= \int \frac{(1 - 1/x^2)\, dx}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t\sqrt{t^2 - 2}} = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right) + C$$

Hence, (a) is the correct answer.

JEE Type Solved Examples : More than One Correct Option Type Questions

• Ex. 20 $\int \frac{\sqrt{4 + x^2}}{x^6} dx = \frac{A(4 + x^2)^{3/2}(Bx^2 - 6)}{x^5} + C$, *then*

(a) $A = \frac{1}{120}$ (b) $B = 1$

(c) $A = -\frac{1}{120}$ (d) $B = -1$

Sol. Here, $I = \int \frac{\sqrt{4 - x^2}}{x^6} dx = \int \frac{\sqrt{1 + \frac{4}{x^2}}}{x^5} dx = \int \frac{\sqrt{1 + \frac{4}{x^2}}}{x^2 \cdot x^3} dx$

Put $t = \sqrt{1 + \frac{4}{x^2}} \Rightarrow t^2 = 1 + \frac{4}{x^2}$

$$\therefore \; 2t\, dt = -\frac{8}{x^3} dx$$

$$\Rightarrow \quad I = \frac{1}{16}\int (t^2 - t^4)\, dx = \frac{1}{16}\left\{\frac{t^3}{3} - \frac{t^5}{5}\right\} + C$$

$$= \frac{1}{120}\cdot\frac{(4 + x^2)^{3/2}}{x^5}(x^2 - 6) + C$$

$$A = \frac{1}{120}, B = 1$$

Hence, (a) and (b) are the correct answers.

Ex. 21 *The value of the integral* $\int e^{\sin^2 x}(\cos x + \cos^3 x)\sin x\, dx$ *is*

(a) $\frac{1}{2}e^{\sin^2 x}(3-\sin^2 x)+C$

(b) $e^{\sin^2 x}\left(1+\frac{1}{2}\cos^2 x\right)+C$

(c) $e^{\sin^2 x}(3\cos^2 x + 2\sin^2 x)+C$

(d) $e^{\sin^2 x}(2\cos^2 x + 3\sin^2 x)+C$

Sol. Put $t=\sin^2 x$

The integral reduces to $I=\frac{1}{2}\int e^t(2-t)\,dt=\frac{3}{2}e^t-\frac{te^t}{2}+C$

$=\frac{1}{2}e^{\sin^2 x}(3-\sin^2 x)+C$ [option (a)]

$=e^{\sin^2 x}\left(1+\frac{1}{2}\cos^2 x\right)+C$ [option (b)]

Hence, (a) and (b) are the correct answers.

Ex. 22 *If* $I=\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx=f(x)+c$, *then* $f(x)$ *is equal to*

(a) $\sqrt{2}\sin^{-1}(\sin x-\cos x)$ (b) $\frac{\pi}{\sqrt{2}}-\sqrt{2}\cos^{-1}(\sin x-\cos x)$

(c) $\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}\sqrt{\tan x}}\right)$ (d) None of these

Sol. $I=\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx=\int\sqrt{2}\cdot\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}\,dx$

If $\sin x-\cos x=p$, then $(\cos x+\sin x)\,dx=dp$

$I=\sqrt{2}\int\frac{dp}{\sqrt{1-p^2}}=\sqrt{2}\sin^{-1}p+c=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$

$=\frac{\pi}{\sqrt{2}}-\sqrt{2}\cos^{-1}(\sin x-\cos x)=\sqrt{2}\tan^{-1}\frac{\sin x-\cos x}{\sqrt{1-(\sin x-\cos x)^2}}$

$=\sqrt{2}\tan^{-1}\frac{\sin x-\cos x}{\sqrt{2\sin x\cos x}}=\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)$

Hence, (a), (b) and (c) are the correct answers.

JEE Type Solved Examples : Passage Based Questions

Passage

(Q. Nos. 23 to 25)

For integral $\int f\left(x-\frac{a}{x}\right)\cdot\left(1+\frac{a}{x^2}\right)dx$, *put* $x-\frac{a}{x}=t$

For integral $\int f\left(x+\frac{a}{x}\right)\cdot\left(1-\frac{a}{x^2}\right)dx$, *put* $x+\frac{a}{x}=t$

For integral $\int f\left(x^2-\frac{a}{x^2}\right)\cdot\left(x+\frac{a}{x^3}\right)dx$, *put* $x^2-\frac{a}{x^2}=t$

For integral $\int f\left(x^2+\frac{a}{x^2}\right)\cdot\left(x-\frac{a}{x^3}\right)dx$, *put* $x^2+\frac{a}{x^2}=t$

many integrands can be brought into above forms by suitable reductions or transformations.

Ex. 23 $\int\frac{x^4-2}{x^2\sqrt{x^4+x^2+2}}\,dx$

(a) $\sqrt{x^2+1+\frac{1}{x^2}}+C$ (b) $\sqrt{x^2+1+\frac{2}{x^2}}+C$

(c) $\sqrt{x^2+\frac{1}{x^2}}+C$ (d) $\sqrt{x^2+\frac{2}{x^2}}+C$

Sol. Here, $I=\int\frac{x-\frac{2}{x^3}}{\sqrt{x^2+1+\frac{2}{x^2}}}\,dx$

Put $x^2+\frac{2}{x^2}+1=t \Rightarrow 2\left(x-\frac{2}{x^3}\right)dx=dt$

$=\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\frac{1}{2}\cdot\frac{t^{1/2}}{1/2}+C=\sqrt{t}+C$

$=\sqrt{x^2+\frac{2}{x^2}+1}+C$

Hence, (b) is the correct answer.

Ex. 24 $\int\frac{(x-1)}{(x+1)\sqrt{x^3+x^2+x}}\,dx$

(a) $\tan^{-1}\left(x+\frac{1}{x}+1\right)+C$ (b) $\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$

(c) $2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$ (d) None of these

Sol. $\int\frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}\,dx=\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}\,dx$

Put $x+\frac{1}{x}+1=t^2 \Rightarrow \left(1-\frac{1}{x^2}\right)dx=2t\,dt$

$=\int\frac{2t\,dt}{(t^2+1)t}=2\int\frac{1}{(t^2+1)}\,dt$

$=2\cdot\tan^{-1}(t)+C=2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right)+C$

Hence, (c) is the correct answer.

• **Ex. 25** $\int \frac{5x^4 + 4x^5}{(x^5 + x + 1)^2} dx$

(a) $x^5 + x + 1 + C$ (b) $\frac{x^5}{x^5 + x + 1} + C$

(c) $x^{-4} + x^{-5} + C$ (d) $\frac{x^5}{x^5 + x + 1} + C$

Sol. Here, $I = \sqrt{\frac{5x^4 + 4x^5}{(x^5 + x + 1)^2}}$

Divide numerator and denominator by x^{10}, we get

$$I = \int \frac{5x^{-6} + 4x^{-5}}{(1 + x^{-4} + x^{-5})^2} dx$$

Put $1 + x^{-4} + x^{-5} = t \Rightarrow (-4x^{-5} - 5x^{-6})\, dx = dt$

$$\therefore \quad I = -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{1 + x^{-4} + x^{-5}} + C$$

$$= \frac{x^5}{x^5 + x + 1} + C$$

Hence, (d) is the correct answer.

JEE Type Solved Examples : Matching Type Questions

• **Ex. 26** *If $x \in (0, 1)$ then match the entries of Column I with Column II considering 'c' as an arbitrary constant of integration.*

Column I	Column II
(A) $\int \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}} - 1}{\sqrt{1+\sqrt{x}} + 1}}\right) dx$	(p) $\frac{4}{3} x^{3/4} + C$
(B) $\int \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}} - \sqrt[4]{x}}{\sqrt{1+\sqrt{x}} + \sqrt[4]{x}}}\right) dx$	(q) $\frac{4}{5} x^{5/4} + C$
(C) $\int \left(\frac{1 - \tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1 + \tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right) dx$	(r) $\frac{2}{3} x^{3/4} + C$
(D) $\int \sqrt{x}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1} - \sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1} + \sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right) dx$	(s) $\frac{2}{5} x^{5/4} + C$

Sol. Let $\sqrt{x} = \tan^2\theta$

$$x = \tan^4\theta \Rightarrow \tan\theta = x^{1/4}$$

$$\therefore \quad dx = 4\tan^3\theta \sec^2\theta\, d\theta \qquad \left[\begin{array}{l}\because x \in (0,1) \\ \therefore \theta \in (0, \pi/4)\end{array}\right]$$

$$\therefore \quad \sqrt{1+\sqrt{x}} = \sec\theta$$

(A) $I = \int \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}} - 1}{\sqrt{1+\sqrt{x}} + 1}}\right) dx$

$$\Rightarrow \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}} - 1}{\sqrt{1+\sqrt{x}} + 1}}\right) = \tan\left(2\tan^{-1}\sqrt{\frac{\sec\theta - 1}{\sec\theta + 1}}\right)$$

$$= \tan\left(2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right) = \tan\theta$$

$$\therefore \quad I = \int \tan\theta \cdot 4\tan^3\theta \sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C$$

(B) $I = \int \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}} - \sqrt[4]{x}}{\sqrt{1+\sqrt{x}} + \sqrt[4]{x}}}\right) dx$

$$\therefore \quad \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}} - \sqrt[4]{x}}{\sqrt{1+\sqrt{x}} + \sqrt[4]{x}}}\right) = \cot\left(2\tan^{-1}\sqrt{\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}}\right)$$

$$= \cot\left(2\tan^{-1}\sqrt{(\sec\theta - \tan\theta)^2}\right)$$

$$= \cot\left(2\tan^{-1}\frac{1 - \sin\theta}{\cos\theta}\right)$$

If $\theta \in (0, \pi/4)$, then $\sec\theta - \tan\theta > 0$

$$= \cot\left(2\tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) = \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\therefore \quad I = \int \tan\theta \cdot 4\tan^3\theta \sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C$$

(C) $\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) = \frac{1}{2}\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$= \frac{1}{2}\sin^{-1}(\cos 2\theta) = \frac{1}{2}\sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{4} - \theta$$

$$\therefore \int \left(\frac{1 - \tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1 + \tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right) dx$$

$$= \int \frac{1 - \tan\left(\frac{\pi}{4} - \theta\right)}{1 + \tan\left(\frac{\pi}{4} - \theta\right)} 4\tan^3\theta \sec^2\theta\, d\theta$$

$$= \int \tan\left(\frac{\pi}{4} - \left(\frac{\pi}{4} - \theta\right)\right) 4\tan^3\theta \sec^2\theta\, d\theta$$

$$= \int 4\tan^4\theta \sec^2\theta\, d\theta$$

$$= \frac{4}{5}\tan^5\theta + C = \frac{4}{5}(x^{5/4}) + C$$

(D) Let $\sqrt{x} = \tan^2\theta \Rightarrow x = \tan^4\theta$

$\therefore \quad dx = 4\tan^3\theta\sec^2\theta\, d\theta \quad \left[\begin{array}{l}\because x \in (0,1)\\ \therefore \theta \in (0, \pi/4)\end{array}\right]$

$\therefore \sqrt{1+\sqrt{x}} = \sec\theta$

$$I = \int \sqrt{x}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1} - \sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1} + \sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right) dx$$

$$\therefore \sqrt{x}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1} - \sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1} + \sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sec\theta+1} - \sqrt{\sec\theta-1}}{\sqrt{\sec\theta+1} + \sqrt{\sec\theta-1}}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(2\tan^{-1}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(2\tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)$$

$$= \tan^2\theta\cdot\tan\left(\frac{\pi}{2} - \theta\right) = \tan^2\theta\cdot\cot\theta = \tan\theta$$

$$\therefore \quad I = \int \tan\theta\cdot 4\tan^3\theta\sec^2\theta\, d\theta = \frac{4\tan^5\theta}{5} + C = \frac{4}{5}(x^{5/4}) + C$$

(A) → (q); (B) → (q); (C) → (q); (D) → (q)

JEE Type Solved Examples : Single Integer Answer Type Questions

• **Ex. 27** *If the primitive of the function* $f(x) = \dfrac{x^{2009}}{(1+x^2)^{1006}}$ *w.r.t. x is equal to* $\dfrac{1}{n}\left(\dfrac{x^2}{1+x^2}\right)^m + C$, *then* $\dfrac{n}{m}$ *is equal to ……*

Sol. $\quad f(x) = \int \dfrac{x^{2009}}{(1+x^2)^{1006}} dx$

Put $\quad 1 + x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{(t-1)^{1004}\,dt}{t^{1006}} = \frac{1}{2}\int\left(1 - \frac{1}{t}\right)^{1004}\cdot\frac{1}{t^2}\,dt$$

Put $\quad 1 - \dfrac{1}{t} = y \Rightarrow \dfrac{1}{t^2}dt = dy$

$$\therefore \quad I = \frac{1}{2}\int y^{1004}dy = \frac{1}{2}\cdot\frac{y^{1005}}{1005} + C$$

$$= \frac{1}{2010}\cdot\left(\frac{t-1}{t}\right)^{1005} + C = \frac{1}{2010}\cdot\left(\frac{x^2}{1+x^2}\right)^{1005} + C$$

$$\Rightarrow \quad m = 1005, n = 2010 \Rightarrow \frac{n}{m} = \frac{2010}{1005} = 2$$

• **Ex. 28** *Suppose* $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x)\end{vmatrix} = 0$ *where f(x) is continuous differentiable function with* $f'(x) \neq 0$ *and satisfies* $f(0) = 1$ *and* $f'(0) = 2$, *then* $f(x) = e^{\lambda x} + k$, *then* $\lambda + k$ *is equal to ……………*

Sol. $f'(x)\cdot f'(x) - f(x)\cdot f''(x) = 0$ or $\dfrac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = 0$

$$\frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right] = 0$$

Integrating, $\dfrac{f(x)}{f'(x)} + C$ …(i)

Put $\quad x = 0, \dfrac{f(0)}{f'(0)} = C \Rightarrow C = \dfrac{1}{2}$.

Hence, $\quad \dfrac{f(x)}{f'(x)} = \dfrac{1}{2}$ …(ii)

From Eq. (i), $2f(x) = f'(x) \therefore \dfrac{f'(x)}{f(x)} = 2$

Again, integrating, $\ln[f(x)] = 2x + k$

Put $x = 0$ to get, $k = 0$

$$f(x) = e^{2x} \Rightarrow \lambda + k = 2 + 0 = 2$$

• **Ex. 29** $\int\{\sin(101x)\cdot\sin^{99}x\}\,dx = \dfrac{\sin(100x)(\sin x)^{\lambda}}{\mu}$, *then* $\dfrac{\lambda}{\mu}$ *is equal to ……*

Sol. (1) $I = \int\{\sin(100x + x)\cdot(\sin x)^{99}\}\,dx$

$$= \int\{\sin(100x)\cos x + \cos 100x\sin x\}(\sin x)^{99}dx$$

$$= \int \underbrace{\sin(100x)}_{\text{I}}\,\underbrace{\cos x\cdot(\sin x)^{99}}_{\text{II}}\,dx + \int\cos(100x)\cdot(\sin x)^{100}dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100}\int\cos(100x)(\sin x)^{100}dx + \int\cos(100x)(\sin x)^{100}dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} + C$$

$$\Rightarrow \lambda = 100, \mu = 100 \Rightarrow \frac{\lambda}{\mu} = \frac{100}{100} = 1$$

Subjective Type Questions

- **Ex. 30** *If I_n denotes $\int z^n e^{1/z}\,dz$, then show that $(n+1)!\,I_n = I_0 + e^{1/z}(1!\,z^2 + 2!\,z^3 + \ldots + n!\,z^{n+1})$.*

Sol. $I_n = \int z^n e^{1/z}\,dz$, applying integration by parts taking $e^{1/z}$ as first function and z^n as second function. We get,

$$I_n = \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} - \int e^{1/z}\left(-\frac{1}{z^2}\right)\cdot\frac{z^{n+1}}{n+1}\,dz$$

$$= \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} + \frac{1}{(n+1)}\int e^{1/z}\cdot z^{n-1}\,dz$$

$$I_n = \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} + \frac{I_{n-1}}{(n+1)}$$

$$= \frac{e^{1/z}\cdot z^{n+1}}{(n+1)} + \frac{1}{(n+1)}\left[\frac{e^{1/z}\cdot z^n}{n} + \frac{1}{n}I_{n-2}\right]$$

$$= \frac{e^{1/z}(z)^{n+1}}{(n+1)} + \frac{e^{1/z}\cdot(z)^n}{(n+1)\,n} + \frac{1}{(n+1)\,n}I_{n-2}$$

$$= \frac{e^{1/z}\cdot(z)^{n+1}}{(n+1)} + \frac{e^{1/z}\cdot(z)^n}{(n+1)n} + \frac{e^{1/z}\cdot(z)^{n-1}}{(n+1)n\cdot(n-1)} + \frac{1}{(n+1)n(n-1)}I_{n-3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{e^{1/z}\cdot(z)^{n+1}}{n+1} + \frac{e^{1/z}\cdot(z)^n}{(n+1)\,n} + \ldots + \frac{e^{1/z}\cdot(z)^1}{(n+1)\,n\ldots 3\cdot 2} + \frac{1}{(n+1)\,n\,(n-1)\ldots 3\cdot 2}I_0$$

Multiplying both the sides by $(n+1)!$. We get,

$$(n+1)!\,I_n = (e^{1/z}\cdot z^{n+1}\cdot n! + e^{1/z}\cdot z^n\,(n-1)! + \ldots + \ldots + e^{1/z}\cdot z^3\cdot(2)! + e^{1/z}\cdot z^2\cdot 1!) + I_0$$

$$\Rightarrow \quad I_n\,(n+1)! = I_0 + e^{1/z}(1!\,z^2 + 2!\,z^3 + \ldots + n!\,z^{n+1})$$

Hence Proved.

