Chapter 9

Sequences and Series

Exercise 9.1

Question 1: Write the first five terms of the sequences whose n^{th} term is $a_n = n(n + 2)$.

Answer 1:

$a_n = n(n + 2)$

substituting n = 1, 2, 3, 4, and 5, we obtain

$a_1 = 1(1 + 2) = 3$

$a_2 = 2(2 + 2) = 8$

$a_3 = 3(3 + 2) = 15$

$a_4 = 4(4 + 2) = 24$

$a_5 = 5(5 + 2) = 35$

Therefore, the required terms are 3, 8, 15, 24, 35.

Question 2: Write the first five terms of the sequences whose n^{th} term is $a_n = \dfrac{n}{n+1}$

Answer 2:

$a_n = \dfrac{n}{n+1}$

Substituting n = 1, 2, 3, 4, 5, we obtain

$a_1 = \dfrac{1}{1+1} = \dfrac{1}{2},\ a_2 = \dfrac{2}{2+1} = \dfrac{2}{3},\ a_3 = \dfrac{3}{3+1} = \dfrac{3}{4},\ a_4 = \dfrac{4}{4+1} = \dfrac{4}{5},\ a_5 = \dfrac{5}{5+1} = \dfrac{5}{6}$

Therefore, the required terms are $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}$, and $\dfrac{5}{6}$

Question 3: Write the first five terms of the sequences whose n^{th} term is $a_n = 2^n$

Answer 3:

$a_n = 2^n$

substituting n = 1, 2, 3, 4, 5, we obtain

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 25$

therefore, the required terms are 2, 4, 8, 16, and 32.

Question 4: Write the first five terms of the sequences whose n^{th} term is $\frac{2n-3}{6}$

Answer 4:

Substituting n = 1, 2, 3, 4, 5, we obtain

$a_1 = \frac{2 \times 1 - 3}{6} = \frac{-1}{6}$

$a_2 = \frac{2 \times 2 - 3}{6} = \frac{1}{6}$

$a_3 = \frac{2 \times 3 - 3}{6} = \frac{1}{2}$

$a_4 = \frac{2 \times 4 - 3}{6} = \frac{5}{6}$

$a_5 = \frac{2 \times 5 - 3}{6} = \frac{7}{6}$

Therefore, the required terms are , $\frac{-1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$

Question 5: Write the first five terms of the sequences whose n^{th} term is $a_n = (-1)^{n-1}5^{n+1}$

Answer 5:

Substituting n = 1, 2, 3, 4, 5, we obtain

$a_1 = (-1)^{1-1}\,5^{1+1} = 5^2 = 25$

$a_2 = (-1)^{2-1}\,5^{2+1} = -5^3 = -125$

$a_3 = (-1)^{3-1}\,5^{3+1} = 5^4 = 625$

$a_4 = (-1)^{4-1}\,5^{4+1} = -5^5 = -3125$

$a_5 = (-1)^{5-1}\,5^{5+1} = 5^6 = 15625$

therefore, the required terms are 25, -125, 625, -3125, and 15625.

Question 6: Write the first five terms of the sequences whose n^{th} term is $a_n = n\dfrac{n^2+5}{4}$

Answer 6:

Substituting n = 1, 2, 3, 4, 5, we obtain

$a_1 = 1.\dfrac{1^2+5}{4} = \dfrac{6}{4} = \dfrac{3}{2}$

$a_2 = 2.\dfrac{2^4+5}{4} = 2.\dfrac{9}{4} = \dfrac{9}{2}$

$a_3 = 3.\dfrac{3^2+5}{4} = 2.\dfrac{14}{4} = \dfrac{21}{2}$

$a_4 = 4.\dfrac{4^2+5}{4} = 21$

$a_5 = 5.\dfrac{5^2+5}{4} = 5.\dfrac{30}{4} = \dfrac{75}{2}$

therefore, the required terms are $\dfrac{3}{2}, \dfrac{9}{2}, \dfrac{21}{2}, 21$, and $\dfrac{75}{2}$.

Question 7: Find the 17th term in the following sequence whose nth term is $a_n = 4n - 3$; a_{17}, a_{24}

Answer 7:

Substituting n = 17, we obtain

$a_{17} = 4\,(17) - 3 = 68 - 3 = 65$

Substituting n = 24, we obtain

$a_{24} = 4\,(24) - 3 = 96 - 3 = 93$

Question 8: Find the 7th term in the following sequence whose nth term is $a_n = \dfrac{n^2}{2n}$; a_7

Answer 8:

Substituting n = 7, we obtain

$a_7 = \dfrac{7^2}{2^7} = \dfrac{49}{128}$

Question 9: Find the 9th term in the following sequence whose nth term is $a_n = (-1)^{n-1}n^3$; a_9

Answer 9:

Substituting n = 9, we obtain

$a_9 = (-1)^{9-1}\,(9)^3 = (9)^3 = 729$

Question 10: Find the 20th term in the following sequence whose nth term is

$a_n = \dfrac{n(n-2)}{n+3}$; a_{20}

Answer 10:

Substituting n = 20, we obtain

$$a_{20} = \frac{20(20-2)}{20+3} = \frac{20(18)}{23} = \frac{360}{23}$$

Question 11: Write the first five terms of the following sequence and obtain the corresponding series:

$a_n = 3a_{n-1} + 2$ for all $n > 1$

Answer 11:

$a_1 = 3$, $a_n = 3a_{n-1} + 2$ for all $n > 1$

$= a_2 = 3a_1 + 2 = 3\,(3) + 2 = 11$

$= a_3 = 3a_2 + 2 = 3\,(11) + 2 = 35$

$= a_4 = 3a_3 + 2 = 3\,(35) + 2 = 107$

$= a_5 = 3a_4 + 2 = 3\,(107) + 2 = 323$

Hence, the first five terms of the sequence are $3, 11, 35, 107, 323$.

The corresponding series is $3 + 11 + 35 + 107 + 323 + \ldots$

Question 12: Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = -1$, $a_n = \frac{a_{n-1}}{n}$, $n \geq 2$

Answer 12:

$a_1 = -1$, $a_n = \frac{a_{n-1}}{n}$, $n \geq 2$

$= a_2 = \frac{a_1}{2}, = \frac{-1}{2}$

$= a_3 = \frac{a_2}{3} = \frac{-1}{6}$

$= a_4 = \frac{a_3}{4} = \frac{-1}{24}$

$= a_5 = \frac{a_4}{4} = \frac{-1}{120}$

Hence, the first terms of the sequence are $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}$ and $\frac{-1}{120}$

The corresponding series is $(1) + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right)$

Question 13: Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$, $n > 2$

answer:

$a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$, $n > 2$

$= a_3 = a_2 - 1 = 2 - 1 = 1$

$= a_4 = a_3 - 1 = 1 - 1 = 0$

$= a_5 = a_4 - 1 = 0 - 1 = -1$

Hence, the first five terms of the sequence are $2, 2, 1, 0,$ and -1.

The corresponding series is $2 + 2 + 1 + 0 + (-1) + \ldots$

Question 14: The Fibonacci sequence is defined by $1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$ find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

answer:

$1 = a_1 = a_2$

$a_n = a_{n-1} + a_{n-2}$, $n > 2$

$= a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

Foe n $= 1, = \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = \frac{1}{1} = 1$

For n = 2, $\frac{a_{n+1}}{a_n} = \frac{a_3}{a_2} = \frac{2}{1} = 2$

For n = 3, $\frac{a_{n+1}}{a_n} = \frac{a_4}{a_3} = \frac{3}{2}$

For n = 4, $\frac{a_{n+1}}{a_n} = \frac{a_5}{a_4} = \frac{5}{3}$

For n = 5, $\frac{a_{n+1}}{a_n} = \frac{a_6}{a_5} = \frac{8}{5}$

Exercise 9.2

Question 1: Find the sum of odd integers from 1 to 2001.

Answer 1:

The odd integers from 1 to 2001 are 1, 3, 5 … 1999, 2001.

This, sequence forms an A.P.

Here, first tern, a = 1

Common difference, d = 2

Here, a + (n – 1) d = 2001

= 1 + (n – 1) (2) = 2001

= 2n – 2 = 2000

= n = 1001

$S_n = \frac{n}{2}$ [2a + (n – 1)d]

$S_n = \frac{1001}{2}$ [2 × 1 + (1001 – 1) × 2]

$= \frac{1001}{2}$ [2 + 1000 × 2]

$= \frac{1001}{2}$ × 2002

= 1001 × 1001

= 1002001

Thus, the sum of odd numbers from 1 to 2001 is 1002001.

Question 2: Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

Answer 2:

The natural numbers lying between 100 and 1000, which are multiples of 5, are 105, 110, …995.

Here, a = 105 and d = 5

a + (n – 1) d = 995

= 105 + (n – 1) 5 = 995

= (n – 1) 5 = 995 – 105 = 890

= n – 1 = 178

= n = 179

$S_n = \dfrac{179}{2} [2 (105) + (179 – 1) (5)]$

$= \dfrac{179}{2} [2 (105) + (178) (5)]$

= (179) (105 + 445)

= (179) (550)

= 98450

Thus, the sum of all natural numbers lying between 100 and 1000, which are multiples of 5, is 98450.

Question 3: In an A.P, the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20th term is –112.

Answer 3:

First term = 2

Let d be the common difference of the A.P.

Therefore, the A.P. is 2, 2 +d, 2 +2d, 2 + 3d

Sum of the first five terms = 10 + 10d

Sum of the next five terms = 10 + 35d

According to given condition

$10 + 10d = \frac{1}{4}(10 + 5d)$

$= 40 + 40d = 10 + 5d$

$= 30 = -5d$

$= d = -6$

$a_{20} = a + (20 - 1)d = 2 + (19)(-6) = 2 - 114 = -112$

thus, the 20th term of the A.P. is - 112

Question 4: How many terms of the A.P. $-6, -\frac{11}{2}, -5$ ... are needed to give the sum -25?

Answer 4:

Let, the sum of n terms of the given A.P. be -25

It is known that

$S_n = \frac{n}{2}[2a + (n-1)d]$

Where n = number of terms, a = first term, and d = common difference

Here, a = - 6

$d = \frac{11}{2} + 6 = \frac{-11+12}{2} = \frac{1}{2}$

therefore, we obtain

$- \quad 25 = \frac{n}{2}[2 \times (-6) + (n-1)\frac{1}{2}]$

$= -50 = n\left[-12 + \frac{n}{2} - \frac{1}{2}\right]$

$= -50 = n\left[-\frac{25}{2} + \frac{n}{2}\right]$

$= -100 = n(-25 + n)$

$= n^2 - 5n - 20n + 100 = 0$

$= n(n-5) - 20(n-5) = 0$

$n = 20$ or 5

Question 5: In an A.P., if *pth* term is 1/q and *qth* term is 1/p, prove that the sum of first p q terms is 1⁄2 (p q +1), where $p \neq q$.

Answer 5:

It is known that the general term of an A.P. is an $= a + (n-1)d$

According to the given information

Pth term $= a_p = a + (p-1)d = \dfrac{1}{q}$ ... (1)

qth term $= a_q = a + (q-1)d = \dfrac{1}{p}$ ... (2)

subtracting (2) from (1), we obtain

$(p-1)d - (q-1)d = \dfrac{1}{q} - \dfrac{1}{p}$

$= (p - 1 - q + 1)d = \dfrac{p-q}{pq}$

$= (p - q)d = \dfrac{p-q}{pq}$

$= d = \dfrac{1}{pq}$

Putting the value of d in (1), we obtain

$a + (p-1)\dfrac{1}{pq} = \dfrac{1}{q}$

$= a = \dfrac{1}{q} - \dfrac{1}{q} + \dfrac{1}{pq} = \dfrac{1}{pq}$

$S_n = \dfrac{pq}{2}[2a + (p\,q - 1)d]$

$= \dfrac{pq}{2}[\dfrac{2}{pq} + (p\,q - 1)\dfrac{1}{pq}]$

$= 1 + \frac{1}{2} (p\,q - 1)$

$= \frac{1}{2} p\,q + 1 - \frac{1}{2} = \frac{1}{2} p\,q + \frac{1}{2}$

$= \frac{1}{2} (p\,q + 1)$

Question 6: If the sum of a certain number of terms of the A.P. 25, 22, 19, ... is 116.

