समाकलन

Ex 9.1

निम्न फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.

(a) $\sqrt[3]{x^2}$ (b) e^{3x}

(c) $\left(\frac{1}{2}\right)^x$ (d) $a^{2\log_a x}$

हल :

(a) $\int \sqrt[3]{x^2}\, dx = \int x^{2/3}\, dx$

$$= \frac{x^{2/3+1}}{\frac{2}{3}+1} + C$$

$$= \frac{x^{5/3}}{5/3} + C$$

$$= \frac{3}{5}x^{5/3} + C$$

(b) $\int e^{3x}\, dx = \frac{e^{3x}}{3} + C$

(c) $\int \left(\frac{1}{2}\right)^x dx = \left(\frac{1}{2}\right)^x \Big/ \log_e \frac{1}{2} + C$

$$= \frac{\left(\frac{1}{2}\right)^x}{\log_e\left(\frac{1}{2}\right)} + C$$

(d) $\int a^{2\log_a x}\, dx = \int a^{\log_a x^2}\, dx$

$$= \int x^2\, dx$$

$$= \frac{x^3}{3} + C$$

प्रश्न 2. निम्न समाकलों के मान ज्ञात कीजिए

$$\int\left(5 \cos x-3 \sin x+\frac{2}{\cos^2 x}\right) dx$$

हल:

$$\int\left(5 \cos x-3 \sin x+\frac{2}{\cos^2 x}\right) dx$$

$= 5 \int\cos x\, dx - 3 \int\sin x\, dx + 2 \int\sec^2 x\, dx$
$= 5 \sin x - 3(-\cos x) + 2 \tan x + c$
$= 5 \sin x + 3 \cos x + 2 \tan x + c$

प्रश्न 3.

$$\int \frac{x^3-1}{x^2} dx$$

हल:

$$\int \frac{x^3-1}{x^2} dx = \int \frac{x^3}{x^2} dx - \int \frac{1}{x^2} dx$$

$$= \int x\, dx - \int x^{-2} dx$$

$$= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1}$$

$$= \frac{x^2}{2} + x^{-1} + c$$

$$= \frac{x^2}{2} + \frac{1}{x} + c$$

प्रश्न 4. $\int(\sec^2 x + \text{cosec}^2 x)\, dx$

हल : $\int(\sec^2 x + \text{cosec}^2 x)\, dx$
$= \int\sec^2 x + \int\text{cosec}^2\, dx$
$= \tan x - \cot x + c$

प्रश्न 5. $\int(1 + x)\sqrt{x}\, dx$

हल: $\int(1 + x)\sqrt{x}\, dx$

$$= \int (x^{1/2} + x^{3/2})\, dx$$

$$= \int x^{1/2}\, dx + \int x^{3/2}\, dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + c$$

$$= \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + c$$

प्रश्न 6. $\int a^x\, da$

हल: $\int a^x\, da$ =

$$= \frac{a^{x+1}}{x+1}$$

प्रश्न 7.

$$\int \frac{x^2}{1+x^2}\, dx$$

हल:

$$\int \frac{x^2}{1+x^2}\, dx = \int \frac{x^2+1-1}{1+x^2}\, dx$$

$$= \int \frac{1+x^2}{1+x^2}\, dx - \int \frac{1}{1+x^2}\, dx$$

$$= \int 1 .dx - \int \frac{1}{1+x^2}\, dx$$

$$= x - \tan^{-1} x + c$$

प्रश्न 8.

$$\int \frac{\cos^2 x}{1+\sin x}\, dx$$

हल:

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{1-\sin^2 x}{1+\sin x}\,dx$$

$$= \int \frac{(1-\sin x)(1+\sin x)}{(1+\sin x)}\,dx$$

= ∫(1 − sin x) dx
= ∫dx − ∫sin x dx
= x − (-cos x) + c
= x + cos x + c

प्रश्न 9. ∫sec x (sec x + tan x) dx

हल: ∫sec x (sec x + tan x) dx
= $\int \sec^2 x\,dx + \int \sec x \tan x\,dx$
= tan x + sec x + c

प्रश्न 10. $\int(\sin^{-1} x + \cos^{-1} x)\,dx$

हल: $\int(\sin^{-1} x + \cos^{-1} x)\,dx$

$$= \int\left(\frac{\pi}{2}\right)dx \quad \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{3}\right]$$

$$= \frac{\pi}{2}\int dx$$

$$= \frac{\pi}{2}x + c$$

प्रश्न 11.

$$\int \frac{x^2-1}{x^2+1}\,dx$$

हल:

$$\int \frac{x^2-1}{x^2+1}\,dx = \int \frac{x^2+1-2}{1+x^2}\,dx$$

$$= \int \frac{x^2+1}{1+x^2}\,dx - 2\int \frac{1}{1+x^2}\,dx$$

$$= \int dx - 2\int \frac{1}{1+x^2}\,dx$$

= $x - 2\tan^{-1} x + c$

प्रश्न 12. $\int \tan^2 x\, dx$

हल: $\int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$
$= \int \sec^2 - \int dx$
$= \tan x - x + c$

प्रश्न 13. $\int \cot^2 x\, dx$

हल: $\int (\text{cosec}^2 x - 1) dx$
$= \int \text{cosec}^2\, dx - \int dx$
$= - \cot x - x + c$

प्रश्न 14.

$$\int \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

हल:

$$\int \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \int \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}\, dx$$

$$= \int \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\, dx$$

$$= \int (1+x)^{1/2}\, dx + \int x^{1/2}\, dx$$

$$= \frac{(1+x)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} + c$$

$$= \frac{2}{3}(1+x)^{3/2} + \frac{2}{3}x^{3/2} + c$$

प्रश्न 15. $\int (\tan^2 x - \cot^2 x)\, dx$

हल: $\int (\tan^2 x - \cot^2 x)\, dx$
$= \int (\sec^2 x - 1 - \text{cosec}^2 x + 1)\, dx$
$= \int \sec^2 x\, dx - \int \text{cosec}^2 x\, dx$
$= \tan x + \cot x + c$

प्रश्न 16.

$$\int \frac{\sin x}{1 + \sin x}\, dx$$

हल:

$$\int \frac{\sin x}{1+\sin x}\, dx$$

$$= \int \frac{1+\sin x-1}{1+\sin x}\, dx$$

$$= \int \frac{1+\sin x}{1+\sin x}\, dx - \int \frac{1}{1+\sin x}\, dx$$

$$= \int dx - \int \frac{1}{1+\sin x} + \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int dx - \int \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$= \int dx - \int \frac{1-\sin x}{\cos^2 x}\, dx$$

$= \int x - \int \sec^2 x\, dx + \int \tan x \sec x\, dx$
$= x - \tan x + \sec x + c$

प्रश्न 17.

$$\int \frac{1}{1-\cos x}\, dx$$

हल:

$$\int \frac{1}{1-\cos x}\, dx$$

$$= \int \frac{1}{1+\cos x} \times \frac{1+\cos x}{1+\cos x}\, dx$$

$$= \int \left(\frac{1+\cos x}{1-\cos^2 x}\right) dx$$

$$= \int \left(\frac{1+\cos x}{\sin^2 x}\right) dx$$

$$= \int \frac{1}{\sin^2 x}\, dx + \int \frac{\cos x}{\sin^2 x}\, dx$$

$= \int \operatorname{cosec}^2 x\, dx + \int \operatorname{cosec} x \cot x\, dx$
$= -\cot x - \operatorname{cosec} x + c$

प्रश्न 18.

$$\int \left[1 + \frac{1}{1+x^2} + \frac{3}{x\sqrt{x^2-1}} + 2^x\right] dx$$

हल:

$$\int\left[1+\frac{1}{1+x^2}+\frac{3}{x\sqrt{x^2-1}}+2^x\right]dx$$

$$= \int 1\,dx+\int\frac{1}{1+x^2}\,dx+3\int\frac{1}{x\sqrt{x^2-1}}\,dx+\int 2^x\,dx$$

$$= \left[x+\tan^{-1}x+3\sec^{-1}x+\frac{2^x}{\log_e 2}+c\right]$$

प्रश्न 19. ∫cot x (tan x – cosec x) dx

हल: ∫cot x (tan x – cosec x) dx
= ∫cot x tan x dx – ∫cot x cosec x dx
= ∫1 dx – ∫cosec x cot x dx
= x + cosec x + c

प्रश्न 20.

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

हल:

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int\left(x+\frac{1}{x}+2\times\sqrt{x}\times\frac{1}{\sqrt{x}}\right)dx$$

$$= \int\left(x+\frac{1}{x}+2\right)dx$$

$$= \int x\,dx+\int\frac{1}{x}\,dx+\int 2\,dx$$

$$= \frac{x^2}{2}+\log|x|+2x+c$$

प्रश्न 21. ∫$\log_x$ x dx

हल: ∫$\log_x$ x dx

$$= \int\frac{\log x}{\log_e x}\,dx$$

=∫1 dx
= x + c

प्रश्न 22.

$$\int \sqrt{1+\cos 2x}\, dx$$

हल:

$$\int \sqrt{1+\cos 2x}\, dx$$

$$= \int \sqrt{1+2\cos^2 x-1}\, dx$$

$$= \int \sqrt{2\cos^2 x}\, dx$$

$= \sqrt{2}\int \cos x\, dx$
$= \sqrt{2} \sin x + c$

प्रश्न 23.

$$\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\, dx$$

हल:

$$\int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \frac{\cos^2 x-\sin^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx - \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx$$

$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx$
$= -\cot x - \tan x + c$

प्रश्न 24.

$$\int \frac{3\cos x+4}{\sin^2 x}\, dx$$

हल:

$$\int \frac{3\cos x+4}{\sin^2 x}\, dx$$

$= 3\int \operatorname{cosec} x \cot x\, dx + 4\int \operatorname{cosec}^2 x\, dx$
$= -3 \operatorname{cosec} x + 4(-\cot x) + c$
$= -3 \operatorname{cosec} x - 4 \cot x + c$

Ex 9.2

निम्न फलनों को x के सापेक्ष समाकलन कीजिए

प्रश्न 1.
(a) $\int x \sin x^2 \, dx$
(b) $\int x\sqrt{x^2 + 1} \, dx$

हल : (a) $\int x \sin x^2 \, dx$

माना $x^2 = t$

$\Rightarrow 2x \, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x \sin x^2 \, dx = \int \sin x^2 . x \, dx$$

$$= \int \sin t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}(-\cos t) + C$$

$$= -\frac{1}{2}\cos x^2 + C$$

(b) $\int x\sqrt{x^2 + 1} \, dx$

माना $x^2 + 1 = t$

$\Rightarrow 2x \, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x\sqrt{x^2+1}\, dx = \int \sqrt{x^2+1} \cdot x\, dx$$

$$= \int t^{1/2} \frac{dt}{2} = \frac{1}{2} t^{3/2} \times \frac{2}{3} + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

प्रश्न 2.