- **Ex. 31** *If $I_n = \int x^n\sqrt{a^2 - x^2}\,dx$, prove that $I_n = -\dfrac{x^{n-1}(a^2 - x^2)^{3/2}}{(n+2)} + \dfrac{(n-1)}{(n+2)}a^2 I_{n-2}$.*

Sol. $I_n = \int x^n\sqrt{a^2 - x^2}\,dx = \int \underset{\text{I}}{x^{n-1}}\cdot\underset{\text{II}}{\{x\sqrt{a^2 - x^2}\}}\,dx$

Applying integration by parts, we get

$$= x^{n-1}\cdot\left\{-\frac{(a^2 - x^2)^{3/2}}{3}\right\} + \int (n-1)\,x^{n-2}\cdot\left\{-\frac{(a^2 - x^2)^{3/2}}{3}\right\}dx$$

$$= -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{3} + \frac{(n-1)}{3}\int x^{n-2}\cdot(a^2 - x^2)\sqrt{a^2 - x^2}\,dx$$

$$\Rightarrow I_n = -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{3} + \frac{(n-1)\,a^2}{3}I_{n-2} - \frac{(n-1)}{3}I_n$$

$$\therefore I_n + \frac{(n-1)}{3}I_n = -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{3} + \frac{(n-1)\,a^2}{3}I_{n-2}$$

$$\left(\frac{n+2}{3}\right)I_n = -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{3} + \frac{(n-1)\,a^2}{3}I_{n-2}$$

$$I_n = -\frac{x^{n-1}(a^2 - x^2)^{3/2}}{(n+2)} + \frac{(n-1)\,a^2}{(n+2)}I_{n-2}$$

Hence Proved.

- **Ex. 32** *If $I_m = \int(\sin x + \cos x)^m\,dx$, then show that $m\,I_m = (\sin x + \cos x)^{m-1}\cdot(\sin x - \cos x) + 2\,(m-1)\,I_{m-2}$*

Sol. $\because I_m = \int(\sin x + \cos x)^m\,dx$

$$= \int(\sin x + \cos x)^{m-1}\cdot(\sin x + \cos x)\,dx,$$

Applying integration by parts

$$= (\sin x + \cos x)^{m-1}(\cos x + \sin x) - \int(m-1)(\sin x + \cos x)^{m-2}\cdot(\cos x - \sin x)\cdot(\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m-1)\int(\sin x + \cos x)^{m-2}(\sin x - \cos x)^2\,dx$$

As we know, $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$,

$$\therefore I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m-1)\int(\sin x + \cos x)^{m-2}\cdot\{2 - (\sin x + \cos x)^2\}\,dx$$

$$= (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m-1)\int 2\,(\sin x + \cos x)^{m-2}\,dx - (m-1)\int(\sin x + \cos x)^m\,dx$$

$$I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2\,(m-1)\,I_{m-2} - (m-1)\,I_m$$

or $(m-1)I_m + I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2(m-1)I_{m-2}$

or $m\,I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2\,(m-1)\,I_{m-2}$

Hence Proved.

- **Ex. 33** *If $I_{m,n} = \int\cos^m x\cdot\cos nx\,dx$, show that $(m+n)\,I_{m,n} = \cos^m x\cdot\sin nx + m\,I_{(m-1,\,n-1)}$*

Sol. We have,

$$I_{m,n} = \int\underset{\text{I}}{\cos^m x}\cdot\underset{\text{II}}{\cos n x}\,dx$$

$$= (\cos^m x)\left[\frac{\sin nx}{n}\right] - \int m\cos^{m-1}x\,(-\sin x)\cdot\frac{\sin nx}{n}\,dx$$

$$= \frac{1}{n}\cos^m x\cdot\sin nx + \frac{m}{n}\int\cos^{m-1}x\,\{\sin x\cdot\sin nx\}\,dx$$

As we have, $\cos(n-1)\,x = \cos nx\cos x + \sin nx\sin x$

$$\therefore I_{m,n} = \frac{1}{n}\cos^m x\cdot\sin x + \frac{m}{n}\int\cos^{m-1}x\,\{\cos(n-1)\,x - \cos nx\cdot\cos x\}\,dx$$

$$= \frac{1}{n}\cos^m x\cdot\sin x + \frac{m}{n}\int\cos^{m-1}x\cdot\cos(n-1)\,x\,dx - \frac{m}{n}\int\cos^m x\cdot\cos n\,x\,dx$$

$$= \frac{1}{n}\cos^m x\cdot\sin nx + \frac{m}{n}I_{m-1,n-1} - \frac{m}{n}I_{m,n}$$

$$I_{m,n}+\frac{m}{n}I_{m,n}=\frac{1}{n}[\cos^m x\cdot\sin nx+m\,I_{m-1,n-1}]$$

$$\left(\frac{m+n}{n}\right)I_{m,n}=\frac{1}{n}[\cos^m x\cdot\sin nx+m\,I_{m-1,n-1}]$$

$$(m+n)\,I_{m,n}=\cos^m x\cdot\sin nx+m\,I_{m-1,n-1}$$

● ***Ex. 34*** *Evaluate* $\int\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx.$

Sol. $I=\int\frac{\tan\left(\frac{\pi}{4}-x\right)dx}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}$

$$=\int\frac{(1-\tan^2 x)\,dx}{(1+\tan x)^2\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}$$

$$I=\int\frac{-\left(1-\frac{1}{\tan^2 x}\right)\sec^2 x\,dx}{\left(\tan x+2+\frac{1}{\tan x}\right)\sqrt{\tan x+1+\frac{1}{\tan x}}}$$

let $\quad y=\sqrt{\tan x+1+\frac{1}{\tan x}}$

$\Rightarrow\quad 2y\,dy=\left(\sec^2 x-\frac{1}{\tan^2 x}\cdot\sec^2 x\right)dx$

$\therefore\quad I=\int\frac{-2y\,dy}{(y^2+1)\cdot y}=-2\int\frac{dy}{1+y^2}=-2\tan^{-1}y+C$

$$=-2\tan^{-1}\left(\sqrt{\tan x+1+\frac{1}{\tan x}}\right)+C$$

● ***Ex. 35*** *Evaluate* $\int\frac{x^2+n(n-1)}{(x\sin x+n\cos x)^2}dx.$

Sol. Here, $\quad I=\int\frac{x^2+n(n-1)}{(x\sin x+n\cos x)^2}dx$

Multiplying and dividing by x^{2n-2}, we get

$$I=\int\frac{\{x^2+n(n-1)\}\,x^{2n-2}}{(x^n\sin x+n\,x^{n-1}\cos x)^2}dx$$

We know $x^n\sin x+n\,x^{n-1}\cos x=t$

$\Rightarrow\quad \{(n\,x^{n-1}\sin x)+(x^n\cos x)+n(n-1)\,x^{n-2}\cos x-(n\,x^{n-1}\sin x)\,dx=dt\}$

$\Rightarrow\quad x^{n-2}\cos x\cdot\{x^2+n(n-1)\}\,dx=dt$

Keeping this in mind, we put

$$I=\int\underset{\text{II}}{\frac{\{x^2+n(n-1)\}\cdot x^{n-2}\cdot\cos x}{(x^n\sin x+n\,x^{n-1}\cos x)^2}}\cdot\underset{\text{I}}{x^n\cdot\sec x}\,dx$$

Applying integration by parts, we get

$$=x^n\sec x\cdot\left(-\frac{1}{(x^n\sin x+n\,x^{n-1}\cos x)}\right)+\int\frac{(x^n\sec x\tan x+n\,x^{n-1}\cdot\sec x)}{(x^n\sin x+n\,x^{n-1}\cos x)}dx$$

$$=-\frac{(x\sec x)}{(x\sin x+n\cos x)}+\int\sec^2 x\,dx$$

$$=-\frac{(x\sec x)}{(x\sin x+n\cos x)}+\tan x+C$$ **Hence Proved.**

● ***Ex. 36*** *If* $\cos\theta>\sin\theta>0$, *then evaluate*

$$\int\left\{\log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta}+\log\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\}d\theta$$

Sol. Here, $I=\int\left\{\log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta}+\log\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\}d\theta$

$$=\int\left\{2\cos^2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}d\theta$$

$$=\int(2\cos^2\theta-1)\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$$

$=\int\underset{\text{II}}{\cos 2\theta}\cdot\underset{\text{I}}{\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)}d\theta$, applying integration by parts

$$=\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\cdot\frac{\sin 2\theta}{2}-\int\frac{2}{\cos 2\theta}\cdot\frac{\sin 2\theta}{2}d\theta$$

$$=\frac{\sin 2\theta}{2}\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+\frac{1}{2}\log|\cos 2\theta|+C$$

● ***Ex. 37*** *Evaluate* $\int\frac{\tan^{-1}x}{x^4}dx.$

Sol. $I=\int\frac{\tan^{-1}x}{x^4}dx=\int\underset{\text{I}}{\tan^{-1}x}\cdot\underset{\text{II}}{\frac{1}{x^4}}dx$

$$=(\tan^{-1}x)\left(-\frac{1}{3x^3}\right)-\int\frac{1}{1+x^2}\cdot\frac{1}{(-3x^3)}dx$$

$$=-\frac{\tan^{-1}x}{3x^3}+\frac{1}{3}\int\frac{dx}{x^3(1+x^2)},$$

Put $1+x^2=t$,

$$2x\,dx=dt=-\frac{\tan^{-1}x}{3x^3}+\frac{1}{6}\int\frac{dt}{(t-1)^2\cdot t}$$

$$I=-\frac{\tan^{-1}x}{3x^3}+\frac{1}{6}I_1 \qquad \text{...(i)}$$

Where, $I_1=\int\frac{1}{(t-1)^2\cdot t}dt=\int\left\{\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t}\right\}dt$

Comparing coefficients, we get

$A=-1,\ B=1,\ C=1$

$\therefore\quad I_1=\int\left\{-\frac{1}{(t-1)}+\frac{1}{(t-1)^2}+\frac{1}{t}\right\}dt$

$$=-\log|t-1|-\frac{1}{(t-1)}+\log|t| \qquad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii), we get

$$I = -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6}\left\{-\log|x^2| - \frac{1}{x^2} + \log|1+x^2|\right\} + C$$

$$= -\frac{\tan^{-1} x}{3x^3} - \frac{1}{6}\log\left|\frac{x^2+1}{x^2}\right| - \frac{1}{6x^2} + C$$

• Ex. 38 *Evaluate* $\int x^2 \log(1-x^2)\,dx$, *and hence prove that* $\frac{1}{1\cdot 5} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 9} + \ldots = \frac{2}{3}\log 2 - \frac{8}{9}$.

Sol. We know, $\log(1-x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\infty\right\}$

Put x^2 instead of x in the above identity,

$$\Rightarrow \quad \log(1-x^2) = -\left\{x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \ldots\infty\right\}$$

$$\Rightarrow \quad x^2\log(1-x^2) = -\left\{x^4 + \frac{x^6}{2} + \frac{x^8}{3} + \frac{x^{10}}{4} + \ldots\infty\right\}$$

Integrating both the sides, we get

$$\int x^2\log(1-x^2)\,dx = -\left\{\frac{x^5}{1\cdot 5} + \frac{x^7}{2\cdot 7} + \frac{x^9}{3\cdot 9} + \ldots\infty\right\} + C$$

Now, to find constant of integration, put $x = 0$

$$\Rightarrow \quad 0 = 0 + C$$

$$\Rightarrow \quad C = 0$$

$$\therefore \quad \int \underset{\text{II}}{x^2}\,\underset{\text{I}}{\log(1-x^2)}\,dx = -\left\{\frac{x^5}{1\cdot 5} + \frac{x^7}{2\cdot 7} + \frac{x^9}{3\cdot 9} + \ldots\infty\right\}$$

Applying integration by parts, and taking limits 0 to 1 for LHS

$$\Rightarrow \quad \left(\frac{x^3}{3}\log(1-x^2)\right)_0^1 - \int_0^1 \frac{x^3}{3}\cdot\frac{1(-2x)}{1-x^2}\,dx$$

$$\Rightarrow \quad \left(\frac{x^3}{3}\log(1-x^2)\right)_0^1 + \frac{2}{3}\left(-\frac{x^3}{3} - x + \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right)_0^1$$

Taking $\log(1-x^2) = \log(1+x) + \log(1-x)$

and $\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$$\Rightarrow \quad \frac{1}{3}\log 2 + \frac{1}{3}\log 2 - \frac{2}{3} - \frac{2}{9} + \lim_{x\to 1}\left(\frac{x^3}{3} - \frac{1}{3}\right)\log(1-x)$$

$$\left[\because \lim_{x\to 1}(x^3-1)\log(1-x) = 0\right]$$

$$\Rightarrow \quad \frac{2}{3}\log 2 - \frac{8}{9} = \text{RHS}$$

$$\therefore \quad \frac{1}{1\cdot 5} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 9} + \ldots = \frac{2}{3}\log(2) - \frac{8}{9}$$

• Ex. 39 *Evaluate* $\int \frac{a + b\sin x}{(b + a\sin x)^2}\,dx$

Sol. Here, $I = \int \frac{a + b\sin x}{(b + a\sin x)^2}\,dx = \frac{b}{a}\int \frac{\frac{a^2}{b} - b + (b + a\sin x)}{(b + a\sin x)^2}\,dx$

$$I = \frac{a^2 - b^2}{a}\int\frac{dx}{(b + a\sin x)^2} + \frac{b}{a}\int\frac{dx}{(b + a\sin x)} \quad \ldots\text{(i)}$$

Now, let $A = \frac{\cos x}{b + a\sin x} \Rightarrow \frac{dA}{dx} = \frac{-b\sin x - a}{(b + a\sin x)^2}$

$$\Rightarrow \quad \frac{dA}{dx} = -\frac{b}{a}\left\{\frac{a\sin x + b + \frac{a^2}{b} - b}{(b + a\sin x)^2}\right\}$$

$$\Rightarrow \quad \frac{dA}{dx} = -\frac{b}{a}\left\{\frac{1}{b + a\sin x} + \frac{a^2 - b^2}{b\,(b + a\sin x)^2}\right\}$$

Integrating both the sides w.r.t. 'x', we get

$$A = -\frac{b}{a}\int\frac{dx}{b + a\sin x} - \frac{(a^2 - b^2)}{a}\int\frac{dx}{(b + a\sin x)^2}$$

$$\Rightarrow \quad \frac{a^2 - b^2}{a}\int\frac{dx}{(b + a\sin x)^2} = -\frac{b}{a}\int\frac{dx}{b + a\sin x} - A \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$I = -\frac{b}{a}\int\frac{dx}{(b + a\sin x)} - A + \frac{b}{a}\int\frac{dx}{(b + a\sin x)}$$

$$\Rightarrow \quad I = -A + C \Rightarrow I = -\left(\frac{\cos x}{b + a\sin x}\right) + C$$

• Ex. 40 *Evaluate* $\int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$.

