Chapter 2

Complex Stresses, Shear Force and Bending Moment Diagrams

CHAPTER HIGHLIGHTS

- *Introduction*
- *Stresses on Inclined planes*
- *Principal planes and stresses*
- *Maximum shear stresses*
- *Strains on inclined plane*
- *Mohr's circle*
- *Strain gauge*
- *Theories of failure*
- *Classification of beams*
- *Shear force and bending moment in beams*

Introduction

State of Stress in 2D System (Biaxial or Plane Stress)

Examples: Beam, shaft.

In 2D system, on any inclined plane there will be two components of stresses.

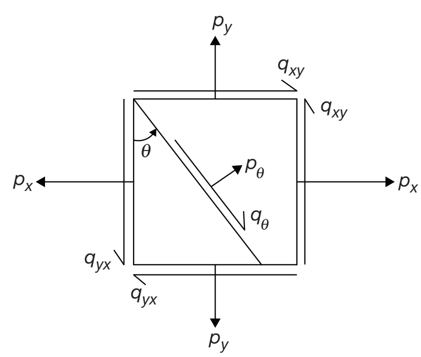
$$\text{Stress tensor} = \begin{bmatrix} p_x & q_{xy} \\ q_{yx} & p_y \end{bmatrix}$$

$q_{xy} = q_{yx}$ for moment equilibrium
Total stress components = 4
Independent stress components = 3.

Stresses on Inclined Planes

A body is subjected to p_x, p_y and q_{xy} as shown in the following figure.

The resultant stress acting on a plane inclined at an angle θ (in anti-clockwise) to the vertical.



Normal stress on inclined plane:

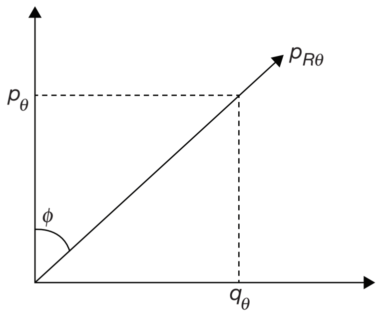
$$p_\theta = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2}\cos 2\theta + q_{xy}\sin 2\theta$$

Shear stress on inclined plane:

$$q_\theta = \left(\frac{p_x - p_y}{2}\right)\sin 2\theta - q_{xy}\cos 2\theta$$

Resultant stress on inclined plane:

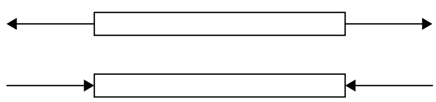
$$p_{R\theta} = \sqrt{p_\theta^{\,2} + q_\theta^{\,2}}$$

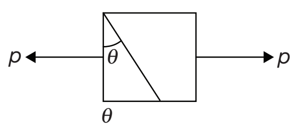


Angle of obliquity, $\phi = \tan^{-1}\left[\dfrac{q_\theta}{p_\theta}\right]$

Special Cases

1. **1D system or uniaxial stress system:**
   Example: Tie or strut (truss members)





$p_x = p;\ p_y = 0,\ q_{xy} = 0$

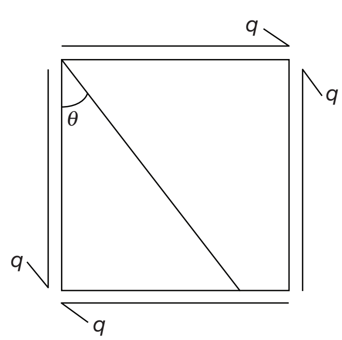
$$p_\theta = \frac{p}{2}(1+\cos 2\theta) = p\cos^2\theta$$

$$q_\theta = \frac{p}{2}(\sin 2\theta) = p\sin\theta\cos\theta$$

$p_{\max} = p$ (at $q = 0°$)

$q_{\max} = \dfrac{p}{2}$ (at $q = 45°$, i.e., diagonal plane)

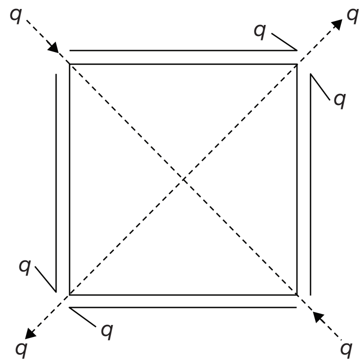
2. **Pure shear:**
   Example: Shafts subjected to torsion.



$p_\theta = q\sin 2\theta = 2q\sin\theta\cos\theta;\ q_\theta = q\cos 2\theta$

$p_{\max} = \pm q$ (at $q = 45°$ or $135°$, i.e., on diagonal planes)

$q_{\max} = q$ (at $q = 90°$ or $0°$)



Principal Planes and Stresses

Principal Stresses

- Maximum or minimum normal stress is principal stress.
- These stresses are used in designs.
- In 2D, two principal stresses and corresponding planes exist.

$$\left.\begin{matrix}\text{Major } p_1 \\ \\ \text{Minor } p_2\end{matrix}\right\} = \boxed{\frac{p_x + p_y}{2} \pm \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q_{xy}^2}}$$

Principal Plane

- The plane on which principal stresses are acting is principal plane.
- On principal plane, shear stress $q = 0$.
- In 2D system, there will be two principal planes separated by 90°.

Let principal plane is making an angle α with vertical.

$$\text{We know } q_\theta = \left(\frac{p_x - p_y}{2}\right)\sin 2\theta - q_{xy}\cos 2\theta$$

$$q_\alpha = 0 = \left(\frac{p_x - p_y}{2}\right)\sin 2\alpha - q_{xy}\cos 2\alpha$$

$$\boxed{\tan 2\alpha = \frac{2q_{xy}}{p_x - p_y}}$$

Maximum Shear Stresses

$$\boxed{q_{\max} = \pm\left(\frac{p_1 - p_2}{2}\right) \text{ or } \pm\sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2_{xy}}}$$

Maximum Shear Stress Planes

- The planes on which q_{max} is acting.
- In 2D system, there will be two q_{max} planes separated by 90°.
- Angle between q_{max} plane and the nearest principal plane will be at 45°.
- On maximum shear stress plane, normal stress will also be acting.

Normal stress on q_{max} plane:

$$p' = \frac{p_1 + p_2}{2} \text{ or } \frac{p_x + p_y}{2}$$

Strains on Inclined Plane

Stress System	Strain System
p_x	ε_x
p_y	ε_y
q_{xy}	$\left(\frac{\gamma_{xy}}{2}\right)$

Normal strain,

$$\varepsilon_\theta = \frac{\varepsilon_x + \varepsilon_y}{2} + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta + \left(\frac{\gamma_{xy}}{2}\right)\sin 2\theta$$

Shear strain,

$$\left(\frac{\gamma_\theta}{2}\right) = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\sin 2\theta - \left(\frac{\gamma_{xy}}{2}\right)2\cos 2\theta$$

Principal strains:

$$\left.\begin{matrix}\text{Major } \varepsilon_1 \\ \\ \text{Minor } \varepsilon_2\end{matrix}\right\} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Principal planes:

$$\tan 2\alpha = \frac{2\left(\frac{\gamma_{xy}}{2}\right)}{\varepsilon_x - \varepsilon_y}$$

α = angle of major principal plane with vertical

NOTE

Principal planes located by stress system (or) by corresponding strain system, both are same.

Maximum Shear Strain

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

$$\gamma_{max} = \varepsilon_1 - \varepsilon_2$$

$$q_{max} = \frac{p_1 - p_2}{2}$$

If in 3D system, $\gamma_{max} = \varepsilon_1 - \varepsilon_3$,

where ε_1 and ε_3 are major and minor strains.

Mohr's Circle

Mohr's circle is used for analysing stresses graphically.

Squaring and adding expressions for normal stress and shear stress leads to the equation of a circle. This principle is used in Mohr's circle.

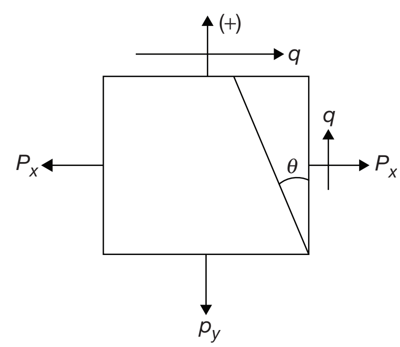
Construction of Mohr's Circle for Complex Stresses

It can be seen that radius of the Mohr's circle represents the maximum shear stress.

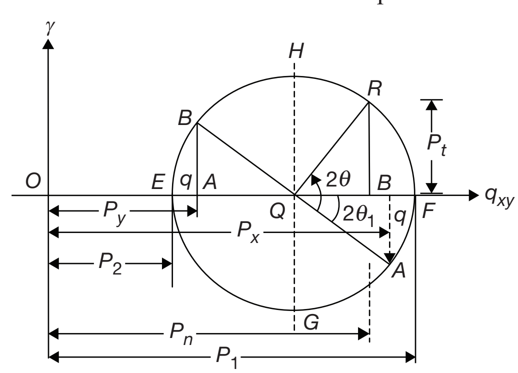
The following sign conventions should be observed while constructing the Mohr's circle.

1. Tensile stress is to be treated as positive and compressive stress negative. Positive normal stresses are to be plotted to the right of the origin and negative normal stresses to the left of the origin.
2. Shear stress producing clockwise moment in element is treated as positive and should be drawn above the *X*-axis.