Find the last term

Answer 6:

Let the sum of n terms of the given A.P. be 116.

$S_n = \frac{n}{2} [2a + (n - 1)d]$

Here, a = 25 and d – 22 – 25 = - 3

$S_n = \frac{n}{2} [2 \times 25 + (n - 1)\,(\text{-}3)]$

$= 116 = \frac{n}{2} [50 - 3n + 3]$

$= 232 = n\,(53 - 3n) = 53n - 3n^2$

$= 3n^2 - 53n + 232 = 0$

$= 3n^2 - 24n - 29n + 232 = 0$

$= 3n\,(n - 8) - 29\,(n - 8) = 0$

$= (n - 8)\,(3n - 29) = 0$

$= n = 8,\ \text{or}\ n = \frac{29}{3}$

However,

n cannot be equal to $\frac{29}{3}$

therefore, n = 8

a_8 = last term = a + (n – 1) d = 25 + (8 – 1) (-3)

= 25 + (7) (-3) = 25 – 21

= 4

Thus, the last term of the A.P. is 4.

Question 7: Find the sum to n terms of the A.P., whose k^{th} term is 5k + 1.

Answer 7:

It is given that the kth term of the A.P. is 5k + 1.

K^{th} term = ak = a + (k – 1) d

= a + (k – 1) d = 5k + 1a + k d – d = 5k + 1

Comparing the coefficient of k, we obtain d = 5a – d = 1

= a – 5 = 1

= a = 6

$S_n = \frac{n}{2}$ [2a + (n – 1) d]

$= \frac{n}{2}$ [2 (6) + (n -1) (5)]

$= \frac{n}{2}$ [12 +5n – 5]

$= \frac{n}{2}$ [5n + 7]

Question 8: If the sum of n terms of an A.P. is (pn + $qn2$), where p and q are constants, find the common difference.

Answer 8:

It is known that:

$S_n = \frac{n}{2}$ [2a + (n – 1)d]

According to the given condition,

$$\frac{n}{2}[2a + (n-1)d] = pn + qn^2$$

$$= \frac{n}{2}[2a + nd - d] = pn + qn^2$$

$$= na + n^2\frac{d}{2} - n\frac{d}{2} = pn + qn^2$$

Comparing the coefficient if n^2 on both sides, we obtain

$$\frac{d}{2} = q$$

$$= d = 2q$$

Thus, the common difference of the A.P. is 2q.

Question 9: The sums of n terms of two arithmetic progressions are in the ratio 5n + 4: 9n + 6. Find the ratio of their 18th terms.

Answer 9:

Let, a_1, a_2, and d_1 d_2 be the first - terms and the common difference of the first and second arithmetic progression respectively.

According to the given condition,

$$\frac{sum\ of\ n\ terms\ of\ first\ A.P.}{sum\ of\ n\ terms\ of\ second\ A.P.} = \frac{5n+4}{9n+6}$$

$$= \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{5n+4}{9n+6}$$

$$= \frac{2a_1(n-1)d_1}{2a_2 + (n-1)d_2} = \frac{5n+4}{9n+6} \quad \cdots (1)$$

Substituting n = 35 in (1), we obtain

$$\frac{2a_1 + 34d_1}{2a_2 + 34d_2} = \frac{5(35)+4}{9(35)+5}$$

$$= \frac{a_1 + 17d_1}{a_2 + 17d_2} = \frac{179}{321} \quad \cdots (2)$$

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$$\frac{18^{th}\,term\,of\,first\,A.P.}{18^{th}\,term\,of\,second\,A.P.} = \frac{a_1 + 17d_1}{a_2 + 17d_2} \quad \dots (3)$$

$$\frac{18_{th}\,term\,of\,first\,A.P.}{18_{th}\,term\,of\,second\,A.P.} = \frac{179}{321}$$

Thus, the ration of 18th term of both the A.P. is 179: 321.

Question 10: If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first (p + q) terms.

Answer 10:

Let, a and d be the first term and the common difference of the A.P. respectively.

Here,

$Sp = \frac{p}{2}$ [2a + (p − 1) d]

$Sq = \frac{q}{2}$ [2a + (q − 1)d]

According to the given condition,

$\frac{p}{2}$ [2a + (p − 1)d] = $\frac{q}{2}$ [2a + (q − 1)d]

= p [2a + (p − 1) d] = q [2a + (q − 1) d]

= 2ap + pd (p − 1) = 2aq + qd (q − 1)

= 2a (p − q) + d [p (p − 1) − q (q − 1)] = 0

= 2a (p − q) + d [p₂ − p − q² + q] = 0

= 2a (p − q) + d [(p − q) (p + q) − (p − q)] = 0

= 2a (p − q) + d [(p −q) (p + q − 1)] = 0

= 2a + d (p + q − 1) = 0

= d = $\frac{-2a}{p+q-1}$ … (1)

$S_{p+q} = \frac{p+q}{2} [2a + (p + q - 1).d]$

$= S_{p+q} = \frac{p+q}{2} \left(2a + (p + q - 1) \left(\frac{-2a}{p+q-1}\right)\right)$ [from (1)]

$= \frac{p+q}{2} [2a - 2a]$

$= 0$

Thus, the sum of first $(p + q)$ terms pf the A.P. is 0.

Question 11: Sum of the first p, q and r terms of an A.P. are a, b and c, respectively.

Prove that $\frac{a}{p} (q - r) + \frac{b}{q} (r - p) + \frac{c}{r} (p - q) = 0$

Answer 11:

Let, a1 and d be the first term and the common difference of the A.P. respectively.

According to the given information,

$S_p = \frac{p}{2} [2a_1 + (p - 1) d] = a$

$= 2a_1 + (p - 1) d = \frac{2a}{p}$ ... (1)

$S_q = \frac{q}{2} [2a_1 + (q - 1) d = b$

$= 2a_1 + (q - 1) d = \frac{2b}{q}$ ... (2)

$S_r = \frac{r}{2} [2a_1 + (r - 1) d] = c$

$= 2a_1 + (r - 1) d = \frac{2c}{r}$ ... (3)

Subtracting (2) from (1), we obtain

$(p - 1) d - (q - 1) d = \frac{2a}{p} - \frac{2b}{q}$

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$= d (p - 1 - q + 1) = \frac{2aq - 2bp}{pq}$

$= d (p - q) = \frac{2aq - 2bp}{pq}$

$= d = \frac{2(aq - bp)}{pq(p-q)} \quad \ldots (4)$

Subtracting (3) from (2), we obtain

$(q - 1) d - (r - 1) d = \frac{2b}{q} - \frac{2c}{r}$

$= d (q - 1 - r + 1) = \frac{2b}{q} - \frac{2c}{r}$

$= d (q - r) = \frac{2br - 2cq}{qr}$

$= d = \frac{2(br - qc)}{qr(q-r)} \quad \ldots (5)$

Equating both the value of d obtained in (4) and (5), we obtain

$\frac{aq - bp}{pq(p-q)} = \frac{br - qc}{qr(q-r)}$

$= qr \, (q - r) \, (aq - bp) \; = \; pq \, (p - a) \, (br - qc)$

$\qquad = \; r \, (aq - bp) \, (q - r) \; = \; p \, (br - qc) \, (p - a$

$\qquad = \; (aqr - bpr) \, (q - r) \; = \; (bpr - pqc) \, (p - q)$

Dividing both sides by pqr, we obtain

$\left(\frac{a}{p} - \frac{b}{q}\right)(q - r) = \left(\frac{b}{q} - \frac{c}{r}\right) (p - q)$

$\left(\frac{a}{p} - \frac{b}{q}\right)(q - r) = \left(\frac{b}{q} - \frac{c}{r}\right)(p - q)$

$= \frac{a}{p}(q - r) - \frac{b}{q}(q - r + p + q) + \frac{c}{r}(p - q) = 0$

$= \frac{a}{p}(q - r) + \frac{b}{q}(r - p) + \frac{c}{r}(p - q) = 0$

Thus, the given result is proved.

Question 12: The ratio of the sums of m and n terms of an A.P. is m²: n². Show that the ratio of mth and nth term is (2m − 1): (2n − 1).

Answer 12:

Let, a and d be the first term and the common difference of the A.P. respectively.

According to the given information,

$$\frac{sum\ of\ m\ terms}{sum\ of\ n\ terms} = \frac{m^2}{n^2}$$

$$= \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2}$$

$$= \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n} \dots (1)$$

Putting m = 2m − 1 and n = 2n − 1 in (1), we obtain

$$\frac{2a+(2m-2)d}{2a+(2n-2)d} = \frac{2m-1}{2n-1}$$

$$= \frac{a+(m-1)d}{a+(n-1)d} = \frac{2m-1}{2n-1} \dots (2)$$

$$\frac{m^{th}\ term\ of\ A.P.}{n_{th}\ term\ of\ A.P.} = \frac{a+(m-1)d}{a+(n-1)d} \dots (2)$$

From (2) and (3), we obtain

$$\frac{m^{th}\ term\ of\ A.P.}{n^{th}\ term\ of\ A.P.} = \frac{2m-1}{2n-1}$$

Thus, the given result is proved.

Question 13: If the sum of n terms of an A.P. is 2n² + 5n and its mth term is 164, find the value of m.

Answer 13:

Let, a and d be the first term and the common difference of the A.P. respectively.

$a_m = a + (m - 1) d = 164 \dots (1)$

sum of n terms:

$s_n = \dfrac{n}{2} [2a + (n - 1)d]$

here,

$= \dfrac{n}{2} [2a + nd - d] = 3n^2 + 5n$

$= na + n^2 \cdot \dfrac{d}{2} = 3n^2 + 5n$

Comparing the coefficient of n^2 on both sides, we obtain

$\dfrac{d}{2} = 3$

$= d = 6$

Comparing the coefficient of n on both sides, we obtain

$= a - \dfrac{d}{2} = 5$

$= a - 3 = 5$

$= a = 8$

Therefore, from (1), we obtain

$8 + (m - 1) 6 = 164$

$= (m - 1) 6 = 164 - 8 = 156$

$= m - 1 = 26$

$= m = 27$

Thus, the value of m is 27.

Question 14: Insert five numbers between 8 and 26 such that the resulting sequence is an A.P.

Answer 14:

Let, A_1, A_2, A_3, A_4, and A_5 be five numbers between 8 and 26 such that 8, A_1, A_2, A_3, A_4, A_5, 26 is an A.P.

Here, a = 8 b = 26, n = 7

Therefore, $26 = 8 + (7 - 1)d$

$= 6d = 26 - 8 = 18$

$= d = 3$

$A_1 = a + d = 8 + 3 = 11$

$A_2 = a + 2d = 8 + 2 \times 3 = 8 + 6 = 14$

$A_3 = a + 3d = 8 + 3 \times 3 = 8 + 9 = 17$

$A_4 = a + 4d = 8 + 4 \times 3 = 8 + 12 = 20$

$A_5 = a + 5d = 8 + 5 \times 3 = 8 + 15 = 23$

Thus, the required five numbers between 8 and 26 are 11, 14, 17, 20, and 23.

Question 15: If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b, then find the value of n.

Answer 15:

A.M. of a and b $= \frac{a+b}{2}$

According to the given condition,

$\frac{a+b}{2} = \frac{a^n + b^n}{a^{n-1} + b^{n-1}}$

$= (a + b)(a^{n-1} + b^{n-1}) = 2(a^n + b^n)$

$= a^n + a^{n-1} + ba^{n-1} + b^n = 2a^n + 2b^n$

$= ab^{n-1} + a^{n-1} b = a^n + b^n$

$= ab^{n-1} - b^n = a^n - a^{n-1} b$

$= b^{n-1} (a - b) = a^{n-1} (a - b)$

$= b^{n-1} = a^{n-1}$

$= \left(\dfrac{a}{b}\right)^{n-1} = 1 = \left(\dfrac{a}{b}\right)^{0}$

$= n - 1 = 0$

$= n = 1$

Question 16: Between 1 and 31, m numbers have been inserted in such a way that the

resulting sequence is an A.P. and the ratio of 7th and $(m - 1)^{th}$ numbers is 5:9. Find the value of m.