Sol. Let $I = \int\frac{dx}{(x-1)^{3/4}(x+2)^{5/4}} = \int\frac{dx}{(x+2)^2\left(\frac{(x-1)}{(x+2)}\right)^{3/4}}$

Let $\quad \frac{x-1}{x+2} = t$

So that, $\frac{3}{(x+2)^2}\,dx = dt$

$$\therefore \quad I = \int\frac{dt}{3\,t^{3/4}} = \frac{1}{3}\int t^{-3/4}\,dt$$

$$= \frac{1}{3}\cdot\frac{t^{1/4}}{1/4} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$$

Indefinite Integral Exercise 1 :
Single Option Correct Type Questions

1. Let $f(x)=\int \frac{x^2}{(1+x^2)(1+\sqrt{1+x^2})}\,dx$ and $f(0)=0$. Then $f(1)$ is equal to

(a) $\log_e(1+\sqrt{2})$ (b) $\log_e(1+\sqrt{2})-\frac{\pi}{4}$

(c) $\log_e(1+\sqrt{2})+\frac{\pi}{4}$ (d) None of these

2. If $\int f(x)\,dx=f(x)$, then $\int \{f(x)\}^2\,dx$ is equal to

(a) $\frac{1}{2}\{f(x)\}^2$ (b) $\{f(x)\}^3$

(c) $\frac{\{f(x)\}^3}{3}$ (d) $\{f(x)\}^2$

3. If $\int f(x)\,dx=F(x)$, then $\int x^3 f(x^2)\,dx$ is equal to

(a) $\frac{1}{2}[x^2\{F(x)\}^2-\int \{F(x)\}^2\,dx]$

(b) $\frac{1}{2}[x^2F(x^2)-\int F(x^2)\,d(x^2)]$

(c) $\frac{1}{2}[x^2F(x)-\frac{1}{2}\int \{F(x)\}^2\,dx]$

(d) None of the above

4. If n is an odd positive integer, then $\int |x^n|\,dx$ is equal to

(a) $\left|\frac{x^{n+1}}{n+1}\right|+C$ (b) $\frac{x^{n+1}}{n+1}+C$

(c) $\frac{|x|^n x}{n+1}+C$ (d) None of these

5. Let $F(x)$ be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r.t. x. If $F(10)=60$, then the value of $F(13)$ is

(a) 66 (b) 132

(c) 248 (d) 264

6. $\int (x^x)^x(2x\log_e x+x)\,dx$ is equal to

(a) $x^{(x^x)}+C$ (b) $(x^x)^x+C$

(c) $x^2\cdot\log_e x+C$ (d) None of these

7. The value of $\int x\log x(\log x-1)\,dx$ is equal to

(a) $2(x\log x-x)^2+C$

(b) $\frac{1}{2}(x\log x-x)^2+C$

(c) $(x\log x)^2+C$

(d) $\frac{1}{2}(x\log x)^3+C$

8. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$ is equal to

(a) $\frac{\sqrt{2x^4-2x^5+1}}{x^2}+C$ (b) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+C$

(c) $\frac{\sqrt{2x^4+2x^2+1}}{x}+C$ (d) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+C$

9. Let $f(x)$ be a polynomial satisfying $f(0)=2$, $f'(0)=3$ and $f''(x)=f(x)$. Then $f(4)$ is equal to

(a) $\frac{5(e^8+1)}{2e^4}$ (b) $\frac{5(e^8-1)}{2e^4}$

(c) $\frac{2e^4}{5(e^8-1)}$ (d) $\frac{2e^4}{5(e^8+1)}$

10. $\int \frac{e^{(x^2+4\ln x)}-x^3e^{x^2}}{x-1}\,dx$ is equal to

(a) $\left(\frac{e^{3\ln x}-e^{\ln x}}{2x}\right)e^{x^2}+C$ (b) $\frac{(x-1)\,xe^{x^2}}{2}+C$

(c) $\frac{(x^2-1)}{2x}-e^{x^2}+C$ (d) None of these

11. $\int \tan^4 x\,dx=A\tan^3 x+B\tan x+f(x)$, then

(a) $A=\frac{1}{3}, B=-1, f(x)=x+C$

(b) $A=\frac{2}{3}, B=-1, f(x)=x+C$

(c) $A=\frac{1}{3}, B=1, f(x)=x+C$

(d) $A=\frac{2}{3}, B=1, f(x)=-x+C$

12. If the anti-derivative of $\int \frac{\sin^4 x}{x}\,dx$ is $f(x)$, then $\int \frac{\sin^4\{(p+q)x\}}{x}\,dx$ in terms of $f(x)$ is

(a) $f\{(p+q)x\}$ (b) $\frac{f\{(p+q)x\}}{p+q}$

(c) $f\{(p+q)x\}(p+q)$ (d) None of these

13. $\int\left(\frac{\sin\theta}{\cos 3\theta}+\frac{\sin 3\theta}{\cos 9\theta}+\frac{\sin 9\theta}{\cos 27\theta}\right)d\theta$ is equal to

(a) $\frac{1}{2}\log\left|\frac{\sec 27\theta}{\sec\theta}\right|+C$ (b) $\frac{1}{2}\log\left|\frac{\sec\theta}{\sec 27\theta}\right|+C$

(c) $\frac{1}{2}\log\left|\frac{\sqrt[27]{\sec 27\theta}}{\sec\theta}\right|+C$ (d) None of these

14. Let $x^2 \neq n\pi - 1, n \in N$. Then, the value of

$\int x \sqrt{\frac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}\, dx$ is equal to

(a) $\log\left|\frac{1}{2}\sec(x^2+1)\right| + C$ (b) $\log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$

(c) $\frac{1}{2}\log|\sec(x^2+1)| + C$ (d) None of these

15. $\int \frac{dx}{\cos(2x)\cos(4x)}$ is equal to

(a) $\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right| - \frac{1}{2}(\log|\sec 2x - \tan 2x|) + C$

(b) $\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1+\sqrt{2}\sin x}\right| - \frac{1}{2}(\log|\sec 2x - \tan 2x|) + C$

(c) $\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right| - \frac{1}{2}(\log|\sec 2x - \tan 2x|) + C$

(d) None of the above

16. $\int \frac{1 - 7\cos^2 x}{\sin^7 x \cos^2 x}\, dx = \frac{f(x)}{(\sin x)^7} + C$, then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\cot x$

17. $\int \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}\, dx$ is equal to

(a) $\tan^{-1}(\sec x + \cos x) + C$ (b) $\log_e|\tan^{-1}(\sec x + \cos x)| + C$

(c) $\frac{1}{(\sec x + \cos x)^2} + C$ (d) None of these

18. The primitive of the function $f(x) = x|\cos x|$, when $\frac{\pi}{2} < x < \pi$ is given by

(a) $\cos x + x\sin x + C$

(b) $-\cos x - x\sin x + C$

(c) $x\sin x - \cos x + C$

(d) None of the above

19. The primitive of the function $f(x) = (2x+1)|\sin x|$, when $\pi < x < 2\pi$ is

(a) $-(2x+1)\cos x + 2\sin x + C$

(b) $(2x+1)\cos x - 2\sin x + C$

(c) $(x^2 + x)\cos x + C$

(d) None of the above

20. Given, $f(x) = \begin{vmatrix} 0 & x^2 - \sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1 - 2x \\ 2 - \cos x & 2x - 1 & 0 \end{vmatrix}$, then

$\int f(x)\, dx$ is equal to

(a) $\frac{x^3}{3} - x^2\sin x + \sin 2x + C$

(b) $\frac{x^3}{3} - x^2\sin x - \cos 2x + C$

(c) $\frac{x^3}{3} - x^2\cos x - \cos 2x + C$

(d) None of the above

Indefinite Integral Exercise 2 :
More than One Option Correct Type Questions

21. $\int \frac{dx}{(x+1)(x-2)} = A\log(x+1) + B\log(x-2) + C$, where

(a) $A + B = 0$ (b) $AB = 0$

(c) $A/B = -1$ (d) None of these

22. If $\int \frac{dx}{(x^2+1)(x^2+4)} = k\tan^{-1}x + l\tan^{-1}\frac{x}{2} + C$, then

(a) $k = \frac{1}{3}$ (b) $l = \frac{2}{3}$ (c) $k = -\frac{1}{3}$ (d) $l = -\frac{1}{6}$

23. If $\int x\log(1+x^2)\, dx = \phi(x)\log(1+x^2) + x(\psi) + C$, then

(a) $\phi(x) = \frac{1+x^2}{2}$ (b) $\psi(x) = \frac{1+x^2}{2}$

(c) $\psi(x) = -\frac{1+x^2}{2}$ (d) $\phi(x) = -\frac{1+x^2}{2}$

24. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx = Ax + B\log_e(9e^{2x} - 4) + C$, then

(a) $A = \frac{3}{2}$ (b) $B = \frac{35}{36}$

(c) C is indefinite (d) $A + B = -\frac{19}{36}$

25. If $\int \tan^5 x\, dx = A\tan^4 x + B\tan^2 x + g(x) + C$, then

(a) $A = \frac{1}{4}, B = -\frac{1}{2}$

(b) $g(x) = \ln|\sec x|$

(c) $g(x) = \ln|\cos x|$

(d) $A = -\frac{1}{4}, B = \frac{1}{3}$

Indefinite Integral Exercise 3 : Statement I and II Type Questions

- **Directions (Q. Nos. 26 to 30) For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :**
  (a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I.
  (b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I.
  (c) Statement I is true, Statement II is false.
  (d) Statement I is false, Statement II is true.

26. Statement I If y is a function of x such that $y(x-y)^2 = x$, then $\int \frac{dx}{x-3y} = \frac{1}{2}[\log(x-y)^2 - 1]$

Statement II $\int \frac{dx}{x-3y} = \log(x-3y) + C$

27. Statement I Integral of an even function is not always an odd function.

Statement II Integral of an odd function is an even function.

28. Statement I If $a > 0$ and $b^2 - 4ac < 0$, then the value of the integral $\int \frac{dx}{ax^2+bx+c}$ will be of the type $\mu \tan^{-1}\frac{x+A}{B} + C$, where A, B, C, μ are constants.

Statement II If $a > 0$, $b^2 - 4ac < 0$, then $ax^2 + bx + C$ can be written as sum of two squares.

29. Statement I $\int \left(\frac{1}{1+x^4}\right) dx = \tan^{-1}(x^2) + C$

Statement II $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

30. Statement I $\int 2^{\tan^{-1} x} d(\cot^{-1} x) = \frac{2^{\tan^{-1} x}}{\ln 2} + C$

Statement II $\frac{d}{dx}(a^x + C) = a^x \ln a$

Indefinite Integral Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 31 to 33)

Let us consider the integral of the following forms

$$f(x_1, \sqrt{mx^2+nx+p})^{1/2}$$

Case I *If $m > 0$, then put* $\sqrt{mx^2+nx+C} = u \pm x\sqrt{m}$

Case II *If $p > 0$, then put* $\sqrt{mx^2+nx+C} = ux \pm \sqrt{p}$

Case III *If quadratic equation $mx^2+nx+p = 0$ has real roots α and β there put* $\sqrt{mx^2+nx+p} = (x-\alpha)u$ or $(x-\beta)u$

31. If $I = \int \frac{dx}{x - \sqrt{9x^2+4x+6}}$ to evaluate I, one of the most proper substitution could be

(a) $\sqrt{9x^2+4x+6} = u \pm 3x$
(b) $\sqrt{9x^2+4x+6} = 3u \pm x$
(c) $x = \frac{1}{t}$
(d) $9x^2+4x+6 = \frac{1}{t}$

32. $\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}} dx$ is equal to

(a) $\frac{(x+\sqrt{1+x^2})^{16}}{10} + C$
(b) $\frac{1}{15(\sqrt{1+x^2}+x)} + C$
(c) $\frac{15}{(\sqrt{1+x^2}-x)} + C$
(d) $\frac{(x+\sqrt{1+x^2})^{15}}{15} + C$

33. To evaluate $\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}}$ one of the most suitable substitution could be

(a) $\sqrt{-x^2+3x-2} = u$
(b) $\sqrt{-x^2+3x-2} = (ux\sqrt{2})$
(c) $\sqrt{-x^2+3x-2} = u(1-x)$
(d) $\sqrt{-x^2+3x-2} = u(x+2)$

Passage II

(Q. Nos. 34 to 36)

Let $I_{n,m} = \int \sin^n x \cos^m x \, dx$. *Then, we can relate* $I_{n,m}$ *with each of the following :*

(i) $I_{n-2,m}$ (ii) $I_{n+2,m}$

(iii) $I_{n,m-2}$ (iv) $I_{n,m+2}$

(v) $I_{n-2,m+2}$ (vi) $I_{n+2,m-2}$

Suppose we want to establish a relation between $I_{n,m}$ *and* $I_{n,m-2}$, *then we get*

$$P(x) = \sin^{n+1} x \cos^{m-1} x \qquad \text{...(i)}$$

In $I_{n,m}$ *and* $I_{n,m-2}$ *the exponent of* $\cos x$ *is* m *and* $m-2$ *respectively, the minimum of the two is* $m-2$, *adding 1 to the minimum we get* $m-2+1=m-1$. *Now, choose the exponent* $m-1$ *of* $\cos x$ *in* $P(x)$. *Similarly, choose the exponent of* $\sin x$ *for* $P(x) = (nH)\sin^n x \cos^m x - (m-1)\sin^{n+2} x \cos^{m-2} x$.

Now, differentiating both the sides of Eq. (i), *we get*

$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x (1-\cos^2 x)\cos^{m-2} x$

$= (n+1)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2} x + (m-1)\sin^n x \cos^n x$

$= (n+m)\sin^n x \cos^m x - (m-1)\sin^n x \cos^{m-2} x$

Now, integrating both the sides, we get

$$\sin^{n+1} x \cos^{m-1} x = (n+m) I_{n,m} - (m-1) I_{n,m-2}$$

Similarly, we can establish the other relations.

34. The relation between $I_{4,2}$ and $I_{2,2}$ is

(a) $I_{4,2} = \frac{1}{6}(-\sin^3 x \cos^3 x + 3 I_{2,2})$

(b) $I_{4,2} = \frac{1}{6}(\sin^3 x \cos^3 x + 3 I_{2,2})$

(c) $I_{4,2} = \frac{1}{6}(\sin^3 x \cos^3 x - 3 I_{2,2})$

(d) $I_{4,2} = \frac{1}{4}(-\sin^3 x \cos^3 x + 2 I_{2,2})$

35. The relation between $I_{4,2}$ and $I_{6,2}$ is

(a) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x + 8 I_{6,2})$

(b) $I_{4,2} = \frac{1}{5}(-\sin^5 x \cos^3 x + 8 I_{6,2})$

(c) $I_{4,2} = \frac{1}{5}(\sin^5 x \cos^3 x - 8 I_{6,2})$

(d) $I_{4,2} = \frac{1}{6}(\sin^5 x \cos^3 x + 8 I_{6,2})$

36. The relation between $I_{4,2}$ and $I_{4,4}$ is

(a) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x + 8 I_{4,4})$

(b) $I_{4,2} = \frac{1}{3}(-\sin^5 x \cos^3 x + 8 I_{4,4})$

(c) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x - 8 I_{4,4})$

(d) $I_{4,2} = \frac{1}{3}(\sin^5 x \cos^3 x + 6 I_{4,4})$

Passage III

(Q. Nos. 37 to 38)

If $f : R \to (0, \infty)$ be a differentiable function $f(x)$ satisfying $f(x+y) - f(x-y) = f(x) \cdot \{f(y) - f(y) - y\}, \forall x, y \in R$, $(f(y) \neq f(-y)$ for all $y \in R)$ and $f'(0) = 2010$.

Now, answer the following questions.

37. Which of the following is true for $f(x)$

(a) $f(x)$ is one-one and into

(b) $\{f(x)\}$ is non-periodic, where {·} denotes fractional part of x.

(c) $f(x) = 4$ has only two solutions.

(d) $f(x) = f^{-1}(x)$ has only one solution.

38. let $g(x) = \log_e(\sin x)$, and $\int f(g(x)) \cos x \, dx = h(x) + c$, (where c is constant of integration), then $h\left(\frac{\pi}{2}\right)$ is equal to

(a) 0 (b) $\frac{1}{2010}$

(c) 1 (d) $\frac{1}{2011}$

Passage IV

(Q. Nos. 39 to 41)

Let $f : R \to R$ *be a function as* $f(x) = (x-1)(x+2)(x-3)(x-6) - 100$. *If* $g(x)$ *is a polynomial of degree* ≤ 3 *such that* $\int \frac{g(x)}{f(x)} dx$ *does not contain any logarithm function and* $g(-2) = 10$. *Then*

39. The equation $f(x) = 0$ has

(a) all four distinct roots

(b) three distinct real roots

(c) two real and two imaginary

(d) all four imaginary roots

40. The minimum value of $f(x)$ is

(a) −136 (b) −100

(c) −84 (d) −68

41. $\int \frac{g(x)}{f(x)} dx$, equals

(a) $\tan^{-1}\left(\frac{x-2}{2}\right) + c$ (b) $\tan^{-1}\left(\frac{x-1}{1}\right) + c$

(c) $\tan^{-1}(x) + c$ (d) None of these

Indefinite Integral Exercise 5 : Matching Type Questions

42. Match the following :

	Column I		Column II
(A)	If $I = \int \frac{\sin x - \cos x}{\lvert \sin x - \cos x \rvert} dx$, where $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then I is equal to	(p)	$\sin x$
(B)	If $\int \frac{x^2}{(x^3+1)(x^3+2)} dx = \frac{1}{3} f\left(\frac{x^3+1}{x^3+2}\right) + C$, then $f(x)$ is equal to	(q)	$x + C$
(C)	If $\int \sin^{-1} x \cdot \cos^{-1} x \, dx = f^{-1}(x)\left[\frac{\pi}{2} x - x f^{-1}(x) - 2\sqrt{1-x^2}\right] + 2x + C$, then $f(x)$ is equal to	(r)	$\ln \lvert x \rvert$
(D)	If $\int \frac{dx}{x f(x)} = f(f(x)) + C$, then $f(x)$ is equal to	(s)	$\sin^{-1} x$

43. Match the following :

	Column I		Column II
(A)	If $\int \left(\frac{x^2 + \cos^2 x}{1+x^2}\right) \operatorname{cosec}^2 x \, dx = A \cot^{-1} x + B \frac{\operatorname{cosec} x}{\sec x}$, then	(p)	$A = 1$
(B)	If $\int \sqrt{x + \sqrt{x^2+2}} \, dx = \frac{A}{3}(x + \sqrt{x^2+2})^{3/2} - \frac{B}{(x + \sqrt{x^2+2})}$, then	(q)	$B = -1$
(C)	If $\int \frac{\sqrt{2-x-x^2}}{x^2} dx = A\sqrt{\frac{2-x-x^2}{x}} + \frac{B}{4\sqrt{2}} \log\left(\frac{4-x+4\sqrt{2-x-x^2}}{x}\right) - \sin^{-1}\left(\frac{2x+1}{3}\right)$, then	(r)	$B = 2$
(D)	If $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = B \cot^{-1}(\tan^2 x)$, then	(s)	$A = -1$

Indefinite Integral Exercise 6 : Single Integer Answer Type Questions

44. If $\int \frac{(2x+3)\,dx}{x(x+1)(x+2)(x+3)+1} = C - \frac{1}{f(x)}$, where $f(x)$ is of the form of $ax^2 + bx + c$, then $(a+b+c)$ equals to ....

