Polynomials

OLYMPIAD EXCELLENCE BOOK **MATHEMATICS**

NOTES

FUNDAMENTALS

- **Polynomial:** A function p(x) of the form $p(x) = a_0 + a_1x^n + ... + a_nx^n$, where $a_0, a_1, ...a_n$ an are real numbers and 'n' is a non-negative (positive) integer is called a polynomial.

  Note: $a_0, a_1, ...a_n$ are called the coefficients of the polynomial.
- If the coefficients of a polynomial are integers, then it is called a polynomial over integers.
- If the coefficients of a polynomial are rational numbers, then it is called a polynomial over rational numbers.
- If the coefficients of a polynomial are real numbers, then it is called a polynomial over real numbers.
- A function $p(x) = a_0 + a_1x + ... + a_nx^n$ is not a polynomial if the power of the variable is either negative or a fractional number.
- **Standard form:** A polynomial is said to be in a standard form if it is written either in the ascending or descending powers of the variable, as $1 + x + 2x^2 + 3x^3 - 6x^5 \times 6x^6$
- **Degree of a polynomial:** The highest power of x in p(x) is the degree of the polynomial.

  Example: $2 - 3x^5 + 6x^4 + 92x^3$: Here, highest term being $-3x^5$: degree of polynomial = 5.

Polynomial	**General Form**	**Coefficients**
Zero polynomial	0	–
Linear polynomial	$ax + b$	$a, b \in R, a \neq 0$
Quadratic polynomial	$ax^2 + bx + c$	$a, b, c \in R, a \neq 0$
Cubic polynomial	$ax^3 + bx^2 + cx + d$	$a, b, c, d \in R, a \neq 0$
Bi-Quadratic polynomial	$ax^4 + bx^3 + cx^2 + dx + c$	$a, b, c, d, e \in R, a \neq 0$

- **Value of a polynomial:** If p(x) is a polynomial in x, and if 'a' is any real number, then the value obtained upon replacing 'x' by 'a' in p(x) is denoted as p(a).
- **Zero of a polynomial:** A real number 'a' for which the value of the polynomial p(x) is zero, is called the zero of the polynomial.
- In other words, a real number 'a' is called a zero of a polynomial p(x) if p(a) = 0.

- **Geometric meaning of the zero of a polynomial:**

  (a) The graph of a linear equation of the form $y = ax + b, a \neq 0$ is a straight line which intersects the X-axis at $\left(\frac{-b}{a}, 0\right)$

  Zero of the polynomial $ax + b$ is the x-coordinate of the point of intersection of the graph with X-axis.

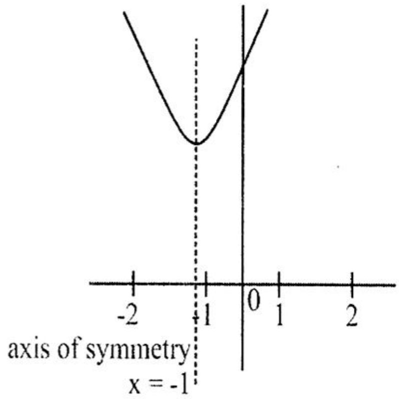
  Note: A linear polynomial $ax + b, a \neq 0$ has exactly one zero, i.e., $\left(\frac{-b}{a}\right)$

(b)  The graph of a quadratic equation $y = ax^2 + bx + c, a \neq 0$ is a curve called parabola that either opens upwards like when the coefficient of x^2 is positive or opens downwards like when the coefficient of x^2 is negative.

The zeros of a quadratic polynomial $ax^2 + bx + c$ are the x-coordinates of the points where the parabola intersects the X-axis.

Example: $p(x) = x^2 + 2x + 4 = 0;$ $b^2 - 4ac = 2^2 - 4.1.4 = -12 < 0$

It has no zeros as the parabola will never intersect X-axis.



Note: For the parabola $ax^2 + bx + c$.

(i) Vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ **where** $D = b^2 - 4ac$

(ii) Axis of symmetry, $x = \frac{-b}{2a}$ **parallel to Y-axis.**

(ii) Zeros are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ **and** $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(c) The graph of a cubic polynomial intersects the X-axis at three points, whose x-coordinates are the zeros of the cubic polynomial.

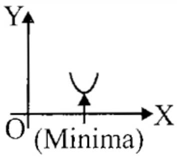
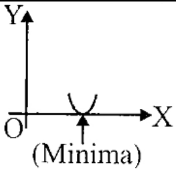
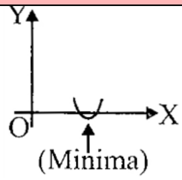
In general, the graph of a polynomial of degree 'n' y = p(x) passes through at most 'n' points on the X-axis. Thus, a polynomial p(x) of degree 'n' has at most 'n' zeros.

- **Relationship between zeros and coefficients of a polynomial.**

Types of Polynomial	General Form	Number of Zeroes	Relationship between zeroes and coefficients	
			Sum of zeroes	Product of zeroes
Linear Polynomial	$ax + b,$ $a \neq 0$	1	*only one zero* $= \frac{-(\text{constant term})}{(\text{coefficient of x})} = \frac{-b}{a}$	
Quadratic Polynomial	$ax^2 + bx + c, a \neq 0$	2	$\frac{-(\text{coefficient of x})}{(\text{coefficient of } x^2)} = \frac{-b}{a}$	$\frac{\text{constant term}}{\text{coefficient of } x^2} = \frac{c}{a}$
Cubic Polynomial	$ax^3 + bx^2 + cx, + d, a \neq 0$	3	$\frac{-(\text{coefficient of x})}{(\text{coefficient of } x^2)} = \frac{-b}{a}$	$\frac{\text{constant term}}{\text{coefficient of } x^2} = \frac{-d}{a}$
			Sum of the product of roots taken two at a time $\frac{\text{coefficient of } x}{\text{coefficient of } x^2} = \frac{c}{a}$	

- To form a quadratic polynomial with the given zeros: If α and β are the zeros of a quadratic polynomial, then the quadratic polynomial is obtained by expanding $(x-\alpha)\,(x-\beta)$. i.e., $(x-\alpha)(x-\beta)=x^2-$ (Sum of the zeros) x + product of zeros.
- To form a cubic polynomial with the given zeros: If α, β and γ are the zeros of a polynomial, then the cubic polynomial is obtained by expanding $(x-\alpha)(x-\beta)(x-\gamma)$.
- Division algorithm of polynomials: If $p(x)$ and $g(x)$ are any two polynomials with $g(x)\neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that $p(x)=g(x)\times q(x)+r(x)$, where either $r(x)=0$ or degree $r(x)<$ degree of $g(x)$.
- If $(x-a)$ is a factor of polynomial $p(x)$ of degree $n>0$, then 'a' is the zero of the polynomial.

Graphical Representation of Different forms of Quadratic Equation

Characteristics of the function	**b^2 - 4ac < 0**	**b^2 - 4ac = 0**	**b^2 - 4ac > 0**
When 'a' is positive i.e. a > 0			
When 'a' is negative i.e. a < 0