Relations and Functions

- A relation R from a set A to a set B is a subset of $A \times B$ obtained by describing a relationship between the first element a and the second element b of the ordered pairs in $A \times B$. That is, $R \subseteq \{(a, b) \in A \times B, a \in A, b \in B\}$
- The domain of a relation R from set A to set B is the set of all first elements of the ordered pairs in R.
- The range of a relation R from set A to set B is the set of all second elements of the ordered pairs in R. The whole set B is called the co-domain of R. Range $\subseteq$ Co-domain
- A relation R in a set A is called an empty relation, if no element of A is related to any element of A. In this case, $R = \Phi \subset A \times A$

  Example: Consider a relation R in set $A = \{3, 4, 5\}$ given by $R = \{(a, b): a^b < 25$, where $a, b \in A\}$. It can be observed that no pair (a, b) satisfies this condition. Therefore, R is an empty relation.

- A relation R in a set A is called a universal relation, if each element of A is related to every element of A. In this case, $R = A \times A$
  Example: Consider a relation R in the set $A = \{1, 3, 5, 7, 9\}$ given by $R = \{(a, b): a + b$ is an even number$\}$.
  Here, we may observe that all pairs (a, b) satisfy the condition R. Therefore, R is a universal relation.

- Both the empty and the universal relation are called trivial relations.
- A relation R in a set A is called reflexive, if $(a, a) \in R$ for every $a \in R$.
  Example: Consider a relation R in the set A, where $A = \{2, 3, 4\}$, given by $R = \{(a, b): a^b = 4, 27$ or $256\}$. Here, we may observe that $R = \{(2, 2), (3, 3)$, and $(4, 4)\}$. Since each element of R is related to itself (2 is related 2, 3 is related to 3, and 4 is related to 4), R is a reflexive relation.

- A relation R in a set A is called symmetric, if $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R, \forall (a_1, a_2) \in R$
  Example: Consider a relation R in the set A, where A is the set of natural numbers, given by $R = \{(a, b): 2 \leq ab < 20\}$. Here, it can be observed that $(b, a) \in R$ since $2 \leq ba < 20$ [since for natural numbers a and b, $ab = ba$]

Therefore, the relation R is symmetric.

- A relation R in a set A is called transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R \Rightarrow (a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in A$
  Example: Let us consider a relation R in the set of all subsets with respect to a universal set U given by $R = \{(A, B): A$ is a subset of $B\}$
  Now, if A, B, and C are three sets in R, such that $A \subset B$ and $B \subset C$, then we also have $A \subset C$. Therefore, the relation R is a symmetric relation.

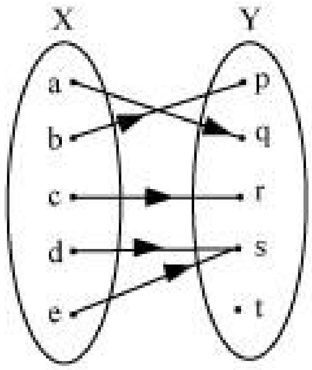
- A relation R in a set A is said to be an equivalence relation, if R is altogether reflexive, symmetric, and transitive.
  Example: Let (a, b) and (c, d) be two ordered pairs of numbers such that the relation between them is given by $a + d = b + c$. This relation will be an equivalence relation. Let us prove this.
  (a, b) is related to (a, b) since $a + b = b + a$. Therefore, $R$is reflexive.
  If (a, b) is related to (c, d), then $a + d = b + c \Rightarrow c + b = d + a$. This shows that (c, d) is related to (a, b). Hence, R is symmetric.
  Let (a, b) is related to (c, d); and (c, d) is related to (e, f), then $a + d = b + c$ and $c + f = d + e$. Now, $(a + d) + (c + f) = (b + c) + (d + e) \Rightarrow a + f = b + e$. This shows that (a, b) is related to (e, f). Hence, R is transitive.
  Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

- Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets Ai called partitions or subdivisions of X satisfying:
  - All elements of Ai are related to each other, for all i.
  - No element of Ai is related to any element of Aj , $i \neq j$
  - $\cup Aj = X$ *and* $Ai \cap Aj = \varnothing$, $i \neq j$

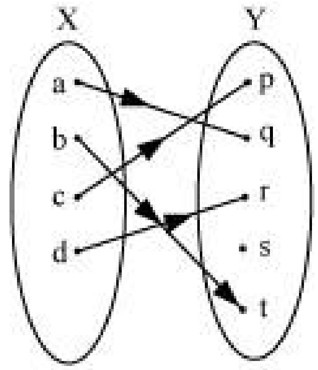
    The subsets Ai are called equivalence classes.

- A function f from set X to Y is a specific type of relation in which every element x of X has one and only one image y in set Y. We write the function f as $f: X \to Y$, where $f(x) = y$
- A function $f: X \to Y$ is said to be one-one or injective, if the image of distinct elements of X under f are distinct. In other words, if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. If the function f is not one-one, then f is called a many-one function.

  The one-one and many-one functions can be illustrated by the following figures:



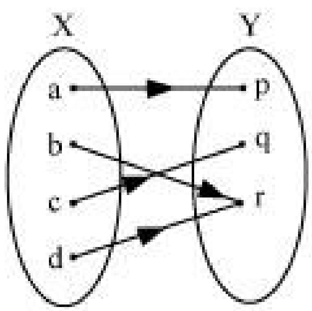
many-one



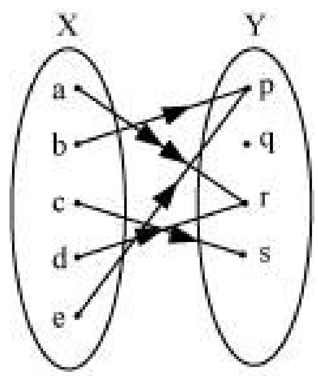
one-one

- A function f: $X \rightarrow Y$ can be defined as an onto (surjective) function, if $\forall\, y \in Y$, there exists $x \in X$ such that $f(x) = y$.