45. Let $F(x)$ be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r.t. x. If $F(10) = 60$, then the sum of digits of the value of $F(13)$, is ....

46. Let $u(x)$ and $v(x)$ are differentiable function such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to ....

47. If $\int \frac{1}{(x^2-1)} \ln\left(\frac{x-1}{x+1}\right) dx = 6A\left[\ln\left(\frac{x-1}{x+1}\right)\right]^2 + C$, then find $24A$.

48. If $\int \frac{e^x (2-x^2)}{(1-x)\sqrt{1-x^2}} dx = \mu e^x \left(\frac{1+x}{1-x}\right)^{\lambda} + C$, then $2(\lambda + \mu)$ is equal to ....

49. If $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x} dx = \ln\{f(x)\} + g(x) + C$, where C is the constant of integrating and $f(x)$ is positive, then $\frac{f(x) + g(x)}{e^x + \sin x}$ is equal to ....

50. Suppose $A = \int \frac{dx}{x^2 + 6x + 25}$ and $B = \int \frac{dx}{x^2 - 6x - 27}$.

If $12(A + B) = \lambda \cdot \tan^{-1}\left(\frac{x+3}{4}\right) + \mu \cdot \ln\left|\frac{x-9}{x+3}\right| + C$, then the value of $(\lambda + \mu)$ is ... .

51. If $\int \frac{\cos 6x + \cos 9x}{1 - 2\cos 5x} dx = -\frac{\sin 4x}{k} - \sin x + C$, then the value of k is ....

52. The value of $\int \frac{\tan x}{1 + \tan x + \tan^2 x} dx = x - \frac{1}{\sqrt{A}} \tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{A}}\right) + C$, then the value of A is......... .

53. $\int \sin^{5/2} x \cos^3 x \, dx = 2\sin^{A/2} x \left[\frac{1}{B} - \frac{1}{C}\sin^2 x\right] + C$, then the value of $(A + B) - C$ is equal to ....

54. If $\int (x^{2010} + x^{804} + x^{402})(2x^{1608} + 5x^{402} + 10)^{1/402} dx = \frac{1}{10a}(2x^{2010} + 5x^{804} + 10x^{402})^{a/402}$. Then $(a - 400)$ is equal to ....

55. If $\int e^{x^3 + x^2 - 1}(3x^4 + 2x^3 + 2x) \, dx = h(x) + c$ Then the value of $h(1) \cdot h(-1)$, is ....

Indefinite Integral Exercise 7 :
Subjective Type Questions

56. Evaluate $e^{(x \sin x + \cos x)} dx \left[\frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x}\right]$

57. Evaluate $\int \sqrt{x + \sqrt{x^2 + 2}} \, dx$.

58. Evaluate $\int \frac{dx}{(\sqrt{(x - \alpha)^2 - \beta^2})(ax + b)}$.

59. Evaluate $\int \frac{\sqrt{1 + \sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$.

60. Evaluate $\int \frac{\sin^3(\theta/2) \, d\theta}{\cos\theta/2 \sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}$.

61. Evaluate $\int \frac{(2\sin\theta + \sin 2\theta) \, d\theta}{(\cos\theta - 1)\sqrt{\cos\theta + \cos^2\theta + \cos^3\theta}}$.

62. Connect $\int x^{m-1}(a + bx^n)^p \, dx$ with $\int x^{m-n-1}(a + bx^n)^p \, dx$ and evaluate $\int \frac{x^8 \, dx}{(1 - x^3)^{1/3}}$.

63. Evaluate $\int \text{cosec}^2 x \ln(\cos x + \sqrt{\cos 2x}) \, dx$.

64. Evaluate $\int \frac{dx}{(\sin x + a \sec x)^2}, a \in N$.

65. Evaluate $\int \frac{dx}{x - \sqrt{x^2 + 2x + 4}}$.

66. Evaluate $\int \frac{dx}{1 + \sqrt{x^2 + 2x + 2}}$.

67. Evaluate $\int \frac{x^4 + 1}{x^6 + 1} dx$.

68. Evaluate $\int \frac{dx}{(1 - x^3)^{1/3}} dx$.

69. Evaluate $\int \frac{(x + \sqrt{1 + x^2})^{15}}{\sqrt{1 + x^2}} dx$.

70. If $y^2 = ax^2 + 2bx + c$, and $u_n = \int \frac{x^n}{y} dx$, prove that $(n+1) a u_{n+1} + (2n+1) b u_n + {}^n c \, y_{n-1} = x^n y$ and deduce that $au_1 = y - b u_0$; $2a^2 u_2 = y(ax - 3b) - (ac - 3b^2) u_0$.

Indefinite Integrals Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

71. $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals to (for some arbitrary constant K) **[Only One Correct Option 2012]**

(a) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(b) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(c) $\frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(d) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

72. If $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$. Then, for an arbitrary constant c, the value of $J - I$ equals **[Only One Correct Option 2008]**

(a) $\frac{1}{2}\log\left|\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right| + C$ (b) $\frac{1}{2}\log\left|\frac{e^{2x} + e^{x} + 1}{e^{2x} - e^{x} + 1}\right| + C$

(c) $\frac{1}{2}\log\left|\frac{e^{2x} - e^{x} + 1}{e^{2x} + e^{x} + 1}\right| + C$ (d) $\frac{1}{2}\log\left|\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right| + C$

(ii) JEE Main & AIEEE

73. The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to **[2016 JEE Main]**

(a) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$ (b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

(c) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$ (d) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$

where, C is an arbitrary constant.

74. The integral $\int \frac{dx}{x^2(x^4 + 1)^{\frac{3}{4}}}$ equals **[2015 JEE Main]**

(a) $\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + C$ (b) $(x^4 + 1)^{\frac{1}{4}} + C$

(c) $-(x^4 + 1)^{\frac{1}{4}} + C$ (d) $-\left(\frac{x^4 + 1}{x^4}\right)^{\frac{1}{4}} + C$

75. The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x + \frac{1}{x}} dx$ is equal to **[2014 JEE Main]**

(a) $(x - 1)\, e^{x + \frac{1}{x}} + C$ (b) $x e^{x + \frac{1}{x}} + C$

(c) $(x + 1)\, e^{x + \frac{1}{x}} + C$ (d) $-x e^{x + \frac{1}{x}} + C$

76. If $\int f(x)\, dx = \psi(x)$, then $\int x^5 f(x^3)\, dx$ is equal to **[2013 JEE Main]**

(a) $\frac{1}{3}[x^3\psi(x^3) - \int x^2\psi(x^3) dx] + C$

(b) $\frac{1}{3} x^3\psi(x^3) - 3\int x^3\psi(x^3)\, dx + C$

(c) $\frac{1}{3} x^3\psi(x^3) - \int x^2\psi(x^3)\, dx + C$

(d) $\frac{1}{3}[x^3\psi(x^3) - \int x^3\psi(x^3)\, dx] + C$

77. If the integral $\int \frac{5 \tan x}{\tan x - 2} dx = x + a \log |\sin x - 2\cos x| + k$, then a is equal to **[2012 AIEEE]**

(a) -1 (b) -2

(c) 1 (d) 2

78. The value of $\sqrt{2}\int \frac{\sin x\, dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is **[2012 AIEEE]**

(a) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$ (b) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

(c) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$ (d) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$

Answers

Exercise for Session 1

1. $\frac{2}{3}(x+1)^{3/2}+\frac{2}{3}x^{3/2}+c$
2. $-\frac{2}{x}+\frac{2}{3x^3}-\frac{3}{5x^5}-2\tan^{-1}x+c$
3. $\log x+2\tan^{-1}x+c$
4. $\frac{x^3}{3}-x+\tan^{-1}x+c$
5. $\frac{1}{2}\left(\frac{x^3}{3}+\tan^{-1}x\right)+c$
6. $\tan x-\tan^{-1}x+c$
7. $\frac{1}{b^3}\left\{bx-2a\log|bx+a|-\frac{a^2}{a+bx}\right\}+c$
8. $\frac{2^x e^x}{1+\log_e 2}+c$
9. $\frac{e^{4x}}{4}+c$
10. $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$
11. $-\frac{1}{8}\cos 4x+C$
12. $\frac{1}{3}\ln|\sec 3x|-\frac{1}{2}\ln|\sec 2x|-\ln|\sec x|+e$
13. $\frac{2}{3}\sin 3x+2\sin x+C$
14. $\frac{3}{4}\sin x+\frac{1}{12}\sin 3x+C$
15. $-\frac{3}{64}\cos 2x+\frac{1}{192}\cos 6x+C$

Exercise for Session 2

1. $\tan x-\sec x+C$
2. $\sin 2x+C$
3. $-\frac{\cos 3x}{3}+C$
4. $\frac{180}{\pi}\sin x^\circ+C$
5. $x+C$
6. $2(\sin x+x\cos\alpha)+C$
7. $\sec x-\operatorname{cosec} x+C$
8. $\tan x-\cot x+C$
9. $(\sin x+\cos x)\sin(\cos x-\sin x)+C$
10. $\tan x-\cot x-3x+C$
11. $-\sqrt{2}\cos\left(\frac{x}{2}\right)+C$
12. $\frac{-\cos 4x}{8}+C$
13. $\frac{1}{2}(x-\sin x)+C$
14. $-2\cos x+C$
15. $\frac{x}{\sqrt{2}}+C$

Exercise for Session 3

1. $\frac{1}{48}\tan^{-1}\left(\frac{3x^4}{4}\right)+C$
2. $\frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right)+C$
3. $\frac{1}{160}\log\left|\frac{4x^4-5}{4x^4+5}\right|+C$
4. $\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+C$
5. $\frac{1}{3}\log|x^3+\sqrt{a^6+x^6}|+C$
6. $\frac{1}{12}\log\left|\frac{2+3e^x}{2-3e^x}\right|+C$
7. $\frac{1}{\log_e 2}\log|2^x+\sqrt{4^x-25}|+C$
8. $-8\sqrt{5+2x-x^2}-3\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$
9. $\frac{1}{2}\log|x^2+2x+2|+\tan^{-1}(x+1)+C$
10. $-\sqrt{3-2x-x^2}-4\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$
11. $\frac{3}{8}\left\{\log|4x^2-4x+17|+\frac{1}{6}\tan^{-1}\left(\frac{2x-1}{4}\right)+C\right\}$
12. $\frac{1}{2}\sqrt{2x^2+3x}-\frac{3}{4\sqrt{2}}\log\left|x+\frac{3}{4}\sqrt{x^2+\frac{3}{2}x}\right|+C$
13. $\sqrt{a-x(x-b)}-(a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}}+C$
14. $\sin^{-1}x+\sqrt{1-x^2}+C$
15. $\frac{(2x+5)}{4}\sqrt{x^2+x+1}+\frac{15}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$
16. $\log\left|\tan\frac{x}{2}+1\right|+C$
17. $\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$
18. $\frac{1}{4}\log|\sin x-\cos x|+\frac{1}{8}\cos 2x+\frac{1}{8}\sin 2x+C$
19. $\sin^{-1}\left(\frac{e^x+2}{3}\right)+C$
20. $-\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+C$
21. $\frac{12}{13}(3\sin x-2\cos x)-\frac{5}{13}\log(3\cos x+2\sin x)+C$
22. $-\frac{2}{3}(4+3x-x^2)^{3/2}-\frac{1}{2}\left[\left(x-\frac{3}{2}\right)\sqrt{4+3x-x^2}+\frac{25}{4}\sin^{-1}\left(\frac{2x-3}{5}\right)\right]+C$
23. $2\sqrt{x^2+x+1}+2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|$ $-6\log\left|\frac{1-x+\sqrt{x^2+x+1}}{2(x+1)}\right|+C$
24. (c)

Exercise for Session 4

1. $x^2e^x-2(xe^x-e^x)+C$
2. $-x^2\cos x+2(x\sin x+\cos x)+C$
3. $x(\log x)-x+C$
4. $x(\log x)^2-2(x\log x-x)+C$
5. $x\tan^{-1}x-\frac{1}{2}\log|1+x^2|+C$
6. $x(\sec^{-1}x)-\log|x+\sqrt{x^2-1}|+C$
7. $\frac{x^2}{2}\tan^{-1}-\frac{1}{2}(x-\tan^{-1}x)+C$
8. $-\frac{1}{x}(1+\log x)+C$
9. $-x\cot\frac{x}{2}+C$
10. $x\log(x^2+1)-2x+2\tan^{-1}x+C$
11. $e^x\cdot\log(\sec x)+C$
12. $e^x\tan x+C$
13. $x\log(\log x)-\frac{x}{\log x}+C$
14. $\frac{1}{2}e^{2x}\tan x+C$
15. $\frac{e^x}{x^2+1}+C$
16. $e^x\cdot\sqrt{\frac{1+x}{1-x}}+C$
17. $\frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c)+b\sin(bx+c)\}+C$
18. $\frac{1}{2}\sec x\tan x+\frac{1}{2}\log|sec\, x+\tan x|+C$
19. $-2(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x})+C$
20. $x(\sin^{-1}x)^2-2(-\sin^{-1}x\cdot\sqrt{1-x^2}+x)+C$
21. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)-(1-x)\tan^{-1}(1-x)+\frac{1}{2}\log\{1+(1+x^2)\}+C$
22. $a\left\{\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right\}+C$
23. $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\cdot\log\left(1+\frac{1}{x^2}\right)+\frac{2}{9}\left(1+\frac{1}{x^2}\right)^{3/2}+C$
24. $\frac{\cos x}{2\cos x-\sin x}-\frac{x}{5}-\frac{2}{5}\log|2\cos x-\sin x|+C$
25. $e^{\sin x}(x-\sec x)+C$

Exercise for Session 5

1. $\frac{1}{2}\log|x-1|-4\log|x-2|+\frac{9}{2}\log|x-3|+C$
2. $\frac{1}{3}\log|1+x|-\frac{1}{6}\log|1-x+x^2|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C.$
3. $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$
4. $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$
5. $\log\left|\frac{1+\sin x}{2+\sin x}\right|+C$
6. $\frac{1}{10}\log|1-\cos x|-\frac{1}{2}\log|1+\cos x|+\frac{2}{5}\log|3+2\cos x|+C$
7. $\frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{1}{2(1+\sin x)}+C$
8. $-\frac{1}{3}\log|1+\tan x|+\frac{1}{6}\log|\tan^2 x-\tan x+1|$ $+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C$
9. $\log\left|\frac{2\log x+1}{3\log x+2}\right|+C$
10. $-\frac{\tan^{-1}x}{x}+\log|x|-\frac{1}{2}\log|1+x^2|+C$

Exercise for Session 6

1. $\sqrt{\frac{x^4+x^2+1}{x^2}}+C,$
2. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}\sqrt{x+1}}\right)+C.$
3. $2t^3-3t^2+6t-6\log|1+t|+C$, where, $t=(x+1)^{1/6}$.
4. $\frac{7}{(a+b)}\left(\frac{x-b}{x+a}\right)^{1/7}+C.$
5. $\sqrt{2}\sec A\,(\sqrt{\tan x\sin A+\cos A})+C$ 6. (a) 7. (d)
8. (c) 9. (d) 10. (c)

Chapter Exercises

1. (b) 2. (a) 3. (b) 4. (c) 5. (b) 6. (b)
7. (b) 8. (d) 9. (b) 10. (d) 11. (a) 12. (a)
13. (c) 14. (b) 15. (a) 16. (c) 17. (b) 18. (b)
19. (b) 20. (d) 21. (a,c) 22. (a, d) 23. (a, c) 24. (b,c,d)
25. (a, b) 26. (c) 27. (c) 28. (a) 29. (d) 30. (d)
31. (a) 32. (d) 33. (c) 34. (a) 35. (a) 36. (b)
37. (b) 38. (d) 39. (c) 40. (c) 41. (a)
42. A → q; B → r; C → p; D → r
43. A → p, q; B → p, r; C → r; D → q 44. (5) 45. (6)
46. (1) 47. (1) 48. (3) 49. (1) 50. (4) 51. (4)
52. (3) 53. (3) 54. (3) 55. (1)
56. $e^{(x\sin x+\cos x)}\cdot\left[\left(x-\frac{1}{x\cos x}\right)\right]+C$
57. $\frac{1}{3}(x+\sqrt{x^2+2})^{3/2}-2\frac{1}{\sqrt{x+\sqrt{x^2+2}}}+C$
58. $\tan\frac{\theta}{2}=t$
59. $2(1+x^{1/3})^{3/2}+C$
60. $\tan^{-1}(\cos\theta+\sec\theta+1)^{1/2}+C$
61. $-\frac{2}{3}\log\left|\frac{\sqrt{\cos\theta+\sec\theta+1}-\sqrt{3}}{\sqrt{\cos\theta+\sec\theta+1}+\sqrt{3}}\right|+C$
62. $-\frac{1}{8}x^6(1-x^3)^{2/3}-\frac{3}{20}x^3(1-x^3)^{2/3}-\frac{9}{40}(1-x^3)^{2/3}+C$
63. $-\cot x\log(\cos x+\sqrt{\cos 2x})-\cot x-x+\sqrt{\cos^2 x-1}+C$
64. $\frac{1}{(4a^2-1)^{3/2}}\left\{2a\sin^{-1}\left(\frac{2a\sin 2x+1}{2a+\sin 2x}\right)+\sqrt{1-\left(\frac{2a\sin 2x+1}{2a+\sin 2x}\right)}\right\}$ $-\frac{1}{(2a+\sin 2x)}+C$
65. $2\ln|\sqrt{x^2+2x+4}-x|-\frac{3}{2(\sqrt{x^2+2x+4}-(x+1))}$ $-\frac{3}{2}\ln\sqrt{x^2+2x+4-x-1+C}$
66. $\log(x+1+\sqrt{x^2+2x+2})+\frac{2}{(x+2)+\sqrt{(x^2+2x+2)}}+C$
67. $\tan^{-1}\left(x-\frac{1}{x}\right)-\frac{2}{3}\tan^{-1}(x^3)+C$
68. $\frac{1}{3}\log\left|\frac{(1-x^3)^{1/3}+x}{x}\right|-\frac{1}{6}\log\left|\frac{(1-x^3)^{2/3}-x(1-x^3)^{1/3}+x^2}{x^2}\right|$ $-\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(1-3)}{\sqrt{3x}}\right\}^{1/3}-x$
69. $\frac{(x+\sqrt{1+x^2})^{15}}{15}+C$ 71. (c) 72. (c) 73. (b)
74. (d) 75. (b) 76. (c) 77. (d) 78. (b)

