Unit 5

Application of Derivative

Teaching-Learning Points

● Concept of derivative originated from the study of rate of change of one quantity with respect to the other. So, the notion of derivatives has a wide range of application in basic sciences, engineering, economics and other field. In this chapter we shall learn, derivative as a Rate measure, tengents and normal, increasing & decreasing function, maxima and minima of function and approximation.

● Derivative as a Rate Measure : If a quantity 'y' varies with another quantity x, satisfying some rule y = f(x) then $\left.\frac{dy}{dx}\right|_{x = x_0}$ represents the rate of change of y w.r.t 'x' at $x = x_0$

Exp. Area of circle depends upon radius

$A = \pi r^2$

$\frac{dy}{dr} = 2\pi r$ will represent rate of change of area w.r.t radius

If two variables x and y varying w.r.t another vareable. If y = f(t) and x = g(t) by chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \frac{dx}{dt} \neq 0$$

Hence rate of change of y w.r.t 'x' can be calculated by using rate of change of both y and x with respect to t.

● Increasing and Decreasing functions–

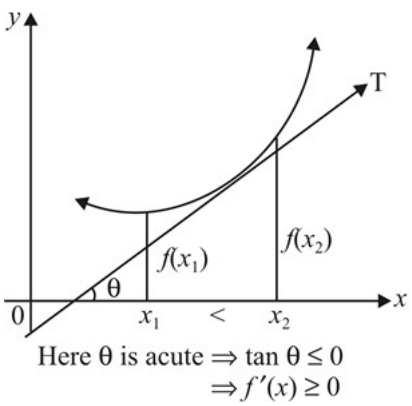
(i) Teacher must explain the pre-idea of increasing and decreasing functions graphically.

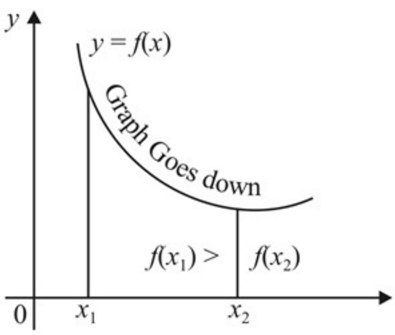
(ii) If f (x) be a real valued continuous function defined on (a, b) is said to be increasing function on (a, b) if

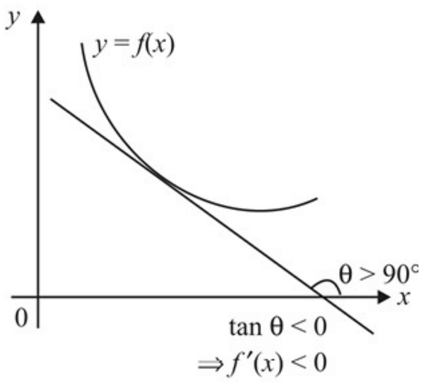
$x_1, x_2 \in (a, b)$ such that $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

OR

$f'(x) > 0 \ \forall \ x \in (a, b)$







if f(x) be real valued continuous function defined on (a, b) is said to be decreasing in (a, b) if $x_1, x_2 \in (a, b)$ such that $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

OR

$$f'(x) < 0 \ \forall \ x \in (a, b)$$

● **Tangents and Normals :**

(i) If we use the Geometrical meaning of the derivative, then f ′(x) at (x_1, y_1) represents the slope of the tangent at the point where f(x) is continuous and differentiable.

(ii) Slope of the Normal to the curve y = f(x) at point (x_1, y_1) is given by $\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$

(iii) Since the tangent to the curve y = f(x) at point (x_1, y_1) is a straight line. Hence tangent will also be represented by an equation. So equation of tangent is given by $y - y_1 = \left.\dfrac{dy}{dx}\right)_{(x_1, y_1)}^{(x - x_1)}$ and normal is given by

$$y - y_1 = \frac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

(iv) Tangent and Normal to any curve y = f(x) at a given point are the lines passing the point and perpendicular to each other.