Answer 16:

Let $A_1, A_2 \ldots A_m$ be m numbers such that 1, A_1, A_2, …A_m, 31 is an A.P.

Here, a = 1, b = 31, n = m + 2

$31 = 1 + (m + 2 - 1) \, d$
$= 30 = (m + 1) \, d$
$= d = \dfrac{30}{m+1}$ … (1)
$A_1 = a + d$
$A_2 = a + 2d$
$A_3 = a + 3d$
$\therefore A_7 = a + 7d$
$A_{m-1} = a + (m - 1) \, d$
According to the given condition,

$\dfrac{a+7d}{a+(m-1)d} = \dfrac{5}{9}$

$$= \frac{1+7\left(\frac{30}{m+1}\right)}{1+(m-1)\left(\frac{30}{m-1}\right)} = \frac{5}{9} \text{ [from (1)]}$$

$$= \frac{m+1+7(30)}{m+1+30(m-1)} = \frac{5}{9}$$

$$= \frac{m+1+210}{m+1+30m-30} = \frac{5}{9}$$

$$= \frac{m+211}{31m-29} = \frac{5}{9}$$

$= 9m + 1899 = 155m - 145$

$= 155m - 9m = 1899 + 145$

$= 146m = 2044$

$= m = 14$

Thus, the value of m is 14.

Question 17: A man starts repaying a loan as first installment of ₹100. If he increases the installment by ₹5 every month, what amount he will pay in the 30th installment?

Answer 17:

The first installment of the loan is ₹100.

The second installment of the loan is ₹105 and so on.

The amount that the man repays every month forms an A.P.

The A.P. is 100, 105, 110 …

First term, a = 100

Common difference, d = 5

$A_{30} = a + (30 - 1) d$

$= 100 + (29) (5)$

$= 100 + 145$

= 245

Thus, the amount to be paid in the 30th installment is ₹245.

Question 18: The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

Answer 18:

The angles of the polygon will form an A.P. with common difference d as 5° and first term an as 120.

It is known that the sum of all angles of the polygon with n sides 180° (n – 2)

$S_n = 180° (n – 2)$

$= \frac{n}{2} [2a + (n -1)d] = 180° (n – 2)$

$= \frac{n}{2} [240° + (n – 1) 5°] = 180° (n – 2)$

$= n [240 + (n – 1) 5] = 360 (n – 2)$

$= 240n + 5n^2 – 5n = 360n – 720$

$= 5n^2 + 235n – 360n + 720 = 0$

$= 5n^2 – 125n + 720 = 0$

$= n^2 – 25n + 144 = 0$

$= n^2 – 16n – 9n + 144 = 0$

$= n (n – 16) – 9 (n – 16) = 0$

$= (n – 9) (n – 16) = 0$

$= n = 9$ or 16

Exercise 9.3

Question 1: Find the 20th and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$

Answer 1:

The given G.P. is $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$

Here, a = first term = $\frac{5}{2}$

r = common ratio = $\frac{\frac{5}{4}}{\frac{5}{2}} = \frac{1}{2}$

$a_{20} = ar^{20-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{2(2)^{19}} = \frac{5}{(2)^{20}}$

$a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{2(2)^{n-1}} = \frac{5}{(2)^n}$

Question 2: Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

Answer 2:

Common ratio, r = 0

Let a be the first term of the G.P.

$a_8 = ar^{8-1} = ar^7 = 192$ a $(2)^7 = (2)^6(3)$

$= a = \frac{(2)^6 \times 3}{(2)^7} = \frac{3}{2}$

$a_{12} = ar^{12-1} = \left(\frac{3}{2}\right)(2)^{11} = (3)(2)^{10} = 3072$

Question 3: The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.

Answer 3:

Let a be the first term and r be the common ratio of the G.P. according to the given condition,

$a_5 = a\ r^{5-1} = a\ r^4 = p$ ... (1)

$a_8 = a\ r^{8-1} = ar^7 = q$ ... (2)

$a_{11} = ar^{11-1} = ar^{10} = s$ ... (3)

dividing eq. (2) by (1), we obtain

$$\frac{ar^7}{ar^4} = \frac{q}{p}$$

$r^3 = \frac{q}{p}$ ... (4)

dividing eq. (3) by (2), we obtain

$$\frac{ar^{10}}{ar^7} = \frac{s}{q}$$ ... (5)

Equation the value of r^3 obtained in (4) and (5), we obtain

$$\frac{q}{p} = \frac{s}{q}$$

$= q^2 = ps$

Thus, the given result is proved.

Question 4: The 4th term of a G.P. is square of its second term, and the first term is –3. Determine its 7th term.

Answer 4:

Let, a be the first term and r be the common ratio of the G.P.

a = - 3

It is known that, an = arn-1

$a_4 = ar^3 = (-3)\ r^3$

$a_2 = a\ r^1 = (-3)\ r$

According to the given condition, $(-3) \, r^3 = [(-3) \, r]^2$

$= - \, 3r^3 = 9 \, r^2 = r = -3 \quad a^7 = a \, r^{7-1} = a$

$r^6 = (-3) \, (-3)^6 = (-3)^7 = -2187$

thus, the 7th term of the G.P. is -2187.

Question 5: Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4, \ldots$ is 128? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729?

(c) $\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \ldots \, i \, s \dfrac{1}{19683}.?$

Answer 5:

(a) the given sequence is $2, 2\sqrt{2}, 4, \ldots$ is 128?

Here, $a = 2$ and $r = (2\sqrt{2})/2 = \sqrt{2}$

Let, the nth term of the given sequence be 128.

$a_n = a \, r^{n-1}$

$= (2) \left(\sqrt{2}\right)^{n-1} = 128$

$= (2) \, (2)^{\frac{n-1}{2}} = (2)^7$

$= (2)^{\frac{n-1}{2}+1} = (2)^7$

$= \dfrac{n-1}{2} + 1 = 7$

$= \dfrac{n-1}{2} = 6$

$= n - 1 = 12$

$= n = 13$

Thus, the 13th term of the given sequence is 128.

(b) the given sequence is $\sqrt{3}, 3, 3\sqrt{3}, \ldots$

$a = \sqrt{3}$ and $r = \frac{3}{\sqrt{3}} = \sqrt{3}$

Let, the nth term of the given sequence be 729.

$a_n = a\ r^{n-1}$

$= a\ r^{n-1} = 729$

$= (\sqrt{3})\left(\sqrt{3}\right)^{n-1} = 729$

$= (3)^{\frac{1}{2}}(3)^{\frac{n-1}{2}} = (3)6$

$= (3)^{\frac{1}{2}+\frac{n-1}{2}} = (3)6$

$= \frac{1}{2} + \frac{n-1}{2} = 6$

$= \frac{1+n-1}{2} = 6$

$= n = 12$

Thus, the 12$^{\text{th}}$ term of the given sequence is 729.

(c) the given sequence is $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

Here, $a = \frac{1}{3}$ and $r = \frac{1}{9} \div \frac{1}{3} = \frac{1}{3}$

Let, the nth term of the given sequence be $\frac{1}{19683}$.

$a_n = ar^{n-1}$

$= a\ r^{n-1} = \frac{1}{19683}$

$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683}$

$= \left(\frac{1}{3}\right)^{n} = \left(\frac{1}{3}\right)^{9}$

$= n = 9$

Thus, the 9th term of the given sequence is $\frac{1}{19683}$.

Question 6: For what values of x, the numbers $\frac{-2}{7}$, x, $\frac{-7}{2}$ are in G.P?

Answer 6:

The given numbers are $\frac{-2}{7}$, x, $\frac{-7}{2}$

Common ratio $= \frac{x}{\frac{-2}{7}} = \frac{-7x}{2}$

Also, common ratio $= \frac{\frac{-7}{2}}{x} = \frac{-7}{2x}$

$= \frac{-7x}{2} = \frac{-7}{2x}$

$= x^2 = \frac{-2\times7}{-2\times7} = 1$

$= x = \sqrt{1}$

$= x = \pm 1$

Thus, for x $= \pm 1$, the given numbers will be in G.P.

Question 7: Find the sum to 20 terms in the geometric progression 0.15, 0.015,

0.0015 ...

Answer 7:

The given G.P. is 0.15, 0.015, 0.00015, ...

Here a = 0.15 and r $= \frac{0.015}{0.15} = 0.1$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{20} = \frac{0.15[1-(0.1)^{20}]}{1-0.1}$

$= \dfrac{0.15}{0.9}[1-(0.1)20]$

$= \dfrac{15}{90}[1-(0.1)20]$

$= \dfrac{1}{6}[1-(0.1)20]$

Question 8: Find the sum to n terms in the geometric progression $\sqrt{7}$, $\sqrt{21}$, $3\sqrt{7}$, ...

Answer 8:

The given G.P. is $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \dots$

Here, a $= \sqrt{7}$ and r $= \dfrac{\sqrt{21}}{7} = \sqrt{3}$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_n = \dfrac{\sqrt{7}\left[1-(\sqrt{3})^n\right]}{1-\sqrt{3}}$

$S_n = \dfrac{\sqrt{7}\left[1-(\sqrt{3})^n\right]}{1-\sqrt{3}} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$

$S_n = \dfrac{\sqrt{7}(\sqrt{3}+1)\left[1-(\sqrt{3})^n\right]}{1-3}$

$S_n = \dfrac{-\sqrt{7}(\sqrt{3}+1)\left[1-(\sqrt{3})^n\right]}{2}$

Question 9: Find the sum to n terms in the geometric progression 1, -a, a^2, $-a^3$ ... (if a $\neq$ - 1)

Answer 9:

The given G.P. is 1, -a, a^2, $-a^3$, ….

Here, first term $= a_1 = 1$

Common ratio $= r = - a$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_n = \frac{1[1-(-a)^n]}{1-(-a)} = \frac{[1-(-a)^n]}{1+a}$$

Question 10: Find the sum to n terms in the geometric progression x^3, x^5, x^7, ... (if $x \neq \pm 1$)

Answer 10:

The given G.P. is x^3, x^5, x^7, ...

Here, $a = x^3$ and $r = x^2$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{x^3\left[1-(x^2)^n\right]}{1-x^2} = \frac{x^3(1-x^{2n})}{1-x^2}$$

Question 11: Evaluate $\sum_{k=1}^{11}(2 + 3^k)$

Answer 11:

$\sum_{k=1}^{11}(2 + 3^k) = \sum_{k=1}^{11}(2) + \sum_{k=1}^{11} 3^k = 2\,(11) + \sum_{k=1}^{11} 3^k = 22 + \sum_{k=1}^{11} 3^k$ ... (1)

$\sum_{k=1}^{11} 3^k = 3^1 + 3^2 + 3^3 + ... + 3^{11}$

The terms of the sequence $3, 3^2, 3^3$ .... Forms a G.P.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{3\left[(3)^{11} - 1\right]}{3 - 1}$$

$$S_n = \frac{3}{2}(3^{11} - 1)$$

$$= \sum_{k=1}^{11} 3^k = \frac{3}{2}(3^{11} - 1)$$

Substituting this value in eq. (1), we obtain

$$\sum_{k=1}^{11}(2 + 3^k) = 22 + \frac{3}{2}(3^{11} - 1)$$

Question 12: The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find

the common ratio and the terms.

Answer 12:

Let, $\frac{a}{r}$, a, ar be the first three terms of the G.P.