Solutions

1. Let $I = \sqrt{\dfrac{x^2}{(1+x^2)1+(1+x^2)}}dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2 d\theta$

Putting, $x = \tan\theta$, $I = \int \dfrac{\tan^2\theta}{1+\sec\theta} d\theta$

$$= \int (\sec\theta - 1)\, d\theta$$
$$= \log(\sec\theta + \tan\theta) - \theta + C$$
$$f(x) = \log_e(x + \sqrt{x^2+1}) - \tan^{-1} x + C$$
$$f(0) = 0 \Rightarrow C = 0$$
$$\Rightarrow f(1) = \log_e(1+\sqrt{2}) - \tan^{-1} 1$$
$$= \log_e(1+\sqrt{2}) - \frac{\pi}{4}$$

2. We have, $\int f(x)\, dx = f(x)$

$$\Rightarrow \frac{d}{dx}\{f(x)\} = f(x) \quad \Rightarrow \frac{1}{f(x)} d\{f(x)\} = dx$$
$$\Rightarrow \log\{f(x)\} = x + \log C \Rightarrow f(x) = Ce^x$$
$$\Rightarrow \{f(x)\}^2 = C^2 e^{2x}$$
$$\Rightarrow \int \{f(x)\}^2 dx = \int C^2 e^{2x} dx = \frac{C^2 e^{2x}}{2} = \frac{1}{2}\{f(x)\}^2$$

3. We have, $\int f(x)\, dx = F(x)$

$$\therefore \int x^3 f(x^2)\, dx = \frac{1}{2}\int \underset{\text{I}}{x^2}\, \underbrace{f(x^2)\, d(x^2)}_{\text{II}}$$
$$= \frac{1}{2}[x^2 F(x^2) - \int F(x^2)\, d(x^2)]$$

4. We have the following cases :

***Case* I** When $x \geq 0$

In this case, we have

$$\int |x^n|\, dx = \int |x|^n dx = \int x^n dx \quad [\because |x| = x]$$
$$= \frac{x^{n+1}}{n+1} + C = \frac{|x|^n x}{n+1} + C \quad [\because x \geq 0 \Rightarrow |x| = x]$$

***Case* II** When $x \leq 0$

In this case, we have $|x| = -x$

$$\therefore \int |x^n|\, dx = \int |x|^n dx$$
$$= \int (-x)^n dx = -\int x^n dx \quad [\because n \text{ in odd}]$$
$$= -\frac{x^{n+1}}{n+1} + C = \frac{(-x)^n x}{n+1} + C = \frac{|x|^n x}{n+1} + C$$

Hence, $\int |x^n|\, dx = \dfrac{|x|^n x}{n+1} + C$

5. $F(x) = \int \dfrac{3x+2}{\sqrt{x-9}} dx$. Let $x - 9 = t^2$

$$\Rightarrow dx = 2t\, dt$$
$$\therefore F(x) = \int \left(\frac{3(t^2+9)+2}{t} \cdot 2t\right) dt$$
$$= 2\int (29 + 3t^2)\, dt = 2[29t + t^3]$$
$$F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}] + C$$

Given, $F(10) = 60 = 2[29+1] + C \Rightarrow C = 0$

$$\therefore F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}]$$
$$F(13) = 2[29 \times 2 + 4 \times 2] = 132$$

6. We have, $\int (x^x)^x (2x \log_e x + x)\, dx$

$$= \int x^{x^2}(2x\log_e x + x)\, dx$$
$$= \int 1 \cdot d(x^{x^2}) = x^{x^2} + C = (x^x)^x + C$$

7. We have, $\int x \log x (\log x - 1)\, dx$

$$= \int \log x (x \log x - x)\, dx$$
$$= \int (x\log x - x)\, d(x\log x - x)$$
$$= \frac{(x\log x - x)^2}{2} + C$$

8. $\int \dfrac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)dx}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}$

Let $2 - \dfrac{2}{x^2} + \dfrac{1}{x^4} = z \Rightarrow \dfrac{1}{4}\int \dfrac{dz}{\sqrt{z}}$

$$\Rightarrow \frac{1}{2}\sqrt{z} + C \Rightarrow \frac{1}{2}\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + C$$

or $\sqrt{\dfrac{2x^4 - 2x^2 + 1}{2x^2}} + C$

9. We have, $f''(x) = f(x)$

$$\Rightarrow 2f'(x) f''(x) = 2f(x) f'(x)$$
$$\Rightarrow \frac{d}{dx}\{f'(x)\}^2 = \frac{d}{dx}\{f(x)\}^2$$
$$\Rightarrow \{f'(x)\}^2 = \{f(x)\}^2 + C \quad \ldots$$

Now, $f(0) = 2$ and $f'(0) = 3$. Therefore, from Eq. (i), we get

$$\{f'(0)\}^2 = \{f(0)\}^2 + C$$
$$\Rightarrow 9 = 4 + C \Rightarrow C = 5$$
$$\therefore \{f'(x)\}^2 = \{f(x)\}^2 + 5$$
$$\Rightarrow f'(x) = \sqrt{5 + \{f(x)\}^2}$$
$$\Rightarrow \int \frac{1}{\sqrt{(\sqrt{5})^2 + \{f(x)\}^2}} d\{f(x)\} = \int dx$$
$$\Rightarrow \log\left| f(x) + \sqrt{5 + \{f(x)\}^2}\right| = x + C_1$$

$\therefore \quad f(0)=2 \Rightarrow \log|2+3|=C_1$

$\Rightarrow \quad C_1=\log 5$

$\therefore \quad \log|f(x)+\sqrt{5+\{f(x)\}^2}|=x+\log 5$

$\Rightarrow \log\left\{\frac{f(x)+\sqrt{5+\{f(x)\}^2}}{5}\right\}=x$

$\Rightarrow f(x)+\sqrt{5+\{f(x)\}^2}=5e^x$

$\Rightarrow \sqrt{5+\{f(x)\}^2}+f(x)=5e^x$ and $\sqrt{5+\{f(x)\}^2}-f(x)=5e^{-x}$

$\Rightarrow \quad 2f(x)=5(e^x-e^{-x})$

$\Rightarrow \quad f(x)=\frac{5}{2}(e^x-e^{-x})$

$\Rightarrow \quad f(4)=\frac{5}{2}(e^4-e^{-4}) \Rightarrow f(4)=\frac{5(e^8-1)}{2e^4}$

10. We have, $\int \frac{e^{x^2+4\ln x}-x^3e^{x^2}}{x-1}dx=\int\frac{e^{x^2}\cdot x^4-x^3e^{x^2}}{x-1}dx$

$=\int x^3e^{x^2}dx=\frac{1}{2}\int te^t dt$, [where $t=x^2$]

$=\frac{1}{2}(t-1)e^t+C=\frac{1}{2}(x^2-1)e^{x^2}+C$

11. $\int \tan^4 x\,dx=\int(\tan^2 x\sec^2 x-\sec^2 x+1)\,dx$

$=\frac{\tan^3 x}{3}-\tan x+x+C$

$\Rightarrow \quad A=\frac{1}{3}, B=-1$ and $f(x)=x+C$

12. $\int \frac{\sin^4 x}{x}dx=f(x) \Rightarrow \int\frac{\sin^4(p+q)x}{(p+q)x}dx=\frac{f\{(p+q)x\}}{p+q}$

$\int\frac{\sin^4(p+q)x}{x}dx=f\{(p+q)x\}$

13. On solving, $\frac{\sin\theta}{\cos 3\theta}=\frac{1}{2}[\tan 3\theta-\tan\theta]$

$\frac{\sin 3\theta}{\cos 9\theta}=\frac{1}{2}[\tan 9\theta-\tan 3\theta]$

$\frac{\sin 9\theta}{\cos 27\theta}=\frac{1}{2}[\tan 27\theta-\tan 9\theta]$

$\therefore \int\left(\frac{\sin\theta}{\cos 3\theta}+\frac{\sin 3\theta}{\cos 9\theta}+\frac{\sin 9\theta}{\cos 27\theta}\right)d\theta=\frac{1}{2}\int(\tan 27\theta-\tan\theta)\,d\theta$

$=\frac{1}{2}\left\{\frac{1}{27}\log(\sec 127\theta)-\log(\sec\theta)\right\}+C$

$=\frac{1}{2}\log\frac{\sqrt[27]{\sec 27\theta}}{\sec\theta}+C$

14. We have, $\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$

$=\int x\sqrt{\frac{2\sin(x^2+1)-2\sin(x^2+1)\cos(x^2+1)}{2\sin(x^2+1)+2\sin(x^2+1)\cos(x^2+1)}}dx$

$=\int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}dx=\int x\tan\left(\frac{x^2+1}{2}\right)dx$

$=\int\tan\left(\frac{x^2+1}{2}\right)d\left(\frac{x^2+1}{2}\right)=\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$

15. $\int\frac{\sin(4x-2x)\,dx}{\sin(2x)\cos(2x)\cos(4x)}=\int\frac{\sin(4x)\,dx}{\sin(2x)\cos(4x)}-\int\sec 2x\,dx$

$=2\int\frac{\cos 2x\,dx}{\cos 4x}-\frac{1}{2}(\log|\sec 2x-\tan 2x|)$

$=2\int\frac{\cos 2x}{(1-2\sin^2 2x)}dx-\frac{1}{2}(\log|\sec 2x-\tan 2x|)$

$=\frac{2}{2\sqrt{2}}\left[\frac{1}{2\times 1}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|\right]-\frac{1}{2}\log|\sec 2x-\tan 2x|+C$

$=\frac{1}{2\sqrt{2}}\left[\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|\right]-\frac{1}{2}\log|\sec 2x-\tan 2x|+C$

16. $\int\frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\int\left(\frac{\sec^2 x}{\sin^7 x}-\frac{7}{\sin^7 x}\right)dx$

$=\int\frac{\sec^2 x}{\sin^7 x}dx-\int\frac{7}{\sin^7 x}dx=I_1+I_2$

Now, $\quad I_1=\int\frac{\sec^2 x}{\sin^7 x}dx=\frac{\tan x}{\sin^7 x}+7\int\frac{\tan x\cdot\cos x}{\sin^8 x}dx$

$=\frac{\tan x}{\sin^7 x}+I_2$

$\therefore \quad I_1+I_2=\frac{\tan x}{\sin^2 x}+C \quad \Rightarrow f(x)=\tan x$

17. We have, $\int\frac{\sin^3 x}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}dx$

$=\int\frac{\frac{\sin^3 x}{\cos^2 x}}{(\cos^2 x+3+\sec^2 x)\tan^{-1}(\sec x+\tan x)}dx$

$=\int\frac{1}{1+(\sec x+\cos x)^2}\times\frac{\sin x(1-\cos^2 x)}{\cos^2 x}\times\frac{1}{\tan^{-1}(\sec x+\tan x)}dx$

$=\int\frac{1}{\tan^{-1}(\sec x+\cos x)}\times\frac{1}{1+(\sec x+\cos x)^2}(\tan x\sec x-\sin x)\,dx$

$=\int\frac{1}{\tan^{-1}(\sec x+\cos x)}d|\tan^{-1}(\sec x+\cos x)|$

$=\log_e|\tan^{-1}(\sec x+\cos x)|+C$

18. We have, $\quad f(x)=x|\cos x|, \frac{\pi}{2}<x<\pi$

$\Rightarrow \quad f(x)=-x\cos x$ [$\because \cos x<0$ for $x\in(\pi/2,\pi)$]

Hence, required primitive is given by

$\int f(x)\,dx=-\int x\cos x\,dx+C=-x\sin x-\cos x+C$

19. We have, $f(x) = (2x+1)|\sin x|, \pi < x < 2\pi$

$\Rightarrow \quad f(x) = -(2x+1)\sin x$

Hence, required primitive is given by

$$-\int \underset{\text{I}}{(2x+1)} \underset{\text{II}}{\sin x}\, dx = -[-(2x+1)\cos x + 2\sin x] + C$$

$$= (2x+1)\cos x - 2\sin x + C$$

20. We have, $f(x) = \begin{vmatrix} 0 & x^2 - \sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1 - 2x \\ 2 - \cos x & 2x - 1 & 0 \end{vmatrix}$

$$\Rightarrow \quad f(x) = \begin{vmatrix} 0 & \sin x - x^2 & 2 - \cos x \\ x^2 - \sin x & 0 & 2x - 1 \\ \cos x - 2 & 1 - 2x & 0 \end{vmatrix}$$

[Interchanging rows and columns]

$$\Rightarrow \quad f(x) = (-1)^3 \begin{vmatrix} 0 & x^2 - \sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1 - 2x \\ 2 - \cos x & 2x - 1 & 0 \end{vmatrix}$$

[Taking (-1) common from each column]

$\Rightarrow \quad f(x) = -f(x)$

$\Rightarrow \quad f(x) = 0$

$\Rightarrow \quad \int f(x)\,dx = 0$

21. $\int \frac{dx}{(x+1)(x-2)} = \int \left(-\frac{1}{3(x+1)} + \frac{1}{3(x-2)}\right) dx$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{3}\log(x-2) + C$$

$\therefore \quad A = -\frac{1}{3}, B = \frac{1}{3}$

$\therefore \quad A + B = 0 \Rightarrow \frac{A}{B} = \frac{-\frac{1}{3}}{\frac{1}{3}} = -1$

22. $\frac{1}{3}\int \left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right) dx$

$\Rightarrow \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} = k\tan^{-1} x + l\tan^{-1}\frac{x}{2}$

$\therefore \quad k = \frac{1}{3}$ and $l = -\frac{1}{6}$

23. $I = \int \underset{\text{I}}{x} \underset{\text{II}}{\log(1+x^2)}\, dx = \log(1+x^2)\cdot\frac{x^2}{2} - \int \frac{1}{1+x^2}\cdot 2x \cdot \frac{x^2}{2}\, dx$

$$= \frac{x^2}{2}\log(1+x^2) - \int \frac{x^3}{x^2+1}\, dx$$

$$I = \left(\frac{x^2+1}{2}\right)\log(1+x^2) - \left(\frac{x^2+1}{2}\right) + C$$

$\therefore \quad \phi(x) = \frac{x^2+1}{2}, \psi(x) = -\left(\frac{1+x^2}{2}\right)$

24. $I = \int \frac{4e^2 + 6^{-x}}{9e^x - 4e^{-x}}\, dx$

$I = \int \frac{4e^{2x}+6}{9e^{2x}-4}\, dx$, put $9e^{2x} - 4 = z \Rightarrow 18e^{2x}dx = dz$

Then, $I = \int \frac{1}{z}\left\{4\cdot\frac{z+4}{9} + 6\right\}\frac{1}{18}\frac{dz}{\frac{z+4}{9}}$

$$= \int \frac{1}{z(z+4)}\left\{\frac{2z+8}{9} + 3\right\} dz$$

$$= \frac{1}{9}\int \frac{2z+35}{z(z+4)}\, dz = \frac{1}{9}\int \frac{2(z+4)+27}{z(z+4)}\, dz$$

$$= \frac{2}{9}\int \frac{dz}{z} + \frac{3}{4}\int\left(\frac{1}{z} - \frac{1}{z+4}\right) dz$$

$$= \left(\frac{2}{9} + \frac{3}{4}\right)\log z - \frac{3}{4}\log(z+4) + C$$

$$= \frac{35}{36}\log(9e^{2x} - 4) = \frac{3}{4}\log(e^{2x}) + C$$

$$= -\frac{3}{2}x + \frac{35}{36}\log(9e^{2x} - 4) - \frac{3}{2}\log 3 + C$$

25. $\int \tan^3 x(\sec^2 x - 1)\, dx = \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx$

$$I_1 = \int \tan^3 x \sec^2 x\, dx = \frac{\tan^4 x}{4} + C_1$$

$$I_2 = \int \tan x(\sec^2 x - 1)\, dx$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C_2$$

26. The Statement II is false since while writing

$$\int \frac{dx}{x-3y} = \log(x-3y) + C,$$

we are assuming that y is a constant. We will know prove the Statement I. From the given relation $(x-y)^2 = \frac{x}{y}$ and

$$2\log(x-y) = \log x - \log y.$$

Also, $\frac{dy}{dx} = \left(-\frac{y}{x}\right)\cdot\frac{x+y}{x-3y}$. To prove the integral relation it is sufficient to show that $\frac{d}{dx}$ RHS $= \frac{1}{x-3y}$.