Measurement of Stresses on a Plane Making an Angle θ with the Plane at Which p_x Acts

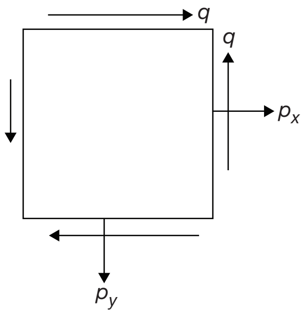


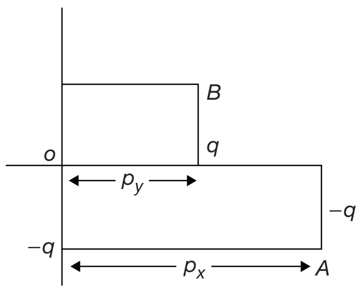
The plane makes angle θ in the anti-clockwise direction from plane at which p_x acts. As per the sign convention, radial vector will be above the *X*-axis in positive direction.



Line AB is a reference line representing p_x, p_y and q. Line QR is drawn at an angle twice θ (i.e., 2θ). Coordinates of R give the values of normal stress p_x and shear stress p_t on the plane.

In the diagram, angle $AQF = 2\theta_1$ represents the position of the principal plane. θ_1 is the angle from the reference plane at which the principal plane is situated.



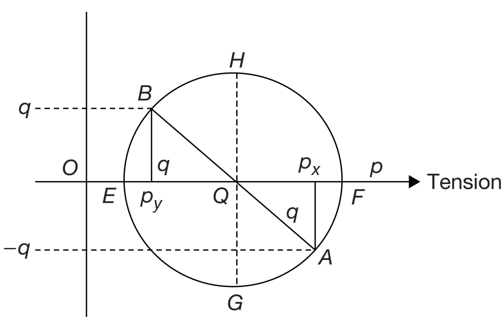


Mohr's circle is drawn on a set of axes representing normal stress (x-direction) and shear stress (y-direction).

Each set of a normal stress and shear stress can be represented by a point.

Points representing (p_x, $-q$) and (p_y, $+q$) are marked as A and B.

The line joining A and B passes through point Q on the horizontal axis. A circle is drawn through A and B with Q as centre. This is known as Mohrs' circle.



$$OQ = \frac{p_x + p_y}{2}$$

Radius $= QB = QA$

$$= \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

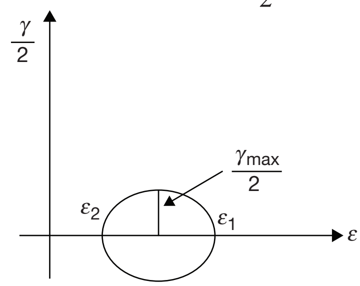
Points E and F represent principal planes where the shear stress is zero.

$\angle AQF = 2\phi_p$ = Angle at which the principal plane exists.

$\angle AQH = 2\phi_s$ = Angle at which the plane of maximum shear stress exists.

Mohr's Circle of Strain

Radius of Mohr's circle of strain $= \frac{\gamma_{max}}{2}$



Strain Gauge

Strain gauge is a small device that is attached to the surface of an object. It contains wires and are stretched or shortened when the object is strained at that point.

The gauges are extremely sensitive and measures strains as small as 1×10^{-6}.

Since each gauge measure the normal strain in only one direction. It is often necessary to use three gauges in combination with each gauge measuring the strain in a different direction.

From three such measurements, it is possible to calculate the strains in any direction on the surface.

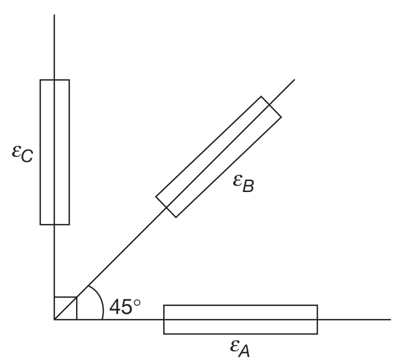
A group of three gauges arranged in a particular pattern are called a strain rosette.

Based on the arrangement of strain gauges. strain rosettes are classified as:

1. Rectangular strain rosettes $\alpha = 45°$
2. Delta strain rosettes $\alpha = 60°$
3. Star strain rosettes $\alpha = 120°$

Where α is angle between strain gauges.

Rectangular Strain Rosettes



Expression for ε_x, ε_y and γ_{xy} in terms of rectangular strain rosettes reading

(ε_A, ε_B, ε_C to be calculated)

We know:

$$\varepsilon_n = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

At, $\theta = 0°$

$(\varepsilon_n)_{\theta=0°} = \in_A = \varepsilon_x$

At, $\theta = 90°$

$(\varepsilon_n)_{\theta=90°} = \varepsilon_c = \varepsilon_y$

$$(\varepsilon_n)_{\theta=45°} = \varepsilon_B = \frac{1}{2}(\varepsilon_A + \varepsilon_C) + \frac{\gamma_{xy}}{2}$$

$$\Rightarrow \gamma_{xy} = 2\varepsilon_B - (\varepsilon_A + \varepsilon_C)$$

In strain tensor form:

$$[\varepsilon_{30}] = \begin{bmatrix} \varepsilon_A & \left(\varepsilon_B - \dfrac{(\varepsilon_A + \varepsilon_C)}{2}\right) \\ \left(\varepsilon_B - \dfrac{(\varepsilon_A + \varepsilon_C)}{2}\right) & \varepsilon_C \end{bmatrix}$$

$\varepsilon_{1,2}$ are principal strains

$$\varepsilon_{1,2} = \frac{1}{2}\left[(\varepsilon_x + \varepsilon_y) \pm \sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\gamma_{xy})^2}\right]$$

Maximum shear strain:

$$\gamma_{max} = \text{Higher of } [|\varepsilon_1| \text{ (or) } |\varepsilon_1 - \varepsilon_2|]$$

Principal stresses:

$$\sigma_1 = \frac{E}{1-\mu^2}[\varepsilon_1 + \mu\varepsilon_2]$$

$$\sigma_2 = \frac{E}{1-\mu^2}[\varepsilon_2 + \mu\varepsilon_1]$$

where E is Young's modulus and μ is Poisson's ratio.

Maximum shear stress:

$$Y_{max} = \text{Higher of } \left[\left|\frac{\sigma_1}{2}\right| \text{or} \left|\frac{\sigma_1 - \sigma_2}{2}\right|\right].$$

Theories of Failure

Strength of a member is based on mechanical properties which are usually determined from simple tension or compression tests. Predicting failure in members subjected to uniaxial stress is both simple and straight forward. But predicting the failure stresses for members subjected to biaxial or triaxial stresses is much more complicated. For that principal theories of failure have been formulated.

Generally, ductile materials fail by yielding, i.e., when permanent deformations occurs in the material and brittle materials fail by fracture. Therefore, for ductile materials, the limiting strength is the stress at yield point and for brittle materials the limiting strength is the ultimate stress in tension or compression.

1. **Maximum principal stress (Rankine) theory:** According to this theory, the failure occurs at a point in a member when the maximum principal stress reaches its limiting strength.

   $p_1 = \sigma_y$ (for tension and compression)

   Since the maximum principal stress theory is based on failure in tension or compression and ignores the possibility of failure due to shear, it is not used for ductile materials.

   But as brittle materials are strong in shear but weak in tension or compression, this theory is generally used.

2. **Maximum shear stress (Guest's or Tresca's) theory:** The failure occurs when the maximum shear stress becomes equal to that at the yield point in tension test.

$$q_{max} = \frac{1}{2}(p_1 - p_2)$$

   The failure occurs when the maximum principal stress reaches a value equal to the shear stress at yield point in tension test. The shear stress at a yield point in simple tension is equal to half of the yield stress in tension.

$$q_{max} = \frac{1}{2}(p_1 - p_2) = \frac{\sigma_y}{2}$$

   This theory is mostly used for designing members of ductile materials which are weak in shear.

NOTE

Aluminium alloys and certain steels are not governed by the Guest's theory.

3. **Maximum principal strain (Saint Venant's) theory:** Failure occurs when the maximum principal strain in a biaxial system reaches the limiting value of strain.

$$\varepsilon_{max} = \frac{p_1}{E} - \frac{\mu p_2}{E}$$

$$\Rightarrow \sigma_y = (p_1 - \mu p_2)$$

   Failure should occur at higher load, because the Poisson's ratio reduces the effect in perpendicular directions.

   This theory is not used in general, because it only gives reliable results in particular cases.

4. **Maximum strain energy (Haigh) theory:** This assumes that failure occurs when total strain energy in the complex stress system is equal to that at the yield point in tensile test.

$$U = \frac{1}{2E}\left[p_1^2 + p_2^2 - 2\mu p_1 p_2\right] = \frac{\sigma_y^2}{2E}$$

   This theory is good for ductile materials.

5. **Maximum distortion energy (Hencky and von Mises) theory:** This assumes that failure occur when shear strain energy (distortion energy) in the complex

system is equal to that at the yield point in tension or compression test.

$$(p_1 - p_2)^2 = 2\sigma_y^2$$

$$(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 = 2\sigma_y^2$$

This theory is mostly used for ductile materials in place of maximum strain energy theory.