$\frac{a}{r} + a + ar = \frac{39}{10}$ ... (1)

$\left(\frac{a}{r}\right)(a)(ar) = 1$ ... (2)

From (2), we obtain a3 = 1

= a = 1 (considering real roots only)

Substituting a = 1 in eq. (1), we obtain

$\frac{1}{r} + 1 + r = \frac{39}{10}$

$= 1 + r + r^2 = \frac{39}{10}r$

$= 10 + 10r + 10r^2 - 39r = 0$

$= 10r^2 - 29r + 10 = 0$

$= 10r^2 - 25r - 4r + 10 = 0$

$= 5r(2r - 5) - 2(2r - 5) = 0$

$= (2r - 5)(5r - 2) = 0$

$= r = \frac{2}{5}$ or $\frac{5}{2}$

Thus, the three terms if G.P. are $\frac{5}{2}$, 1 and $\frac{2}{5}$

Question 13: How many terms of G.P. 3, 3^2, 3^3 ... are needed to give the sum 120?

Answer 13:

The given G.P. is $3, 3^2, 3^3 \dots$

Let n terms of this G.P. be required to obtain the sum as 120.

$S_n = \dfrac{a(1-r^n)}{1-r}$

Here, $a = 3$ and $r = 3$

$S_n = 120 = \dfrac{3(3^n-1)}{3-1}$

$= 120 = \dfrac{3(3^n-1)}{2}$

$= \dfrac{120 \times 2}{3} = 3n - 1$

$= 3n - 1 = 80$

$= 3n = 81$

$= 3n = 3^4$

$= n = 4$

Thus, four terms of the given G.P. are required to obtain the sum as 120.

Question 14: The sum of first three terms of a G.P. is 16 and the sum of the next three

terms are 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

Answer 14:

Let, the G.P. be a, a r, ar^2, ar^3, … According to the given condition,

$a + a\,r + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$= a\,(1 + r + r^2) = 16 \dots (1)$

$= ar^3\,(1 + r + r^2) = 128 \dots (2)$

Dividing eq. (2) by (1), we obtain

$$\frac{ar^3(1+r+r^2)}{a(1+r+r^2)} = \frac{128}{16}$$

$$= r^3 = 8$$

$$= r = 2$$

Substituting $r = 2$ in (1), we obtain a $(1 + 2 + 4) = 16$

$$= a\ (7) = 16$$

$$= a = \frac{16}{7}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{16}{7}\frac{(2^n - 1)}{2 - 1} = \frac{16}{7}(2^n - 1)$$

Question 15: Given a G.P. with a = 729 and 7_{th} term 64, determine S7.

Answer 15:

A = 729 a^7 = 64

Let, r be the common ratio of the G.P. it is known that,

$$= a_n = a\ rn^{-1}$$

$$= a_7 = ar^{7-1} = (729)\ r^6$$

$$= 64 = 729\ r^6$$

$$= r^6 = \frac{64}{729}$$

$$= r^6 = \left(\frac{2}{3}\right)^6$$

$$= r = \frac{2}{3}$$

Also, it is known that,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_7 = \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-\frac{2}{3}}$$

$$= 3 \times 729 \left[1 - \left(\frac{2}{3}\right)^7\right]$$

$$= (3)^7 \left[\frac{(3)^7-(2)^7}{(3)^7}\right]$$

$$= (3)^7 - (2)^7$$

$$= 2187 - 128$$

$$= 2059$$

Question 16: Find a G.P. for which sum of the first two terms is –4 and the fifth term is 4 times the third term.

Answer 16:

Let, a be the first term and r be the common ratio of the G.P.

According to the given condition,

$$S_2 = -4 = \frac{a(1-r^2)}{1-r} \ldots (1)$$

$$= a_5 = 4 \times a^3$$

$$= a\,r^4 = 4\,ar^2 = r^2 = 4$$

$$= r = \pm\, 2$$

From (1), we obtain

$$-4 = \frac{a\left[1-(2)^2\right]}{1-2} \text{ for } r = 2$$

$$= -4 = \frac{a(1-4)}{-1}$$

$$= -4 = a\,(3)$$

$= a = \dfrac{-4}{3}$

Also, $- 4 = \dfrac{a[1-(-2)^2]}{1-(-2)}$ for $r = - 2$

$= - 4 = \dfrac{a(1-4)}{1+2}$

$= - 4 = \dfrac{a(-3)}{3}$

$= a = 4$

Thus, the required G.P. is $\dfrac{-4}{3}, \dfrac{-8}{3}, \dfrac{-16}{3}, \ldots$ or 4, -8, 16, -32, …

Question 17: If the 4th, 10th and 16th terms of a G.P. are x, y and z, respectively. Prove that x, y, z is in G.P.

Answer 17:

Let, a be the first term and r be the common ratio of the G.P.

According to the given condition,

$= a_4 = a\, r^3 = x$ …. (1)

$= a_{10} = a\, r^9 = y$ …. (2)

$= a_{16} = a\, r^{15} = z$ … (3)

Dividing (2) by (1), we obtain

$\dfrac{y}{x} = \dfrac{ar^9}{ar^3} = \dfrac{y}{x} = r^6$

Dividing (3) by (2), we obtain

$\dfrac{z}{y} = \dfrac{ar^{15}}{ar^9} = \dfrac{z}{y} = r^6$

$= \dfrac{y}{x} = \dfrac{z}{y}$

Thus, x, y, z is in G.P.

Question 18: Find the sum to n terms of the sequence, $8, 88, 888, 8888\ldots$

Answer 18:

The given sequence is $8, 88, 888, 8888\ldots$

This sequence is not a G.P. however, it can be changed to G.P.by writing the terms as

$S_n = 8 + 88 + 888 + 8888 + \ldots. + n \text{ terms}$

$= \dfrac{8}{9} [9 + 99 + 999 + 9999 + \ldots \text{ to n terms}]$

$= \dfrac{8}{9} [(10 - 1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \ldots \text{ to n terms}]$

$= \dfrac{8}{9} [(10 + 10^2 + \ldots. \text{ n terms}) - (1 + 1 + 1 + \text{n terms})]$

$= \dfrac{8}{9} \left(\dfrac{10(10^n - 1)}{10 - 1} - n \right)$

$= \dfrac{8}{9} \left[\dfrac{10(10^n - 1)}{9} - n \right]$

$= \dfrac{80}{81} (10^n - 1) - \dfrac{8}{9} n$

Question 19: Find the sum of the products of the corresponding terms of the sequences $2, 4, 8, 16, 32$ and $128, 32, 8, 2, 1/2$.

Answer 19:

Required sum $= 2 \times 128 + 4 \times 32 + 8 \times 8 + 16 \times 2 + 32 \times \dfrac{1}{2}$

$= 64 \left[4 + 2 + 1 + \dfrac{1}{2} + \dfrac{1}{2^2} \right]$

Here, $4, 2, 1, \dfrac{1}{2}, \dfrac{1}{2^2}$ is a G.P.

First term, $a = 4$

Common ratio $r = \dfrac{1}{2}$

It is known that,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{4\left[1-\left(\frac{1}{2}\right)^5\right]}{1-\frac{1}{2}} = \frac{4\left[1-\frac{1}{32}\right]}{\frac{1}{2}} = 8\left(\frac{32-1}{32}\right) = \frac{31}{4}$$

Required sum $= 64 \times \frac{31}{4} = (16)(31) = 496$

Question 20: Show that the products of the corresponding terms of the sequences form

a, a r, ar², … arⁿ⁻¹ and A, AR, AR², … ARⁿ⁻¹ a G.P, and find the common ratio.

Answer 20:

It has to be proved that the sequence: aA, a r AR, ar²AR², …arⁿ⁻¹ARⁿ⁻¹, forms a G.P.

$$\frac{second\ term}{first\ term} = \frac{ar\,AR}{aA} = Rr$$

$$\frac{third\ term}{secind\ term} = \frac{ar^2\,AR^2}{ar\,AR} = Rr$$

Thus, the above sequence forms a G.P. and the common ratio is rR.

Question 21:Find four numbers forming a geometric progression in which third term is greater than the first term by 9, and the second term is greater than the 4ᵗʰ by 18.

Answer 21:

Let, a be the first term and r be the common ratio of the G.P.

$= a_1 = a$, $a_2 = a$ r, $a_3 = ar^2$, $a_4 = ar^3$

By the given condition

$= a_3 = a_1 + 9 = ar^2 = a + 9$ … (1)

$= a_2 = a_4 + 18 = a\,r = ar^3 + 18$ … (2)

From eq. (1) and (2), we obtain

$= a\,(r^2 - 1) = 9$ … (3)

$= a\,r\,(1 - r^2) = 18$ … (4)

Dividing (4) by (3), we obtain

$$\frac{ar\left(1 - r^2\right)}{a\left(r^2 - 1\right)} = \frac{18}{9}$$

$= -r = 2$

$= r = -2$

Substituting the value of r in (1), we obtain

$4a = a + 9$

$= 3a = 9$

$= a = 3$

Thus, the first four numbers of the G.P. are 3, 3 (-2), $3(-2)^2$ and $3(-2)^3$ i.e., 3, -6, 12, and -24.

Question 22: If p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that

$A^{q-r}.b^{r-p}.\,c^{\,p-q} = 1$.

Answer 22:

Let, A be the first term and R be the common ratio of the G.P.

According to the given information,

$AR^{p-1} = a$

$AR^{q-1} = b$

$AR^{r-1} = c$

$= a^{q-r} . b^{r-p} . c^{p-q}$

$= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)}$

$= A^{q-r+r-p+p-q} \times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)}$

$= A^0 \times R^0$

$= 1$

Thus, the given result is proved.

Question 23: If the first and the nth term of a G.P. are a ad b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

Answer 23:

The first terms of the A.P. is a and the last term is b.

Therefore, the G.P. is a, a r, ar^2, ar^3 … ar^{n-1}, where r is common ratio

$= b = ar^{n-1}$ … (1)

$= p =$ product of n terms

$= (a) (ar) (ar^2) … (ar^{n-1})$

$= (a \times a \times a) (r \times r^2 \times …r^{n-1})$

$= an \; r \; 1 + 2 + … (n-1)$ … (2)

Here, 1, 2, … $(n-1)$ is an A.P.

$= 1 + 2 + … + (n-1)$

$= \frac{n-1}{2}[2 + (n-1-1) \times 1] = \frac{n-1}{2}[2 + n - 2] = \frac{n(n-1)}{2}$

$P = a^n \, r^{\frac{n(n-1)}{2}}$

$P^2 = a^{2n} \, r^{n(n-1)}$

$= \left[a^2 r^{(n-1)} \right]^n$

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$= [a \times ar^{n-1}]^n$

$= (ab)^n$ [using eq. (1)]

Thus, the given result id proved.

Question 24: Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\dfrac{1}{r^n}$.

Answer 24:

Let, a be the first term and r be the common ratio of the G.P.

Sum of first n terms $= \dfrac{a(1-r^n)}{1-r}$

Since there are n terms from $(n + 1)^{th}$ to $(2n)^{th}$ term,

Sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term

$S_n = \dfrac{a_{n+1}(1-r^n)}{1-r}$

Thus, required ratio $= \dfrac{a(1-r^n)}{1-r} \times \dfrac{1-r}{ar^n(1-r^n)} = \dfrac{1}{r^n}$

Thus, the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ terms is $\dfrac{1}{r^n}$

Question 25: If a, b, c and d are in G.P. show that:

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc - cd)^2$

Answer 25:

a, b, c and d are in G.P. therefore,

$= bc = ad$ … (1)

$= b2 = ac$ … (2)

$= c2 = bd$ … (3)

It has to be proved that,

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc - cd)^2$

R.H.S.