Now, $\text{RHS} = \frac{1}{2}\log\left[\frac{x}{y} - 1\right] \qquad \left[\because (x-y)^2 = \frac{x}{y}\right]$

$$= \frac{1}{2}[\log(x-y) - \log y]$$

$$= \frac{1}{2}\left[\frac{\log x - \log y}{2} - \log y\right] = \frac{1}{4}[\log x - 3\log y]$$

$$\Rightarrow \frac{d}{dx}\text{RHS} = \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\frac{dy}{dx}\right]$$

$$= \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\left(-\frac{y}{x}\right)\frac{x+y}{x-3y}\right] = \frac{1}{x-3y}$$

Thus, Statement I is true. Hence, choice (c) is correct.

27. Let $g(x) = f(x) + f(-x)$

Assuming, $\int f(x)\, dx = F(x) + C$

$$\int g(x)\,dx = \int \{f(x) + f(-x)\}\,dx$$
$$= \int f(x)\,dx + \int f(-x)\,dx$$
$$= F(x) + C + \{-F(-x) + C'\}$$
$$= F(x) - F(-x) + C + C'$$

which may be an odd function, if $C + C' = 0$.

Similarly, integral of an odd function is not always an even function.

Hence, Statement I is true and Statement II is false.

28. If $a > 0$ and $b^2 - 4ac < 0$, then

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$\Rightarrow \int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a\left(x + \frac{b}{2a}\right)^2 + k^2}$$

where $k^2 = \frac{4ac - b^2}{4a} > 0$. which will have an answer of the type

$$\frac{1}{a}\cdot\frac{1}{k/\sqrt{a}}\tan^{-1}\frac{x + \frac{b}{2a}}{k/\sqrt{a}} + C \text{ or } \mu\tan^{-1}\frac{x + A}{B} + C$$

Thus, choice (a) is correct.

29. $I = \int \frac{dx}{x^4 + 1} \Rightarrow I = \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2 + \frac{1}{x^2}}\,dx$

$$= \frac{1}{2}\int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2}\,dx - \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2}\,dx$$

$$I = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x - \frac{1}{x}}{2}\right) + \frac{1}{2}\cdot\frac{1}{2}\log\left(\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right) + C$$

$\therefore$ Statement I is false.

30. Since, $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$,

$\therefore \quad d(\cot^{-1} x) = -d(\tan^{-1} x)$

Thus, $\int 2^{\tan^{-1} x}\, d(\cot^{-1} x) = -\int 2^{\tan^{-1} x}\, d(\tan^{-1} x)$

$$= -\frac{2^{\tan^{-1} x}}{\ln 2} + C$$

$\therefore$ Statement I is false. Statement II is true.

31. As $m = 9 > 0$, hence, we can substitute

$$\sqrt{9x^2 + 4x + 6} = u \pm 3x$$

32. Here, as per notations given, we can substitute

$$\sqrt{1 + x^2} = (u - x)$$

As $m = 1 > 0$ and $p = 1 > 0$

$$\Rightarrow \quad I = \int \frac{u^{15}}{u}\,du = \int u^{14}du = \frac{1}{15}u^{15} + C$$
$$= \frac{1}{15}(x + \sqrt{1 + x^2})^{15} + C$$

33. Here, $m = -1 < 0$

$p = -2 < 0$

Also, $-x^2 + 3x - 2 = -(x - 1)(x - 2)$

We can use case III

$\Rightarrow$ Putting, $\sqrt{-x^2 + 3x - 2} = u(x - 2)$

or $(x - 1)u$ or $u(1 - x)$

34. Let $P = \sin^3 x \cos^3 x$

$$\frac{dP}{dx} = 3\sin^2 x\cos^4 x - 3\sin^4 x\cos^2 x$$
$$= 3\sin^2 x(1 - \sin^2 x)\cos^2 x - 3\sin^4 x\cos^2 x$$
$$= 3\sin^2 x\cos^2 x - 6\sin^4 x\cos^2 x$$

$\therefore \quad P = 3I_{2,2} - 6I_{4,2}$

$\therefore \quad I_{4,2} = \frac{1}{6}(-P + 3I_{2,2})$

35. Let $P = \sin^5 x\cos^3 x$

$\therefore \quad \frac{dP}{dx} = 5\sin^4 x\cos^4 x - 3\sin^6 x\cos^2 x$
$$= 5\sin^4 x(1 - \sin^2 x)\cos^2 x - 3\sin^6 x\cos^2 x$$
$$= 5\sin^4 x\cos^2 x - 8\sin^6 x\cos^2 x$$

$\therefore \quad P = 5I_{4,2} - 8I_{6,2}$

$\therefore \quad I_{4,2} = \frac{1}{5}(P + 8I_{6,2})$

36. Let $P = \sin^5 x\cos^3 x$

$\therefore \quad \frac{dP}{dx} = 5\sin^4 x\cos^4 x - 3\sin^6 x\cos^2 x$
$$= 5\sin^4 x\cos^4 x - 3\sin^4 x(1 - \cos^2 x)\cos^2 x$$
$$= 8\sin^4 x\cos^4 x - 3\sin^4 x\cos^2 x$$

$\therefore \quad P = 8I_{4,4} - 3I_{4,2}$

$\therefore \quad I_{4,2} = \frac{1}{3}(-P + 8I_{4,4})$

37. Here, $2f'(x) = \lim_{h\to 0}\left(\frac{f(x + h) - f(x)}{h} + \frac{f(x\,x - h) - f(x)}{-h}\right)$

$$= \lim_{h\to 0}\left(\frac{f(x + h) - f(x - h)}{h}\right) \quad \text{...(i)}$$

$\therefore \quad 2f'(0)\lim_{h\to 0}\left(\frac{f(h) - f(0)}{h} + \frac{f(-h) - f(0)}{-h}\right)$

$$= \lim_{h\to 0}\frac{f(h) - f(-h)}{h} \quad \text{...(ii)}$$

Now by given relation, we have

$f(h) - f(-h) = \frac{f(x + h) - f(x - h)}{-h}$ and $f(0) = 1$

From Eqs. (i) and (ii), we have $\frac{f'(x)}{f(x)} = 2010$

$\Rightarrow \quad f(x) = e^{2010x}, f(0) = 1$

$\therefore$ $\{f(x)\}$ is non-periodic.

38. Here, $\int f(g(x)) \cos x \, dx = \int f(\log (\sin x)) . \cos x \, dx$

$$= \int e^{2010 \log (\sin x)} . \cos x \, dx$$

$$= \int (\sin x)^{2010} . \cos x \, dx$$

$$= \int (\sin x)^{2010} . \cos x \, dx$$

$$= \frac{(\sin x)^{2011}}{2011} + C$$

$$\therefore \quad h(x) = \frac{(\sin x)^{2011}}{2011}$$

$$\Rightarrow \quad h\left(\frac{\pi}{2}\right) = \frac{1}{2011}$$

Sol. (Q.Nos. 39 to 41)

Here, $f(x) = (x-1)(x+2)(x-3)(x-6) - 100$

$$= (x^2 - 4x + 3)(x^2 - 4x - 12) - 100$$

$$= (x^2 - 4x)^2 - 9(x^2 - 4x) - 136$$

$$= (x^2 - 4x + 8)(x^2 - 4x - 17)$$

39. $\therefore f(x) = 0 \Rightarrow \underbrace{(x^2 - 4x + 8)}_{D>0} \underbrace{(x^2 - 4x + 17}_{D<0} = 0$

$\therefore$ Equation has two distinct and two imaginary roots.

40. $f(x) = (x^2 - 4x - 17)(x^2 - 4x + 8)$

$$= \{((x-2)^2 - 21\} \{(x-2)^2 + 4\}$$

$\because \quad (f(x))_{\min} = (-21)(4) = -84$

which occurs at $x = 2$

41. $\because \quad \int \frac{g(x)}{f(x)} = \frac{g(x)}{(x^2 - 4x - 17)(x^2 - 4x + 8)}$

$$= \frac{Ax + B}{x^2 - 4x - 17} + \frac{Cx + D}{x^2 - 4x + 8}$$

Clearly, A, B and C must be zero.

$$\therefore \quad \frac{g(x)}{(x^2 - 4x - 17)(x^2 - 4x + 8)} = \frac{D}{x^2 - 4x + 8}$$

$$\therefore \quad g(x) = D(x^2 - 4x - 17)$$

$$g(-2) = D(4 + 8 - 17) = -10 \quad \text{[given]}$$

$$\Rightarrow \quad \frac{g(x)}{f(x)} = \frac{2(x^2 - 4x - 17)}{(x^2 - 4x - 17)(x^2 - 4x + 8)} = \frac{2}{x^2 - 4x + 8}$$

$$\therefore \quad \int \frac{g(x)}{f(x)} dx = \int \frac{2}{x^2 - 4x + 8} dx = 2 \int \frac{dx}{(x-2)^2 + (2)^2}$$

$$= 2 \cdot \frac{1}{2} \tan^{-1}\left(\frac{x-2}{2}\right) + C = \tan^{-1}\left(\frac{x-2}{2}\right) + C$$

42. (A) If $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then $\sin x > \cos x$

$$\therefore \quad \int \frac{\sin x - \cos x}{|\sin x - \cos x|} dx = \int 1 \, dx = x + C$$

(B) $\int \frac{x^2 dx}{(x^3 + 1)(x^3 + 2)} = \frac{1}{3} \int 3x^2 \left(\frac{1}{x^3 + 1} - \frac{1}{x^3 + 2}\right) dx$

$$= \frac{1}{3} \ln \left|\frac{x^3 + 1}{x^3 + 2}\right| + C$$

(C) $\int \sin^{-1} x \cos^{-1} x \, dx = \int \left[\frac{\pi}{2} \sin^{-1} x - (\sin^{-1} x)^2\right] dx$

$$\Rightarrow \frac{\pi}{2}(x \sin^{-1} x + \sqrt{1 - x^2}) - \{x (\sin^{-1} x)^2$$

$$+ \sin^{-1} x \sqrt{1 - x^2} - x\} + C \text{ (by parts)}$$

$$\Rightarrow \sin^{-1} x \left[\frac{\pi}{2} x - x \sin^{-1} x - 2\sqrt{1 - x^2}\right] + \frac{\pi}{2} \sqrt{1 - x^2} + 2x + C$$

$\therefore \quad f^{-1}(x) = \sin^{-1} x, f(x) = \sin x$

(D) $\int \frac{dx}{x \ln |x|} = \ln |\ln |x|| + C$

$\therefore \quad f(x) = \ln |x|$

43. (A) $I = \int \left(\frac{x^2 + \cos^2 x}{1 + x^2}\right) \cdot \operatorname{cosec}^2 x \, dx$

$$= \int \left(\frac{x^2 + 1 - \sin^2 x}{1 + x^2}\right) \cdot \operatorname{cosec}^2 x \, dx$$

$$= \int \left(\operatorname{cosec}^2 x - \frac{1}{1 + x^2}\right) dx = -\cot x + \cot^{-1} x + k$$

$\Rightarrow \quad A = 1, B = -1$

(B) $I = \int \sqrt{x + \sqrt{x^2 + 2}} \, dx$

Put $\sqrt{x^2 + 2} + x = t \Rightarrow \sqrt{x^2 + 2} - x = \frac{2}{t}$

$$\therefore \quad 2x = t - \frac{2}{t} \Rightarrow 2 \, dx = \left(1 + \frac{2}{t^2}\right) dt$$

$$= \frac{1}{2} \int \left(\sqrt{t} + \frac{2}{t^{3/2}}\right) dt$$

$$\Rightarrow \quad I = \frac{1}{2} \cdot \frac{t^{3/2}}{\frac{3}{2}} + \frac{1}{-\frac{1}{2} t^{1/2}} + k$$

$$= \frac{1}{3}(x + \sqrt{x^2 + 2})^{3/2} - \frac{2}{\sqrt{x + \sqrt{x^2 + 2}}} + k$$

$\Rightarrow \quad A = 1$ and $B = 2$

(C) $I = \int \frac{2 - x - x^2}{x^2 \sqrt{2 - x - x^2}} dx$

$$= -\int \frac{dx}{\sqrt{2 - x - x^2}} + 2 \int \frac{dx}{x^2 \sqrt{2 - x - x^2}} - \int \frac{dx}{x \sqrt{2 - x - x^2}}$$

$$= -\int \frac{dx}{\sqrt{\frac{9}{4} - \left(x + \frac{1}{2}\right)^2}} + 2\int -\frac{dt}{\sqrt{2 - \frac{1}{t} - \frac{1}{t^2}}} + \int \frac{\frac{1}{t}dt}{\sqrt{2 - \frac{1}{t} - \frac{1}{t^2}}}$$

$$\left(\text{put } x = \frac{1}{t}\right)$$

$$= -\sin^{-1}\left(\frac{2x+1}{3}\right) - \frac{1}{2}\int \frac{(4t-1)dt}{\sqrt{2t^2 - t - 1}} + \frac{1}{2}\int \frac{dt}{\sqrt{2t^2 - t - 1}}$$

$$= -\sin^{-1}\left(\frac{2x+1}{3}\right) - \frac{1}{2}\frac{\sqrt{2t^2 - t - 1}}{\frac{1}{2}} + \frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{t^2 - \frac{t}{2} - \frac{1}{2} + \frac{1}{16} - \frac{1}{16}}}$$

$$= -\sin^{-1}\left(\frac{2x+1}{3}\right) - \frac{\sqrt{2 - x - x^2}}{x} + \frac{1}{2\sqrt{2}}\log\left|\left(t - \frac{1}{4}\right) + \sqrt{2t^2 - t - 1}\right| + K$$

$$= -\frac{\sqrt{2 - x - x^2}}{x} + \frac{1}{2\sqrt{2}}\log\left|\frac{(4-x) + \sqrt{2 - x - x^2}}{4x}\right| - \sin^{-1}\left(\frac{2x+1}{3}\right) + K$$

(D) $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$, dividing N^r and D^r by $\cos^4 x$

$$I = \int \frac{2\tan x \cdot \sec^2 x}{\tan^4 x + 1}\,dx, \text{ put } \tan^2 x = t$$

$$\Rightarrow 2\tan x \cdot \sec^2 x\,dx = dt = \int \frac{dt}{t^2 + 1} = -\cot^{-1}(t) + C$$

$$\Rightarrow -\cot^{-1}(\tan^2 x) + C \quad \therefore B = -1$$

44. $\int \frac{2x+3}{(x^2 + 3x)(x^2 + 3x + 2) + 1}\,dx$

Put $x^2 + 3x = t \Rightarrow (2x + 3)\,dx = dt$

$$\int \frac{dt}{t(t+2) + 1} = \int \frac{dt}{(t+1)^2} = C - \frac{1}{t+1} = C - \frac{1}{x^2 + 3x + 1}$$

$\Rightarrow \quad a = 1, b = 3, c = 1 \Rightarrow a + b + c = 5$

45. $F(x) = \int \frac{3x+2}{\sqrt{x-9}}\,dx.$

Let $\quad x - 9 = t^2 \Rightarrow dx = 2t\,dt$

$\therefore \quad F(x) = \int \left(\frac{3(t^2+9)+2}{t} \cdot 2t\right) dt$

$$= 2\int (29 + 3t^2)\,dt = 2[29t + t^3]$$

$$F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}] + C$$

Given, $\quad F(10) = 60 = 2[29 + 1] + C$

$\Rightarrow \quad C = 0$

$\therefore \quad F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}]$

$$F(13) = 2[29 \times 2 + 4 \times 2]$$

$$= 4 \times 33 = 132$$

Hence, sum of digits $= 1 + 3 + 2 = 6$

46. $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x)$

$\Rightarrow \quad p = 7$ (given)

Again, $\quad \frac{u(x)}{v(x)} = 7 \Rightarrow \left(\frac{u(x)}{v(x)}\right)' = 0$

$\Rightarrow \quad q = 0$

Now, $\quad \frac{p+q}{p-q} = \frac{7+0}{7-0} = 1$

47. Let $t = \ln\left(\frac{x-1}{x+1}\right) \Rightarrow \frac{dt}{dx} = \frac{2}{x^2 - 1}$

$$I = \int \frac{1}{2}t\,dt = \frac{1}{4}t^2 + C$$

$$I = \frac{1}{4}\left[\ln\left(\frac{x-1}{x+1}\right)\right]^2 + C$$

$\Rightarrow \quad 6A = \frac{1}{4} \Rightarrow 24A = 1$

48. $I = \int e^x \left(\frac{1}{(1-x)\sqrt{1-x^2}} + \sqrt{\frac{1+x}{1-x}}\right) dx$

As $\quad \frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{(1-x)\sqrt{1-x^2}}$