Classification of Beams

A structural member on which forces act at right angles to its axis is called a 'beam'. Beam can be classified depending upon the types of supports as follows:

1. **Cantilever:** If one end of the beam is fixed and the other end is free, it is called a 'cantilever'.
2. **Simply supported beam:** When both ends of the beam is supported, it is called a simply supported beam.
3. **Fixed beam:** When both ends are rigidly fixed, it is called a fixed beam.
4. **Overhanging beam:** In overhanging beams, supports are not provided at the ends.
5. **Continuous beam:** If more than two supports are provided, it is called a continuous beam.

Shear Force and Bending Moment in Beams

Statically determinate beam: In statically determinate beams, the reaction at supports can be determined by applying the equation of static equilibrium. The values of reactions are not affected by the deformation of the beam.

The various types of loading are:

1. Point load or concentrated loads.
2. Uniformly distributed loads.
3. Uniformly varying loads.

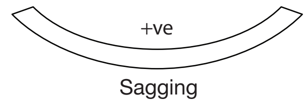
Shear force and bending moment: Shear force is the force that is trying to shear off a section of the beam and is obtained by the algebraic sum of all the forces and reactions acting normal to the axis of the beam acting either to the left or right of the section.

Bending moment acting at a section of a beam is the moment that is trying to bend it and is obtained by the algebraic sum of all the moments and reactions about the section, either to the right or left of the section.

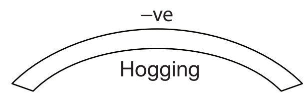
Shear force is treated as positive if it leads to move the left portion upward in relation to the right portion.

Bending moment is treated as positive if tries to sag the beam. The moment will be clockwise if the left portion of the beam is considered.

Sign conventions: Positive bending moment produces concavity upwards.

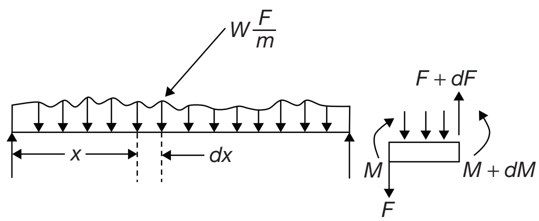


Negative bending moment produces convexity upwards.



Point of contra flexure: Bending moment in a beam varies depending upon the loads. Bending moment at a point may be positive, negative, or zero. The point at which bending moment changes its sign is called 'point of contra flexure'. Bending moment is zero at this point. At point of contra flexure beam curvature is changed from sagging to hogging or vice versa.

Relation between Load Intensity, Shear Force and Bending Moment



Considering an elemental length dx.

The shear force 'F' acts on the left side of the element and at the right side, it is $F + dF$.

The bending moment M acts on the left side of the element and at the right side it is $M + dM$.

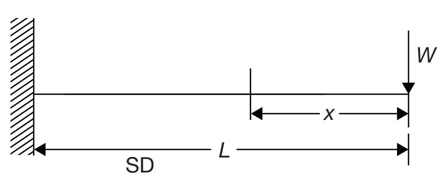
Since dx is very small, applied load may be taken as uniform and equal to $W\frac{N}{m}$.

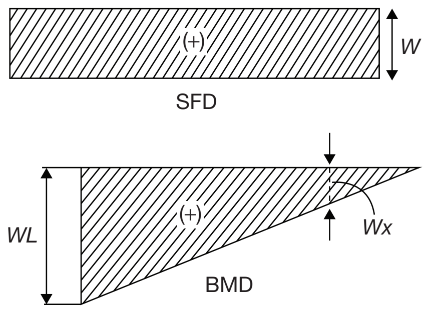
Taking moment about the right face and neglecting small quantity of higher order, we get: $\frac{dM}{dx} = -F$.

Shear Force and Bending Moment Diagrams

Cantilever Subjected to Central Concentrated Load

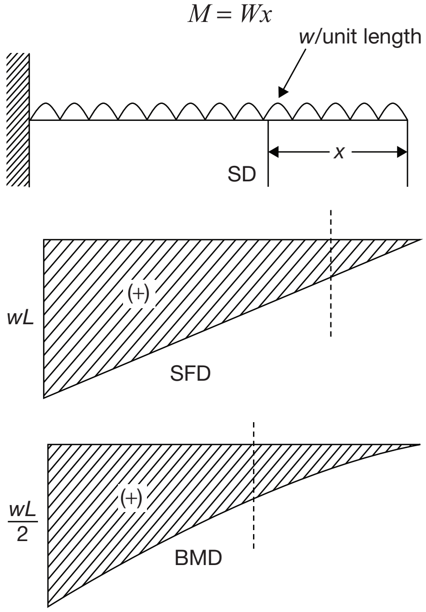
SD = Space diagram
SFD = Shear force diagram
BMD = Bending moment diagram



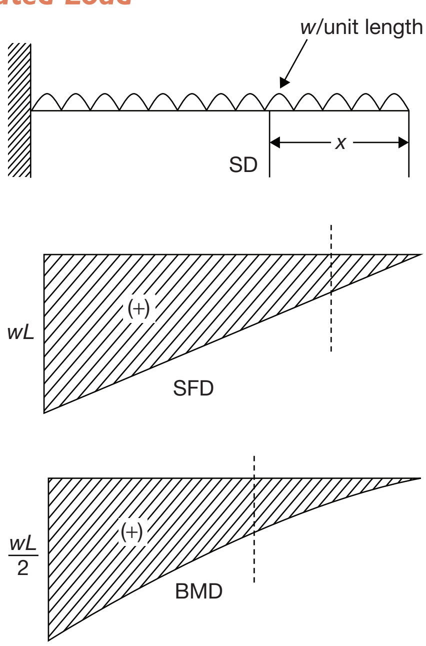


Shear force is constant throughout the beam. Bending moment varies linearly.

$$F = W$$
$$M = Wx$$



Cantilever Subjected to Uniformly Distributed Load



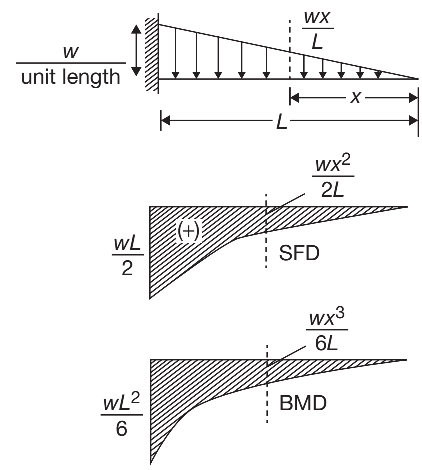
Shear force has got linear variation.

$$F = wx$$

Bending moment varies parabolically.

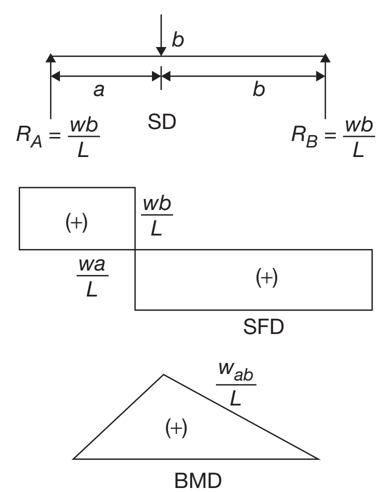
$$M = \frac{Wx^2}{2}$$

Cantilever Subjected to Uniformly Varying Load

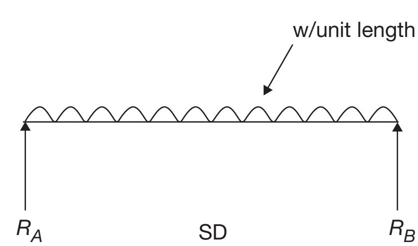


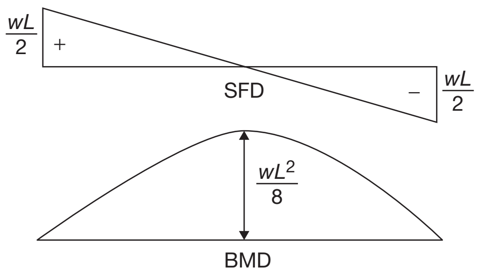
Shear force has a parabolic variation and bending moment has a cubic variation.

Simply Supported Beam with Concentrated Load

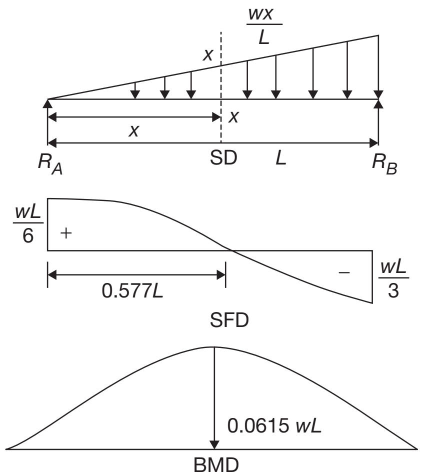


Simply Supported Beam with Uniformly Distributed Load





Simply Supported Beam with Uniformly Varying Load



Total load $= \dfrac{wL}{2}$

This acts at the centroid $\left(\dfrac{L}{3}\right)$

$$R_A \cdot L - \frac{wL}{2}\frac{L}{3} = 0$$

$$R_A = \frac{wL}{6}$$

$$R_B = \frac{wL}{2} - \frac{wL}{6} = \frac{wL}{3}$$

Total load on LHS of xx.

$$\frac{wx}{L} \times \frac{x}{2} = \frac{wx^2}{2L}$$

$$F = R_A - \frac{wx^2}{2L} \quad = \frac{wL}{6} - \frac{wx^2}{2L}$$

$$Alx = 0,\ F = \frac{wL}{6}$$

$$Alx = L,\ F = -\frac{wL}{3}$$

Moment at section xx:

$$M = \frac{wLx}{6} - \frac{wx^2}{2L}\frac{x}{3}$$

Maximum value of moment occurs at $x = \dfrac{L}{\sqrt{3}}$

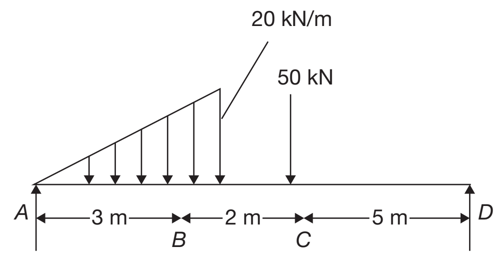
Some Important Points

1. Algebraic sum of all forces (including reactions) is zero.
2. Algebraic sum of all moments about any point in zero.
3. Moment at hinged joint is zero.
4. Moment is zero at the free end of a beam.
5. Shear force and bending moment are maximum at the fixed end of a cantilever.
6. Moment is zero at simply supported ends.