$= (ab + bc + cd)^2$

$= (ab + ad + cd)^2$ [using (1)]

$= [ab + d(a + c)]^2$

$= a^2b^2 + 2abd(a + c) + d^2(a + c)^2$

$= a^2b^2 + 2a^2bd + 2acbd + d^2(a^2 + 2ac + c^2)$

$= a^2b^2 + 2a^2c^2 + 2b^2c^2 + d^2a^2 + 2d^2b^2 + d^2c^2$ [using (1) and (2)]

$= a^2b^2 + a^2c^2 + a^2c^2 + b^2c^2 + b^2c^2 + d^2a^2 + d^2b^2 + d^2b^2 + d^2c^2$

$= a^2b^2 + a^2c^2 + a^2d^2 + b^2 \times b^2 + b^2c^2 + b^2d^2 + c^2d^2 + c^2 \times c^2 + c^2d^2$

[using (2) and (3) and rearranging terms]

$= a^2(b^2 + c^2 + d^2) + b^2(b^2 + c^2 + d^2) + c^2(b^2 + c^2 + d^2)$

$= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = $ L.H.S.

L.H.S. $=$ R.H.S.

$= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc - cd)^2$

Question 26: Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Answer 26:

Let, G_1 and G_2 be two numbers between 3 and 81 such that the series, 3, G_1, G_2, 81, forms a G.P.

Let, a be the first term and r be the common ratio of the G.P.

$81 = (3)(r)^3$

$= r^3 = 27$

$= r = 3$ (taken real roots only)

For r = 3

$G_1 = a\,r = (3)\,(3) = 9$

$G_2 = a\,r^2 = (3)\,(3)^2 = 27$

Thus, the required two numbers are 9 and 27.

Question 27: Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Answer 27:

A.M. is a and b is $\sqrt{ab}$

By the given condition: $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab}$

Squaring both sides, we obtain

$= \dfrac{(a^{n+1}+b^{n+1})^2}{(a^n+b^n)^2} = ab$

$= a^{2n+2} + 2a^{n+1}\,b^{n+1} + b^{2n+2} = (ab)\,(a^{2n} + 2a^n b^n + b^{2n})$

$= a^{2n+2} + 2a^{n+1}\,b^{n+1} + b^{2n+2} = a^{2n+1}\,b + 2a^{n+1}\,b^{n+1} + ab^{2n+1}$

$= a^{2n+2} + b^{2n+2} = a^{2n+1}\,b + ab^{2n+1}$

$= a^{2n+2} - a^{2n+1}\,b = ab^{2n+1} - b^{2n+2}$

$= a^{2n+1}\,(a - b) = b^{2n+1}\,(a - b)$

$= \left(\dfrac{a}{b}\right)^{2n+1} = 1 = \left(\dfrac{a}{b}\right)^{0}$

$= 2n + 1 = 0$

$= n = \dfrac{-1}{2}$

Question 28: The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$

Answer 28:

Let, the two numbers be a and b.

G.M. $= \sqrt{ab}$

According to the given condition,

$= a + b = 6\sqrt{ab}$ … (1)

$= (a + b)^2 = 36 \, (ab)$

Also, $(a - b)^2 = (a + b)^2 - 4ab = 36ab - 4ab = 32ab$

$= a - b = \sqrt{32}\sqrt{ab}$

$= 4\sqrt{2}\sqrt{ab}$ … (2)

Adding (1) and (2), we obtain

$2a = (6 + 4\sqrt{2})\sqrt{ab}$

$= a = (3 + 2\sqrt{2})\sqrt{ab}$

Substituting the value of a in (1), we obtain

$= b = 6\sqrt{ab} - (3 + 2\sqrt{2})\sqrt{ab}$

$= b = (3 - 2\sqrt{2})\sqrt{ab}$

$= \dfrac{a}{b} = \dfrac{(3+2\sqrt{2})\sqrt{ab}}{(3-2\sqrt{2})\sqrt{ab}} = \dfrac{3+2\sqrt{2}}{3-2\sqrt{2}}$

Thus, the required ratio is $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$

Question 29: If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A + G)(A - G)}$

Answer 29:

It is given that A and G are A.M. and G.M. between two positive numbers.

Let, these two positive numbers be a and b.

= A.M. = A = $\frac{a+b}{2}$ ... (1)

G.M. = G = $\sqrt{ab}$ ... (2)

From (1) and (2), we obtain

= a + b = 2A ... (3)

= ab = G^2 .... (4)

Substituting the value of a and b from (3) and (4) in the identity

$(a - b)^2 = (a + b)^2 - 4ab$. We obtain

$(a - b)^2 = 4A^2 - 4G^2 = 4(A^2 - G^2)$

$(a - b)^2 = 4(A + G)(A - G)$

$(a - b) = 2\sqrt{(A + G)(A - G)}$ ... (5)

From (3) and (5), we obtain

$2a = 2A + 2\sqrt{(A + G)(A - G)}$

$= a = A + \sqrt{(A + G)(A - G)}$

Substituting the value of a in (3), we obtain

$= b = 2A - A - \sqrt{(A + G)(A - G)} = A - \sqrt{(A + G)(A - G)}$

Thus, the two numbers are A $\pm\sqrt{(A + G)(A - G)}$

Question 30: The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and an hour?

Answer 30:

It is given that the number of bacteria doubles every hour. Therefore, the numbers of bacteria after every hour will form a G.P.

Here, $a = 30$ and $r = 2$

$= a_3 = ar^2 = (30)(2)^2 = 120$

Therefore, the numbers of bacteria at the end of 2nd hour will be 120.

$= a_5 = ar^4 = (30)(2)^4 = 480$

The number of bacteria at the end of 4th hour will be 480.

$= a_{n+1} = a\, r^n = (30)\, 2^n$

Thus, numbers of bacteria at the end of nth hour will be $30\,(2)^n$

Question 31: What will ₹500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Answer 31:

The amount deposited in the bank is ₹500.

At the end of first year, amount $= ₹500\left(1 + \frac{1}{10}\right) = rs500(1.1)$

At the end of 2nd year, amount $= ₹500\,(1.1)\,(1.1)$

At the end of 3rd year, amount $= ₹500\,(1.1)\,(1.1)\,(1.1)$ and so on.

Amount at the end of 10 years $= ₹500\,(1.1)\,(1.1)\,\dots\,(10 \text{ times}) = Rs500\,(1.1)^{10}$

Question 32: If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Answer 32:

Let, the root of the quadratic equation be a and b.

According to the given condition,

A.M. $= \frac{a+b}{2} = 8 = a + b = 16 \dots (1)$

G.M. $= \sqrt{ab} = 5 = ab = 25 \dots$ (2)

The quadratic equation is given by,

$x^2 - x$ (sum of roots) + (product of roots) = 0

$x^2 - x (a + b) + (ab) = 0$

$x^2 - 16x + 25 = 0$ [using (1) and (2)]

thus, the required quadratic equation is $x^2 - 16x + 25 = 0$

Exercise 9.4

Question 1: Find the sum to n terms of the series $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$

Answer 1:

The given series is $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$ n^{th} term, $a_n = n(n + 1)$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k + 1)$

$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2} \left[\frac{2n+1}{3} + 1 \right]$

$= \frac{n(n+1)}{2} \left(\frac{2n+4}{3} \right)$

$= \frac{n(n+1)(n+2)}{3}$

Question 2: Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + ...$

Answer 2:

The given series is $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + ...$ n^{th} term

$= a_n = n(n + 1)(n + 2)$

$= (n^2 + n)(n + 2)$

$= n^3 + 3n^2 + 2n$

$S_n = \sum_{k=1}^{n} a_k$

$= \sum_{k=1}^{n} k^3 + 3 \sum_{k=1}^{n} k^2 + 2 \sum_{k=1}^{n} k$

$= \left[\frac{n(n+1)}{2} \right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n + 1 + 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n^2+n+4n+6}{2}\right]$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)}{4}(n^2 + 2n + 3n + 6)$$

$$= \frac{n(n+1)[n(n+2)+3(n+2)]}{4}$$

$$= \frac{n(n+1)(n-2)(n-3)}{4}$$

Question 3: Find the sum to n terms of the series $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + ...$

Answer 3:

The given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + ...$ n^{th} term

$a_n = (2n + 1) n^2 = 2n^3 + n^2$

$S_n = \sum_{k=1}^{n} a_k$

$$= \sum_{k=1}^{n}(2k^3 + k^2) = 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2} \left[\frac{3n^2 + 5n + 1}{3} \right]$$

$$= \frac{n(n+1)(3n^2 + 5n + 1)}{6}$$

Question 4: Find the sum to n terms of the series $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots$

Answer 4:

The given series is $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots$

n^{th} term, an $= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ [By partial fractions]

$a_1 = \frac{1}{1} - \frac{1}{2}$

$a_2 = \frac{1}{2} - \frac{1}{3}$

$a_3 = \frac{1}{3} - \frac{1}{4}$ ....

$a_n = \frac{1}{n} - \frac{1}{n+1}$

Adding the above terms column wise, we obtain

$$= a_1 + a_2 + \dots + a_n = \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n} \right] - \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \frac{1}{n+1} \right]$$

$S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$

Question 5: Find the sum to n terms of the series $5^2 + 6^2 + 7^2 + \dots + 20^2$

Answer 5:

The given series is $5^2 + 6^2 + 7^2 + \dots + 20^2$ nth term

$a_n = (n + 4)^2 = n^2 + 8n + 16$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2 + 8k + 16)$

$= \sum_{k=1}^{n} k^2 + 8 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 16$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n$$

16th term is $(1+ + 4)2 = 202$

$$S_n = \frac{16(16+1)(2\times16+1)}{6} + \frac{8\times16\times(16+1)}{2} + 16 \times 16$$

$$= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 16 \times 16$$

$$= 1496 + 1088 + 256$$

$$= 2840$$

$$5^2 + 6^2 + 7^2 + \dots + 20^2 = 2840$$

Question 6: Find the sum to n terms of the series $3 \times 8 + 6 \times 11 + 9 \times 14 +\dots$

Answer 6:

The given series is $3 \times 8 + 6 \times 11 + 9 \times 14 +\dots$ an

$= (n^{th} \text{ term of } 3, 6, 9 \dots) \times (n^{th} \text{ term of } 8, 11, 14\dots)$

$= (3n)(3n + 5)$

$= 9n^2 + 15n$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(9k^2 + 15k)$

$= 9 \sum_{k=1}^{n} 2 + 15 \sum_{k=1}^{n} k$

$= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2}$

$= \frac{3n\,(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$

$= \frac{3n(n+1)}{2}(2n + 1 + 5)$

$= \frac{3n(n+1)}{2}(2n + 6)$

$= 3n\,(n + 1)\,(n + 3)$

Question 7: Find the sum to n terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$

Answer 7:

The given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^3) + \ldots$ an

$$= (1^2 + 2^2 + 3^3 + \ldots + n^2)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6} = \frac{2^3 + 3n^2 + n}{6}$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} \left[\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k \right]$$

$$= \frac{1}{3}\sum_{k=1}^{n} k^3 + \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$$

$$= \frac{1}{3}\frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2} \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \times \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6} \left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{n^2 + n + 2n + 1 + 1}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{n^2 + n + 2n + 2}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{n(n+1) + 2(n+1)}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[\frac{(n+1)(n+2)}{2} \right]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

Question 8: Find the sum to n terms of the series whose n^{th} term is given by n (n + 1) (n + 4).

Answer 8:

$a_n = n\,(n + 1)\,(n + 4) = n\,(n^2 + 5n + 4) = n^3 + 5n^2 + 4n$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^3 + 5 \sum_{k=1}^{n} k^2 + 4 \sum_{k=1}^{n} k$

$= \dfrac{n^2(n+1)^2}{4} + \dfrac{5n(n+1)(2n+1)}{6} + \dfrac{4n(n+1)}{2}$

$= \dfrac{n(n+1)}{2} \left[\dfrac{n(n+1)}{2} + \dfrac{5(2n+1)}{3} + 4 \right]$

$= \dfrac{n(n+1)}{2} \left[\dfrac{3n^2+3n+20n+10+24}{6} \right]$

$= \dfrac{n(n+1)}{2} \left[\dfrac{3n^2+23n+34}{6} \right]$

$= \dfrac{n(n+1)(3n^2+23n+34)}{12}$

Question 9: Find the sum to n terms of the series whose nth terms is given by $n^2 + 2^n$

Answer 9:

$a_n = n^2 + 2^n$

$S_n = \sum_{k=1}^{n} k^2 + 2^k = \sum_{k=1}^{n} k^2 + \sum_{k-1}^{n} 2^k$ … (1)

Consider $\sum_{k=1}^{n} 2^k = 2^1 + 2^2 + 2^3 + \dots$

The above series $2, 2^2, 2^3, \dots$ is a G.P. with both the first term and common ratio equal to 2.