$\therefore \quad I = e^x\sqrt{\frac{1+x}{1-x}} + C \Rightarrow I = e^x\left(\frac{1+x}{1-x}\right)^{1/2} + C$

$\Rightarrow \quad \mu = 1, \lambda = \frac{1}{2}$

$\Rightarrow \quad 2(\mu + \lambda) = 2\left(1 + \frac{1}{2}\right) = 2 \times \frac{3}{2} = 3$

49. $I = \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x}\,dx$

$$= \ln(e^x + \sin x + x) - x + C$$

$\therefore \quad f(x) = e^x + \sin x + x$ and $g(x) = -x$

$\Rightarrow \quad f(x) + g(x) = e^x + \sin x$

$\Rightarrow \quad \frac{f(x) + g(x)}{e^x + \sin x} = 1$

50. $12\left[\frac{1}{4}\tan^{-1}\frac{x+3}{4} + \frac{1}{2 \cdot 6}\ln\left|\frac{x-9}{x+3}\right|\right] = 3\tan^{-1}\left(\frac{x+3}{4}\right) + \ln\left|\frac{x-9}{x+3}\right|$

$\Rightarrow \quad \lambda = 3, \mu = 1$

$\Rightarrow \quad \lambda + \mu = 4$

51. $\frac{2\cos\frac{15}{2}x\cos\frac{3}{2}x}{1 - 2\left(2\cos^2\frac{5x}{2} - 1\right)} = \frac{2\left(4\cos^3\frac{5x}{2} - 3\cos\frac{5x}{2}\right)\cos\frac{3x}{2}}{3 - 4\cos^2\frac{5x}{2}}$

$$= -2\cos\frac{5x}{2}\cos\frac{3x}{2}$$

$$= -(\cos 4x + \cos x)$$

$$I = -\frac{\sin 4x}{4} - \sin x + C$$

52. $I = \int \frac{\tan x}{1 + \tan x + \tan^2 x} dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x}} dx$

$= \int \frac{\sin 2x}{2 + \sin 2x} dx = \int dx - 2 \int \frac{dx}{2 + \sin 2x}$

$= x - 2\int \frac{\sec^2 x}{2 \sec^2 x + 2 \tan x} dx$

Let $t = \tan x, dt = \sec^2 x\, dx$

$= x - \frac{2}{2} \int \frac{dx}{t^2 + t + 1} = x - \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$I = x - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2 \tan x + 1}{\sqrt{3}}\right) + C$

53. $\int \sin^{5/2} x \cos^3 x\, dx = \int \sin^2 x \sin^{1/2} x \cos^2 x \cos x\, dx$

$= \int \sin^2 x \sin^{1/2} x (1 - \sin^2 x) \cos x dx$

$= \int t^2 t^{1/2} (1 - t^2)\, dt = \int [t^{5/2} - t^{9/2}] dt$

$= \int t^{5/2} dt - \int t^{9/2} dt = \frac{t^{7/2}}{7/2} - \frac{t^{11/2}}{11/2} + C$

$= \frac{2}{7} t^{7/2} - \frac{2}{11} t^{11/2} + C = \frac{2}{7} \sin^{7/2} x - \frac{2}{11} \sin^{11/2} x + C$

$= 2 \sin^{7/2} x \left[\frac{1}{7} - \frac{1}{11} \sin^2 x\right] + C$

$\Rightarrow \quad A = 7, B = 7, C = 11$

$\Rightarrow \quad (A + B) - C = (7 + 7) - 11 = 3$

54. Let $I = \int (x^{2010} + x^{804} + x^{402}) (2x^{1608} + 5x^{402} + 10)^{1/402} dx$

$= \int x (x^{2009} + x^{508} + x^{401}) \cdot (2x^{1608} + 5x^{402} + 10)^{1/402} dx$

$= \int (x^{2009} + x^{803} + x^{401}) \cdot (2x^{2010} + 5x^{804} + 10^{402})^{1/402} dx$

Put $2x^{2010} + 5x^{804} + 10^{x^{402}} = t$

$\Rightarrow \quad 4020 (x^{2009} + x^{803} + x^{401})\, dx = dt$

$\therefore \quad I = \int \frac{1}{4020} \cdot (t)^{1/402} dt = \frac{1}{4020} \cdot \frac{t^{1/420 + 1}}{1/402 + 1}$

$= \frac{1}{4020} \cdot \frac{t^{403/402}}{403/402}$

$= \frac{1}{4030} (2x^{2010} + 5x^{804} + 10^{402})^{403/402}$

$\therefore\ a - 400 = 3$

55. Let $e^{x^3 + x^2 - 1} (3x^4 + 2x^3 + 2x)\, dx$

$= \int x^2 \cdot e^{x^3 + x^2 - 1} \cdot (3x^2 + 2x)\, dx + \int e^{x^3 + x^2 - 1} \cdot (2x)\, dx$

I II

Applying by parts in first internal, we get

$I = x e^{x^3 + x^2 - 1} - \int 2x \cdot e^{x^3 + x^2 - 1} dx + \int e^{x^3 - x^2 - 1} (2x)\, dx$

$= x^2 \cdot e^{x^3 + x^2 - 1} + C = h(x) + C$

$\therefore \quad h(x) = x^2 \cdot e^{x^3 + x^2 - 1}$

$\Rightarrow \quad h(1) \cdot h(-1) = e^1 \cdot e^{-1} = 1$

56. $I = \int e^{(x \sin x + \cos x)} \left\{ \frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x} \right\} dx$

$= \int e^{(x \sin x + \cos x)} \cdot (x^2 \cos x)\, dx$

$- \int e^{(x \sin x + \cos x)} \cdot \frac{d}{dx}\left(\frac{1}{x \cos x}\right) dx$

$= e^{(x \sin x + \cos x)} \cdot \left(x - \frac{1}{x \cos x}\right) + C$

57. $I = \int \sqrt{x + \sqrt{x^2 + 2}}\, dx$

Let $x + \sqrt{x^2 + 2} = p$ or $x^2 + 2 = p^2 + x^2 - 2px$

$\Rightarrow \quad x = \frac{p^2 - 2}{2p}$ or $dx = \frac{(p^2 + 2)\, dp}{2p^2}$

$I = \int \frac{p^{1/2} (p^2 + 2)}{2p^2} dp = \frac{1}{2} \int p^{1/2} dp + \int p^{-3/2} dp$

$= \frac{1}{3} (x + \sqrt{x^2 + 2})^{3/2} - 2 \frac{1}{\sqrt{x + \sqrt{x^2 + 2}}} + C$

58. $I = \int \frac{dx}{(ax + b) \sqrt{(x - \alpha)^2 - \beta^2}}$

Put $(x - \alpha) = \beta \sec \theta \Rightarrow dx = \beta \sec \theta \tan \theta\, d\theta$

$\therefore \quad I = \int \frac{d\theta}{a (\alpha \cos \theta + \beta) + b \cos \theta}$

$= \int \frac{d\theta}{(a\alpha + b) \cos \theta + a\beta}$

$= \frac{1}{(a\alpha + b)} \int \frac{d\theta}{\cos \theta + \left(\frac{a\beta}{a\alpha + b}\right)}$

if $\left|\frac{a\beta}{a\alpha + b}\right| < 1$ ...(i)

Then, $I = \frac{1}{(a\alpha + b)} \cdot \operatorname{cosec} \alpha^1 \log \left| \frac{\cot \frac{\alpha^1}{2} + 1}{\cot \frac{\alpha^1}{2} - 1} \right| + C$

where $\frac{a\beta}{a\alpha + b} = \cos \alpha^1$, $\tan \frac{\theta}{2} = t$

Again, if $\left|\frac{a\beta}{a\alpha + b}\right| > 1$

$I = \frac{1}{a\alpha + b} \cot \alpha^1 \tan^{-1}\left(t \tan \frac{\alpha^1}{2}\right)$

where $\sec \alpha^1 = \frac{a\beta}{a\alpha + b} \Rightarrow \tan \frac{\theta}{2} = t$

59. $I = \int \frac{\sqrt{1 + \sqrt[3]{x}}}{\sqrt[3]{x^2}} dx = \int x^{-2/3} (1 + x^{1/3})^{1/2} dx,$

$m = -\frac{2}{3}, n = \frac{1}{3}, p = \frac{1}{2}$

$\therefore \quad \dfrac{m+1}{n} = 1$ ie, integer

$\therefore$ Let us make the substitution,

$$1 + x^{1/3} = t^2 \quad \therefore \quad \frac{1}{3} x^{-2/3}\, dx = 2t\, dt$$

Hence, $\quad I = 6 \int t^2\, dt = 2t^3 + C = 2(1 + x^{1/3})^{3/2} + C$

60. $I = \displaystyle\int \frac{\sin^3(\theta/2)}{\cos\theta/2\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\, d\theta$

$$= \frac{1}{2}\int \frac{2\sin\theta/2 \cdot \cos\theta/2 \cdot 2\sin^2\theta/2}{2\cos^2\theta/2\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\, d\theta$$

$$= \frac{1}{2}\int \frac{\sin\theta(1 - \cos\theta)}{(1 + \cos\theta)\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\, d\theta$$

Put $\quad \cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{(t-1)}{(t+1)\sqrt{t^3 + t^2 + t}}\, dt$$

$$= \frac{1}{2}\int \frac{t^2 - 1}{(t+1)^2\sqrt{t^3 + t^2 + t}}\, dt$$

$$= \frac{1}{2}\int \frac{(1 - 1/t^2)}{\left(t + \frac{1}{t} + 2\right)\sqrt{t + 1 + \frac{1}{t}}}\, dt$$

Put $\quad t + 1 + \dfrac{1}{t} = u^2 \quad \Rightarrow \left(1 - \dfrac{1}{t^2}\right) dt = 2u du$

$$I = \frac{1}{2}\int \frac{2u\, du}{(u^2+1)\cdot u} = \int \frac{du}{u^2 + 1} = \tan^{-1}(u) + C$$

$$= \tan^{-1}\left(t + 1 + \frac{1}{t}\right)^{1/2} + C$$

$$= \tan^{-1}(\cos\theta + \sec\theta + 1)^{1/2} + C$$

61. $I = \displaystyle\int \frac{(2\sin\theta + \sin 2\theta)\, d\theta}{(\cos\theta - 1)\sqrt{\cos\theta + \cos^2\theta + \cos^3\theta}}$

Put $\cos\theta = x^2$

$\Rightarrow \quad -\sin\theta\, d\theta = 2x\, dx$

$$= 2\int \frac{(1 + x^2)}{(1 - x^2)} \cdot \frac{2x\, dx}{\sqrt{x^2 + x^4 + x^6}}$$

$$= 4\int \frac{(1 + 1/x^2)\, dx}{(1/x - x)\sqrt{(1/x - x)^2 + 3}},$$

Put $\quad \dfrac{1}{x} - x = t$

$\Rightarrow \left(-\dfrac{1}{x^2} - 1\right) dx = dt$

$$= -4\int \frac{dt}{t\sqrt{t^2 + 3}}$$

Again, put $\; t^2 + 3 = u^2$

$\Rightarrow \quad 2t\, dt = 2u\, du$

$$\therefore \quad I = 4\int \frac{-u\, du}{u(u^2 - 3)} = -4\int \frac{du}{u^2 - 3}$$

$$= -\frac{2}{\sqrt{3}}\log\left|\frac{u - \sqrt{3}}{u + \sqrt{3}}\right| + C$$

$$= -\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{t^2 + 3} - \sqrt{3}}{\sqrt{t^2 + 3} + \sqrt{3}}\right| + C$$

$$= -\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{x^2 + 1/x^2 + 1} - \sqrt{3}}{\sqrt{x^2 + 1/x^2 + 1} + \sqrt{3}}\right| + C$$

$$= -\frac{2}{3}\log\left|\frac{\sqrt{\cos\theta + \sec\theta + 1} - \sqrt{3}}{\sqrt{\cos\theta + \sec\theta + 1} + \sqrt{3}}\right| + C$$

62. Let $\quad I_{m-1} = \int x^{m-1}(a + bx^n)^p\, dx$

and $\quad I_{m-n-1} = \int x^{m-n-1}(a + bx^n)^p\, dx$

Let $\quad p = x^{\lambda+1}(a + bx^n)^{\mu+1}$

where λ and μ are the smaller indices of x and $(a + bx^n)$.

Here, $\lambda = m - n - 1, \mu = p$

$\therefore \quad p = x^{m-n}(a + bx^n)^{p+1}$

Differentiating w.r.t. x, we have

$$\frac{dp}{dx} = x^{m-n}(p+1)(a + bx^n)^p(n\, bx^{n-1}) + (a + bx^n)^{p+1}(m-n)\, x^{m-n-1}$$

$$= nb(p+1)\, x^{m-1}(a + bx^n)^p + (m-n)\, x^{m-n-1}(a + bx^n)^p\, \{a + bx^n\}$$

$$= nb(p+1)\, x^{m-1}(a + bx^n)^p + a(m-n)\, x^{m-n-1}(a + bx^n)^p + b(m-n)\, x^{m-1}(a + bx^n)^p$$

$= b(np + m)\, x^{m-1}(a + bx^n)^p + a(m-n)\, x^{m-n-1}(a + bx^n)^p$Int
egrating both the sides w.r.t. x, we get

$$p = b(np + m)\, I_{m-1} + a(m-n)\, I_{m-n-1}$$

$$\therefore x^{m-n}(a + bx^n)^{p+1} = b(np + m)\, I_{m-1} + a(m-n)\, I_{m-n-1}$$

or $\quad I_{m-1} = \dfrac{x^{m-n}(a + bx^n)^{p+1}}{b(np + m)} - \dfrac{a(m-n)}{b(np + m)} \cdot I_{m-n-1}$

Hence Proved.

Again, $\displaystyle\int \frac{x^8}{(1 - x^3)^{1/3}}\, dx = \int x^{9-1}(1 - x^3)^{-1/3}\, dx$

Here, $m = 9, \quad b = -1, \quad n = 3, \quad p = -1/3, \quad a = 1$

$\therefore \quad I_8 = \dfrac{(x^6(1 - x^3)^{2/3})}{-8} + \dfrac{6}{8} I_5 \qquad$ ...(i)

$\Rightarrow \quad I_5 = \dfrac{x^3(1 - x^3)^{2/3}}{-5} + \dfrac{3}{5} I_2 \qquad$ (here $m = 6$)

$\Rightarrow \quad I_2 = \dfrac{(1 - x^3)^{2/3}}{2}$

Hence, $I_8 = -\dfrac{1}{8}x^6(1 - x^3)^{2/3} - \dfrac{3}{20}x^3(1 - x^3)^{2/3} - \dfrac{9}{40}(1 - x^3)^{2/3} + C$

63. Let $I = \int \operatorname{cosec}^2 x \ln(\cos x + \sqrt{\cos 2x})\, dx$

$$= \int \operatorname{cosec}^2 x \cdot \ln\{\sin x(\cot x + \sqrt{\cot^2 x - 1})\}\, dx$$

$$= \int \operatorname{cosec}^2 x \ln(\sin x)\, dx + \int \operatorname{cosec}^2 x \cdot \ln(\cot x + \sqrt{\cot^2 x - 1})\, dx$$

In second integral put $\cot x = 1$

$\therefore \quad \operatorname{cosec}^2 x\, dx = dt$

$\therefore \quad I = \int \operatorname{cosec}^2 x \cdot \ln(\sin x)\, dx - \int \ln(t + \sqrt{t^2 - 1})\, dt$

In first integral (integrating by parts taking $\operatorname{cosec}^2 x$ as second integral) and in second integral (integration by parts taking unity as second function).