(a) $\int \frac{e^x - \sin x}{e^x + \cos x}$ **(b)** $\int \frac{e^x}{\sqrt{1+e^x}}$

हल :

(a)

$$\int \frac{e^x - \sin x}{e^x + \cos x} dx$$

माना $e^x + \cos x = t$

$\Rightarrow (e^x - \sin x) = dt$

$$\int \frac{(e^x - \sin x)}{(e^x + \cos x)} dx = \int \frac{dt}{t}$$

$= \log |t| + C$

$= \log |e^x + \cos x| + C$

(b)

$$\int \frac{e^x}{\sqrt{1+e^x}} dx$$

माना

$1 + e^x = t$

$e^x\, dx = dt$

$$\int \frac{e^x}{\sqrt{1+e^x}} dx = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt$$

$$= \frac{t^{1/2}}{1/2} + C = 2(1 + e^x)^{1/2} + C$$

$$= 2\sqrt{1+e^x} + C$$

प्रश्न 3.

(a) $\int \sqrt{e^x + 1}\, dx$ **(b)** $\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} dx$

हल :

(a)

$$\int \sqrt{e^x + 1}\ dx$$

माना $\sqrt{e^x + 1} = y$

$\Rightarrow$ $e^x + 1 = y^2$

$\Rightarrow$ $e^x = y^2 - 1$

$\Rightarrow$ $e^x\, dx = 2y\, dy$

$\Rightarrow$ $dx = \dfrac{2y}{e^x} dy$

$\Rightarrow$ $dx = \dfrac{2y}{y^2 - 1} dy$

$$\int \sqrt{e^x + 1}\ dx = \int y \cdot \frac{2y}{y^2 - 1}\, dy$$

$$= 2\int \frac{y^2}{y^2 - 1}\, dy$$

$$= 2\int \frac{(y^2 - 1) + 1}{y^2 - 1}\, dy$$

$$= 2\int 1\, dy + 2\int \frac{1}{y^2 - 1}\, dy$$

$$= 2y + 2 \times \frac{1}{2} \log\left(\frac{y - 1}{y + 1}\right) + C$$

$$= 2y + \log\left(\frac{y - 1}{y + 1}\right) + C$$

$$= 2\sqrt{e^x + 1} + \log\left(\frac{\sqrt{e^x + 1} - 1}{\sqrt{e^x + 1} + 1}\right) + C$$

$$= 2\sqrt{e^x + 1} + \log\left[\frac{(\sqrt{e^x + 1} - 1)\,(\sqrt{e^x + 1} + 1)}{(\sqrt{e^x + 1} + 1)\,(\sqrt{e^x + 1} + 1)}\right] + C$$

$$= 2\sqrt{(e^x + 1)} + \log\left[\frac{e^x + 1 - 1}{e^x + 1 + 1 + 2\sqrt{e^x + 1}}\right] + C$$

$$= 2\sqrt{e^x + 1} + \log\left[\frac{e^x}{e^x + 2 + 2\sqrt{e^x + 1}}\right] + C$$

(b)

$$\int \frac{e^{\sqrt{x}} \cos (e^{\sqrt{x}})}{\sqrt{x}} dx$$

माना $\qquad e^{\sqrt{x}} = t$

$\Rightarrow \qquad e^{\sqrt{x}} . \frac{d}{dx}(\sqrt{x}) = dt$

$\Rightarrow \qquad e^{\sqrt{x}} . \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \qquad \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2dt$

$$\int \frac{\cos (e^{\sqrt{x}}) . e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int \cos t \, dt = 2 \sin t + C$$

$$= 2 \sin (e^{\sqrt{x}}) + C$$

प्रश्न 4.

(a) $\frac{1}{x(1+\log x)}$ **(b)** $\frac{(1+\log x)^3}{x}$

हल :

(a) $\int \frac{1}{x(1+\log x)} dx$

माना $\qquad 1 + \log x = t$

$\Rightarrow \qquad \frac{1}{x} dx = dt$

$$\int \frac{1}{(1+\log x)} \cdot \frac{1}{x} dx = \int \frac{1}{t} dt$$

$$= \log | t | + C$$

$$= \log | 1 + \log x | + C$$

(b) $\int \frac{(1+\log x)^3}{x} dx$

माना $\qquad 1 + \log x = t$

$$\Rightarrow \qquad \frac{1}{x}\,dx = dt$$

$$\int (1+\log x)^3 \cdot \frac{1}{x}\,dx = \int t^3 .dt = \frac{1}{4}t^4 + C$$

$$= \frac{1}{4}(1+\log x)^4 + C$$

प्रश्न 5.

(a) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ **(b)** $\int \frac{\sin^p x}{\cos^{p+2} x}\,dx$

हल :

(a) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$

माना $\quad m\tan^{-1}x = t$

$$\Rightarrow \qquad \tan^{-1}x = \frac{t}{m}$$

$$\Rightarrow \qquad \frac{1}{1+x^2}\,dx = \frac{dt}{m}$$

$$\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx = \int e^t \frac{dt}{m}$$

$$= \frac{1}{m}e^t + C$$

$$= \frac{1}{m}e^{m\tan^{-1}x} + C$$

(b) $\int \frac{\sin^p x}{\cos^{p+2} x}\,dx = \int \frac{\sin^p x}{\cos^p x.\cos^2 x}\,dx$

$$= \int \tan^p x \sec^2 x\,dx$$

माना $\quad \tan x = t$

$$\sec^2 x\,dx = dt$$

$$= \int t^p\,dt = \frac{t^{p+1}}{p+1} + C$$

$$= \frac{(\tan x)^{p+1}}{p+1} + C$$

प्रश्न 6.

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$ (b) $\int \frac{1+\cos x}{\sin x \cos x} dx$

हल :

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$

$= \int \frac{1}{\sqrt{1+2\cos^2 x - 1}} dx$

$= \int \frac{1}{\sqrt{2\cos^2 x}} dx$

$= \frac{1}{\sqrt{2}} \int \sec x \, dx$

$= \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + C$

(b) $\int \frac{1+\cos x}{\sin x \cos x} dx$

$= \int \frac{1}{\sin x \cos x} dx + \int \frac{\cos x}{\sin x \cos x} dx$

$= 2\int \frac{1}{\sin 2x} dx + \int \frac{1}{\sin x} dx$

$= 2\int \operatorname{cosec} 2x \, dx + \int \operatorname{cosec} x \, dx$

$= \frac{2\log|\operatorname{cosec} 2x - \cot 2x|}{2} + \log|\operatorname{cosec} x - \cot x| + C$

= log|cosec 2x – cot 2x| + log |cosec x – cot x |+ C

प्रश्न 7.

(a) ∫sin 3x sin 2x dx

(b) ∫$\sqrt{1-sin x dx}$

हल : (a) ∫sin 3x sin 2x dx

$$= \int \frac{1}{2}[\cos(3x-2x)-\cos(3x+2x)]\,dx$$

$$= \frac{1}{2}\int[\cos x-\cos 5x]\,dx$$

$$= \frac{1}{2}\left(\sin x-\frac{\sin 5x}{5}\right)+C$$

$$= \frac{1}{2}\left(\sin x-\frac{1}{5}\sin 5x\right)+C$$

(b) $\int\sqrt{1-\sin x}\,dx$

$$= \int\sqrt{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}\right)}\,dx$$

$$= \int\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}\,dx$$

$$= \int\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)dx$$

$$= \int\sin\frac{x}{2}\,dx-\int\cos\frac{x}{2}\,dx$$

$$= \frac{-\cos\frac{x}{2}}{\left(\frac{1}{2}\right)}-\frac{\sin\frac{x}{2}}{\left(\frac{1}{2}\right)}+C$$

$$= \pm 2\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+C$$

प्रश्न 8. (a) $\int\cos^4 x\,dx$
(b) $\int\sin^3 x\,dx$

हल : (a) $\int\cos x^4\,dx$
$= -\int(\cos^2 x)^2\,dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \quad \left(\because \cos^2 A = \frac{1+\cos 2A}{2}\right)$$

$$= \frac{1}{4}\int (1+\cos 2x)^2\, dx$$

$$= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx$$

$$= \frac{1}{4}\int 1+2\cos 2x+\frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{8}\int (2+4\cos 2x+1+\cos 4x)\, dx$$

$$= \frac{1}{8}\int (\cos 4x+4\cos 2x+3)\, dx$$

$$= \frac{1}{8}\left[\frac{\sin 4x}{4}+\frac{4\sin 2x}{2}+3x\right]+C$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 4x+2\sin 2x+3x\right]+C$$

(b) $\int \sin^3 x\, dx$

$$= \int \left(\frac{3}{4}\sin x-\frac{1}{4}\sin 3x\right) dx$$

$$(\because \sin 3x = 3\sin x - 4\sin^3 x)$$

$$= \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$

$$= \frac{-3}{4}\cos x + \frac{\cos 3x}{4\times 3} + C$$

$$= \frac{-3}{4}\cos x + \frac{\cos 3x}{12} + C$$

प्रश्न 9.