$= \sum_{k=1}^{n} 2^k = \dfrac{(2)[(2)^n - 1]}{2-1} = 2\,(2^n - 1)$ … (2)

Therefore, from (1) and (2), we obtain

$S_n = \sum_{k=1}^{n} k^2 + 2\,(2^n - 1) = \dfrac{n\,(n+1)(2n+1)}{6} + 2\,(2^n - 1)$

Question 10: Find the sum to n terms of the series whose n^{th} terms is given by

$(2n - 1)^2$

Answer 10:

$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$

$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(4k^2 - 4k + 1)$

$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= \dfrac{4n(n+1)(2n+1)}{6} - \dfrac{4n(n+1)}{2} + n$

$= \dfrac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$

$= n\left[\dfrac{2(2n^2+3n+1)}{3} - 2(n+1) + 1\right]$

$= n\left[\dfrac{4n^2+6n+2-6n-6+3}{3}\right]$

$= n\left[\dfrac{4n^2-1}{3}\right]$

$= \dfrac{n(2n+1)(2n-1)}{3}$

Miscellaneous Exercise

Question 1: Show that the sum of $(m + n)^{\text{th}}$ and $(m - n)^{\text{th}}$ terms of an A.P. is equal to twice the mth term.

Answer 1:

Let a and d be the first term and the common difference of the A.P. respectively. It is known that the kth term of an A. P. is given by

a_k = a + (k −1) d

∴ a_{m+n} = a + (m + n −1) d

a_{m-n} = a + (m − n −1) d

a_m = a + (m −1) d

∴ a_{m+n} + a_{m-n} = a + (m + n −1) d + a + (m − n −1) d

= 2a + (m + n −1 + m − n −1) d

= 2a + (2m − 2) d

= 2a + 2 (m − 1) d

=2 [a + (m − 1) d]

= 2am

Thus, the sum of $(m + n)^{\text{th}}$ and $(m - n)^{\text{th}}$ terms of an A.P. is equal to twice the mth term.

Question 2: If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Answer 2:

Let the three numbers in A.P. be a − d, a, and a + d.

According to the given information,

$(a - d) + (a) + (a + d) = 24$ ... (1)

$\Rightarrow 3a = 24$

$\therefore a = 8$

$(a - d)\, a\, (a + d) = 440$ ... (2)

$\Rightarrow (8 - d)\,(8)\,(8 + d) = 440$

$\Rightarrow (8 - d)\,(8 + d) = 55$

$\Rightarrow 64 - d^2 = 55$

$\Rightarrow d^2 = 64 - 55 = 9$

$\Rightarrow d = \pm 3$

Therefore, when $d = 3$, the numbers are 5, 8, and 11 and when $d = -3$, the numbers are 11, 8, and 5.

Thus, the three numbers are 5, 8, and 11.

Question 3: Let the sum of n, 2n, 3n terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3 (S_2 - S_1)$

Answer 3:

Let a and b be the first term and the common difference of the A.P. respectively. Therefore,

$S_1 = \dfrac{n}{2} [2a + (n - 1)\, d)]\ldots$ (1)

$S_2 = \dfrac{2n}{2} [2a + (2n - 1)\, d)] = n [2a + (2n - 1)\, d)]$ ... (2)

$S_3 = \dfrac{3n}{2} [2a + (3n - 1)\, d)]$ ... (3)

From (1) and (2), we obtain

$S_2 - S_1 = n [2a + + (2n - 1)\, d)] - \dfrac{n}{2} [2a + (n - 1)\, d)]$

$$= n \left\{ \frac{4a + 4nd - 2d - 2a - nd + d}{2} \right\}$$

$$= n \left\{ \frac{2a + 3nd - d}{2} \right\}$$

$$= \frac{n}{2} [2a + 93n - 1)d]$$

$$\therefore 3\,(S_2 - S_1) = \frac{3n}{2} [2a + (3n - 1)d] = S_3 \text{ [from (3)]}$$

Hence, the given result is proved.

Question 4: Find the sum of all numbers between 200 and 400 which are divisible by 7.

Answer 4:

The numbers lying between 200 and 400, which are divisible by 7, are

203, 210, 217 ... 399

∴ First term, a = 203

Last term, l = 399

Common difference, d = 7

Let the number of terms of the A.P. be n.

∴ a_n = 399 = a + (n −1) d

⇒ 399 = 203 + (n −1) 7

⇒ 7 (n −1) = 196

⇒ n −1 = 28

⇒ n = 29

$$S_{29} = \frac{29}{2} (203 + 399)$$

$$= \frac{29}{2} (602)$$

= (29) (602)

= 8729

Thus, the required sum is 8729.

Question 5: Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Answer 5:

The integers from 1 to 100, which are divisible by 2, are 2, 4, 6... 100.

This forms an A.P. with both the first term and common difference equal to 2.

$\Rightarrow 100 = 2 + (n-1)\,2$

$\Rightarrow n = 50$

$\therefore 2 + 4 + 6 + \ldots + 100 = \dfrac{50}{2}\,[2(2) + (50-1)\,(2)]$

$= \dfrac{50}{2}\,[4 + 98]$

$= (25)\,(102)$

$= 2550$

The integers from 1 to 100, which are divisible by 5, are 5, 10... 100.

This forms an A.P. with both the first term and common difference equal to 5.

$\therefore 100 = 5 + (n-1)\,5$

$\Rightarrow 5n = 100$

$\Rightarrow n = 20$

$\therefore 5 + 10 + \ldots + 100 = \dfrac{20}{2}\,[2\,(5) + (20-1)\,5]$

$= 10\,[10 + (19)\,5]$

$= 10\,[10 + 95] = 10 \times 105$

$= 1050$

The integers, which are divisible by both 2 and 5, are 10, 20, ... 100.

This also forms an A.P. with both the first term and common difference equal to 10.

$\therefore 100 = 10 + (n - 1)\,(10)$

$\Rightarrow 100 = 10n$

$\Rightarrow n = 10$

$\therefore 10 + 20 + \ldots + 100 = \dfrac{10}{2}\,[2\,(10) + (10 - 1)\,(10)]$

$= 5\,[20 + 90] = 5\,[110] = 550$

Required sum $= 2550 + 1050 - 550 = 3050$

Thus, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

Question 6: Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

Answer:

The two-digit numbers, which when divided by 4, yield 1 as remainder, are 13, 17, ... 97.

This series forms an A.P. with first term 13 and common difference 4.

Let n be the number of terms of the A.P.

It is known that the nth term of an A.P. is given by, $a_n = a + (n - 1)\,d$

$\therefore 97 = 13 + (n - 1)\,(4)$

$\Rightarrow 4\,(n - 1) = 84$

$\Rightarrow n - 1 = 21$

$\Rightarrow$ n = 22

Sum of n terms of an A.P. is given by,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{22} = \frac{22}{2}[22(13) + (22 - 1)(4)]$

= 11 [26 + 84]

= 1210

Thus, the required sum is 1210.

Question 7: If f is a function satisfying f (x + y) = f(x). f(y) for all x, y ∈ N, such that f (1) = 3 and $\sum$ f(x) = 120 n1, find the value of n.

Answer 7:

It is given that,

f (x + y) = f (x) × f (y) for all x, y ∈ N .... (1)

f (1) = 3

Taking x = y = 1 in (1),

we obtain f (1 + 1) = f (2) = f (1) f (1) = 3 × 3 = 9

Similarly,

f (1 + 1 + 1) = f (3) = f (1 + 2) = f (1) f (2) = 3 × 9 = 27

f (4) = f (1 + 3) = f (1) f (3) = 3 × 27 = 81

∴ f (1), f (2), f (3), ..., that is 3, 9, 27, ..., forms a G.P. with both the first

term and common ratio equal to 3.

It is known that, Sn = $\frac{a(r^n - 1)}{r - 1}$

It is given that, $\sum_{x=1}^{n} f(x) = 120$

$= 120 = \frac{3(3^n - 1)}{3 - 1}$

$= 120 = \frac{3}{2}(3^n - 1)$

$= 3n - 1 = 80$

$= 3n = 81 = 3^4$

$= n = 4$

Thus, the value of n is 4.

Question 8: The sum of some terms of G.P. is 315 whose first term and the common

ratio is 5 and 2, respectively. Find the last term and the number of terms.

Answer 8:

Let the sum of n terms of the G.P. be 315.

It is known that, $S_n = \frac{a(r^n - 1)}{r - 1}$

It is given that the first term a is 5 and common ratio r is 2.

$\therefore 315 = \frac{5(2^n - 1)}{2 - 1}$

$= 2n - 1 = 63$

$= 2n = 64 = (2)^6$

$= n = 6$

Last term of the G.P = 6th term $= ar^{6-1} = (5)(2)^5 = (5)(32) = 160$

Thus, the last term of the G.P. is 160.

Question 9: The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Answer 9:

Let a and r be the first term and the common ratio of the G.P. respectively.

$a = 1$ $a_3 = ar^2 = r^2$ $a^5 = ar^4 = r^4$

$\therefore r^2 + r^4 = 90$

$\Rightarrow r^4 + r^2 - 90 = 0$

$= r^2 = \dfrac{-1 \pm \sqrt{1+360}}{2} = \dfrac{-1 \pm \sqrt{361}}{2} = \dfrac{-1 \pm 19}{2} = $ - 10 or 9

$= r \pm 3$ (taking real roots)

Thus, the common ratio of the G.P. is ± 3.

Question 10: The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Answer 10:

Let the three numbers in G.P. be a, ar, and ar^2

From the given condition,

$a + ar + ar^2 = 56$

$\Rightarrow a(1 + r + r^2) = 56$ .... (1)

$a - 1, ar - 7, ar2 - 21$ forms an A.P.

$\therefore (ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$

$\Rightarrow ar - a - 6 = ar^2 - ar - 14$

$\Rightarrow ar^2 - 2ar + a = 8$

$\Rightarrow ar^2 - ar - ar + a = 8$

$\Rightarrow a(r^2 + 1 - 2r) = 8$

$\Rightarrow a(r - 1)^2 = 8$ .... (2)

From (1) and (2), we get

$\Rightarrow 7(r^2 - 2r + 1) = 1 + r + r^2$

$\Rightarrow 7r^2 - 14r + 7 - 1 - r - r^2 = 0$

$\Rightarrow 6r^2 - 15r + 6 = 0$

$\Rightarrow 6r^2 - 12r - 3r + 6 = 0$

$\Rightarrow 6r(r - 2) - 3(r - 2) = 0$

$\Rightarrow (6r - 3)(r - 2) = 0$

When r = 2, a = 8

When

Therefore, when r = 2, the three numbers in G.P. are 8, 16, and 32.

When, r = 1/2, the three numbers in G.P. are 32, 16, and 8.

Thus, in either case, the three required numbers are 8, 16, and 32.

Question 11: A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Answer 11:

Let the G.P. be T_1, T_2, T_3, T_4 ... T_{2n}.

Number of terms = 2n

According to the given condition,

$T_1 + T_2 + T_3 + ... + T_{2n} = 5 [T_1 + T_3 + ... + T_{2n-1}]$

$\Rightarrow T_1 + T_2 + T_3 + ... + T_{2n} - 5 [T_1 + T_3 + ... + T_{2n-1}] = 0$

$\Rightarrow T_2 + T_4 + ... + T_{2n} = 4 [T_1 + T_3 + ... + T_{2n-1}]$

Let the G.P. be a, ar, ar^2, ar^3 ...

$$\therefore \frac{ar(r^n-1)}{r-1} = \frac{4 \times a(r^n-1)}{r-1}$$

$= ar = 4a$

$= r = 4$

Thus, the common ratio of the G.P. is 4.