SOLVED EXAMPLES

Example 1

Determine the shear force and bending moment variation for the simply supported beam as shown in the figure. Indicate values of salient points.

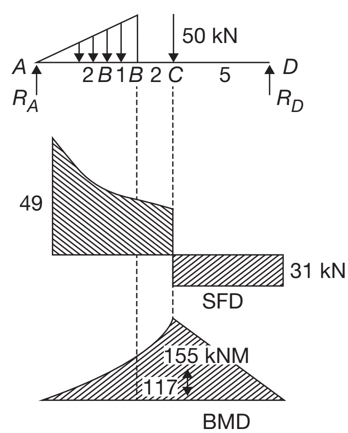


Solution

Shear force is taken as +ve if it tends to move the left portion upward.

If the moment on the left side is clockwise, it is treated as +ve.

Total value of the uniformly varying load on AB is $\dfrac{20 \times 3}{2} = 30$ kN



$$R_A + R_D = 30 + 50 = 80$$

Taking moments about A:

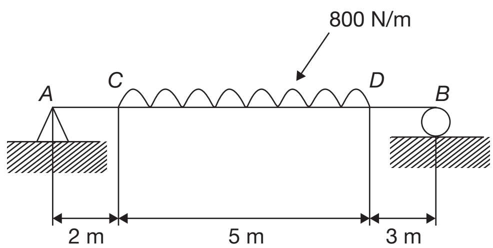
$30 \times 2 + 50 \times 5 - R_D \times 10 = 0$

$$R_D = \frac{310}{10} = 31 \text{ kN}$$

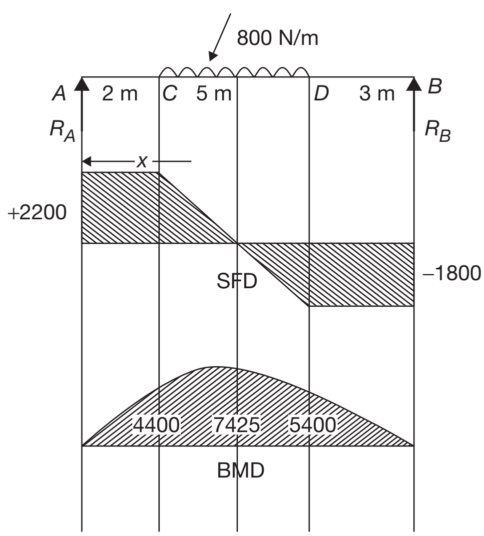
$\therefore R_A = 80 - 31 = 49$ kN
Shear force at A: $F_A = R_A = 49$ kN
Moment at A, $M_A = 0$ (as simply supported)
(Also, $M_A = 30 \times 2 + 50 \times 5 - 31 \times 10 = 0$)
Shear force at B: $F_B = 49 - 30 = 19$ kN
$M_B = 49 \times 3 - 30 \times 1 = 117$ kN-m
F_C (left) $49 - 30 = 19$ kN
$M_c = 49 \times 5 - 30 \times 3 = 155$ kN-m
F_c (right) $= 19 - 50 = -31$ kN
$F_D = R_D = 31$ kN
$M_D = 0.$

Example 2

Determine the shear force and bending moment variation for a simply supported beam shown in the following figure.



Solution



Taking moment about A.
$R_B \times 10 = (800 \times 5) \times 4.5 = 1800$ N
$R_A = (800 \times 5) - 1800 = 2200$ N
Shear force at $A = R_A = +2200$ N

Portion *AC*: Measuring x from A and taking all these forces to the left of section.
Shear force, $F = +R_A = +2200$ N. (constant)
Bending moment $M = R_A x = 2200x$

$$M_A = 0$$
$$M_C = 2200 \times 2 = 4400 \text{ Nm}$$

Portion *CD*: Measuring x from A

$$F = +2200 - (x - 2) \times 800$$
$$= +2200 - 800x + 1600$$
$$= +3800 - 800x \text{ (Linear variation)}$$
$$F = 0$$

When $800x = 3800$

or, $x = \dfrac{3800}{800} = 4.75M$

$$M = R_A \times x - (x-2)800 \times \frac{(x-2)}{2}$$
$$= R_A x - (x-2)^2 \times 400$$

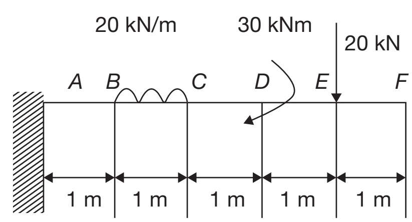
$F_D = +3800 - 800 \times 7 = -1800$ N
$M_D = 2200 \times 7 - (7 - 2)^2 \times 400 = 5400$ N
Maximum bending moment occurs, when $F = 0$.
That is, at $x = 4.75$ m
So, $M_{max} = 2200 \times 4.75 - (4.75 - 2)^2$
$400 = 7425$ Nm.

Portion *DB*: Taking x from B, and considering the right-hand side forces:

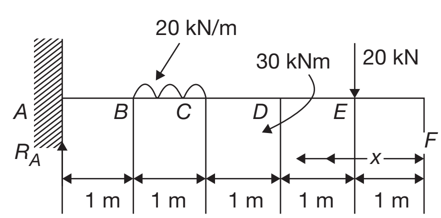
$$F = -R_B = -1800 \text{ N (constant)}$$
$$M = R_B \times x = 1800x$$
$$M_B = 0$$
$$M_D = 1800 \times 3 = 5400 \text{ Nm.}$$

Example 3

Draw shear force and bending moment variation for the cantilever beam loaded as shown in the following figure.



Solution



$R_A = 20 \times 1 + 20 = 40$ kN
Measuring x from F

Portion *FE*:

$$F_E = 0 \text{ upto } E$$
$$M_F = 0 \text{ upto } E$$

Portion *ED*:

$$F = 20 \text{ constant}$$
$$M = [20(x - 1)] \text{ linear}$$

$$F_E = 20 \text{ kN}$$
$$M_E = [20(1-1)] = 0$$
$$M_P = -20(2-1) = -20$$

Portion *DC*:

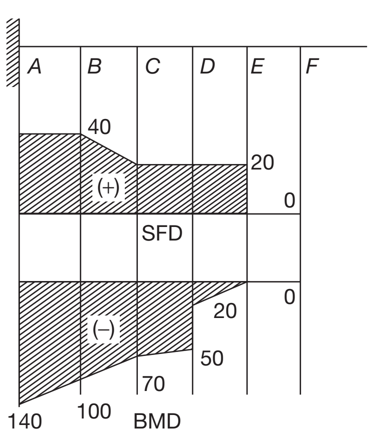
$$F = 20 \text{ constant}$$
$$M = -[20(x-1)+30] \text{ linear}$$
$$F_D = 20 \text{ kN}$$
$$M_D = [20(2-1)+30]$$
$$= -[20+30]$$
$$= -50 \text{ kN-m}$$
$$F_c = 20$$
$$M_c = -[20(3-1)+30]$$
$$= -50 \text{ kN-m}$$
$$F_c = 20$$
$$M_c = [20(3-1)+30]$$
$$= (20 \times 2 + 30)$$
$$= (40+30) = -70 \text{ kN-m.}$$

Portion *CB*:

$$F = 20 + (x-3)\,20 \text{ linear}$$
$$M = \left[20(x-1)+30+\frac{(x-3)^2 20}{2}\right] \text{ parabolic}$$
$$F_c = 20 + 0 = 20 \text{ kN}$$
$$M_c = [40+70+0] = -70 \text{ kN-m}$$
$$F_B\ 20 + 1 \times 20 = 40$$
$$M_B = \left[20(4-1)+30+\frac{(4-3)^2 20}{2}\right]$$
$$= [60+30+10]$$
$$= -100 \text{ kN-m.}$$

Portion *BA*:

$$F = 20 + 20 = 40 \text{ constant}$$
$$M = -[20(x-1)+30+20(x-3.5)] \text{ linear}$$
$$F_B = 40$$
$$M_B = -(20 \times 3 + 30 + 20 \times 0.5)$$
$$= (60+30+10) = -100$$
$$F_A = 40$$
$$M_A = (20 \times 4 + 30 + 20 \times 1.5)$$
$$= -(80+30+30)$$
$$= -140 \text{ kN-m.}$$



Example 4

Show that sum of normal stresses in any two mutually perpendicular directions is constant.