(a) $\int \frac{1}{\sin x \cos^3 x}\, dx$ (b) $\int \frac{(1+x)e^x}{\cos^2(xe^x)}\, dx$

हल :

(a) $\int \frac{1}{\sin x \cos^3 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x.\cos^3 x} dx$$

$$= \int \left[\frac{\sin^2 x}{\sin x.\cos^3 x} + \frac{\cos^2 x}{\sin x.\cos^3 x}\right] dx$$

$$= \int \left[\frac{\tan x}{\cos^2 x} + \frac{1}{\tan x.\cos^2 x}\right] dx$$

$$= \int \left[\tan x.\sec^2 + \frac{1}{\tan x} \cdot \sec^2 x\right] dx$$

$$= \int \left(\tan x + \frac{1}{\tan x}\right) \sec^2 x \, dx$$

माना $\tan x = t$

$\sec^2 x \, dx = dt$

$$= \int \left(t + \frac{1}{t}\right) dt$$

$$= \frac{1}{2}t^2 + \log|t| + C$$

$$= \frac{1}{2} \tan^2 x + \log|\tan x| + C$$

$$= \log|\tan x| + \frac{1}{2} \tan^2 x + C$$

(b) $\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx$

माना $xe^x = t$

$\Rightarrow (xe^x + e^x.1)dx = dt$

$\Rightarrow (1 + x)e^x \, dx = dt$

$$\therefore \quad \int \frac{(1+x)e^x}{\cos^2(xe^x)}\,dx = \int \frac{1}{\cos^2 t}\,dt$$

$= \int \sec^2 t\, dt = \tan t + c$

$= \tan (xe^x) + c$

प्रश्न 10.

(a) $\int \frac{1}{1-\tan x}\,dx$ **(b)** $\int \frac{1}{1+\cot x}\,dx$

हल :

(a) $\int \frac{1}{1-\tan x}\,dx$

$$= \int \frac{1}{1-\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{\cos x - \sin x + \cos x + \sin x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{\sin x + \cos x}{\cos x - \sin x}\,dx$$

माना cosx – sinx = t

⇒ (sin x + cos x) dx = dt

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\cos x - \sin x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

(b) $\int \frac{1}{1+\cot x}\,dx$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}}\,dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{(\sin x - \cos x)}{(\sin x + \cos x)}\,dx$$

माना sin x + cos x = t

⇒ -cos x + sin x dx = dt

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

प्रश्न 11.

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx$ (b) $\int \frac{1-\tan x}{1+\tan x} dx$

हल :

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx$

माना tan x = t

⇒ $\sec^2 x\, dx = dt$

$$= \int \frac{\sec^2 x.\sec^2 x\, dx}{\sqrt{\tan x}}$$

$$= \int \frac{(1+\tan^2 x)}{\sqrt{\tan x}} \sec^2 x\, dx$$

$$= \int \frac{1+t^2}{t^{1/2}} dt$$

$$= \int (t^{-1/2} + t^{2-1/2})\, dt$$

$$= \int t^{-1/2}\, dt + \int t^{3/2}\, dt$$

$$= \frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2} + C$$

$$= 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + C$$

(b) $$\int \left(\frac{1-\tan x}{1+\tan x}\right) dx = \int \left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right) dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

माना sin x + cos x = t

(cos x − sin x) dx = dt

$= \int \frac{dt}{t}$

= log|t| + C

= log |sin x + cos x| + C

प्रश्न 12.

(a) $\int \frac{\sin (x-a)}{\sin (x+a)} dx$ (b) $\int \frac{\sin x}{\sin (x-a)} dx$

हल :

(a) $\int \frac{\sin (x-a)}{\sin (x+a)} dx$

माना x + a = t

dx = dt

x = t − a

$$\therefore \int \frac{\sin (x-a)}{\sin (x+a)} dx$$

$$= \int \frac{\sin (t-a-a)}{\sin t} dt = \int \frac{\sin (t-2a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos 2a - \cos t \sin 2a}{\sin t} dt$$

$$= \int \frac{\sin t \cos 2a}{\sin t} dt - \int \sin 2a \cdot \frac{\cos t}{\sin t} dt$$

= cos 2a ∫ dt − sin 2a ∫ cot dt

= (cos 2a)t − sin 2a log |sin t| + C1

= (x + a) cos 2a − sin 2a log |sin (x + a) | + C1

= x cos 2a − sin 2a log |sin(x + a)| + a cos 2a + C1

= x cos 2a − sin 2a log |sin (x + a)| + C

(जहाँ C = a cos 2a + C1)

(b) $\int \frac{\sin x}{\sin (x-a)} dx$

माना x − a = t

x = t + a

dx = dt

$$= \int \frac{\sin (t+a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos a + \sin a \cos t}{\sin t}$$

$$= \int \left(\frac{\sin t \cos a}{\sin t} + \frac{\sin a \cos t}{\sin t} \right) dt$$

= ∫cos a dt + ∫sin a cot dt

= cos a.t + sin a log |sin t|

= (x – a) cos a + sin a log |sin (x – a)| + C1
= x cos a + sin a log |sin (x – a)| + (-a cos a + C1)
= x cos a + sin a log |sin (x -a)| + C
(जहाँ C = -a cos a + C1)

प्रश्न 13.

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

(b) $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} dx$

हल :

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

$$= \int \frac{\sin (5x-3x)}{\sin 5x \sin 3x} dx$$

$$= \int \left[\frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} - \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x}\right] dx$$

$$= \int \cot 3x \, dx - \int \cot 5x \, dx$$

$$= \frac{\log|\sin 3x|}{3} - \frac{\log|\sin 5x|}{5} + C$$

$$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + C$$

(b) $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} dx$

$$= \int \frac{\sin\left\{\left(x-\frac{\pi}{6}\right)+\left(x+\frac{\pi}{6}\right)\right\}}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} dx$$

$$= \int \left[\frac{\sin\left(x-\frac{\pi}{6}\right)\cos\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} + \frac{\cos\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} \right] dx$$

$$= \int \left[\cot\left(x+\frac{\pi}{6}\right) dx + \cot\left(x-\frac{\pi}{6}\right) \right] dx$$

$$= \log\left|\sin\left(x+\frac{\pi}{6}\right)\right| + \log\left|\sin\left(x-\frac{\pi}{6}\right)\right| + C$$

$$= \log\left[\sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)\right] + C$$

प्रश्न 14.

(a) $\int \frac{1}{3\sin x + 4\cos x}\, dx$

(b) $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$

हल :

(a) $\int \frac{1}{3\sin x + 4\cos x}\, dx$

माना 4 = sin θ तथा 3 = r cos θ

तब्र $r^2 \sin^2\theta + r^2\cos^2\theta = 3^2 + 4^2 = 5^2$

$\Rightarrow r = 5$

तथा $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{4}{3} \Rightarrow \tan\theta = \dfrac{4}{3}$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\therefore \qquad I = \int \frac{1}{3\sin x + 4\cos x}\,dx$$

$$= \int \frac{1}{r\cos\theta\sin x + r\sin\theta\cos x}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(\theta + x)}\,dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(\theta + x)\,dx$$

$$= \frac{1}{5}\log|\operatorname{cosec}(\theta + x) - \cot(\theta + x)| + C$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{\theta + x}{2}\right)\right|$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{x + \tan^{-1}\left(\frac{4}{3}\right)}{2}\right)\right| + C$$

(b) $\displaystyle\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b-x+a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)} \right] dx$$

$$= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)]\, dx$$

$$= \frac{1}{\sin(a-b)} [\log|\sin(x-a)| - \log|\sin(x-b)| + C$$

$$= \frac{1}{\sin(a-b)} \left[\log \frac{|\sin(x-a)|}{|\sin(x-b)|} \right] + C$$

$$= \operatorname{cosec}(a-b) \log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| + C$$

प्रश्न 15.

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

(b) $\int \frac{\sec x}{\sin(2x+\alpha) + \sin\alpha}\, dx$

हल :

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

माना $a\cos^2 x + b\sin^2 x = t$

$\Rightarrow (-2a\cos x \sin x + 2b \sin x \cos x)\, dx = dt$

$\Rightarrow (2b \sin \cos x - 2a \sin x \cos x)\, dx = dt$

$\Rightarrow 2(b - a) \sin x \cos x\, dx = dt$

$$\Rightarrow \quad \sin x \cos x \, dx = \frac{1}{2(b-a)} \, dt$$

$$\int \frac{\sin x \cos x}{a \cos^2 x + b \sin^2 x} dx$$

$$= \int \frac{1}{t} \times \frac{1}{2(b-a)} dt$$

$$= \frac{1}{2(b-a)} \log |t| + C$$

$$= \frac{1}{2(b-a)} \log |a \cos^2 x + b \sin^2 x| + C$$

(b) $\int \frac{\sec x}{\sqrt{\sin(2x+\alpha) + \sin \alpha}} dx$

माना $I = \int \frac{\sec x}{\sqrt{\sin(2x+\alpha) + \sin \alpha}} dx$

$$= \int \frac{\sec x}{\sqrt{2 \sin\left(\frac{2x+\alpha+\alpha}{2}\right) \cos\left(\frac{2x+\alpha-\alpha}{2}\right)}} dx$$

$$\left[\because \sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)\right]$$

$$= \int \frac{\sec x}{\sqrt{2 \sin(x+\alpha) \cos x}} dx$$

$$= \int \frac{\sec x}{\sqrt{2 \cos x (\sin x \cos \alpha + \cos x \sin \alpha)}} dx$$

$$= \int \frac{\sec x}{\sqrt{2 \cos^2 x (\tan x \cos \alpha + \sin \alpha)}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec x}{\cos x \sqrt{\tan x \cos \alpha + \sin \alpha}} dx$$

$$I = \frac{1}{\sqrt{2}} \int \frac{\sec^2 x}{\sqrt{\tan x \cos \alpha + \sin \alpha}} dx \qquad \ldots(1)$$

माना $\left.\begin{aligned} \tan x \cos \alpha + \sin \alpha &= t \\ \Rightarrow \quad \sec^2 x \cos \alpha \, dx &= dt \\ \Rightarrow \quad \sec^2 x \, dx &= \frac{dt}{\cos \alpha} \end{aligned}\right\} \quad \ldots(2)$

समी. (2) का उपयोग (1) में करने पर,

$$I = \frac{1}{\sqrt{2}} \int \frac{dt}{\cos \alpha \sqrt{t}}$$

$$= \frac{1}{\sqrt{2}.\cos \alpha} \int \frac{1}{\sqrt{t}} \, dt$$

$$= \frac{1}{\sqrt{2}.\cos \alpha} \left(\frac{t^{-1/2+1}}{-1/2+1} \right) + C$$

$$= \frac{2}{\sqrt{2} \cos \alpha} t^{1/2} + C$$

$$= \frac{\sqrt{2}}{\cos \alpha} (\tan x \cos \alpha + \sin \alpha)^{1/2} + C$$

$$\Rightarrow \quad I = \sqrt{2} \sec \alpha (\tan x \cos \alpha + \sin \alpha)^{1/2} + C$$

$$\therefore \int \frac{\sec x}{\sqrt{\sin (2x + \alpha) + \sin \alpha}} dx = \sqrt{2} \sec \alpha (\tan x \cos \alpha + \sin \alpha)^{1/2} + C$$

प्रश्न 16.