● **Approximations :**

Let $f : D \to R$, $D \subset R$, R set of real number

and y = f(x)

$\Delta y = f(x + \Delta x) - f(x)$

and also $\dfrac{dy}{dx} = f'(x)$

$\Rightarrow \Delta y = \left(\dfrac{dy}{dx}\right).\Delta x$

if $dx \approx \Delta x$ then $dy \approx \Delta y$

Hence we get the approximate value of Δy

● **Maximum and minium :**

Before attempting the exercise the students must know the following facts :

(i) $|ax + b| \geq 0 \ \forall \ x \in R$, where a, b any real number

(ii) $-1 \le \sin(ax+b) \le 1, \forall x \in R$

(iii) $0 \le |\sin(ax+b)| \le 1, \forall\ x \in R$

(iv) $-a \le |\,a\sin(\lambda x) \le a, \forall\ x \in R$ and $a > 0$

(v) $(ax+b)^2 \ge 0\ \forall\ x \in R$

First derivative test – let f(x) be a differentiable function defined on Interval I and x = C ∈ I, then

(a) x = c is said to be a point of local maxima, if

(i) f ′(x) = 0, and

(ii) f ′(x) changes sign from (+) ive to (–) ive as increases through c i.e., f ′(x) < 0 at every point sufficiently close to and the left of C, and f ′(x) < 0 at every point sufficiently close to and to the right of C.

(b) x = c is called the point of local minima, if

(i) f ′(c) = 0 and

(ii) f ′(x) changes sign from (–)ive to (+)ive as x increases through c i.e., f ′(x) < 0 at every point sufficiently close to and the left of c and f ′(x) > 0 at every point sufficiently close to and to the right to c.

(c) f ′(x) = 0 and f ′(x) does not change signs as x increases through c i.e., f ′(x) has the same sign in the complete neighbourbood of c, then point c is neither a point of local maxima and nor a point of local minima. Such point is said to be a point of inflexion.

● **Second derivative test :**

Theorem let f he a real valued function having second derivative at c such that

(i) f ′(c) = 0 and f ″(c) > 0 then f has a local minimum value at c

(ii) f ′(c) = 0 and f ″(c) < 0 then f has a local maximum value at c

(iii) f ′(c) = 0 and f ″(c) = 0 test fail

Question for Practice

Very Short Answer Type Questions (1 Mark)

Q1. Find the rate of change of area of circle with respect to its radius r when r = 3cm.

Q2. If the radius of a circle is increasing at the rate of .7 cm/sec at what rate its circumference is increasing.

Q3. What is the point on curve $y = 3x^2 - 1$ at which slope of tangent is 3.

Q4. At what point on curve $y = \dfrac{\sqrt{3}x^2}{2}$ tangents makes 60° with x axis.

Q5. What is the slope of the normal to the curve

$x = a\cos^3\theta$ $y = a\sin^3\theta$ at $\theta = \pi/4$

Q6. What is the point on the curve $y = x^2 - 2x + 3$ at which tangent is || to x-axis.

Q7. What is maximum value of |sin 2x + 3|

Q8. What is the minimum value of |4 sin 2x + 3|

Q9. What is the max value of sin x + cos x

Q10. Find an angle which increases twice as fast as its sine.

Short Answer Type Questions (4 Mark)

Q1. A man of height 2 m walks at a uniform speed of 5 km/hr away from a lamp post which is 6 m high. Find the rate at which shadow increases.

Q2. The two equal sides of an isoceles triangles with fixed base b are decreasing at the rate of 3cm/sec.

How fast is the area decreasing when two equal sides are equal to the base.

Q3. Water is leaking from a conical funnel at the rate of 5cm^3/sec. If the radius of the base of the funnel is 10 cm and its height is 20 cm. Find the rate at which the water level is dropping when it is 5 cm away from the top.

Q4. Water is dripping out from a conical funnel at the uniform rate of 4 cm^3/sec through a tiny hole at the vertex in the bottom. When the slant height of the water is 3 cm. find the rate of decrease in slant height of the water, given that the vertical, angle of the cone is 120°.

Q5. Find the points on the curve $9y^2 = x^3$ where the normal to the curve makes equal intercepts on its coordinate axis.

Q6. Show that the line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y = be^{-x/a}$ at a pt where is crosses y-axis.

Q7. Find the points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which tangents are

(i) parallel to x-axis (ii) parallel to y-axis

Q8. A kite is 120 m height and 130 m of string is out. If the kite is moving horizontally at the rate of 5.2 m/s. Find the rate at which string is being paid out at that instant.

Q9. Using differential find approximate values of

(i) $\sqrt{.037}$ (ii) $\sqrt{.0037}$

Q10. Find the intervals in which the following function are strictly increasing or strictly decreasing:

(i) f(x) = $20-9x+6x^2-x^3$

(ii) $f(x)=x^3-12x^2+36x+17$

(iii) $f(x)=\frac{3}{10}x^4-\frac{4}{5}x^3-3x^2+\frac{36}{5}x+11$

(iv) $f(x)=x^4-2x^2$

(v) $f(x)=(x-2)^3(x-1)^2$

(vi) $f(x) = (x+2)e^{-x}$

(vii) $f(x) = x^2(x-2)^2$

(viii) $f(x) = \log(1+x) - \frac{x}{1+x} \quad x > -1$

(ix) $f(x) = \sin x + \cos x \quad (0, 2\pi)$

(x) $f(x) + \sin 3x \quad (0, \pi/2)$

Q11. Find the least value of 'a' so that the function f(x) = x^2 + ax + 1 is strictly increasing on (1, 2).

