Sequences and Series (central node)

Definitions

Sequence is a function whose domain is the set of N natural numbers.

Real sequence: A sequence whose range is a subset of R is called a real sequence

Series: If $a_1, a_2, a_3, \ldots a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots a_n + \ldots$ is a series. A series is finite or infinite according to the no. of terms in the corresponding sequence as finite or infinite.

Progression: Those sequences whose terms follow certain patterns are called progression.

Geometric Progression (G.P.)

A sequence is said to be a geometric progression or G.P., if the ratio of any term to its preceding term is same throughout. This constant factor is called **'Common ratio'**. Usually we denote the first term of a G.P. by 'a' and its common ratio by 'r'. The general or n^{th} term of G.P. is given by $a_n = ar^{n-1}$

The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad \frac{a(1 - r^n)}{1 - r} \quad \text{if } r \neq 1$$

Sum of infinite terms of G.P. is given by

$$S_n = \frac{a}{1 - r}; \ |r| < 1$$

Geometric Mean (GM)

- If a, b, c are in G.P., b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0
- If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in G.P., then $G_1, G_2, G_3 \ldots G_n$ are n GMs between a & b.

$$G_1 = a \cdot \left(\frac{b}{a}\right)^{1/n+1} G_2 = a \cdot \left(\frac{b}{a}\right)^{2/n+1} \ldots G_n = a \cdot \left(\frac{b}{a}\right)^{n/n+1}$$

or

$$G_1 = ar, \ G_2 = ar^2, \ \ldots G_n = ar^{n-1}$$

where $r = \left(\frac{b}{a}\right)^{1/n+1}$,

Arithmetico Geometric Series

In a series, where each term is formed by multiplying the corresponding term of an AP & GP is called Arithmetico-Geometric Series.

Eg: $1 + 3x + 5x^2 + 7x^3 + \ldots$

Here, 1, 3, 5, … are in AP and $1, x, x^2, \ldots$ are in GP.

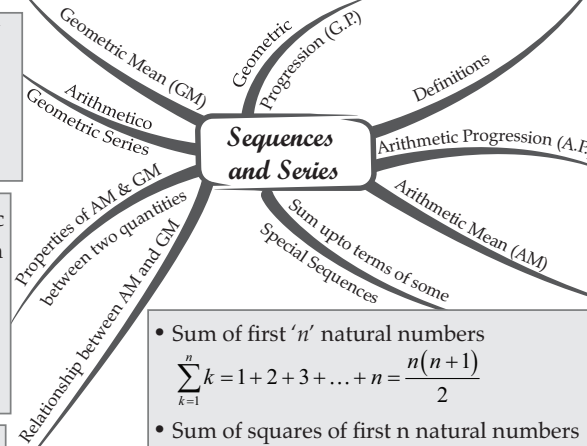## Properties of AM & GM between two quantities

- If A and G are respectively arithmetic and geometric means between two positive numbers a and b; then the quadratic equation have a, b as it's root is $x^2 - 2Ax + G^2 = 0$
- If A and G be the AM & GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$

Relationship between AM and GM

Let A and G be the AM and GM of two given positive real numbers a & b, respectively. Then,

$$A = \frac{a + b}{2} \quad \text{and} \quad G = \sqrt{ab}$$

Thus, we have , $A - G = \frac{a + b}{2} - \sqrt{ab} = \frac{a + b - 2\sqrt{ab}}{2}$

$$= \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} \geq 0 \quad \ldots\ldots\ldots\ldots (1)$$

From (1), we obtain the relationship $A \geq G$

Special Sequences / Sum upto n terms of some

- Sum of first 'n' natural numbers

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

- Sum of squares of first n natural numbers

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Sum of cubes of first n natural numbers

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$

- Sum of first 'n' odd natural numbers

$$\sum_{k=1}^{n} (2k - 1) = 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Arithmetic Progression (A.P)

An A.P. is a sequence in which terms increase or decrease regularly by the fixed number (same constant). This fixed number is called 'common difference' of the A.P. If 'a' is the first term & 'd' is the common difference and 'l' is the last term of A.P., then general term or the nth term of the A.P. is given by $a_n = a + (n-1)d$.

The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$$

Arithmetic Mean (AM)

If three numbers are in A.P, then the middle term is called AM between the other two, so if a, b, c, are in A.P., b is AM of a and c.

AM for any 'n' +ve numbers $a_1, a_2, a_3, \ldots, a_n$ is

$$\text{AM} = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

n-Arithmetic Mean between Two Numbers: If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in A.P. then $A_1, A_2, \ldots, A_n$ are n AM's between a & b.

$$A_1 = a + \frac{b - a}{n+1}, A_2 = a + \frac{2(b - a)}{n+1} \ldots A_n = a + \frac{n(b - a)}{n+1}$$

or $A_1 = a + d, A_2 = a + 2d, \ldots, A_n = a + nd$,

where $d = \frac{b - a}{n+1}$