Solution

Equation for normal stress is

$$p_n = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2}\cos 2\theta + q\sin 2\theta$$

on a plane at angle $\theta + 90°$,

$$= \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \times \cos(2\theta + 180) + q\sin(2\theta + 180)$$

$$= \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2}\cos 2\theta - q\sin 2\theta$$

By adding, $p_n + p_n' = p_x + p_y = \text{constant}$.

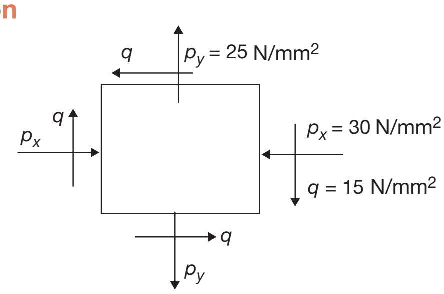
Example 5

The components of stresses on a rectangular element are

$$p_x = -30 \text{ N/mm}^2$$
$$p_y = +25 \text{ N/mm}^2$$
$$q = +15 \text{ N/mm}^2$$

Determine the magnitude of the two principal stresses and the angle between p_x and the major principal stress.

Solution



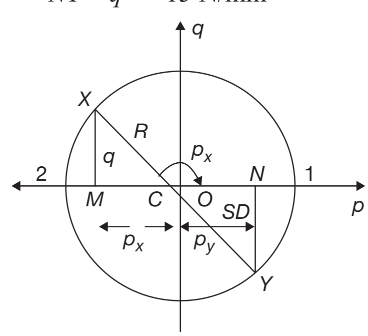
To draw the Mohr's circle, first draw a horizontal line representing the normal stress and then a vertical line representing the shearing stress. The point of intersection of these lines is the origin O, the point from where the stress values are plotted.

Locate the point 'X', such that $OM = p_x$

$$= -30 \text{ N/mm}^2, \text{ and } Mx = q = +15 \text{ N/mm}^2$$

Similarly, locate point Y such that:

$$ON = p_y = +25 \text{ N/mm}^2 \text{ and}$$
$$NY = q = -15 \text{ N/mm}^2$$



Draw line XY and locate the mid-point C the centre of Mohr's circle with C as the centre and radius equal to CX or CY, draw a circle which is the Mohr's circle.

$$OC = \frac{p_x + p_y}{2}$$

$$= \frac{-30+25}{2} = -2.5 \text{ N/mm}^2$$

Radius of Mohr's circle = R

$$= \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

$$= \sqrt{\left(\frac{-30-25}{2}\right)^2 + 15^2} = 31.33 \text{ N/mm}^2$$

Draw a horizontal diameter passing through the centre 'O'. Locate the extreme points 1 and 2 on this diameter. Then, $O1$ is the maximum principal stress and $O2$ is the minor principal stress.

Now,
$p_1 = O1$ = The maximum principal stress
$p_2 = O2$ = The minor principal stress
From Mohr's circle, we have:
$p_1 = OC + R = 2.5 + 31.33$
$= 28.82$ N/mm^2 (Tensile).
$p_2 = OC - R = -2.5 - 31.33$
$= -33.82$ N/mm^2 (Compressive)

$$\tan\alpha = \frac{XM}{CM} = \frac{15}{OM - OC}$$

$$= \frac{15}{30-2.5} = 0.545$$

$\therefore\ \alpha = 28.6°$
Angle $XC1 = 2\phi_p = 180 - 28.6° = 151.4°$
$\therefore\ \phi_p = 75.7°$
Here, the major principal stress.

$p_1 = 28.82$ N/mm^2 (Tensile) acts on a plane making an angle $\phi_p = 75.7°$ in the clockwise direction from the diameter xy to the diameter 1–2.

That is, principal planes lie at an angle ϕ_p from the x-direction.

Example 6

At a point in a material, the principal stresses are 800 N/cm^2 and 300 N/cm^2 where both are tensile. Find the normal, tangential and resultant stresses on a plane inclined at 50° to the major principal plane.

Solution

$p_1 = 800$ N/cm^2 (tensile)
$p_2 = 300$ N/cm^2 (tensile)
Angle with major principal plane = 50°
Let, p_n = Normal stress at the point

$$p_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2}\cos 2\phi$$

$$= \frac{800+300}{2} + \frac{800-300}{2}\cos 100°$$

$$= 550 + 250(-0.1736)$$

$$= 506.6 \text{ N/cm}^2$$

Let, q be the tangential stress at the point,

$$q = \frac{p_1 - p_2}{2}\sin 2\theta$$

$$= \frac{800-300}{2}\sin 100$$

$$= 250(0.9848)$$
$$= 246.20 \text{ N/cm}^2$$

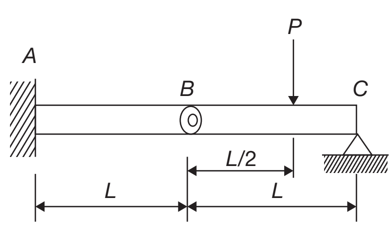
Resultant stress = $p_r = \sqrt{p_n^2 + q^2}$

$$= \sqrt{(506.6)^2 + (246.20)^2}$$

$$= 563.26 \text{ N/cm}^2.$$

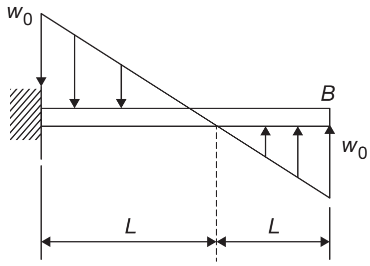
Exercises

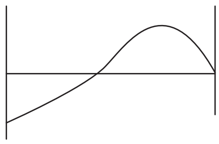
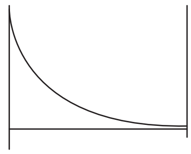
1. A beam is made up of two identical bars AB and BC, by hinging them together at B. The end A is built-in (cantilevered) and the end C is simply supported. With the load P acting as shown, the bending moment at A is

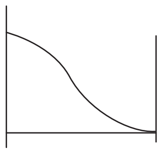
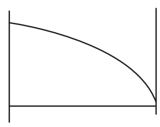


(A) zero
(B) $\frac{PL}{2}$
(C) $\frac{3PL}{2}$
(D) indeterminate

2. A cantilever beam carries the anti-symmetric load shown, where w_0 is the peak intensity of the distributed load. Qualitatively, the correct bending moment diagram for this beam is



(A)  (B) 

(C)  (D) 

3. The symmetry of stress tensor at a point in the body under equilibrium is obtained from
(A) conservation of mass
(B) force equilibrium equations
(C) moment equilibrium equations
(D) conservation of energy

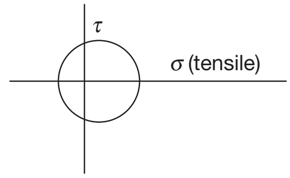
4. The components of strain tensor at a point in the plane strain case can be obtained by measuring longitudinal strain in following directions:
(A) Along any two arbitrary directions
(B) Along any three arbitrary directions
(C) Along two mutually orthogonal directions
(D) Along any arbitrary directions

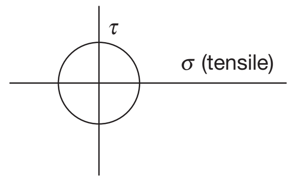
5. Mohr's circle for the state of stress defined by $\begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix}$ MPa is a circle with
(A) centre at (0, 0) and radius 30 MPa.
(B) centre at (0, 0) and radius 60 MPa.
(C) centre at (30, 0) and radius 30 MPa.
(D) centre at (30, 0) and zero radius.

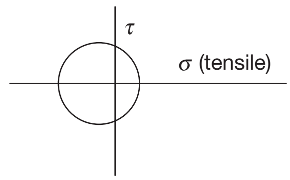
6. A small element at the critical section of component is in a bi-axial state of stress with the two principal stresses being 360 MPa and 140 MPa. The maximum shear stress is
(A) 110 MPa
(B) 180 MPa
(C) 314 MPa
(D) 330 MPa

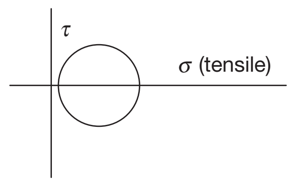
7. The magnitude of the only shear stresses acting at a point in plane stress situation is 7.5 N/mm^2. The magnitudes of the principle stresses will be
(A) +15.0 N/mm^2 and –7.5 N/mm^2
(B) +7.5 N/mm^2 and –15.0 N/mm^2
(C) +7.5 N/mm^2 and –7.5 N/mm^2
(D) +10.0 N/mm^2 and –7.5 N/mm^2

8. Which of the following Mohr's circles shown, qualitatively and correctly represents the state of plane stress at a point in a beam above the neutral axis, where it is subjected to combine shear and bending compressive stresses

(A) 

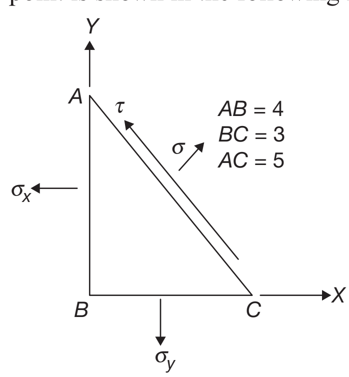
(B) 

(C) 

(D) 

9. The state of two dimensional stresses acting on a concrete lamina consists of a direct tensile stress, σ_x = 1.5 N/mm^2, and shear stress, τ = 1.20 N/mm^2, which cause cracking of concrete. Then the tensile strength of the concrete in N/mm^2 is
(A) 1.50 (B) 2.08
(C) 2.17 (D) 2.29

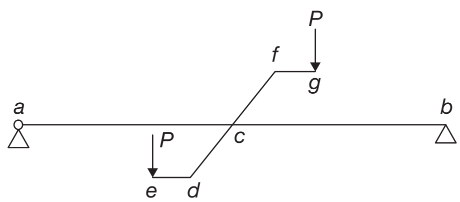
10. In a two dimensional analysis, the state of stress at a point is shown in the following figure.