(a) $\int \frac{1}{\sqrt{\cos^3 x \sin (x + a)}} dx$

(b) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

हल :

(a) $\int \frac{1}{\sqrt{\cos^3 x . \sin (x + a)}} dx$

$$= \int \frac{1}{\cos^2 x \sqrt{\left(\frac{\sin (x + a)}{\cos x} \right)}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\left(\frac{\sin x \cos a}{\cos x} + \frac{\cos x \sin a}{\cos x}\right)}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x . \cos a + \sin a}} dx$$

माना $\tan x \cos a + \sin a = t$

$$\cos a \sec^2 x \, dx = dt$$

$$\sec^2 x \, dx = \frac{1}{\cos a} dt$$

$$= \int \frac{1}{\sqrt{t}} \times \frac{1}{\cos a} dx$$

$$= \frac{1}{\cos a}\left(\frac{t^{-1/2+1}}{-1/2+1}\right) + C$$

$$= \frac{1}{\cos a} \times \frac{t^{1/2}}{1/2} + C$$

$$= \frac{2}{\cos a}\sqrt{\tan x \cos a + \sin a} + C$$

(b) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2 \alpha + 1)}{\cos x - \cos \alpha} dx$$

$$= 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha} dx$$

$$= 2\int \frac{(\cos x + \cos \alpha)(\cos x - \cos \alpha)}{(\cos x - \cos \alpha)} dx$$

$= 2\int (\cos x + \cos \alpha)\, dx$

$= 2\int \cos x \, dx + 2\int \cos \alpha \, dx$

$= 2 \sin x + 2 \cos \alpha \int dx$

$= 2 \sin x + 2x \cos \alpha + C$

$= 2(\sin x + x \cos \alpha) + C$

Ex 9.1

निम्न फलनों को x के सापेक्ष समाकलन कीजिए

प्रश्न 1.
(a) $\int x \sin x^2 \, dx$
(b) $\int x\sqrt{x^2 + 1} \, dx$

हल : (a) $\int x \sin x^2 \, dx$

माना $x^2 = t$

$\Rightarrow 2x \, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x \sin x^2 \, dx = \int \sin x^2 . x \, dx$$

$$= \int \sin t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}(-\cos t) + C$$

$$= -\frac{1}{2}\cos x^2 + C$$

(b) $\int x\sqrt{x^2 + 1} \, dx$

माना $x^2 + 1 = t$

$\Rightarrow 2x \, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1} \cdot x\,dx$$

$$= \int t^{1/2}\frac{dt}{2} = \frac{1}{2}t^{3/2} \times \frac{2}{3} + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

प्रश्न 2.

(a) $\int \frac{e^x - \sin x}{e^x + \cos x}$ **(b)** $\int \frac{e^x}{\sqrt{1+e^x}}$

हल :

(a)

$$\int \frac{e^x - \sin x}{e^x + \cos x} dx$$

माना $e^x + \cos x = t$

$\Rightarrow (e^x - \sin x) = dt$

$$\int \frac{(e^x - \sin x)}{(e^x + \cos x)} dx = \int \frac{dt}{t}$$

= log |t| + C

= log $|e^x + \cos x|$+C

(b)

$$\int \frac{e^x}{\sqrt{1+e^x}} dx$$

माना

$1 + e^x = t$

$e^x\, dx = dt$

$$\int \frac{e^x}{\sqrt{1+e^x}} dx = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt$$

$$= \frac{t^{1/2}}{1/2} + C = 2(1 + e^x)^{1/2} + C$$

$$= 2\sqrt{1+e^x} + C$$

प्रश्न 3.

(a) $\int \sqrt{e^x + 1}\, dx$ **(b)** $\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} dx$

हल :

(a)

$$\int \sqrt{e^x + 1}\, dx$$

माना $\sqrt{e^x + 1} = y$

$\Rightarrow e^x + 1 = y^2$

$\Rightarrow e^x = y^2 - 1$

$\Rightarrow e^x\, dx = 2y\, dy$

$\Rightarrow dx = \frac{2y}{e^x} dy$

$$\Rightarrow \qquad dx = \frac{2y}{y^2-1}dy$$

$$\int\sqrt{e^x+1}\,dx = \int y\cdot\frac{2y}{y^2-1}dy$$

$$= 2\int\frac{y^2}{y^2-1}dy$$

$$= 2\int\frac{(y^2-1)+1}{y^2-1}dy$$

$$= 2\int 1\,dy + 2\int\frac{1}{y^2-1}dy$$

$$= 2y + 2\times\frac{1}{2}\log\left(\frac{y-1}{y+1}\right)+C$$

$$= 2y + \log\left(\frac{y-1}{y+1}\right)+C$$

$$= 2\sqrt{e^x+1}+\log\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right)+C$$

$$= 2\sqrt{e^x+1}+\log\left[\frac{(\sqrt{e^x+1}-1)(\sqrt{e^x+1}+1)}{(\sqrt{e^x+1}+1)(\sqrt{e^x+1}+1)}\right]+C$$

$$= 2\sqrt{(e^x+1)}+\log\left[\frac{e^x+1-1}{e^x+1+1+2\sqrt{e^x+1}}\right]+C$$

$$= 2\sqrt{e^x+1}+\log\left[\frac{e^x}{e^x+2+2\sqrt{e^x+1}}\right]+C$$

(b)

$$\int\frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}dx$$

माना $\qquad e^{\sqrt{x}} = t$

$$\Rightarrow \qquad e^{\sqrt{x}}.\frac{d}{dx}(\sqrt{x}) = dt$$

$$\Rightarrow \qquad e^{\sqrt{x}}.\frac{1}{2\sqrt{x}}dx = dt$$

$$\Rightarrow \quad \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2dt$$

$$\int \frac{\cos(e^{\sqrt{x}}).e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \cos t\;dt = 2\sin t + C$$

$$= 2\sin(e^{\sqrt{x}}) + C$$

प्रश्न 4.

(a) $\frac{1}{x(1+\log x)}$ **(b)** $\frac{(1+\log x)^3}{x}$

हल :

(a) $\int \frac{1}{x(1+\log x)}\,dx$

माना $1 + \log x = t$

$$\Rightarrow \quad \frac{1}{x}\,dx = dt$$

$$\int \frac{1}{(1+\log x)}\cdot\frac{1}{x}\,dx = \int \frac{1}{t}\,dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

(b) $\int \frac{(1+\log x)^3}{x}\,dx$

माना $1 + \log x = t$

$$\Rightarrow \quad \frac{1}{x}\,dx = dt$$

$$\int (1+\log x)^3\cdot\frac{1}{x}\,dx = \int t^3.dt = \frac{1}{4}t^4 + C$$

$$= \frac{1}{4}(1+\log x)^4 + C$$

प्रश्न 5.

(a) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ **(b)** $\int \frac{\sin^p x}{\cos^{p+2} x}\,dx$

हल :

(a) $\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

माना $m \tan^{-1} x = t$

$\Rightarrow \tan^{-1} x = \frac{t}{m}$

$\Rightarrow \frac{1}{1+x^2} dx = \frac{dt}{m}$

$$\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx = \int e^t \frac{dt}{m}$$

$$= \frac{1}{m} e^t + C$$

$$= \frac{1}{m} e^{m \tan^{-1} x} + C$$

(b) $\int \frac{\sin^p x}{\cos^{p+2} x} dx = \int \frac{\sin^p x}{\cos^p x . \cos^2 x} dx$

$$= \int \tan^p x \sec^2 x \, dx$$

माना $\tan x = t$

$\sec^2 x \, dx = dt$

$$= \int t^p \, dt = \frac{t^{p+1}}{p+1} + C$$

$$= \frac{(\tan x)^{p+1}}{p+1} + C$$

प्रश्न 6.

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$ (b) $\int \frac{1+\cos x}{\sin x \cos x} dx$

हल :

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$

$$= \int \frac{1}{\sqrt{1+2\cos^2 x - 1}} dx$$

$$= \int \frac{1}{\sqrt{2}\cos^2 x} dx$$

$$= \frac{1}{\sqrt{2}} \int \sec x \, dx$$

$$= \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + C$$

(b) $\int \frac{1+\cos x}{\sin x \cos x} dx$

$$= \int \frac{1}{\sin x \cos x} dx + \int \frac{\cos x}{\sin x \cos x} dx$$

$$= 2\int \frac{1}{\sin 2x} dx + \int \frac{1}{\sin x} dx$$

$$= 2\int \text{cosec } 2x \, dx + \int \text{cosec } x \, dx$$

$$= \frac{2 \log|\text{cosec } 2x - \cot 2x|}{2} + \log|\text{cosec } x - \cot x| + C$$

= log|cosec 2x – cot 2x| + log |cosec x – cot x |+ C

प्रश्न 7.
(a) ∫sin 3x sin 2x dx
(b) ∫$\sqrt{1 - sin x dx}$

हल : (a) ∫sin 3x sin 2x dx

$$= \int \frac{1}{2}[\cos(3x-2x) - \cos(3x+2x)] dx$$

$$= \frac{1}{2}\int [\cos x - \cos 5x] dx$$

$$= \frac{1}{2}\left(\sin x - \frac{\sin 5x}{5}\right) + C$$

$$= \frac{1}{2}\left(\sin x - \frac{1}{5}\sin 5x\right) + C$$

(b) $\int \sqrt{1-\sin x} \, dx$

$$= \int \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)} dx$$

$$= \int \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}\, dx$$

$$= \int \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) dx$$

$$= \int \sin\frac{x}{2}\, dx - \int \cos\frac{x}{2}\, dx$$

$$= \frac{-\cos\frac{x}{2}}{\left(\frac{1}{2}\right)} - \frac{\sin\frac{x}{2}}{\left(\frac{1}{2}\right)} + C$$

$$= \pm\, 2\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + C$$

प्रश्न 8. (a) $\int \cos^4 x\, dx$
(b) $\int \sin^3 x\, dx$

हल : (a) $\int \cos x^4\, dx$
$= -\int (\cos^2 x)^2\, dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \quad \left(\because \cos^2 A = \frac{1+\cos 2A}{2}\right)$$

$$= \frac{1}{4}\int (1+\cos 2x)^2\, dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{8}\int (2 + 4\cos 2x + 1 + \cos 4x)\, dx$$

$$= \frac{1}{8}\int (\cos 4x + 4\cos 2x + 3)\, dx$$

$$= \frac{1}{8}\left[\frac{\sin 4x}{4} + \frac{4\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 4x + 2\sin 2x + 3x\right] + C$$

(b) $\int \sin^3 x\, dx$

$$= \int\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right) dx$$

$$(\because \sin 3x = 3\sin x - 4\sin^3 x)$$

$$= \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$

$$= \frac{-3}{4}\cos x + \frac{\cos 3x}{4\times 3} + C$$

$$= \frac{-3}{4}\cos x + \frac{\cos 3x}{12} + C$$

प्रश्न 9.