Q12. Find the equation of tangent lines to the curve $y = 4x^3 - 3x + 5$ which are perpendicular to the line $9y + x + 3 = 0.$

Q13. Show that the curves $x = y^2 \quad xy = k$ cut orthogonally if $8k^2 = 1$

Q14. If the tangents to the curve $y = x^3 + ax + b$ at P(1, –6) 11 to the line y – x = 5. Find the values of a and b.

Q15. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which y coordinate is changing 8 times as fast as x coordinate.

Q16. Find the absolute maximum and munimum value of $f(x) = 2x^3 - 9x^2 + 12x - 5 \quad [0,3]$

Q17. Find the local maximum and local minimum of $f(x) = \sin 2x - x \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

Q18. Find the local maximum and local minimum $f(x) = \sin^4 x + \cos^4 x \quad 0 < x < \pi$

Q19. If $y = a\log|x| + bx^2 + x$ has extreme value at x = –1 and x = 2 find a and b.

Q20. If $y = \frac{ax-b}{(x-1)(x-4)}$ has a turning point (2, –1) find the value of a and b.

Long Answer Type Questions (6 Mark)

Q1. A rectangle is inscribed in a semicircle of radius r with one of its side on the diameter of semicircle. Find the dimensions of the rectangle so that its area is maximum. Also find the max area

Q2. A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Q3. A wire of length 36 m is cut into two pieces. One of the piece is turned into the form of a square and other in the form of equilateral triangle. Find the length of each piece so that sum of areas of two figures be minimum.

Q4. Given the perimeters of circle and square, show that sum of area is least when side of square is double the radius of circle.

Q5. Find the point on the parabola $y^2 = 4x$ which is nearest to the point (2, –8).

Q6. Prove that semivertical angle of right circular cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Q7. Prove that of all rectangles with given area, the square has the smallest perimeter.

Q8. Prove that all the rectangles with given perimeter, the square has largest area.

Q9. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangles is isosceles.

Q10. Show that of all rectangles inscribed in a given fixed circle. The square has the maximum area.

Q11. An open box with a square box is to be made out of a given quantity of sheet of area c^2. Show that max volume of the box is $\frac{c^3}{6\sqrt{3}}$.

Q12. If the lengths of three sides of a trapezium other than base are equal to 10 cm each, then find the area of the trapezium when it is maximum.

Q13. A point on the hypotenuse of right angles triangle is at a distance a and b from the sides. Show that the minimum length of the hypotenuse is $(a^{a/3} + b^{2/3})^{3/2}$.

Q14. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of sphere.

Q15. Show that maximum volume of cylinder which can be inscribed in a cone of height 'h' and semivertical 'd' is $\frac{4}{27}\pi h^3 \tan^2 d$.

Q16. Find the area of greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one extremity of major axis.

Q17. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semivertical angle 45° is $\frac{4}{27}\pi h^3$.

Q18. Show that the right circular cone of least curved surface area and given voluem has an altitude equal to $\sqrt{2}$ times the radius of base.

Q19. Show that a cylindrical vessel of given volume has the least surface area when its height is twice its radius.

Q20. Find the equation of the line through the point (3, 4) which cuts from the first quadrant a triangle of minimum area.

Answers

Answer (1 Mark)

1. 6 π cm^2/sec

2. 1.4 π cm/se

3. $x = \frac{1}{2}$ $y = -1/4$

4. $\left(1, \frac{\sqrt{3}}{2}\right)$

5. 1

6. (1, 2)