Question 12: The sum of the first four terms of an A.P. is 56. The sum of the last four terms are 112. If its first term is 11, then find the number of terms.

Answer 12:

Let the A.P. be a, a + d, a + 2d, a + 3d ... a + (n – 2) d, a + (n – 1) d.

Sum of first four terms = a + (a + d) + (a + 2d) + (a + 3d) = 4a + 6d

Sum of last four terms

= [a + (n – 4) d] + [a + (n – 3) d] + [a + (n – 2) d] + [a + n – 1) d]

= 4a + (4n – 10) d

According to the given condition,

4a + 6d = 56

⇒ 4(11) + 6d = 56 [Since a = 11 (given)]

⇒ 6d = 12

⇒ d = 2

∴ 4a + (4n –10) d = 112

⇒ 4(11) + (4n – 10)² = 112

⇒ (4n – 10)² = 68

⇒ 4n – 10 = 34

⇒ 4n = 44

$\Rightarrow n = 11$

Thus, the number of terms of the A.P. is 11.

Question 13: If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$ then show that a, b, c and d are in G.P.

Answer 13:

It is given that,

$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$

$= (a + bx) (b - cx) = (b + cx) (a - bx)$

$= ab - acx + b^2x - bcx^2 = ab - b^2x + acx - bcx^2$

$= 2b^2x = 2acx$

$= b^2 = ac$

$= \frac{b}{a} = \frac{c}{b} \ldots (1)$

Also, $\frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$

$= (b + cx) (c - dx) = (b - cx) (c + dx)$

$= bc - bdx + c^2x - cdx^2 = bc + bdx - c^2x + cdx^2$

$= 2c^2x = 2bdx$

$= c^2 = bd$

$= \frac{c}{d} = \frac{b}{c} \ldots (2)$

From (1) and (2), we obtain

$= \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

Thus, a, b, c, and dare in G.P.

Question 14: The pth, q^{th} and r^{th} terms of an A.P. are a, b, c respectively. Show that $(q - r)\,a + (r - p)\,b + (p - q)\,c = 0$

Answer 14:

Let, t and d be the first term and the common difference of the A.P. respectively.

The nth term of an A.P. is given by, an $= t + (n - 1)\,d$

Therefore,

$= a_p = t + (p - 1)\,d = a$ ... (1)

$= a_q = t + (p - 1)\,d = b$ ... (2)

$= a_r = t + (r - 1)\,d = c$ ... (3)

Subtracting eq. (2) from (1), we obtain

$(p - 1 - q + 1)\,d = a - b$

$= (p - q)\,d = a - b$

$= d = \dfrac{a-b}{p-q}$ .... (4)

Subtracting eq. (3) from (2), we obtain

$(q - 1 - r + 1)\,d = b - c$

$= (q - r)\,d = b - c$

$= d = \dfrac{b-c}{q-r}$ .... (5)

Equating both the values of d obtained in (4) and (5), we obtain

$= \dfrac{a-b}{p-q} = \dfrac{b-c}{q-r}$

$= (a - b)\,(q - r) = (b - c)\,(p - q)$

$= aq - ar - bq + br = bp - bq - cp + cq$

$= bp - cp + cq - aq + ar - br = 0$

$= (-aq + ar) + (bp - br) + (-cp + cq) = 0$ [by rearranging terms]

$= -a (q - r) - b (r - p) - c (p - q) = 0$

$= a (q - r) + b (r - p) + c (p - q) = 0$

Thus, the given result is proved.

Question 15: If $a \left(\frac{1}{b} + \frac{1}{c}\right), b \left(\frac{1}{c} + \frac{1}{a}\right), c \left(\frac{1}{a} + \frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

Answer 15:

It is given that a $\left(\frac{1}{b} + \frac{1}{c}\right), b \left(\frac{1}{c} + \frac{1}{a}\right), c \left(\frac{1}{a} + \frac{1}{b}\right)$ are in A.P.

$b \left(\frac{1}{c} + \frac{1}{a}\right) - a \left(\frac{1}{b} + \frac{1}{c}\right) = c \left(\frac{1}{a} + \frac{1}{b}\right) - b \left(\frac{1}{c} + \frac{1}{a}\right)$

$= \frac{b(a+c)}{ac} - \frac{a(b+c)}{bc} = \frac{c(a+b)}{ab} - \frac{b(a+c)}{ac}$

$= \frac{b^2 a + b^2 c - a^2 b - a^2 c}{abc} = \frac{c^2 a + c^2 b - b^2 a - b^2 c}{abc}$

$= b^2 a - a^2 b + b^2 c - a^2 c = c^2 a - b^2 a + c^2 b - b^2 c$

$= ab (b - a) + c (b^2 - a^2) = a (c^2 - b^2) + bc (c - b)$

$= ab (b - a) + c (b - a) (b + a) = a (c - b) (c + b) + bc (c - b)$

$= (b - a) (ab + cb + ca) = (c - b) (ac + ab + bc)$

$= b - a = c - b$

Thus, a, b, and c, are in A.P.

Question 16: If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

Answer 16:

It is given that a, b, c and d are in G.P.

$b^2 = ac$ ... (1)

$c^2 = bd$ .... (2)

$ad = bc$ ... (3)

It has to be proved that $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P. i.e.,

$= (b^n + c^n)^2 = (an + b^n)(c^n + d^n)$

Consider L.H.S.

$= (b^n + c^n)^2 = b^2n + 2b^2cn + c^2n$

$= (ac)^n + 2b^nc^n + (bd)^n$ [using (1) and (2)]

$= a^n c^n + b^n c^n + b^n c^n + b^n d^n$

$= a^n c^n + b^n c^n + a^n d^n + b^n d^n$ [using (3)]

$= c^n (a^n + b^n) + d^n (a^n + b^n)$

$= (a^n + b^n)(c^n + d^n)$ R.H.S.

Thus, $(a^n + b^n)$, $(b^n + c^n)$, and $(c^n + d^n)$ are in G.P.

Question 17: If a and b are the roots of are roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d, form a G.P.

Prove that $(q + p) : (q - p) = 17:15$.

Answer 17:

It is given that a and b are the roots of $x^2 - 3x + p = 0$

$a + b = 3$ and $ab = p$ .... (1)

Also, c and d are the roots of $x^2 - 12x + q = 0$

$c + d = 12$ and $cd = q$ ... (2)

It is given that a, b, c, d, are in G.P.

Let $a = x$, $b = xr$, $c = xr^2$, $d = xr^3$ from (1) and (2)

We obtain $x + xr = 3 = x(1 + r) = 3$

$= xr^2 + xr^3 = 12$

$= xr^2(1 + r) = 12$

On dividing, we obtain

$= \dfrac{xr^2(1+r)}{x(1+r)} = \dfrac{12}{3}$

$= r^2 = 4$

$= r = \pm 2$

When $r = 2$, $x = \dfrac{3}{1+2} = \dfrac{3}{3} = 1$

Where $r = -2$, $x = \dfrac{3}{1-2} = \dfrac{3}{-1} = -3$

Case I

When $r = 2$ and $x = 1$, $ab = x^2r = 2$, $cd = x^2r^5 = 32$

$= \dfrac{q+p}{q-p} = \dfrac{32+2}{32-2} = \dfrac{34}{30} = \dfrac{17}{15}$

i.e., $(q + p): (q - p) = 17:15$

case II

When $r = -2$, $x = -3$, $ad = x^2r = -18$, $cd = x^2r^5 = -288$

$= \dfrac{q+p}{q-p} = \dfrac{-288-18}{-288+18} = \dfrac{-306}{-270} = \dfrac{17}{15}$

Thus, in both the cases, we obtain $(q + p): (q - p) = 17: 15$

Question 18: The ratio of the A.M and G.M. of two positive numbers a and b, is m: n. Show that

$$a: b = \left(m + \sqrt{m^2 + n^2}\right): \left(m - \sqrt{m^2 - n^2}\right)$$

Answer 18:

Let the two numbers be a and b.

A.M. $= \dfrac{a+b}{2}$ and G.M. $= \sqrt{ab}$

According to the given condition,

$$\dfrac{a+b}{2\sqrt{ab}} = \dfrac{m}{n}$$

$$= \dfrac{(a+b)^2}{4(ab)} = \dfrac{m^2}{n^2}$$

$$= (a+b)^2 = \dfrac{4ab\,m^2}{n^2}$$

$$= (a+b) = \dfrac{2\sqrt{ab}\,m}{n} \quad \dots (1)$$

$$= (a-b)^2 = \dfrac{4ab\,m^2}{n^2} - 4ab = \dfrac{4ab\,(m^2-n^2)}{n^2}$$

$$= (a-b) = \dfrac{2\sqrt{ab}\,\sqrt{m^2+n^2}}{n} \quad \dots (2)$$

Adding (1) and (2), we obtain

$$2a = \dfrac{2\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$$

$$= a = \dfrac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$$

Substituting the value of a in (1), we obtain

$$= b = \dfrac{2\sqrt{ab}}{n}m - \dfrac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$$

$$= \dfrac{\sqrt{ab}}{n}m - \dfrac{\sqrt{ab}}{n}\sqrt{m^2 - n^2}$$

$$= \dfrac{\sqrt{ab}}{n}\left(m - \sqrt{m^2 - n^2}\right)$$

$$= a : b = \dfrac{\dfrac{\sqrt{ab}}{n}(m+\sqrt{m^2+n^2})}{\dfrac{\sqrt{ab}}{n}(m-\sqrt{m^2-n^2})} = \dfrac{(m+\sqrt{m^2+n^2})}{(m-\sqrt{m^2-n^2})}$$

Thus, a; b = $(m + \sqrt{m^2 + n^2}) : (m - \sqrt{m^2 - n^2})$

Question 19: If a, b, c are in A.P; b, c, d are in G.P and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that

a, c, e is in G.P.

Answer 19:

It is given that a, b, c is in A.P.

b – a = c – b … (1)

It is given that b, c, d, are in G.P.

= $c^2 = bd$ … (2)

Also, $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P.

= $\frac{1}{d} - \frac{1}{c} = \frac{1}{e} - \frac{1}{d}$

= $\frac{2}{d} = \frac{1}{c} + \frac{1}{e}$ … (3)

It has to be proved that a, c, e, are in G.P. i.e., c2 = ae

From (1), we obtain

2b = a + c

= b = $\frac{a+c}{2}$

From (2), we obtain

= d = $\frac{c^2}{b}$

Subtracting this value in (3), we obtain

= $\frac{2b}{c^2} = \frac{1}{c} + \frac{1}{e}$

= $\frac{2(a+c)}{2c^2} = \frac{1}{c} + \frac{1}{e}$

$$= \frac{a+c}{c^2} = \frac{e+c}{ec}$$

$$= \frac{a+c}{c} = \frac{e+c}{e}$$

$$= (a + c)\, e = (e + c)\, c$$

$$= ae + ce = ec + c^2$$

$$= c^2 = ae$$

Thus, a, c, and e are in G.P.

Question 20: Find the sum of the following series up to n terms:

(i) $5 + 55 + 555 + \ldots$ (ii) $.6 + .66 + .666 + \ldots$

Answer 20:

(1) $5 + 55 + 555 + \ldots$

Let, $Sn = 5 + 55 + 555 + \ldots$ to n terms

$$= \frac{5}{9}\,[9 + 99 + 999 + \ldots \text{ to n terms}]$$

$$= \frac{5}{9}\,[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots \text{ to n terms}]$$

$$= \frac{5}{9}\,[(10 + 10^2 + 10^3 + \ldots \text{ n terms}) - (1 + 1 + \ldots \text{ n terms})]$$

$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{9}\right]$$

$$= \frac{50}{81}\,(10^n - 1) - \frac{5n}{9}$$

(ii) $.6 + .66 + .666 + \ldots$

Let, $Sn = 0.6 + 0.66 + 0.666 + \ldots$ to n terms

$$= 6\,[0.1 + 0.11 + 0.111 + \ldots \text{ to n terms}]$$

$$= \frac{6}{9} [0.9 + 0.99 + 0.999 + \text{to n terms}]$$

$$= \frac{6}{9} \left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \cdots to\ n\ terms \right]$$

$$= \frac{2}{3} \left[(1 + 1 + \cdots n\ terms) - \frac{1}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots nterms\right) \right]$$

$$= \frac{2}{3} \left[n - \frac{1}{10}\left(\frac{1 - \left(\frac{1}{10}\right)^n}{1 - \frac{1}{10}} \right) \right]$$

$$= \frac{2}{3} n - \frac{2}{30} \times \frac{10}{9} (1 - 10^{-n})$$

$$= \frac{2}{3} n - \frac{2}{27} (1 - 10^{-n})$$

Question 21: Find the 20th term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + ... +$ n terms.