(a) $\int \frac{1}{\sin x \cos^3 x} dx$ **(b)** $\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx$

हल :

(a) $\int \frac{1}{\sin x \cos^3 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x.\cos^3 x} dx$$

$$= \int\left[\frac{\sin^2 x}{\sin x.\cos^3 x} + \frac{\cos^2 x}{\sin x.\cos^3 x}\right] dx$$

$$= \int\left[\frac{\tan x}{\cos^2 x} + \frac{1}{\tan x.\cos^2 x}\right] dx$$

$$= \int\left[\tan x.\sec^2 + \frac{1}{\tan x}\cdot \sec^2 x\right] dx$$

$$= \int\left(\tan x + \frac{1}{\tan x}\right)\sec^2 x\, dx$$

माना tan x = t

$\sec^2 x \, dx = dt$

$$= \int \left(t + \frac{1}{t} \right) dt$$

$$= \frac{1}{2} t^2 + \log |t| + C$$

$$= \frac{1}{2} \tan^2 x + \log |\tan x| + C$$

$$= \log |\tan x| + \frac{1}{2} \tan^2 x + C$$

(b) $\int \frac{(1+x)e^x}{\cos^2 (xe^x)} dx$

माना $xe^x = t$

$\Rightarrow (xe^x + e^x.1)dx = dt$

$\Rightarrow (1 + x)e^x \, dx = dt$

$$\therefore \int \frac{(1+x)e^x}{\cos^2 (xe^x)} dx = \int \frac{1}{\cos^2 t} dt$$

$= \int \sec^2 t \, dt = \tan t + c$

$= \tan (xe^x) + c$

प्रश्न 10.

(a) $\int \frac{1}{1 - \tan x} dx$ **(b)** $\int \frac{1}{1 + \cot x} dx$

हल :

(a) $\int \frac{1}{1 - \tan x} dx$

$$= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} \int \frac{2\cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} \int \frac{\cos x - \sin x + \cos x + \sin x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} \int 1\, dx + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2}x + \frac{1}{2} \int \frac{\sin x + \cos x}{\cos x - \sin x}\, dx$$

माना cosx – sinx = t

⇒ (sin x + cos x) dx = dt

$$= \frac{1}{2}x + \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2} \log|\cos x - \sin x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

(b) $\int \frac{1}{1+\cot x}\, dx$

$$= \int \frac{1}{1 + \frac{\cos x}{\sin x}}\, dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)}\, dx$$

माना sin x + cos x = t

⇒ -cos x + sin x dx = dt

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

प्रश्न 11.

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$ **(b)** $\int \frac{1 - \tan x}{1 + \tan x}\,dx$

हल :

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$

माना tan x = t

⇒ $\sec^2 x$ dx = dt

$$= \int \frac{\sec^2 x . \sec^2 x\,dx}{\sqrt{\tan x}}$$

$$= \int \frac{(1 + \tan^2 x)}{\sqrt{\tan x}}\ \sec^2 x\,dx$$

$$= \int \frac{1 + t^2}{t^{1/2}}\,dt$$

$$= \int (t^{-1/2} + t^{2-1/2})\,dt$$

$$= \int t^{-1/2}\,dt + \int t^{3/2}\,dt$$

$$= \frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2} + C$$

$$= 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + C$$

(b) $\int\left(\frac{1-\tan x}{1+\tan x}\right)dx = \int\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)dx$

$= \int\frac{\cos x-\sin x}{\cos x+\sin x}\,dx$

माना sin x + cos x = t

(cos x − sin x) dx = dt

$= \int \frac{dt}{t}$

= log|t| + C

= log |sin x + cos x| + C

प्रश्न 12.

(a) $\int\frac{\sin(x-a)}{\sin(x+a)}dx$ (b) $\int\frac{\sin x}{\sin(x-a)}dx$

हल :

(a) $\int\frac{\sin(x-a)}{\sin(x+a)}dx$

माना x + a = t

dx = dt

x = t − a

$\therefore \int\frac{\sin(x-a)}{\sin(x+a)}dx$

$= \int\frac{\sin(t-a-a)}{\sin t}dt = \int\frac{\sin(t-2a)}{\sin t}dt$

$= \int\frac{\sin t\cos 2a-\cos t\sin 2a}{\sin t}dt$

$= \int\frac{\sin t\cos 2a}{\sin t}dt - \int\sin 2a\cdot\frac{\cos t}{\sin t}\,dt$

= cos 2a ∫ dt − sin 2a ∫ cot dt

= (cos 2a)t − sin 2a log |sin t| + C1

= (x + a) cos 2a − sin 2a log |sin (x + a) | + C1

= x cos 2a − sin 2a log |sin(x + a)| + a cos 2a + C1

= x cos 2a − sin 2a log |sin (x + a)| + C

(जहाँ C = a cos 2a + C1)

(b) $\int \frac{\sin x}{\sin (x-a)} dx$

माना x − a = t

x = t + a

dx = dt

$= \int \frac{\sin (t+a)}{\sin t} dt$

$= \int \frac{\sin t \cos a + \sin a \cos t}{\sin t}$

$= \int \left(\frac{\sin t \cos a}{\sin t} + \frac{\sin a \cos t}{\sin t} \right) dt$

= ∫cos a dt + ∫sin a cot dt

= cos a.t + sin a log |sin t|

= (x − a) cos a + sin a log |sin (x − a)| + C1

= x cos a + sin a log |sin (x − a)| + (-a cos a + C1)

= x cos a + sin a log |sin (x -a)| + C

(जहाँ C = -a cos a + C1)

प्रश्न 13.

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

(b) $\int \frac{\sin 2x}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x + \frac{\pi}{6} \right)} dx$

हल :

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

$$= \int \frac{\sin (5x-3x)}{\sin 5x \sin 3x} dx$$

$$= \int \left[\frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} - \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x} \right] dx$$

$$= \int \cot 3x \, dx - \int \cot 5x \, dx$$

$$= \frac{\log |\sin 3x|}{3} - \frac{\log |\sin 5x|}{5} + C$$

$$= \frac{1}{3} \log |\sin 3x| - \frac{1}{5} \log |\sin 5x| + C$$

(b) $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$

$$= \int \frac{\sin\left\{\left(x-\frac{\pi}{6}\right)+\left(x+\frac{\pi}{6}\right)\right\}}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$$

$$= \int\left[\frac{\sin\left(x-\frac{\pi}{6}\right)\cos\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}+\frac{\cos\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}\right]dx$$

$$= \int\left[\cot\left(x+\frac{\pi}{6}\right)dx+\cot\left(x-\frac{\pi}{6}\right)\right]dx$$

$$= \log\left|\sin\left(x+\frac{\pi}{6}\right)\right|+\log\left|\sin\left(x-\frac{\pi}{6}\right)\right|+C$$

$$= \log\left[\sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)\right]+C$$

प्रश्न 14.

(a) $\int \frac{1}{3\sin x+4\cos x}\,dx$

(b) $\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

हल :

(a) $\int \frac{1}{3\sin x+4\cos x}\,dx$

माना 4 = sin θ तथा 3 = r cos θ

तब्र $r^2 \sin^2\theta + r^2\cos^2\theta = 3^2 + 4^2 = 5^2$

$\Rightarrow r = 5$

तथा $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{4}{3} \Rightarrow \tan\theta = \dfrac{4}{3}$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\therefore \qquad I = \int \frac{1}{3\sin x + 4\cos x}\,dx$$

$$= \int \frac{1}{r\cos\theta\sin x + r\sin\theta\cos x}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(\theta + x)}\,dx$$

$$= \frac{1}{r}\int \text{cosec}\,(\theta + x)\,dx$$

$$= \frac{1}{5}\log|\,\text{cosec}\,(\theta + x) - \cot(\theta + x)\,| + C$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{\theta + x}{2}\right)\right|$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{x + \tan^{-1}\left(\frac{4}{3}\right)}{2}\right)\right| + C$$

(b) $\displaystyle\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b-x+a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int\left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]dx$$

$$= \frac{1}{\sin(a-b)}\int[\cot(x-a) - \cot(x-b)]\,dx$$

$$= \frac{1}{\sin(a-b)}[\log|\sin(x-a)| - \log|\sin(x-b)| + C$$

$$= \frac{1}{\sin(a-b)}\left[\log\frac{|\sin(x-a)|}{|\sin(x-b)|}\right] + C$$

$$= \operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$$

प्रश्न 15.