If σ = 120 MPa and τ_{xy} = 70 MPa, σ_x and σ_y are respectively
(A) 26.7 MPa and 172.5 MPa
(B) 54 MPa and 128 MPa
(C) 67.5 MPa and 213.3 MPa
(D) 16 MPa and 138 MPa

11. If principal stresses in a two-dimensional case are –10 MPa and 20 MPa respectively, then maximum shear stress at the point is
(A) 10 MPa
(B) 15 MPa
(C) 20 MPa
(D) 30 MPa

12. A beam having a double cantilever attached at mid span is shown in the figure. The nature of forces in beam '*ab*' is

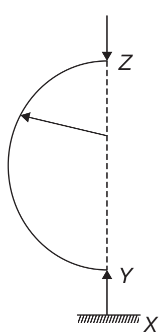


lengths
$cd = cf$
$de = fg$
ac = cb

(A) bending and shear.
(B) bending, shear and torsion.
(C) pure torsion.
(D) torsion and shear.

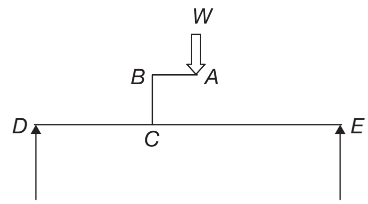
13. A cantilever beam curved in plan and subjected to lateral loads will develop at any section
(A) bending moment and shearing force.
(B) bending moment and twisting moment.
(C) twisting moment and shearing force.
(D) bending moment, twisting moment and shearing force.

14. A curved member with a straight vertical leg is carrying a vertical load at *Z*, as shown in the figure. The stress resultants in the *XY* segment are. Bending moment, shear force and axial force.



(A) Bending moment and axial force only
(B) Bending moment and shear force only
(C) Axial force only
(D) Bending moment only

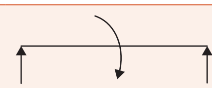
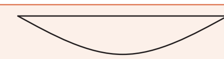
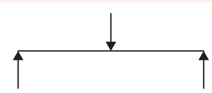
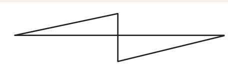
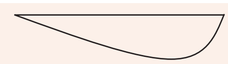
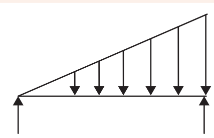
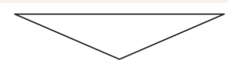
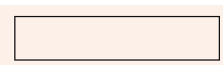
15. For the loading given in the figure below, two statements (I and II) are made



I. Member *AB* carries shear force and bending moment.
II. Member *BC* carries axial load and shear force.

Which of the following is true?
(A) Statement I is true but II is false
(B) Statement I is false but II is true
(C) Both statements I and II are true
(D) Both statements I and II are false

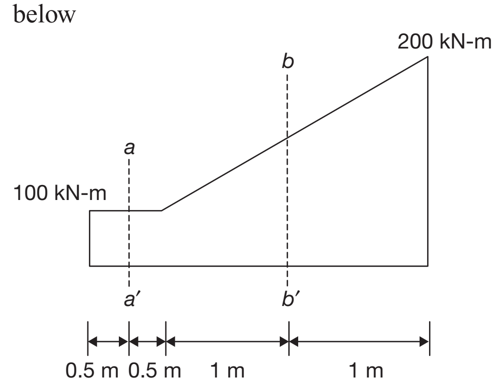
16. List I shows different loads acting on a beam and List II shows different bending moment distributions. Match the load with the corresponding bending moment diagram.

	List I		List II
a.		1.	
b.		2.	
c.		3.	
d.		4.	
		5.	

Codes:

	a	b	c	d		a	b	c	d
(A)	4	2	1	3	(B)	5	4	1	3
(C)	2	5	3	1	(D)	2	4	1	3

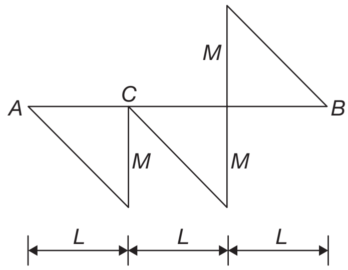
17. The bending moment diagram for a beam is given below



The shear force at sections *aa*′ and *bb*′ respectively are of the magnitude

(A) 100 kN, 150 kN
(B) zero, 100 kN
(C) zero, 50 kN
(D) 100 kN, 100 kN

18. A simply supported beam *AB* has the bending moment diagram as shown in the following figure:



The beam is possibly under the action of following loads:

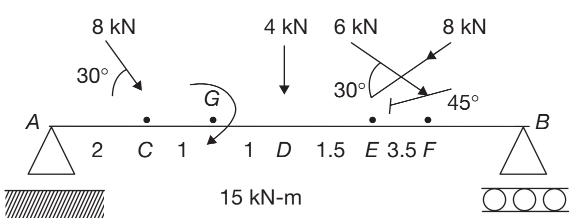
(A) Couples of *M* at *C* and 2*M* at *D*
(B) Couples of 2*M* at *C* and M at *D*
(C) Concentrated loads of *M*/*L* at *C* and 2*M*/*L* at *D*
(D) Concentrated load of *M*/*L* at *C* and couple of 2*M* at *D*

19. If failure in shear along 45° planes is to be avoided, then a material subjected to uniaxial tension should have its shear strength equal to at least the

(A) tensile strength.
(B) compressive strength.
(C) half the difference between the tensile and compressive strength.
(D) half the tensile strength.

Direction for questions 20 and 21:

A horizontal beam *AB* 10 m long is hinged at '*A*' and simply supported at '*B*'. The beam is loaded as shown in the figure.



20. The value of reaction force at '*A*' will be

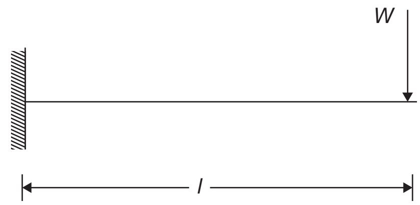
(A) 6.47 kN
(B) 34.78 kN
(C) 8.58 kN
(D) 9.49 kN

21. The maximum bending moment will be

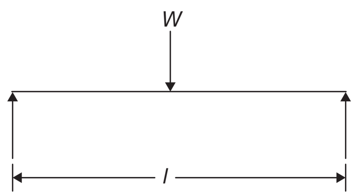
(A) 31.82 kN-m
(B) 34.78 kN-m
(C) 33.17 kN-m
(D) 38.25 kN-m

22. Constant bending moment over span/will occur in

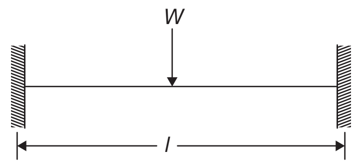
(A)



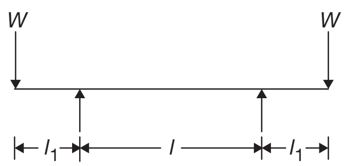
(B)



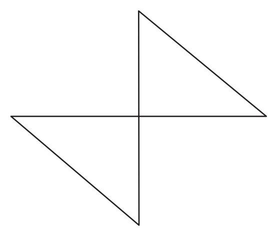
(C)



(D)

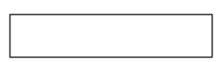


23. The bending moment diagram shown in the figure corresponds to the shear force diagram in

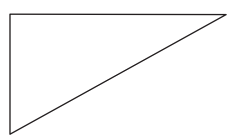


(A)

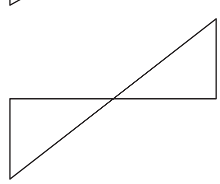
(B)



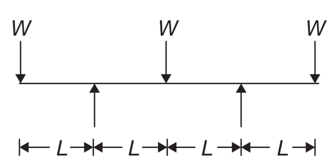
(C)



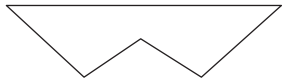
(D)

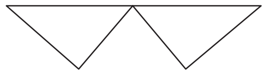


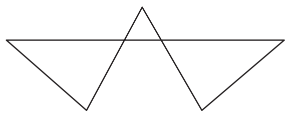
24. A loaded beam is shown in the following figure

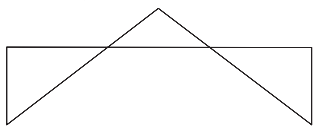


The bending moment diagram of the beam is best represented as

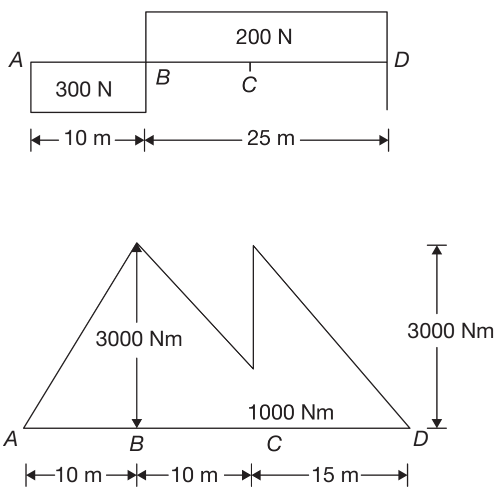
(A) 

(B) 

(C) 

(D) 

25. Shear force and bending moment diagrams for a beam *ABCD* are shown in the figure. It can be concluded that



(A) the beam has three supports.
(B) end *A* is fixed.
(C) a couple 2000 Nm acts at *C*.
(D) a uniformly distributed load is confined to portion *BC* only.

