CHAPTER XII.

Mathematical Induction.

158. Many important mathematical formulæ are not easily demonstrated by a direct mode of proof; in such cases we frequently find it convenient to employ a method of proof known as **mathematical induction**, which we shall now illustrate.

Example 1. Suppose it is required to prove that the sum of the cubes of the first n natural numbers is equal to $\left\{\frac{n(n+1)}{2}\right\}^2$.

We can easily see by trial that the statement is true in simple cases, such as when $n=1$, or 2, or 3; and from this we might be led to *conjecture* that the formula was true in all cases. Assume that it is true when n terms are taken; that is, suppose

$$1^3+2^3+3^3+\ldots\ldots \text{ to } n \text{ terms}=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Add the $(n+1)^{\text{th}}$ term, that is, $(n+1)^3$ to each side; then

$$\begin{aligned}1^3+2^3+3^3+\ldots\ldots \text{ to } n+1 \text{ terms} &= \left\{\frac{n(n+1)}{2}\right\}^2+(n+1)^3 \\ &= (n+1)^2\left(\frac{n^2}{4}+n+1\right) \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \left\{\frac{(n+1)(n+2)}{2}\right\}^2;\end{aligned}$$

which is *of the same form* as the result we assumed to be true for n terms, $n+1$ taking the place of n; in other words, if the result is true when we take a certain number of terms, whatever that number may be, it is true when we increase that number by one; but we see that it is true when 3 terms are taken; therefore it is true when 4 terms are taken; it is therefore true when 5 terms are taken; and so on. Thus the result is true universally.

Example 2. To determine the product of n binomial factors of the form $x+a$.

By actual multiplication we have

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+bc+ca)x+abc;$$

$$\begin{aligned}(x+a)(x+b)(x+c)(x+d)=x^4&+(a+b+c+d)x^3\\ &+(ab+ac+ad+bc+bd+cd)x^2\\ &+(abc+abd+acd+bcd)x+abcd.\end{aligned}$$

In these results we observe that the following laws hold:

1. The number of terms on the right is one more than the number of binomial factors on the left.

2. The index of x in the first term is the same as the number of binomial factors; and in each of the other terms the index is one less than that of the preceding term.

3. The coefficient of the first term is unity; the coefficient of the second term is the sum of the letters a, b, c, ...; the coefficient of the third term is the sum of the products of these letters taken two at a time; the coefficient of the fourth term is the sum of their products taken three at a time; and so on; the last term is the product of all the letters.

Assume that these laws hold in the case of $n-1$ factors; that is, suppose

$$(x+a)(x+b)\ldots(x+h)=x^{n-1}+p_1x^{n-2}+p_2x^{n-3}+p_3x^{n-4}+\ldots+p_{n-1},$$

where
$$\begin{aligned}p_1&=a+b+c+\ldots h;\\ p_2&=ab+ac+\ldots+ah+bc+bd+\ldots\ldots;\\ p_3&=abc+abd+\ldots\ldots;\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ p_{n-1}&=abc\ldots h.\end{aligned}$$

Multiply both sides by another factor $x+k$; thus

$$(x+a)(x+b)\ldots(x+h)(x+k)$$
$$=x^n+(p_1+k)x^{n-1}+(p_2+p_1k)x^{n-2}+(p_3+p_2k)x^{n-3}+\ldots+p_{n-1}k.$$

Now $p_1+k=(a+b+c+\ldots+h)+k$
= sum of all the n letters a, b, c, ...k;

$p_2+p_1k=p_2+k(a+b+\ldots+h)$
= sum of the products taken two at a time of all the n letters a, b, c, ... k;

$p_3+p_2k=p_3+k(ab+ac+\ldots+ah+bc+\ldots)$
= sum of the products taken three at a time of all the n letters a, b, c, ... k;

$p_{n-1}k$ = product of all the n letters a, b, c, ... k.

If therefore the laws hold when $n-1$ factors are multiplied together, they hold in the case of n factors. But we have seen that they hold in the case of 4 factors; therefore they hold for 5 factors; therefore also for 6 factors; and so on; thus they hold universally. Therefore

$$(x+a)(x+b)(x+c)\ldots(x+k)=x^n+S_1x^{n-1}+S_2x^{n-2}+S_3x^{n-3}+\ldots+S_n$$

where S_1 = the sum of all the n letters $a, b, c \ldots k$;

S_2 = the sum of the products taken two at a time of these n letters.

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S_n = the product of all the n letters.

159. Theorems relating to divisibility may often be established by induction.

Example. Shew that x^n-1 is divisible by $x-1$ for all positive integral values of n.

By division $$\frac{x^n-1}{x-1}=x^{n-1}+\frac{x^{n-1}-1}{x-1};$$

if therefore $x^{n-1}-1$ is divisible by $x-1$, then x^n-1 is also divisible by $x-1$. But x^2-1 is divisible by $x-1$; therefore x^3-1 is divisible by $x-1$; therefore x^4-1 is divisible by $x-1$, and so on; hence the proposition is established.

Other examples of the same kind will be found in the chapter on the *Theory of Numbers.*

160. From the foregoing examples it will be seen that the only theorems to which induction can be applied are those which admit of successive cases corresponding to the order of the natural numbers 1, 2, 3, ... n.

EXAMPLES. XII.

Prove by Induction:

1. $1+3+5+\ldots\ldots+(2n-1)=n^2.$

2. $1^2+2^2+3^2+\ldots\ldots+n^2=\frac{1}{6}n(n+1)(2n+1).$

3. $2+2^2+2^3+\ldots\ldots+2^n=2(2^n-1).$

4. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots$ to n terms $=\frac{n}{n+1}$.

5. Prove by Induction that x^n-y^n is divisible by $x+y$ when n is even.