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

(b) $\int \frac{\sec x}{\sin(2x+\alpha) + \sin\alpha}\, dx$

हल :

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

माना $a\cos^2 x + b\sin^2 x = t$

$\Rightarrow (-2a\cos x \sin x + 2b \sin x \cos x)\, dx = dt$

$\Rightarrow (2b \sin \cos x - 2a \sin x \cos x)\, dx = dt$

$\Rightarrow 2(b-a)\sin x \cos x\, dx = dt$

$$\Rightarrow \sin x \cos x\, dx = \frac{1}{2(b-a)}\, dt$$

$$\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$$

$$= \int \frac{1}{t} \times \frac{1}{2(b-a)}\, dt$$

$$= \frac{1}{2(b-a)}\log|t| + C$$

$$= \frac{1}{2(b-a)}\log|a\cos^2 x + b\sin^2 x| + C$$

(b) $\int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\,dx$

माना $I = \int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\,dx$

$$= \int \frac{\sec x}{\sqrt{2\sin\left(\frac{2x+\alpha+\alpha}{2}\right)\cos\left(\frac{2x+\alpha-\alpha}{2}\right)}}\,dx$$

$$\left[\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= \int \frac{\sec x}{\sqrt{2\sin(x+\alpha)\cos x}}\,dx$$

$$= \int \frac{\sec x}{\sqrt{2\cos x(\sin x\cos\alpha+\cos x\sin\alpha)}}\,dx$$

$$= \int \frac{\sec x}{\sqrt{2\cos^2 x(\tan x\cos\alpha+\sin\alpha)}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec x}{\cos x\sqrt{\tan x\cos\alpha+\sin\alpha}}\,dx$$

$$I = \frac{1}{\sqrt{2}}\int \frac{\sec^2 x}{\sqrt{\tan x\cos\alpha+\sin\alpha}}\,dx \qquad \ldots(1)$$

माना $\tan x\cos\alpha+\sin\alpha = t$ $\qquad \ldots(2)$

$\Rightarrow \quad \sec^2 x\cos\alpha\,dx = dt$

$\Rightarrow \quad \sec^2 x\,dx = \frac{dt}{\cos\alpha}$

समी. (2) का उपयोग (1) में करने पर,

$$I = \frac{1}{\sqrt{2}} \int \frac{dt}{\cos \alpha \sqrt{t}}$$

$$= \frac{1}{\sqrt{2}.\cos \alpha} \int \frac{1}{\sqrt{t}} dt$$

$$= \frac{1}{\sqrt{2}.\cos \alpha} \left(\frac{t^{-1/2+1}}{-1/2+1} \right) + C$$

$$= \frac{2}{\sqrt{2} \cos \alpha} t^{1/2} + C$$

$$= \frac{\sqrt{2}}{\cos \alpha} (\tan x \cos \alpha + \sin \alpha)^{1/2} + C$$

$$\Rightarrow \quad I = \sqrt{2} \sec \alpha (\tan x \cos \alpha + \sin \alpha)^{1/2} + C$$

$$\therefore \int \frac{\sec x}{\sqrt{\sin (2x + \alpha) + \sin \alpha}} dx = \sqrt{2} \sec \alpha (\tan x \cos \alpha + \sin \alpha)^{1/2} + C$$

प्रश्न 16.

(a) $\displaystyle \int \frac{1}{\sqrt{\cos^3 x \sin (x + a)}} dx$

(b) $\displaystyle \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

हल :

(a) $\displaystyle \int \frac{1}{\sqrt{\cos^3 x . \sin (x + a)}} dx$

$$= \int \frac{1}{\cos^2 x \sqrt{\left(\frac{\sin (x + a)}{\cos x} \right)}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\left(\frac{\sin x \cos a}{\cos x} + \frac{\cos x \sin a}{\cos x} \right)}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x.\cos a + \sin a}}\, dx$$

माना $\tan x \cos a + \sin a = t$

$$\cos a \sec^2 x\, dx = dt$$

$$\sec^2 x\, dx = \frac{1}{\cos a}\, dt$$

$$= \int \frac{1}{\sqrt{t}} \times \frac{1}{\cos a}\, dx$$

$$= \frac{1}{\cos a}\left(\frac{t^{-1/2+1}}{-1/2+1}\right) + C$$

$$= \frac{1}{\cos a} \times \frac{t^{1/2}}{1/2} + C$$

$$= \frac{2}{\cos a}\sqrt{\tan x \cos a + \sin a} + C$$

(b) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha}\, dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2 \alpha + 1)}{\cos x - \cos \alpha}\, dx$$

$$= 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha}\, dx$$

$$= 2\int \frac{(\cos x + \cos \alpha)(\cos x - \cos \alpha)}{(\cos x - \cos \alpha)}\, dx$$

$= 2\int (\cos x + \cos \alpha)\, dx$

$= 2\int \cos x\, dx + 2\int \cos \alpha\, dx$

$= 2 \sin x + 2 \cos \alpha \int dx$

$= 2 \sin x + 2x \cos \alpha + C$

$= 2(\sin x + x \cos \alpha) + C$

Ex 9.3

निम्न फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.

(a) $\int \frac{1}{50+2x^2}\, dx$ (b) $\int \frac{1}{\sqrt{32-2x^2}}\, dx$

हल :

(a) $\int \frac{1}{50+2x^2}\, dx$

$$= \frac{1}{2}\int \frac{1}{25+x^2}\, dx$$

$$= \frac{1}{2}\int \frac{1}{5^2+x^2}\, dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+5^2}\, dx$$

$$= \frac{1}{2}\times\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

$$= \frac{1}{10}\tan^{-1}\left(\frac{x}{5}\right)+C$$

(b) $\int \frac{1}{\sqrt{32-2x^2}}\, dx = \int \frac{1}{\sqrt{2}\sqrt{(4^2-x^2)}}\, dx$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x}{4}\right)+C$$

प्रश्न 2.

(a) $\int \frac{1}{\sqrt{1-e^{2x}}}\, dx$ (b) $\frac{1}{\sqrt{1+4x^2}}\, dx$

हल :

(a) $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$

$= \int \frac{1}{\sqrt{(1)^2-(e^x)^2}}\,dx$

माना $e^x = t$

$\Rightarrow \quad e^x\,dx = dt$

$\Rightarrow \quad dx = \frac{dt}{e^x}$

$\Rightarrow \quad dx = \frac{dt}{t}$

$= \int \frac{1}{\sqrt{1-t^2}}\,\frac{dt}{t}$

$= \int \frac{1}{t\sqrt{1-t^2}}\,dt$

$= \log|1-\sqrt{1-t^2}| + C$

$= \log|1-\sqrt{1-e^{2x}}| + C$

(b) $\int \frac{1}{\sqrt{1+4x^2}}\,dx$

$= \int \frac{1}{\sqrt{(1)^2+(2x)^2}}\,dx$

$= \frac{1}{2}\log|2x+\sqrt{1+(2x)^2}| + C$

$= \frac{1}{2}\log|2x+\sqrt{4x^2+1}| + C$

प्रश्न 3.

(a) $\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx$ (b) $\int \frac{1}{\sqrt{(2-x)^2+1}}\,dx$

हल :

(a) $\int \frac{1}{\sqrt{a^2 - b^2x^2}} dx$

$$= \int \frac{1}{\sqrt{a^2 - (bx)^2}} dx$$

$$= \frac{1}{b} \sin^{-1}\left(\frac{bx}{a}\right) + C$$

(b) $\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx$

$$= -[\log|(2-x) + \sqrt{(2-x)^2 + 1}|] + C$$

$$= -[\log|(2-x) + \sqrt{4 + x^2 - 4x + 1}|] + C$$

$$= -[\log|(2-x) + \sqrt{x^2 - 4x + 5}|] + C$$

प्रश्न 4.

(a) $\int \frac{x^2}{\sqrt{x^6 + 4}} dx$ **(b)** $\int \frac{x^4}{\sqrt{1 - x^{10}}} dx$

हल :

(a) $\int \frac{x^2}{\sqrt{x^6 + 4}} dx$

$$= \int \frac{x^2 \, dx}{\sqrt{(x^3)^2 + 2^2}}$$

माना $x^3 = t$

$\Rightarrow$ $3x^2 \, dx = dt$

$\Rightarrow$ $x^2 \, dx = \frac{dt}{3}$

$$= \frac{1}{3} \int \frac{1}{\sqrt{t^2 + 2^2}}$$

$$= \frac{1}{3} \log|t + \sqrt{t^2 + 2^2}| + C$$

$$= \frac{1}{3} \log|x^3 + \sqrt{x^6 + 4}| + C$$

(b) $\int \frac{x^4}{\sqrt{1-x^{10}}}\,dx$

$$= \int \frac{x^4}{\sqrt{1-(x^5)^2}}\,dx$$

माना $\quad x^5 = t$

तो $\quad 5x^4\,dx = dt$

या $\quad x^4\,dx = \frac{dt}{5}$

$$= \frac{1}{5}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{5}\sin^{-1} t + C$$

$$= \frac{1}{5}\sin^{-1}(x^5) + C$$

प्रश्न 5.

(a) $\int \frac{1}{x^2+6x+8}\,dx$ **(b)** $\frac{1}{\sqrt{2x^2-x+2}}\,dx$

हल :

(a) $\int \frac{1}{x^2+6x+8}\,dx$

$$= \int \frac{1}{x^2+2\times 3x+3^2-1}\,dx$$

$$= \int \frac{1}{(x+3)^2-1}\,dx$$

$$= \frac{1}{2}\log\left|\frac{(x+3)-1}{(x+3)+1}\right| + C$$

$$\left(\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right)$$

$$= \frac{1}{2}\log\left|\frac{x+2}{x+4}\right| + C$$

(b) $\int \frac{1}{\sqrt{2x^2 - x + 2}} dx = \int \frac{1}{\sqrt{2\left(x^2 - \frac{1}{2}x + 1\right)}} dx$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} + 1}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}} dx$$

सूत्र $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + C$ से

$$= \frac{1}{\sqrt{2}} \log \left| \left(x - \frac{1}{4}\right) + \sqrt{x^2 - \frac{1}{2}x + 1} \right| + C$$

प्रश्न 6.

(a) $\int \frac{e^x}{e^{2x} + 2e^x \cos x + 1} dx$

(b) $\int \frac{1 + \tan^2 x}{\sqrt{\tan^2 x + 3}} dx$

हल :

(a) $\int \frac{e^x}{e^{2x} + 2e^x \cos x + 1} dx$

$$= \int \frac{e^x}{\{(e^x)^2 + 2.e^x \cos x + \cos^2 x\} + \sin^2 x} dx$$

$$= \int \frac{e^x}{(e^x + \cos x)^2 + \sin^2 x} dx$$

$$= \frac{1}{\sin x} \tan^{-1} \left(\frac{e^x + \cos x}{\sin x}\right) + C$$

(b) $\int \frac{1 + \tan^2 x}{\sqrt{\tan^2 x + 3}} dx$

$$= \int \frac{\sec^2 x \, dx}{\sqrt{(\sqrt{3})^2 + \tan^2 x}}$$

माना tanx = t

$\sec^2 x\, dx = dt$

$$= \int \frac{dt}{\sqrt{(\sqrt{3})^2 + t^2}}$$

$$= \log |t + \sqrt{t^2 + 3}| + C$$

$$= \log |\tan x + \sqrt{\tan^2 x + 3}| + C$$

प्रश्न 7.