26. According to maximum shear stress failure criterion, yielding in material occurs when maximum shear stress is

(A) $\frac{1}{2}$ yield stress
(B) $\sqrt{2}$ yield stress
(C) $\frac{\sqrt{2}}{3}$ yield stress
(D) 2 yield stress

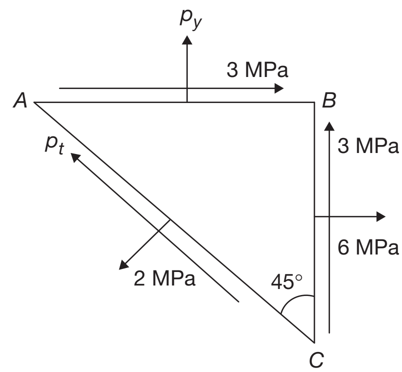
27. When a material is subjected to uniaxial tension, to avoid failure due to shear in 45° planes, the shear strength of the material should be atleast

(A) half the tensile strength.
(B) $\frac{1}{\sqrt{2}}$ times tensile strength.
(C) tensile strength.
(D) $\frac{3}{4}$ times tensile strength.

28. At a point in a strained material, direct stresses 120 N/mm^2 (tensile) and 100 N/mm^2 (compressive) are acting. If major principal stress is 150 N/mm^2, maximum shearing stress at the point is

(A) 87 N/mm^2
(B) 140 N/mm^2
(C) 130 N/mm^2
(D) 280 N/mm^2

29. At a point in a stressed body stresses acting are as shown in the figure. Value of p_y is



(A) –8 MPa
(B) 8 MPa
(C) –4 MPa
(D) 4 MPa

30. A body is subjected to direct stress σ_x, σ_y and σ_z in the *x*, *y* and *z* directions. If *E* = modulus of elasticity and μ = Poisson's ratio, direct strain ε_x in the *x* direction is

(A) $\varepsilon_x = \frac{1}{E}[\sigma_x + \mu(\sigma_y + \sigma_z)]$

(B) $\varepsilon_x = \frac{1}{E}[\sigma_x + \mu(\sigma_y - \sigma_z)]$

(C) $\varepsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)]$

(D) $\varepsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y - \sigma_z)]$

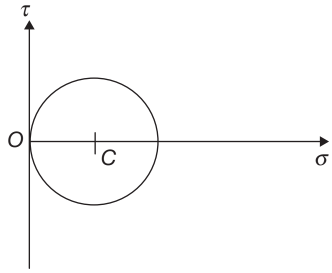
31. Slope of a beam under load is

(A) rate of change of deflection.
(B) rate of change of bending moment.
(C) rate of change of bending moment *x* flexural rigidity.
(D) rate of change of deflection *x* flexural rigidity.

32. If two principal strains at a point are 1000×10^{-6} and -500×10^{-6} then maximum shear strain is

(A) 500×10^{-6}
(B) 750×10^{-6}
(C) 1500×10^{-6}
(D) $1500\sqrt{2} \times 10^{-6}$

33. Figure shows Mohr's circle for a state of stress at a point in a strained material



Consider the following statements:

I. Major principal stress is equal to twice the maximum shear stress.

II. Minor principal stress is equal to the maximum shear stress.

III. Major principal stress is equal to maximum shear stress.

IV. Plane of maximum shear stress is at 90° to the major principal plane.

The correct statements are

(A) I and IV (B) II and III
(C) I alone (D) IV alone

34. During tension test of a bar, its cross-sectional area became half of the original cross sectional area. Values of true strain and engineering strain are

(A) 0.693 and 1 (B) 1 and 2
(C) 0.5 and 0.5 (D) 0.5 and 1

35. In a uniaxial stress system, the principal plane is defined as one on which

I. Shear stress is zero.

II. Normal stress is zero.

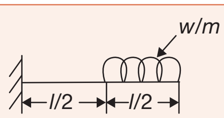
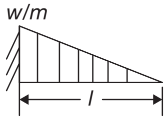
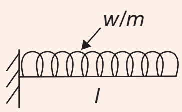
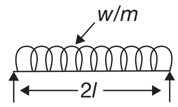
III. Shear stress is maximum.

IV. Normal stress is maximum.

Of the above statements,

(A) I and II are correct (B) II and III are correct
(C) III and IV are correct (D) I and IV are correct

36. Match the following List I (Loaded beam) and List II (Maximum bending moment).

List I		List II	
a.		1.	$\frac{wl^2}{2}$
b.		2.	$\frac{wl^2}{6}$
c.		3.	$\frac{3}{8}wl^2$
d.		4.	$\frac{wl^2}{4}$

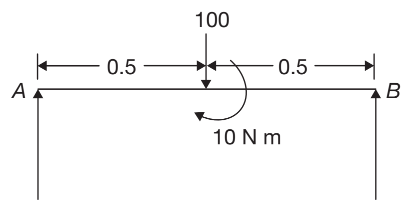
Codes:

	a	b	c	d		a	b	c	d
(A)	3	2	1	1	(B)	3	4	2	1
(C)	1	2	3	4	(D)	4	3	2	2

37. A simply supported beam with span 6 m has a rectangular cross section with depth 350 mm. If it is to be loaded centrally with a concentrated load of 30 kN, The width required at a distance of 2 m from end for uniform strength of 8 N/mm^2 is

(A) 176 mm (B) 167 mm
(C) 192 mm (D) 184 mm

38. A simply supported beam *AB* is centrally loaded by a concentrated load of 100 N and a moment of 10 Nm as shown in figure. Reaction (in N) at *A* and *B* respectively are



(A) 55,55 (B) 60,40
(C) 40, 60 (D) 50, 50

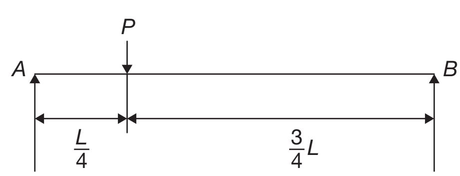
39. State of stress at a point is as follows. σ_x = 900 MPa, σ_y = 300 MPa. Maximum shear stress = 500 MPa. Value of maximum principal stress (in MPa) is

(A) 900 (B) 1000
(C) 1100 (D) 1200

40. At a point in a strained body, normal stresses are zero and shear stresses are 100 MPa. Value of principal stress are

(A) 50 MPa (B) 100 MPa
(C) 150 MPa (D) 20 MPa

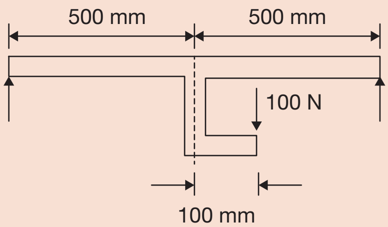
41. A simply supported beam of uniform cross-section of width *b* and depth *d* and length *L* is loaded as shown in the figure. The ratio of maximum allowable stress in bending to that in shear is



(A) $\frac{L}{2d}$ (B) $\frac{3}{4}\frac{L}{d}$ (C) $\frac{L}{D}$ (D) $\frac{3}{2}\frac{L}{D}$

Previous Years' Questions

1. In a simply-supported beam loaded as shown in the following figure, the maximum bending moment in Nm is **[GATE, 2007]**



(A) 25 (B) 30
(C) 35 (D) 60

2. An axially loaded bar is subjected to a normal stress of 173 MPa. The shear stress in the bar is **[GATE, 2007]**

(A) 75 MPa (B) 86.5 MPa
(C) 100 MPa (D) 122.3 MPa

3. Consider the following statements:

I. On a principal plane, only normal stress acts.
II. On a principal plane, both normal and shear stresses act.
III. On a principal plane, only shear stress acts.
IV. Isotropic state of stress is independent of frame of reference.

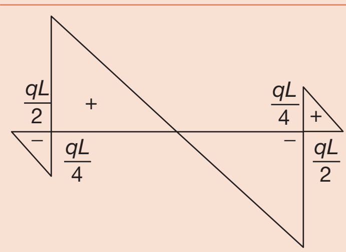
Which of the above statements is/are correct? **[GATE, 2009]**

(A) I and IV (B) II only
(C) II and IV (D) II and III

4. Match List I (Shear force diagrams) beams with List II (Diagrams of beams with supports and loading) and select the correct answer by using the codes given below the lists: **[GATE, 2009]**

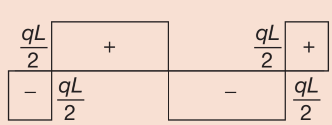
List I

a.



b.

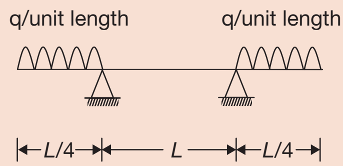
c.