  The onto and many-one (not onto) functions can be illustrated by the following figures:

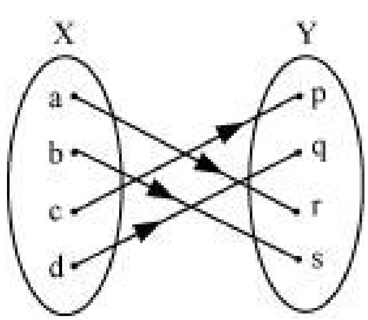


Onto



Not onto

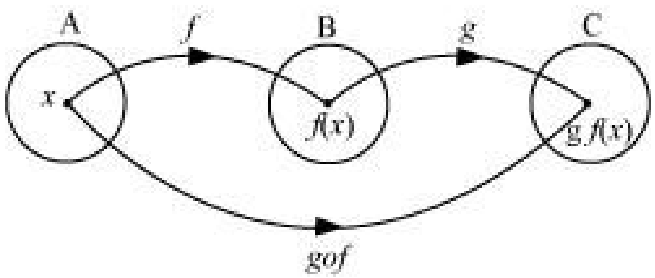
- A function f: $X \rightarrow Y$ is said to be bijective, if it is both one-one and onto. A bijective function can be illustrated by the following figure:



Example: Show that the function f: $\mathbf{R} \rightarrow \mathbf{N}$ given by $f(x) = x^3 - 1$ is bijective.

Solution:
Let $x_1, x_2 \in \mathbf{R}$
For $f(x_1) = f(x_2)$, we have
$x_1^3 - 1 = x_2^3 - 1$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1 = x_2$
Therefore, f is one-one.
Also, for any y in **N**, there exists $\sqrt[3]{y+1}$ in **R** such that
$f\left(\sqrt[3]{y+1}\right) = \left(\sqrt[3]{y+1}\right)^3 - 1 = y$.
Therefore, f is onto.
Since f is both one-one and onto, f is bijective.

- **Composite function:** Let $f: A \to B$ and $g: B \to C$ be two functions. The composition of f and g, i.e. gof, is defined as a function from A to C given by $gof(x) = g(f(x)), \forall x \in A$



Example: Find gof and fog, if $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are given by $f(x) = x^2 - 1$ and $g(x) = x^3 + 1$.

Solution:

$gof(x) = g(f(x))$

$= g\left(x^2 - 1\right)$

$= \left(x^2 - 1\right)^3 + 1$

$= x^6 - 1 - 3x^4 + 3x^2 + 1$

$= x^2\left(x^4 - 3x^2 + 3\right)$

$$fog(x) = f(g(x))$$
$$= f\left(x^3 + 1\right)$$
$$= \left(x^3 + 1\right)^2 - 1$$
$$= x^6 + 2x^3 + 1 - 1$$
$$= x^3\left(x^3 + 2\right)$$

- A function f: $X \to Y$ is said to be invertible, if there exists a function g: $Y \to X$ such that $gof = I_X$ and $fog = I_Y$. In this case, g is called inverse of f and is written as $g = f^{-1}$
- A function f is invertible, if and only if f is bijective.

Example: Show that f: $\mathbf{R}^+ \cup \{0\} \to \mathbf{N}$ defined as $f(x) = x^3 + 1$ is an invertible function. Also, find f^{-1}.

Solution:

Let $x_1, x_2 \in \mathbf{R}^+ \cup \{0\}$ and $f(x_1) = f(x_2)$

$$\therefore x_1^3 + 1 = x_2^3 + 1$$
$$\Rightarrow x_1^3 = x_2^3$$
$$\Rightarrow x_1 = x_2$$

Therefore, f is one-one.

Also, for any y in $\mathbf{N}$, there exists $3\sqrt{y-1} \in \mathbf{R}^+ \cup \{0\}$ such that $f\left(3\sqrt{y-1}\right) = y$.

$\therefore f$ is onto.

Hence, f is bijective.

This shows that, f is invertible.

Let us consider a function g: $\mathbf{N} \to \mathbf{R}^+ \cup \{0\}$ such that $g(y) = 3\sqrt{y-1}$

Now,

$$gof(x) = g(f(x)) = g\left(x^3 + 1\right) = 3\sqrt{\left(x^3 + 1\right) - 1} = x$$
$$fog(y) = f(g(y)) = f\left(3\sqrt{y-1}\right) = \left(3\sqrt{y-1}\right)^3 + 1 = y$$

Therefore, we have

$gof(x) = I_{R^+ \cup \{0\}}$ and $fog(y) = I_{\mathbf{N}}$

$\therefore f^{-1}(y) = g(y) = 3\sqrt{y-1}$

- **Relation:** A relation *R* from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element *x* and the second element *y* of the ordered pairs (*x*, *y*) in A × B.
- The image of an element *x* under a relation *R* is *y*, where (*x*, *y*) ∈ R
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation *R*. Range ⊆Co-domain

Example: In the relation X from **W** to **R**, given by X = {(*x*, *y*): *y* = 2*x* + 1; *x* ∈ W, *y* ∈ R}, we obtain X = {(0, 1), (1, 3), (2, 5), (3, 7) …}. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 …}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 …}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.