Answer 21:

The given series is $2 \times 4 + 4 \times 6 + 6 \times 8 + ... +$ n terms

nth term $= a_n = 2n \times (2n + 2) = 4n^2 + 4n$

$a_{20} = 4\ (20)^2 + 4\ (20) = 4\ (400) + 80 = 1600 + 80 = 1680$

Thus, the 20th term of the series is 1680.

Question 22:

Find the sum of the first n terms of the series: $3 + 7 + 13 + 21 + 31 + ...$

Answer 22:

The given series is $3 + 7 + 13 + 21 + 31 + ...$

$S = 3 + 7 + 13 + 21 + 31 + ... + a_{n-1} + a_n$

$S = 3 + 7 + 13 + 21 + ... + a_{n-2} + a_{n-1} + a_n$

On subtracting both the eq., we obtain

$S - S = [3 + (7 + 13 + 21\ 31 + \ldots + a_{n-1} + a_n)] - [(3 + 7 + 13 + 21 + 31 + \ldots + a_{n-1} + a_n)]$

$S - S = 3 + [(7 - 3) + (13 - 7) + (21 - 13) + \ldots + (a_n - a_{n-1}] - a_n$

$0 = 3 + [4 + 6 + 8 + \ldots (n - 1)\ \text{terms}] - a_n$

$a_n = 3 + [4 + 6 + 8 + \ldots (n - 1)\ \text{terms}]$

$a_n = 3 + \left(\frac{n-1}{2}\right) [2 \times 4 + (n - 1 - 1)\,2]$

$= 3 + \left(\frac{n-1}{2}\right) [8 + (n - 2)\,2]$

$= 3 + \left(\frac{n-1}{2}\right) [2n + 4]$

$= 3 + (n - 1)(n - 2)$

$= 3 + (n^2 + n - 2)$

$= n^2 + n + 1$

$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$

$= n\left[\frac{(n+1)(2n+1) + 3(n+1) + 6}{6}\right]$

$= n\left[\frac{2n^2 + 3n + 1 + 3n + 3 + 6}{6}\right]$

$= n\left(\frac{2n^2 + 6n + 10}{6}\right)$

$= \frac{n}{3}(n^2 + 3n + 5)$

Question 23: If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3\,(1 + 8S_1)$

Answer 23:

From the given information

$$S_1 = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n^2(n+1)^2}{4}$$

Here, $S_3 (1 + 8S_1) = \frac{n^2(n+1)^2}{4}\left[1 + \frac{8n(n+1)}{2}\right]$

$$= \frac{n^2(n+1)^2}{4}[4n^2 + 4n + 1]$$

$$= \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= \frac{[n(n+1)(2n+1)]^2}{4} \quad \ldots (1)$$

Also, $9S_2^2 = 9\frac{[n(n+1)(2n+1)]^2}{6^2}$

$$= \frac{9}{36}[n(n+1)(2n+1)]^2$$

$$= \frac{[n(n+1)(2n+1)]^2}{4} \quad \ldots. (2)$$

Thus, from (1) and (2), we obtain $9S_2^2 = S_3 (1 + 8S_1)$

Question 24: Find the sum of the following series up to n terms:

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$$

Answer 24:

The nth term of the given series is $\frac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+\cdots+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+\cdots+(2n-1)}$

Here, $1, 3, 5, \ldots (2n - 1)$ is an A.P. with first term a, last term $(2n - 1)$ and number of terms as n

$= 1 + 3 + 5 + \ldots + (2n - 1) = \frac{n}{2}[2 \times 1 + (n - 1)\, 2] = n^2$

$a_n = \frac{n^2(n+1)^2}{4n^2} = \frac{(n+1)^2}{4} = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}$

$$S_n = \sum_{k=1}^{a} a_k = \sum_{k=1}^{n} \left(\frac{1}{4} k^2 + \frac{1}{2} k + \frac{1}{4} \right)$$

$$= \frac{1}{4} \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \frac{n(n+1)}{2} + \frac{1}{4} n$$

$$= \frac{n[(n+1)(2n+1)+6(n+1)+6]}{24}$$

$$= \frac{n[2n^2+3n+1+6n+6+6]}{24}$$

$$= \frac{n(2n^2+9n+12)}{24}$$

Question 25: Show that $\dfrac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \dfrac{3n+5}{3n+1}$

Answer 25:

n^{th} term of the numerator $= n\,(n + 1)^2 = n^3 + 2n^2 + n$

n^{th} term of the denominator $= n^2\,(n + 1) = n^3 + n^2$

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{\sum_{k=1}^{n} n_k}{\sum_{k=1}^{n} n_k} = \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)} \quad \cdots\ (1)$$

Here, $\sum_{k=1}^{n}(k^3 + 2k^2 + k)$

$$= \frac{n^n(n+1)^2}{6} = \frac{2n(n+1)(2n+1)}{4} = \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1 \right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6} \right]$$

$$= \frac{n(n+1)}{12}[3n^2 + 11n + 10]$$

$$= \frac{n(n+1)}{12}[3n^2 + 6n + 5n + 10]$$

$$= \frac{n(n+1)}{12}[3n(n + 2) + 5(n + 2)]$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \quad \cdots\ (2)$$

Also, $\sum_{k=1}^{n}(k^2 + k^2) = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$

$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2n+1}{3}\right]$

$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$

$= \frac{n(n+1)}{12}\left[3n^2 + 7n + 2\right]$

$= \frac{n(n+1)}{12}\left[3n^2 + 6n + n + 2\right]$

$= \frac{n(n+1)}{12}\left[3n(n + 2) + 1(n + 2)\right]$

$= \frac{n(n+1)(n+2)(3n+1)}{12} \ldots (3)$

From (1), (2) and (3), we obtain

$= \frac{1\times2^2+2\times3^2+\cdots+n\times(n+1)^2}{1^2\times2+2^2\times3+\cdots+n^2\times(n+1)} = \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$

$= \frac{3n+5}{3n+1}$

Hence, proved.

Question 26: A farmer buys a used tractor for ₹12000. He pays ₹6000 cash and agrees to pay the balance in annual installments of ₹500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?

Answer 26:

It is given the farmer pays ₹6000 in cash.

Therefore, unpaid amount = ₹12000 - ₹6000 = ₹6000

According to the given conditions, the interest paid annually is

12% of 6000, 12% of 5500, 12% of 5000 … 12% of 500

Thus, total interest to be paid

= 12% of 6000 + 12% of 5500 + 12% of 5000 + … + 12% of 500

= 12% of (6000 + 5500 + 5000 + … + 500)

= 12% of (500 + 1000 + 1500 + … + 6000)

Now, the series 500, 1000, 1500, 6000 is an A.P. with both the first term and common difference equal to 500

Let, the number of terms of the A.P. be n.

6000 = 500 + (n − 1) 500

= 1 + (n − 1) = 12

= n = 12

Sum of A.P.

$$= \frac{12}{2}[2(500) + (12 − 1)(500)] = 6[1000 + 5500] = 6[6500] = 39000$$

Thus total interest to be paid

= 12% of (500 + 1000 + 1500 + … + 6000))

= 12% of 39000 = ₹4680

Thus, cost of tractor = (₹12000 + ₹4680) = ₹16680

Question 27: Sham shed Ali buys a scooter for ₹22000. He pays ₹4000 cash and agrees to pay the balance in annual installment of ₹1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Answer:

It is given that sham shed Ali buys a scooter for ₹22000 and pays ₹4000 in cash.

Unpaid amount = ₹22000 - ₹4000 = ₹18000

According to the given condition, the interest paid annually is 10% of 18000, 10% of 17000, 10% of 16000 …. 10% of 1000.

Thus, total interest to be paid

= 10% of 18000, 10% of 17000, 10% of 16000 …. 10% of 1000

= 10% of (18000 + 17000 + 16000 + … + 1000)

= 10% of (1000 + 200 + 3000 + … + 18000)

Here, 1000, 2000, 3000 … 18000 forms an A.P. with first term and common difference both equal to 1000.

Let, the numbers of terms be n.

= 18000 = 1000 + (n − 1) (1000)

= n = 18

$1000 + 2000 + … + 18000 = \frac{18}{2} [2 (1000) + (18 − 1) (1000)]$

= 9 [2000 + 17000]

= 171000

Total interest paid = 10% of (18000 + 17000 + 16000 + … + 1000)

= 10% of ₹171000 = ₹17100

Cost of scooter = ₹22000 + ₹17100 = ₹39100

Question 28: A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paisa to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

Answer:

The numbers of letters mailed forms a G.P. 4, 42, …48

First term = 4

Common ratio $= 4$

Numbers of item $= 8$

It is known that the sum of n terms of a G.P. is given by

$$S_n = \frac{a(r^n - 1)}{1 - r}$$

$$S_8 = \frac{4(4^8 - 1)}{4 - 1} = \frac{4(65536 - 1)}{3} = \frac{4(65535)}{3} = 4(21845) = 87380$$

It is given that the cost to mail one letter is 50 paisa.

Cost of mailing 87380 letters $= ₹87380 \times \dfrac{50}{100} = ₹43690$

Thus, the amount spent when 8th set of letter is mailed is ₹43690.

Question 29: A man deposited ₹10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

Answer:

It is given that the man deposited ₹10000 in a bank at the rate of 5%

Simple interest annually $= \dfrac{5}{100} \times ₹10000 = ₹500$

Interest in first year $= 10000 + 500 + 500 + \ldots + 500$

Amount in 15th year $= ₹$

$= ₹10000 + 14 \times ₹500$

$= ₹10000 + ₹7000$

$= ₹17000$

Amount after 20 years $= ₹10000 + 500 + 500 + \ldots + 500$

$= ₹10000 + 20 \times ₹500$

$= ₹10000 + ₹10000$

= ₹20000

Question 30: A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Answer:

Cost of machine = ₹15625

Machine depreciates by 20% every year.

Therefore, its value after every year is 80% of the original cost i.e., $\frac{4}{5}$ of the original cost.

Value at the end of 5 years = $15625 \times \frac{4}{5} \times \frac{4}{5} \dots \times \frac{4}{5} = 5 \times 1024 = 5120$

Thus, the value of the machine at the end of 5 years is ₹= 5120.

Question 31: 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Answer:

Let, x be the numbers of days in which 150 workers finish the work.

According to the given information,

150x = 150 + 146 + 142 + … (x + 8) terms

The series 150 + 146 + 142 + … (x + 8) terms is an A.P. with first term 146, common differences − 4 and number of terms as (x + 8)

$= 150x = \frac{(x+8)}{2} [2\,(150) + (x + 8 - 1)\,(-4)\,]$

$= 150x = (x + 8)\,[150 + (x + 7)\,(-2)]$

$= 150x = (x + 8)\,(150 - 2x - 14)$

$= 150x = (x + 8)(136 - 2x)$

$= 75x = 69x - x^2 + 544 - 8x$

$= x^2 + 15x - 544 = 0$

$= x^2 + 32x - 17x - 544 = 0$

$= x(x + 32) - 17(x + 32) = 0$

$= (x + 32)(x - 17) = 0$

$= x = 17$ or $x = -32$

However, x cannot be

$=$ negative $x = 17$

Therefore, originally, the number of days in which the work was completed is 17. Thus, required number of days $= (17 + 8) = 25$