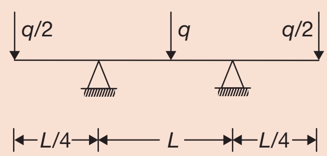
d.

List II

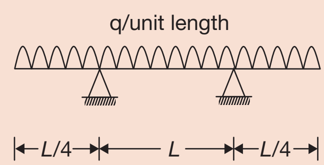
1.



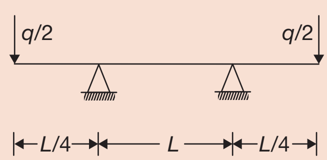
2.



3.



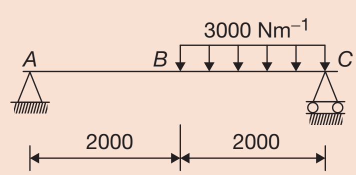
4.



Codes:

	a	b	c	d		a	b	c	d
(A)	3	1	2	4	(B)	3	4	2	1
(C)	2	1	4	3	(D)	2	4	3	1

5. A mass less beam has a loading pattern as shown in the figure. The beam is of rectangular cross-section with a width of 30 mm and height of 100 mm.



The maximum bending moment occurs at **[GATE, 2010]**

(A) location *B*
(B) 2675 mm to the right of *A*
(C) 2500 mm to the right of *A*
(D) 3225 mm to the right of *A*

6. The major and minor principal stresses at a point are 3 MPa and –3MPa respectively. The maximum shear stress at the point is **[GATE, 2010]**

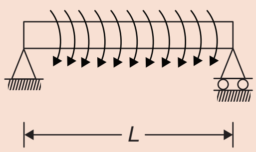
(A) zero (B) 3 MPa
(C) 6 MPa (D) 9 MPa

7. Two people weighing W each are sitting on a plank of length L floating on water at $L/4$ from either end. Neglecting the weight of the plank, the bending moment at the centre of the plank is **[GATE, 2010]**

(A) $\dfrac{WL}{8}$ (B) $\dfrac{WL}{16}$

(C) $\dfrac{WL}{32}$ (D) zero

8. For the simply supported beam of length L, subjected to a uniformly distributed moment M kN-m per unit length as shown in the figure, the bending moment (in kN-m) at the mid-span of the beam is **[GATE, 2010]**



(A) zero (B) M
(C) ML (D) M/L

9. The state of stress at a point under plane stress condition is $\sigma_{xx} = 40$ MPa, $\sigma_{yy} = 100$ MPa and $\tau_{xy} = 40$ MPa. The radius of the Mohr's circle representing the given state of stress in MPa is **[GATE, 2012]**

(A) 40 (B) 50
(C) 60 (D) 100

10. If a small concrete cube is submerged deep in still water in such a way that the pressure exerted on all faces of the cube is p, then the maximum shear stress developed inside the cube is **[GATE, 2012]**

(A) 0 (B) $\dfrac{p}{2}$

(C) p (D) $2p$

11. The following statements are related to bending of beams: **[GATE, 2012]**

I. The slope of the bending moment diagram is equal to the shear force.
II. The slope of the shear force diagram is equal to the load intensity.
III. The slope of the curvature is equal to the flexural rotation.
IV. The second derivative of the deflection is equal to the curvature.

The only FALSE statements is

(A) I (B) II
(C) III (D) IV

12. The state of 2D-stress at a point is given by the following matrix of stresses:

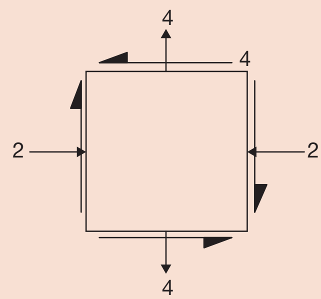
$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 100 & 30 \\ 30 & 20 \end{bmatrix} \text{MPa}$$

What is the magnitude of maximum shear stress in MPa? **[GATE, 2013]**

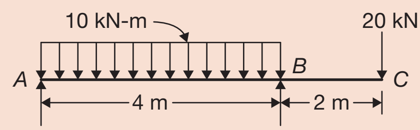
(A) 50 (B) 75
(C) 100 (D) 110

13. The state of stress at a point is given by $\sigma_x = -6$ MPa, $\sigma_y = 4$ MPa, and $\tau_{xy} = -8$ MPa. The maximum tensile stress (in MPa) at the point is _______. **[GATE, 2014]**

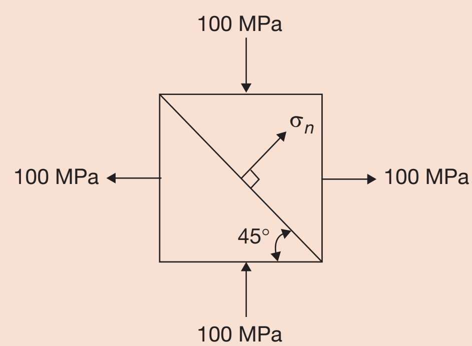
14. For the state of stress (in MPa) shown in the following figure, the maximum shear stress (in MPa) is _______. **[GATE, 2014]**



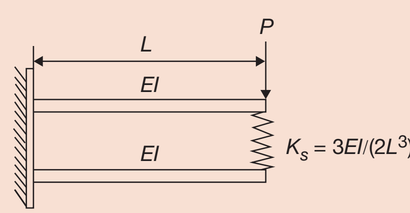
15. For the overhanging beam shown in figure, the magnitude of maximum bending moment (in kN-m) is _______. **[GATE, 2015]**



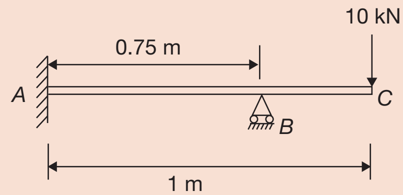
16. Two triangular wedges are glued together as shown in the following figure. The stress acting normal to the interface, σ_n is _______ MPa. **[GATE, 2015]**



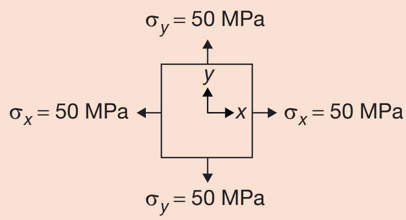
17. Two beams are connected by a linear spring as shown in the following figure. For a load P as shown in the figure, the percentage of the applied P carried by the spring is _________. **[GATE, 2015]**



18. A horizontal beam *ABC* is loaded as shown in the figure. The distance of the point of contraflexure from end *A* (in m) is ________. **[GATE, 2015]**

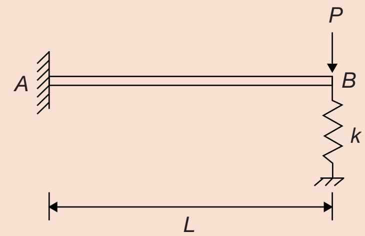


19. For the plane stress situation shown in the figure, the maximums shear stress and the plane on which it acts are **[GATE, 2015]**

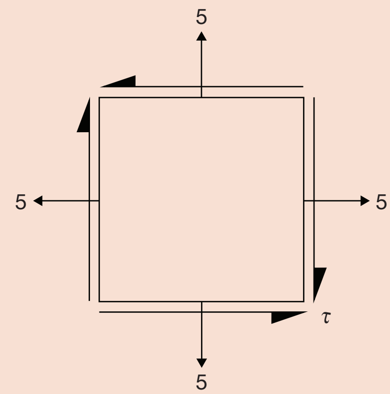


(A) −50 MPa, on a plane 45° clockwise wrt *x*-axis.
(B) −50 MPa, on a plane 45° anti-clockwise wrt *x*-axis.
(C) 50 MPa, at all orientations.
(D) zero, at all orientations.

20. A steel strip of length, $L = 200$ mm is fixed at end *A* and rests at *B* on a vertical spring of stiffness, $k = 2$ N/mm. The steel strip is 5 mm wide and 10 mm thick. A vertical load, $P = 50$ N is applied at *B*, as shown in the figure. Considering $E = 200$ GPa, the force (in N) developed in the spring is ________. **[GATE, 2015]**



21. For the stress state (in MPa) shown in the figure, the major principal stress is 10 MPa. **[GATE, 2016]**



The shear stress τ is

(A) 10.0 MPa (B) 5.0 MPa
(C) 2.5 MPa (D) 0.0 MPa

Answer Keys

Exercises

1. B	**2.** C	**3.** C	**4.** B	**5.** D	**6.** A	**7.** C	**8.** A	**9.** C	**10.** C
11. B	**12.** A	**13.** D	**14.** C	**15.** A	**16.** D	**17.** C	**18.** A	**19.** D	**20.** D
21. B	**22.** D	**23.** B	**24.** A	**25.** C	**26.** A	**27.** A	**28.** B	**29.** A	**30.** C
31. A	**32.** B	**33.** C	**34.** A	**35.** D	**36.** A	**37.** D	**38.** C	**39.** C	**40.** B
41. C									

Previous Years' Questions

1. B **2.** B **3.** A **4.** A **5.** C **6.** B **7.** D **8.** A **9.** B **10.** A
11. C **12.** A **13.** 8.4 to 8.5 **14.** 5 **15.** 40 **16.** 0 (Zero) **17.** 33.33%
18. –0.25 **19.** D **20.** 3.2 **21.** B