(a) $\int \frac{1}{\sqrt{3x - 2 - x^2}} dx$

(b) $\int \frac{1}{\sqrt{4 + 8x - 5x^2}} dx$

हल :

(a) $\int \frac{1}{\sqrt{3x - 2 - x^2}} dx$

$$\therefore \quad 3x - 2 - x^2 = -(x^2 - 3x + 2)$$

$$= -\left(x^2 - 2 \times \frac{3}{2} \times x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2\right)$$

$$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 2\right]$$

$$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}\right] = \frac{1}{4} - \left(x - \frac{3}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{3x - 2 - x^2}} dx = \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} dx$$

$$= \sin^{-1}\left(\frac{x - 3/2}{1/2}\right) + C$$

$$= \sin^{-1}(2x - 3) + C$$

(b) $\int \frac{1}{\sqrt{4+8x-5x^2}}\,dx$

$\therefore\ 4 + 8x - 5x^2 = -(5x^2 - 8x - 4)$

$= -5\left(x^2 - \frac{8}{5}x - \frac{4}{5}\right)$

$= -5\left(x^2 - 2\times\frac{4}{5}\times x + \left(\frac{4}{5}\right)^2 - \left(\frac{4}{5}\right)^2 - \frac{4}{5}\right)$

$= -5\left[\left(x-\frac{4}{5}\right)^2 - \frac{16}{25} - \frac{4}{5}\right]$

$= -5\left[\left(x-\frac{4}{5}\right)^2 - \left(\frac{6}{5}\right)^2\right] \quad = 5\left(\frac{6}{5}\right)^2 - 5\left(x-\frac{4}{5}\right)^2$

$\int \frac{1}{\sqrt{4+8x-5x^2}}\,dx = \int \frac{1}{\sqrt{5\left(\frac{6}{5}\right)^2 - 5\left(x-\frac{4}{5}\right)^2}}\,dx$

$= \frac{1}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{6}{5}\right)^2 - \left(x-\frac{4}{5}\right)^2}}\,dx$

$= \frac{1}{\sqrt{5}} \sin^{-1}\left(\frac{x-4/5}{6/5}\right) + C$

$= \frac{1}{\sqrt{5}} \sin^{-1}\left(\frac{5x-4}{6}\right) + C$

प्रश्न 8.

(a) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

(b) $\int \frac{1}{\sqrt{x^2 + 2ax + b^2}}\,dx$

हल :

(a) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx = \int \frac{\sin x + \cos x}{1-(1-\sin 2x)}\,dx$

$= \int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$

माना sin x − cos x = t
(cosx + sin x)dx = dt

$$= \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sin^{-1} t + C$$

$$= \sin^{-1} (\sin x - \cos x) + C$$

(b) $\int \frac{1}{\sqrt{x^2 + 2ax + b^2}} dx$

$$= \int \frac{1}{\sqrt{(x^2 + 2ax + a^2) - a^2 + b^2}} dx$$

$$= \int \frac{1}{\sqrt{(x+a)^2 + (b^2 - a^2)}} dx$$

$$= \log \left| (x+a) + \sqrt{x^2 + 2ax + b^2} \right| + C$$

प्रश्न 9.

(a) $\int \sqrt{\frac{a-x}{x}} dx$ **(b)** $\int \sqrt{\frac{a+x}{a-x}} dx$

हल :

(a) $\int \sqrt{\frac{a-x}{x}} dx$

माना x = a cos²θ
dx = a 2cos θ (-sin θ) dθ
dx = − 2a sin θ cosθ dθ

$$\therefore \quad \cos^2 \theta = \frac{x}{a} \text{ तथा } \sin^2 \theta = \frac{a^2 - x^2}{a^2}$$

$$\therefore \quad \cos \theta = \sqrt{\frac{x}{a}}$$

$$\Rightarrow \quad \theta = \cos^{-1} \sqrt{\frac{x}{a}}$$

$$\int \sqrt{\frac{a-x}{x}} dx = \int \sqrt{\frac{a - a\cos^2 \theta}{a\cos^2 \theta}} (-2a \sin \theta \cos \theta) d\theta$$

$$= \int \sqrt{\frac{1-\cos^2\theta}{\cos^2\theta}} (-2a \sin\theta \cos\theta)\, d\theta$$

$$= \int -2a \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \times \sin\theta \cos\theta\, d\theta$$

$$= -2a \int \frac{\sin\theta}{\cos\theta} \sin\theta \cos\theta\, d\theta$$

$$= -2a \int \sin^2\theta\, d\theta = -2a \int \frac{1-\cos 2\theta}{2} d\theta$$

$$= -2a \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right] + C$$

$$= \frac{a}{2} \sin 2\theta - a\theta + C$$

$$= \frac{1}{2} a [\sin 2\theta - 2\theta] + C$$

$$= \frac{1}{2} a \left[\sin 2\left(\cos^{-1}\sqrt{\frac{x}{a}}\right) - 2\cos^{-1}\sqrt{\frac{x}{a}}\right] + C$$

(b) $\int \sqrt{\left(\frac{a+x}{a-x}\right)} dx$

माना x = a cos θ

dx = -a sin θ dθ

$$\therefore \int \sqrt{\left(\frac{a+x}{a-x}\right)} dx = \int \sqrt{\left(\frac{a+a\cos\theta}{a-a\cos\theta}\right)} \cdot (-a \sin\theta\, d\theta)$$

$$= -a \int \sqrt{\left(\frac{1+\cos\theta}{1-\cos\theta}\right)} \cdot \sin\theta\, d\theta$$

$$= -a \int \sqrt{\left[\frac{1+(2\cos^2\theta/2-1)}{1-(1-2\sin^2\theta/2)}\right]} \cdot 2\sin\theta/2 \cos\theta/2\, d\theta$$

$$= -2a \int \frac{\cos\theta/2}{\sin\theta/2} \cdot \sin\theta/2 \cos\theta/2\, d\theta$$

$$= -a \int 2\cos^2\theta/2\, d\theta = -a \int (1+\cos\theta)\, d\theta$$

$$= -a[\theta + \sin\theta] + c = -a\theta - a\sqrt{1-\cos^2\theta} + c$$

$$= -a \cos^{-1}(x/a) - a\sqrt{[1-(x/a)^2]} + c$$

$$= -a \cos^{-1}(x/a) - \sqrt{a^2-x^2} + c$$

प्रश्न 10.

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\, dx$$

हल :

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\, dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\, dx$$

$$= \frac{2}{3}\int \frac{1}{\sqrt{(a^{3/2})^2 - t^2}}\, dt \qquad \text{माना } x^{3/2} = t$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + c \qquad \frac{3}{2}\, x^{1/2}\, dx = dt$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c \qquad \sqrt{x}\, dx = \frac{2}{3}\, dt$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$$

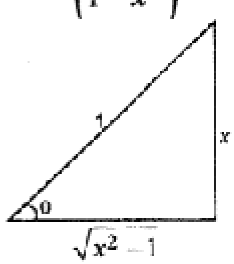
प्रश्न 11.

(a) $\int \frac{1}{\left(1 - x^2\right)^{3/2}}\, dx$

(b) $\int \frac{x + 1}{\sqrt{x^2 + 1}}\, dx$

हल :

(a) $\int \frac{1}{\left(1 - x^2\right)^{3/2}}\, dx$



माना x = sin θ

dx = cosθ dθ

$$= \int \frac{1}{(1-\sin^2\theta)^{3/2}} \cos\theta \, d\theta$$

$$= \int \frac{1}{\cos^2\theta} \cos\theta \, d\theta$$

$$= \int \frac{1}{\cos^3\theta} d\theta = \int \sec^2\theta d\theta$$

$$= \tan\theta + C$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$

(b) $$\int \frac{x+1}{\sqrt{x^2+1}} dx = \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= I_1 + I_2$$

$$I_1 = \int \frac{x}{\sqrt{x^2+1}} dx$$

माना $x^2 + 1 = t$

$2x\, dx = dt$

$$x\,dx = \frac{1}{2} dt$$

$$\Rightarrow \quad I_1 = \frac{1}{2} \int \frac{1}{t^{1/2}} dt$$

$$\Rightarrow \quad I_1 = \frac{1}{2} \left[\frac{t^{-1/2+1}}{-1/2+1}\right] + C_1$$

$$\Rightarrow \quad I_1 = \frac{1}{2} \left[\frac{t^{1/2}}{1/2}\right] + C_1$$

$$\Rightarrow \quad I_1 = \sqrt{t} + C_1$$

$$\Rightarrow \quad I_1 = \sqrt{x^2+1} + C_1 \qquad \text{...(ii)}$$

$$I_2 = \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= \log\left|x + \sqrt{x^2+1}\right| + C_2$$

$$\int \frac{x+1}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} + \log \left| x + \sqrt{x^2+1} \right| + (C_1 + C_2)$$

$$= \sqrt{x^2+1} + \log (x + \sqrt{x^2+1}) + C,$$

जहाँ $(C_1 + C_2) = C$

प्रश्न 12.