We have, $(\ln \sin x)(-\cot x) - \int \cot x(-\cot x)\, dx$

$$- \ln(t + \sqrt{t^2 - 1})\, t + \int \frac{t\, dt}{\sqrt{t^2 - 1}}$$

$$= -\cot x(\ln \sin x) - \cot x - x - t \ln(t + \sqrt{t^2 - 1}) + \sqrt{t^2 - 1} + C$$

$$= -\cot x(\ln \sin x) - \cot x - x - \cot x\{\ln(\cot x + \sqrt{\cot^2 x - 1})\} + \sqrt{\cot^2 x - 1} + C$$

$$= -\cot x \cdot \ln(\cos x + \sqrt{\cos 2x}) - \cot x - x + \sqrt{\cot^2 x - 1} + C$$

64. $I = \int \frac{dx}{(\sin x + a \sec x)^2}\ a \in N = \int \frac{\cos^2 x\, dx}{(a + \sin x \cos x)^2}$

$$= \int \frac{\cos^2 x\, dx}{(a^2 + \sin^2 x \cdot \cos^2 x + 2a \sin x \cdot \cos x)}$$

$$= \int \frac{4\cos^2 x\, dx}{4a^2 + \sin^2 2x + 4a \sin 2x} = 2\int \frac{1 + \cos 2x}{(2a + \sin 2x)^2} dx$$

$$= 2\int \frac{1}{(2a + \sin 2x)^2} dx + 2\int \frac{\cos 2x}{(2a + \sin 2x)^2} dx$$

$$= 2I_1 - \frac{1}{(2a + \sin 2x)} \quad \text{...(i)}$$

$I_1 = \int \frac{dx}{(2a + \sin 2x)^2}$,

we know

$$\therefore \quad I_1 = \int \frac{du}{(4a^2 - 1)\frac{\sqrt{4a^2 - 1}\sqrt{1 - u^2}}{(2a - u)}}$$

$$u = 2a\frac{\sin 2x + 1}{2a + \sin 2x} = \frac{1}{(4a^2 - 1)^{3/2}} \int \frac{(2a - u)}{\sqrt{1 - u^2}} du$$

$$\Rightarrow \frac{du}{dx} = \frac{(4a^2 - 1)\cos 2x}{(2a + \sin 2x)^2}$$

$$\Rightarrow \frac{1}{(4a^2 - 1)^{3/2}}[2a \sin^{-1} u + \sqrt{1 - u^2}] = I_1 \text{ and } \sin 2x = \frac{2au - 1}{2a - u}$$

$$\therefore \quad I = \frac{1}{(4a^2 - 1)^{3/2}}\left[2a \sin^{-1}\left(\frac{2a \sin 2x + 1}{2a + \sin 2x}\right) + \sqrt{1 - \left(\frac{2a \sin 2x + 1}{2a + \sin 2x}\right)}\right] - \frac{1}{(2a + \sin 2x)} + C$$

65. $I = \int \frac{dx}{x - \sqrt{x^2 + 2x + 4}}$

Put $\sqrt{x^2 + 2x + 4} = t + x$

$\Rightarrow \quad x^2 + 2x + 4 = t^2 + x^2 + 2tx$

$\Rightarrow \quad 2x - 2tx = t^2 - 4$

$$\Rightarrow \quad x = \frac{t^2 - 4}{2 - 2t} = \frac{1}{2}\frac{(t^2 - 4)}{(1 - t)}$$

$$\Rightarrow \quad dx = -\frac{1}{2}\left[\frac{t^2 - 2t + 4}{(1 - t)^2}\right]$$

$$\therefore \quad I = -\frac{1}{2}\int \frac{t^2 - 2t + 4}{-t(1 - t)^2} dt$$

$$= \frac{1}{2}\int\left[\frac{4}{t} + \frac{3}{(1 - t)} + \frac{3}{(1 - t)^2}\right] dt$$

$$= \frac{1}{2}\left[4 \log|t| - 3 \log|1 - t| + \frac{3}{(1 - t)}\right]$$

$$= 2 \log|\sqrt{x^2 + 2x + 4} - x| - \frac{3}{2} \log |1 - \sqrt{x^2 + 2x + 4} + x| + \frac{3}{2(1 - \sqrt{x^2 + 2x + 4} + x)}$$

$$= 2 \log|\sqrt{x^2 + 2x + 4} - x| - \frac{3}{2} \log |\sqrt{x^2 + 2x + 4} - 1 - x| - \frac{3}{2(\sqrt{x^2 + 2x + 4} - x - 1)}$$

66. $I = \int \frac{dx}{1 + \sqrt{x^2 + 2x + 2}}$

$\sqrt{x^2 + 2x + 2} = t - x$, squaring both the sides, we get

$$x^2 + 2x + 2 = t^2 + x^2 - 2tx$$

$$2x + 2tx = t^2 - 2$$

$$x = \frac{t^2 - 2}{2(1 + t)}$$

$$\Rightarrow \quad dx = \frac{t^2 + 2t + 2}{2(1 + t)^2} dt$$

$$\therefore \quad 1 + \sqrt{x^2 + 2x + 2} = 1 + t - \frac{t^2 - 2}{2(1 + t)} = \frac{t^2 + 4t + 4}{2(1 + t)}$$

$$\Rightarrow \quad I = \int \frac{2(1 + t)(t^2 + 2t + 2)}{(t^2 + 4t + 4)\cdot 2 \cdot (1 + t)^2} dt = \int \frac{(t^2 + 2t + 2)}{(1 + t)(t + 2)^2} dt$$

Using partial fractions, we get

$$I = \int \frac{dt}{t + 1} - 2\int \frac{dt}{(t + 2)^2}$$

$$= \ln|t + 1| + \frac{2}{(t + 2)} + C$$

$$I = \ln(x + 1 + \sqrt{x^2 + 2x + 2}) + \frac{2}{(x + 2) + \sqrt{x^2 + 2x + 2}} + C$$

67. $$I=\int \frac{x^4+1}{x^6+1}dx=\int \frac{(x^2+1)^2-2x^2}{(x^2+1)(x^4-x^2+1)}dx$$

$$=\int \frac{(1+1/x^2)\,dx}{(x^2+1/x^2-1)}-2\int \frac{x^2\,dx}{(x^3)^2+1}$$

$$=\int \frac{(1+1/x^2)\,dx}{(x-1/x)^2+1}-2\int \frac{x^2\,dx}{(x^3)^2+1}$$

In first integral put $x-1/x=t$ and in second integral put $x^3=u$

$$=\int \frac{dt}{t^2+1}-\frac{2}{3}\int \frac{du}{u^2+1}$$

$$=\tan^{-1}(t)-\frac{2}{3}\tan^{-1}(u)+C$$

$$=\tan^{-1}(x-1/x)-\frac{2}{3}\tan^{-1}(x^3)+C$$

68. $I=\int \frac{dx}{(1-x^3)^{1/3}}$

Put $x=1/t,\ dx=-1/t^2\,dt$

$$\therefore\quad I=-\int \frac{dt}{t^2(1-1/t^3)^{1/3}}=-\int \frac{dt}{t(t^3-1)^{1/3}}$$

Again, put $t^3-1=u^3\Rightarrow 3t^2\,dt=3u^2\,du$

$$=-\int \frac{u^2\,du}{(1+u^3)\cdot u}=-\int \frac{u\,du}{(1+u)(1-u+u^2)}$$

$$=\frac{1}{3}\int \frac{du}{1+u}-\frac{1}{3}\int \frac{u+1}{u^2-u+1}du$$

(using partial fractions)

$$=\frac{1}{3}\int \frac{du}{u+1}-\frac{1}{3}\int \frac{1/2\,(2u-1)+3/2}{(u^2-u+1)}du$$

$$=\frac{1}{3}\int \frac{du}{u+1}-\frac{1}{6}\int \frac{2u-1}{u^2-u+1}du-\frac{1}{2}\int \frac{du}{(u-1/2)^2+3/4}$$

$$=\frac{1}{3}\log|u+1|-\frac{1}{6}\log|u^2-u+1|-\frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left\{\frac{2u-1}{\sqrt{3}}\right\}+C$$

$$=\frac{1}{3}\log|(t^3-1)^{1/3}+1|-\frac{1}{6}\log|(t^3-1)^{2/3}-(t^3-1)^{1/3}+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(t^3-1)^{1/3}-1}{\sqrt{3}}\right\}+C$$

$$=\frac{1}{3}\log\left|\frac{(1-x^3)^{1/3}+x}{x}\right|-\frac{1}{6}\log\left|\frac{(1-x^3)^{2/3}-(1-x^3)^{1/3}+x^2}{x^2}\right|-\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2(1-x^3)^{1/3}-x}{\sqrt{3}x}\right\}+C$$

69. $I=\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}}dx$

Put $(x+\sqrt{1+x^2})=t$

$$\therefore\quad \left(1+\frac{x}{\sqrt{1+x^2}}\right)dx=dt\ \text{ or }\ \frac{t\,dx}{\sqrt{1+x^2}}=dt$$

$$\therefore\quad I=\int t^{15}\cdot\frac{dt}{t}=\int t^{14}\,dt=\frac{t^{15}}{15}+C=\frac{(x+\sqrt{1+x^2})^{15}}{15}+C$$

70. $$\because\ u_{n+1}=\int \frac{x^{n+1}}{y}dx=\int \frac{x^{n+1}}{\sqrt{ax^2+2bx+c}}dx$$

$$=\frac{1}{2a}\int \frac{x^n(2ax+2b)-2bx^n}{\sqrt{ax^2+2bx+c}}dx$$

$$=\frac{1}{2a}\int x^n\frac{(2ax+2b)\,dx}{\sqrt{ax^2+2bx+c}}-\frac{b}{a}\int \frac{x^n}{\sqrt{ax^2+2bx+c}}dx$$

$$u_{n+1}=\frac{1}{2a}\int \underset{\text{I}}{x^n}\ \underset{\text{II}}{\frac{(2ax+2b)}{\sqrt{ax^2+2bx+c}}}-\frac{b}{a}u_n$$

$$u_{n+1}=\frac{1}{2a}\left[x^n\cdot 2\sqrt{ax^2+2bx+c}-\int nx^{n-1}\cdot 2\sqrt{(ax^2+bx+c)}\,dx\right]-\frac{b}{a}u_n$$

$$=\frac{1}{a}x^n y-\frac{n}{a}\int x^{n-1}\cdot\sqrt{ax^2+2bx+c}\,dx-\frac{b}{a}u_n$$

$$au_{n+1}=x^n y-n\int \frac{x^{n-1}(ax^2+2bx+c)}{\sqrt{ax^2+2bx+c}}dx-\frac{b}{a}u_n$$

$$au_{n+1}=x^n y-n[au_{n+1}+2bu_n+cu_{n-1}]-bu_n$$

$$\Rightarrow\ (n+1)\,a\,u_{n+1}+(2n+1)\,bu_n+nc\cdot u_{n-1}=x^n y \qquad \text{...(i)}$$

Now, putting, $n=0$ in both the sides, we get

$$au_1+bu_0=x^0 y$$

$$au_1=y-bu_0 \qquad \text{...(ii)}$$

Putting $n=1$ in Eq. (i), we get

$$2au_2+3bu_1+cu_0=xy$$

$$2au_2+3b\left(\frac{y-bu_0}{a}\right)+cu_0=xy \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow\ 2a^2u_2+3by-3b^2u_0+acu_0=axy$$

$$\Rightarrow\ 2a^2u_2=y(ax-3b)+(3b^2-ac)\,u_0$$

71. **Plan** Integration by Substitution

i.e. $I=\int f\{g(x)\}\cdot g'(x)dx$

Put $g(x)=t\ \Rightarrow g'(x)dx=dt$

$\therefore$ $I=\int f(t)dt$

Description of Situation Generally, students gets confused after substitution, i.e. $\sec x+\tan x=t$.

Now, for $\sec x$, we should use

$$\sec^2 x-\tan^2 x=1$$

$$\Rightarrow\ (\sec x-\tan x)(\sec x+\tan x)=1$$

$$\Rightarrow\ \sec x-\tan x=\frac{1}{t}$$

Here, $I=\int \frac{\sec^2 dx}{(\sec x+\tan x)^{9/2}}$

Put $\sec x+\tan x=t$

$\Rightarrow \quad (\sec x\tan x+\sec^2 x)dx=dt$

$\Rightarrow \quad \sec x\cdot t\, dx=dt$

$\Rightarrow \quad \sec x dx=\frac{dt}{t}$

$\therefore \quad \sec x-\tan x=\frac{1}{t} \Rightarrow \sec x=\frac{1}{2}\left(t+\frac{1}{t}\right)$

$\therefore \quad I=\int \frac{\sec x\cdot \sec x dx}{(\sec x+\tan x)^{9/2}}$

$$\Rightarrow \quad I=\int \frac{\frac{1}{2}\left(t+\frac{1}{t}\right)\cdot\frac{dt}{t}}{t^{9/2}}=\frac{1}{2}\int\left(\frac{1}{t^{9/2}}+\frac{1}{t^{13/2}}\right)dt$$

$$=-\frac{1}{2}\left\{\frac{2}{7t^{7/2}}+\frac{2}{11t^{11/2}}\right\}+K$$

$$=-\left[\frac{1}{7(\sec x+\tan x)^{7/2}}+\frac{1}{11(\sec x+\tan x)^{11/2}}\right]+K$$

$$=\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$$

72. Since, $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx$ and $J=\int\frac{e^{3x}}{1+e^{2x}+e^{4x}}dx$

$\therefore \quad J-I=\int\frac{(e^{3x}-e^x)}{1+e^{2x}+e^{4x}}dx$

Put $e^x=u \Rightarrow e^x dx=du$

$$\therefore \quad J-I=\int\frac{(u^2-1)}{1+u^2+u^4}du=\int\frac{\left(1-\frac{1}{u^2}\right)}{1+\frac{1}{u^2}+u^2}du$$

$$=\int\frac{\left(1-\frac{1}{u^2}\right)}{\left(u+\frac{1}{u}\right)^2-1}du$$

Put $u+\frac{1}{u}=t \Rightarrow \left(1-\frac{1}{u^2}\right)du=dt$

$$=\int\frac{dt}{t^2-1}=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{u^2-u+1}{u^2+u+1}\right|+C=\frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+C$$

73. Let $I=\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx=\int\frac{2x^{12}+5x^9}{x^{15}(1+x^{-2}+x^{-5})^3}dx$

$$=\int\frac{2x^{-3}+5x^{-6}}{(1+x^{-2}+x^{-5})^3}dx$$

Now, put $1+x^{-2}+x^{-5}=t$

$\Rightarrow \quad (-2x^{-3}-5x^{-6})\,dx=dt$

$\Rightarrow \quad (2x^{-3}+5x^{-6})\,dx=-dt$

$\therefore \quad I=-\int\frac{dt}{t^3}=-\int t^{-3}\,dt$

$$=-\frac{t^{-3+1}}{-3+1}+C=\frac{1}{2t^2}+C=\frac{x^{10}}{2(x^5+x^3+1)^2}+C$$

74. $\int\frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}=\int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}}$

Put $1+\frac{1}{x^4}=t^4 \Rightarrow -\frac{4}{x^5}dx=4t^3dt$

$\Rightarrow \quad \frac{dx}{x^5}=-t^3dt$

$$\therefore \quad I=\int\frac{-t^3dt}{t^3}=-\int dt=-t+C=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

75. $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx=\int e^{x+\frac{1}{x}}dx+\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int\frac{d}{dx}(x)e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int e^{x+\frac{1}{x}}dx$$

$$\left[\because \int\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx=e^{x+\frac{1}{x}}\right]$$

$$=\int e^{x+\frac{1}{x}}dx+xe^{x+\frac{1}{x}}-\int ex^{x+\frac{1}{x}}dx$$

$$=xe^{x+\frac{1}{x}}+C$$

76. Given, $\int f(x)\,dx=\psi(x)$

Let $\quad I=\int x^5 f(x^3)\,dx$

Put $\quad x^3=t$

$\Rightarrow \quad x^2dx=\frac{dt}{3}$ ...(i)

$\therefore \quad I=\frac{1}{3}\int t\,f(t)\,dt$

$$=\frac{1}{3}\left[t\int f(t)\,dt-\int\left\{\frac{d}{dt}(t)\,dt\right\}dt\right]$$

[Integration by parts]

$$=\frac{1}{3}[t\,\psi(t)-\int\psi(t)\,dt]$$

$$=\frac{1}{3}[x^3\psi(x^3)-3\int x^2\psi(x^3)\,dx]+C \quad \text{[from Eq. (i)]}$$

$$=\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)\,dx+C$$

77. **Given** Integral is $\int \frac{5 \tan x}{\tan x - 2} dx$

To find The value of a, if

$\int \frac{5 \tan x}{\tan x - 2} dx = x + a \log | \sin x - 2 \cos x | + k$ ...(i)

Now, let us assume that $I = \int \frac{5 \tan x}{\tan x - 2} dx$

Multiplying by $\cos x$ in numerator and denominator, we get

$$I = \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

This special integration requires special substitution of type

$$N^r = A(D^r) + B\left(\frac{dD^r}{dx}\right)$$

$\Rightarrow$ Let $5 \sin x = A (\sin x - 2 \cos x) + B (\cos x + 2 \sin x)$

$\Rightarrow 0 \cos x + 5 \sin x = (A + 2B) \sin x + (B - 2A) \cos x$

Comparing the coefficients of $\sin x$ and $\cos x$, we get

$$A + 2B = 5 \text{ and } B - 2A = 0$$

Solving the above two equations in A and B, we get

$$A = 1 \text{ and } B = 2$$

$\Rightarrow 5 \sin x = (\sin x - 2 \cos x) + 2 (\cos x + 2 \sin x)$

$\Rightarrow I = \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$

$$= \int \frac{(\sin x - 2 \cos x) + 2 (\cos x + 2 \sin x)}{(\sin x - 2 \cos x)} dx$$

$\Rightarrow I = \int \frac{\sin x - 2 \cos x}{\sin x - 2 \cos x} dx + 2 \int \frac{(\cos x + 2 \sin x)}{(\sin x - 2 \cos x)} dx$

$\Rightarrow I = \int 1 \, dx + 2 \int \frac{d(\sin x - 2 \cos x)}{(\sin x - 2 \cos x)}$

$\Rightarrow I = x + 2 \log | (\sin x - 2 \cos x) | + k$ ...(ii)

where, k is the constant of integration.

Now, by comparing the value of I in Eqs. (i) and (ii), we get $a = 2$.

78. Let $I = \sqrt{2} \int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)} dx$

Put $x - \frac{\pi}{4} = t \Rightarrow dx = dt$

$\therefore$ $$I = \sqrt{2} \int \frac{\sin\left(\frac{\pi}{4} + t\right) dt}{\sin t}$$

$$= \sqrt{2} \int \left[\frac{1}{\sqrt{2}} \cot t + \frac{1}{\sqrt{2}}\right] dt$$

$$= 1 + \log | \sin t | + C$$

$$= x + \log \left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$$