7. 4

8. 1

9. $\sqrt{2}$

10. $\frac{d\theta}{dt}$ = $2\frac{d}{dt}\sin\theta$

$\frac{d\theta}{dt}$ = $2\cos\theta\frac{d\theta}{dt}$

cos θ = $\frac{1}{2}$

θ = π/3

Answer and Hints of 4 Mark Question

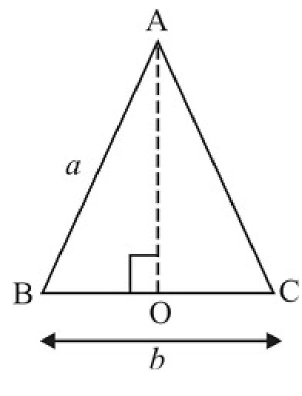
1. $\frac{5}{2}\text{km/hr}$

2. Let ABC is isoceles Δ

AB = AC

At any time AS = AC = a

$AD^2 = AB^2 - BD^2 =$ $a^2 - \left(\frac{b}{2}\right)^2$



A = $\frac{1}{2}\text{BC} \times \text{AD}$

A = $\frac{1}{2}\frac{b\sqrt{4a^2-b^2}}{2} = \frac{b}{4}\sqrt{4a^2-b^2}$

$\frac{dA}{dt}$ = $\frac{b}{4}\times\frac{1}{2}(4a^2-b^2)^{-1/2}4\ \ 2a\frac{da}{dt} = \frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$

when a = b $\frac{dA}{dt} = \frac{b\times b}{\sqrt{4b^2-b^2}}(-3) = -\sqrt{3}b\ \text{cm}^2/\text{s}$

–ve sign indicate the area of isosceles Δ is decreasig at rate of $\sqrt{3}\ \text{cm}^2/\text{sec}$

3. $\frac{4}{45\pi}\text{cm/sec}$

4. V he the volume of cone & l be the slant height of water at any time

V = $\frac{\pi}{3}(l\sin 60°)^2 l\cos 60° = \frac{\pi}{8}l^3$

Find $\frac{dl}{dt}$ when $\frac{dV}{dt} = -4\ \text{cm}^3/\text{s}\quad l = 3$

5. $9y^2 = x^3$

Diff w.r.t. 'x' we get

$9\times 2y\frac{dy}{dx}$ = $3x^2$ ⇒ $\frac{dy}{dx} = \frac{x^2}{6y}$ scape of normal to curve at x, y, = $\frac{-6y_1}{x_1^2}$

Normal to the curve makes equal intercepts on coordinate axes its slope will be = ± 1

$\frac{-6y_1}{x_1^2}$ = ± 1

(x_1, y_1) lies on curve (1) $9y_1^2 = x_1^3$

$9\left(\pm\frac{x_1^2}{6}\right)^2$ = x_1^3 ⇒ $x_1^4 = 4x_1^3 \Rightarrow x_1 = 0, 4$

when x_1 = 0 y_1 = 0 when x_1 = 4 $y_1 = \pm\frac{8}{3}$ as normal makes equal intercept it cannot passing through origin

Ans. $\left(4, \frac{8}{3}\right) (4, -8/3)$

7. (i) for no real pt tangent is || to x-axis

(ii) at (± 3, 0) tangent is || to y-axis

8. 2 cm/sec.

9. (i) Hint : Let x = .04 Δx = – .003 Ans. .1925 approximate

(ii) Hint : Let x = .0036 Δx = .0001 Ans. .0608 approximate

10. (i) f is strictly increasing in (1, 3)

f is strictly decreasing in $(-\infty, 1) \cup (3, \infty)$

(ii) f is strictly decreasing in (2, 6)

f is strictly increasing in $(-\infty\ 2) \cup (6, \infty)$

(iii) f is strictly decreasing in $(-\infty\ 2) \cup (1, 3)$

f is strictly increasing in $(-2, 1) \cup (3, \infty)$

(iv) f is strictly increasing in $(-1, 0) \cup (1, \infty)$

f is strictly decreasing in $(-\infty\ -1) \cup (0, 1)$

(v) f is strictly decreasing in $\left(1, \frac{7}{5}\right)$

f is strictly increasing in $(-\infty - 1) \cup \left(\frac{7}{5}, \infty\right)$

(vi) f is strictly increasing in $(-\infty\ -1)$

f is strictly decreasing in $(-1, \infty)$

(vii) f is strictly increasing in $(0, 1) \cup (2, \infty)$

f is strictly decreasing in $(-\infty, 0) \cup (1, 2)$

(viii) f is strictly decreasing in (–1, 0)

f is strictly increasing in $(0, \infty)$

(ix) f is strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(x) f is strictly decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

f is strictly increasing in $\left(0, \frac{\pi}{6}\right)$

11. a = – 2

12. 9x – y – 3 = 9x – y + 13 = 0

14. **Hints** : Slpe of tangent = slope of line

$\frac{dy}{dx}$ = $3x^2$ + a slop of line = 1

$\left(\frac{dy}{dx}\right)_{(1,-6)}$ = 3 + a = 1 a = – 2

Curve passes through (1, –6) $-6 = 1^3 + a + b \Rightarrow b = -5$

Ans. a = – 2 b = – 5

15. (4, 11) $\left(-4, \frac{-31}{3}\right)$

16. pt of maxima is 3 and absolute max value is 4

pt of maxima is 0 and absolute man value is – 5

17. f has local min at $\frac{-\pi}{6}$ local min value $\frac{-\sqrt{3}}{2} + \frac{\pi}{2}$

f has local max at $\frac{\pi}{6}$ local max value $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$

18. f has local min at $\frac{\pi}{4}$ and min value is $\frac{1}{2}$

f has local max at $\frac{\pi}{2}$ and mix volue is 1

f has local min value at $\frac{3\pi}{4}$ and min valueis $\frac{1}{2}$

19. a = 2 b = $\frac{-1}{2}$

20. y = $\frac{ax-b}{(x-1)(x-y)} = \frac{ax-b}{x^2-5x+4}$ Domain x ∈ R – {1, 4}

$$\frac{dy}{dx} = \frac{(x^2-5x+4)a-(ax-b)(2x-5)}{(x^2-5x+4)^2}$$

$\frac{dy}{dx}$ = 0 and x = 2 we get b = 0

trining pt (2 – 1) lies on curve also

$-1 = \frac{2a-b}{4-10+4}$ ⇒ 2a – b = 2

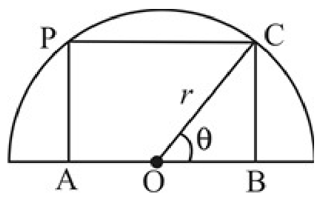
Ans. a = 1 b = 0

Answer and Hints of Six Marks Questions

1. Let ∠BO C = θ OC = r/radius of semicircle

Area of rectangle = AB × BC AB = 20B = 2r cosθ

A = r^2 2sinθ cosθ = r^2 sin 2θ BC = r sin θ

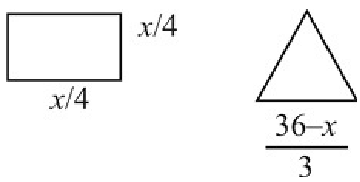


2. Radius of semi circle is $\frac{10}{4+\pi}$ side of rectangle

are $\frac{20}{4+\pi}$ m and $\frac{10}{4+\pi}$ cm

3. Let the length of the piece bent in the form of a square be x cm, then the length of the piece bent in the form of equilateral Δ is 36 – x cm. Let s be the combined are of two figure

$$S = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$$



Now find $\frac{d5}{dt}$ = 0

Ans. x = $\frac{144\sqrt{3}}{9+4\sqrt{3}}$

12. $75\sqrt{3}\ \text{cm}^2$

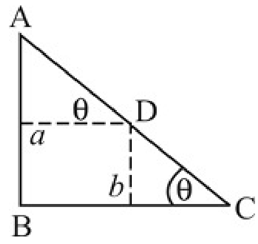
13. AC = AD + DC

l = a sec θ + b cosec θ

$\frac{dl}{d\theta}$ = a sec θ tan θ – b cosec θ cot θ AC = l

$\frac{dl}{d\theta}$ = 0 $\frac{a\sin\theta}{\cos\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0$

tan^{3}θ = $\frac{b}{a}$ $\tan\theta = \left(\frac{b}{a}\right)^{1/3}$



$\frac{d^2 l}{d\theta^2}$ =

$a[\sec^3\theta + \sec\theta\tan^2\theta] + b[\mathrm{cosec}^3\theta + \mathrm{cosec}\theta\cot^2\theta] > 0$ for $0 < \theta < \frac{\pi}{2}$

l = $a[1+\tan^2\theta]^{1/2} + b[1+\cot^2\theta]^{1/2} \Rightarrow a\left[1+\frac{b^{2/3}}{a^{2/3}}\right]^{1/2} + b\left[1+\frac{a^{2/3}}{b^{2/3}}\right]^{1/2}$

l = $a^{2/3}[a^{2/3}+b^{2/3}]^{1/2} + b^{2/3}[a^{2/3}+b^{2/3}]^{1/2} \Rightarrow (a^{2/3}+b^{2/3})^{3/2}$

20. eq of line $\frac{x}{a} + \frac{y}{b} = 1$ It passes through (3, 4)

$\frac{3}{a} + \frac{4}{b} = 1$ ⇒ 3b + 4a = ab ⇒ b = $\frac{49}{a-3}$

A = Area of Δ = $\frac{1}{2}ab = \frac{1}{2}\frac{a \times 49}{a-3} = \frac{2a^2}{a-3}$ a ≠ 3

find $\frac{dA}{da}$ = 0 A is munimum at a = 6 b = 8