(a) $\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx$

(b) $\int \frac{1}{\sqrt{2x-x^2}} dx$

हल :

(a) $\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx$

$x = \alpha \cos^2\theta + \beta \sin^2\theta$

$dx = (\beta - \alpha) \sin 2\theta \, d\theta$

$\alpha(1 - \sin^2 \theta) + \beta \sin^2\theta = x$

$\alpha + \beta \sin^2\theta - \alpha \sin^2\theta = x$

$(\beta - \alpha) \sin^2\theta = x - \alpha$

$$\Rightarrow \quad \sin^2 \theta = \frac{x-\alpha}{\beta-\alpha}$$

$$\Rightarrow \quad \sin \theta = \sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

$$\Rightarrow \quad \theta = \sin^{-1} \sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

प्रश्नानुसार,

$$\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}} dx$$

$$= \int \frac{(\beta-\alpha) \sin 2\theta}{\sqrt{\alpha \cos^2 \theta + \beta \sin^2 \theta - \alpha) (\beta - \alpha \cos^2 \theta - \beta \sin^2 \theta)}} d\theta$$

$$\Rightarrow \qquad \sin^2\theta = \frac{x-\alpha}{\beta-\alpha}$$

$$\Rightarrow \qquad \sin\theta = \sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

$$\Rightarrow \qquad \theta = \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

प्रश्नानुसार,

$$\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$$

$$= \int \frac{(\beta-\alpha)\sin 2\theta}{\sqrt{(\alpha\cos^2\theta+\beta\sin^2\theta-\alpha)(\beta-\alpha\cos^2\theta-\beta\sin^2\theta)}}\,d\theta$$

$$= \int \frac{(\beta-\alpha)\sin 2\theta}{\sqrt{(\beta\sin^2\theta-\alpha\sin^2\theta)(\beta\cos^2\theta-\alpha\cos^2\theta)}}\,d\theta$$

$$= \int \frac{(\beta-\alpha)\cdot\sin 2\theta}{\sqrt{(\beta-\alpha)\sin^2\theta\times(\beta-\alpha)\cos^2\theta}}\,d\theta$$

$$= \int \frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\sin\theta\cos\theta}\,d\theta$$

$$= \int 2\,d\theta = 2\theta + C = \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C$$

(b) $$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{-(x^2-2x+1-1)}}\,dx$$

$$= \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$= \sin^{-1}(x-1) + C$$

प्रश्न 13.

(a) $\int \frac{1}{\sqrt{(x-1)(x-2)}}\,dx$

(b) $\int \frac{\cos x}{\sqrt{4-\sin 2x}}\,dx$

हल :

(a) $\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{x^2-3x+2}} dx$

$$= \int \frac{1}{\sqrt{x^2 - 2 \times \frac{3}{2} \times x + \frac{9}{4} - \frac{9}{4} + 2)}} dx$$

$$= \int \frac{1}{\left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

$$= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C$$

(b) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} \; dx$

माना $\quad \sin x = t$

$\cos x \, dx = dt$

$$= \int \frac{dt}{\sqrt{2^2 - t^2}}$$

$$= \sin^{-1} \frac{t}{2} + C$$

$$= \sin^{-1} \left(\frac{\sin x}{2}\right) + C$$

Ex 9.4

निम्नलिखित फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.

$$\int \frac{1}{16-9x^2}\,dx$$

हल:

$$\int \frac{1}{16-9x^2}\,dx$$

$$= \int \frac{1}{4^2-(3x)^2}\,dx$$

$$= \frac{1}{2} \times \frac{1}{4\times 3} \log\left[\frac{4+3x}{4-3x}\right] + C$$

$$\left(\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right] + C\right)$$

$$= \frac{1}{24} \log\left[\frac{4+3x}{4-3x}\right] + C$$

प्रश्न 2.

$$\int \frac{1}{x^2-36}\,dx$$

हल :

$$\int \frac{1}{x^2-36}\,dx = \int \frac{1}{x^2-6^2}\,dx$$

$$= \frac{1}{2\times 6} \log\left|\frac{x-6}{x+6}\right| + C$$

$$= \frac{1}{12} \log\left|\frac{x-6}{x+6}\right| + C$$

प्रश्न 3.

$$\int \frac{3x}{(x+1)(x-2)}\,dx$$

हल : $\int \frac{3x}{(x+1)(x-2)} dx$

$\therefore \frac{3x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$\Rightarrow 3x = Ax - 2A + Bx + B$

$\Rightarrow 3x = (A + B)x - 2A + B$

तुलना करने पर, $A + B = 3$ ….(1)

तथा $-2A + B = 0$

$B = 2A$ …….(2)

समी. (2) से B का मान समी. (1) में रखने पर,

$\Rightarrow A + 2A = 3 \Rightarrow A - 1$

माना $B = 2 \times 1 = 2$

$\therefore \frac{3x}{(x+1)(x-2)} = \frac{1}{x+1} + \frac{2}{x-2}$

$\int \frac{3x}{(x+1)(x-2)} dx = \int \frac{1}{x+1} dx + 2 \int \frac{1}{(x-2)} dx$

$= \log | x + 1 | + 2 \log | x - 2| + C$

प्रश्न 4.

$\int \frac{3x-2}{(x+1)^2(x+3)} dx$

हल :

$\int \frac{3x-2}{(x+1)^2(x+3)} dx$

$\therefore \frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$\Rightarrow 3x - 2 = A (x + 1) (x + 3) + B (x + 3) + C (x + 1)^2$

$\Rightarrow 3x - 2 = A (x^2 + 4x + 3) + Bx + 3B + C (x^2 + 2x + 1)$

$\Rightarrow 3x - 2 = (A + C)x^2 + (4A + B + 2C)x + 3A + 3B + C$

तुलना से,

$A + C = 0$ तो $A = - C$ ….(i)

$4A + B + 2C = 3$

$4A + B - 2A = 3$

$24 + B = 3$ …(ii)

$3A + 3B + C = - 2$

$3A + 3B - A = - 2$

2A + 3B = − 2 ...(iii)

समी. (iii) में से (ii) को घटाने पर

(2A + 3B) − (2A + B) = − 2 − 3

$2B = -5 \Rightarrow B = -\frac{5}{2}$

2A + B = 3

$$\therefore \qquad 2A = 3 - B \Rightarrow 2A = 3 + \frac{5}{2}$$

$$\Rightarrow \qquad 2A = \frac{11}{2}, \quad \Rightarrow A = \frac{11}{4}$$

अत: $A = \frac{11}{4},\ B = -\frac{5}{2}, C - \frac{11}{4}$

$$\therefore \int \frac{3x - x}{(x+1)^2\ (x+3)} dx$$

$$= \int \frac{A}{x+1} dx + \int \frac{B}{(x+1)^2} dx + \int \frac{C}{x+3} dx$$

$$= \frac{11}{4} \int \frac{1}{x+1} dx - \frac{5}{2} \int \frac{1}{(x+1)^2} dx - \frac{11}{4} \int \frac{1}{x+3} dx$$

$$= \frac{11}{4} \log|x+1| + \frac{5}{2} \times \frac{1}{(x+1)} - \frac{11}{4} \log|x+3| + C$$

$$= \frac{11}{4} \log\left|\frac{x+1}{x+3}\right|_2 + \frac{5}{2}\frac{1}{(x+1)} + C$$

प्रश्न 5.

$$\int \frac{x^2}{(x+1)(x-2)(x-3)} dx$$

हल :

$$\int \frac{x^2}{(x+1)(x-2)(x-3)} dx$$

माना

$$\frac{x^2}{(x+1)(x-2)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$x^2 = A\ (x - 2)(x - 3) + B\ (x + 1)\ (x - 3) + C\ (x + 1]\ (x - 2)$

$x^2 = A(x^2 - 2x - 3x + 6) + B\ (x^2 + x - 3x - 3) + C\ (x^2 + x - 2x - 2)$

$x^2 = A(x^2 - 5x + 6) + B(x^2 - 2x - 3) + C(x^2 - x - 2)$
$x^2 = (A + B + C)x^2 + (-5A - 2B - C)x + (6A - 3B - 2C)$
दोनों पक्षों में x के गुणांकों की तुलना करने पर,
$A + B + C = 1$ ...(1)
$-5A - 2B - C = 0$ ...(2)
$6A - 3B - 2C = 0$ ...(3)
समी. (1), (2) व (3) को हल करने पर

$A = \frac{1}{12}, B = -\frac{4}{3}$ और $C = \frac{9}{4}$

$$\therefore \int \frac{x^2}{(x+1)(x-2)(x-3)} dx$$

$$= \frac{1}{12} \int \frac{1}{x+1} dx - \frac{4}{3} \int \frac{1}{x-2} dx + \frac{9}{4} \int \frac{1}{x-3} dx$$

$$= \frac{1}{12} \log |x+1| - \frac{4}{3} \log |x-2| + \frac{9}{4} \log |x-3| + C$$

प्रश्न 6.

$$\int \frac{x^2}{x^4 - x^2 - 12} dx$$

हल:

$$\int \frac{x^2}{x^4 - x^2 - 12} dx$$

माना $x^2 = y$

$$\frac{x^2}{x^4 - x^2 + 12} = \frac{y}{y^2 - y - 12} = \frac{y}{y(y-4) + 3(y-4)}$$

$$= \frac{y}{y(y-4) + 3(y-4)}$$

$$= \frac{y}{(y+3)(y-4)}$$

$$\frac{y}{(y+3)(y-4)} = \frac{A}{(y+3)} + \frac{B}{(y-4)}$$

$y = A(y - 4) + B(y + 3)$
$y = Ay - 4A + By + 3B$

y = Ay + By + (-4A + 3B)
y = (A + B)y + (-4A + 3B)

तुलना करने पर A + B = 1, − 4A + 3B = 0

हल करने पर, A = 3/2, B = 4/7

$$\int \frac{x^2}{x^4 - x^2 - 12} dx = \frac{3}{7} \int \frac{1}{x^2 + 3} dx + \frac{4}{7} \int \frac{1}{x^2 - 4} dx$$

$$= \frac{3}{7}\left[\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}\right] + \frac{4}{7}\left[\frac{1}{2 \times 2} \log \left|\frac{x-2}{x+2}\right|\right] + c$$

$$= \frac{\sqrt{3}}{7} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{7} \log \left|\frac{x-2}{x+2}\right| + c$$

प्रश्न 7.

$$\int \frac{1}{x^3 - x^2 - x + 1} dx$$

हल:

$$\int \frac{1}{x^3 - x^2 - x + 1} dx$$

$$\frac{1}{x^3 - x^2 - x + 1} = \frac{1}{x^2(x-1) - (x-1)}$$

$$= \frac{1}{(x^2 - 1)(x-1)}$$

$$= \frac{1}{(x-1)^2(x+1)}$$

$$\text{माना} \frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

1 = A (x − 1) (x + 1) + B (x + 1} + C $(x - 1)^2$
1 = A $(x^2 - 1)$ + B (x + 1) + C$(x^2 - 2x + 1)$
1 = (A + C)x^2 + (B − 2C)x − A + B + C

तुलना करने पर , A + C = 0,B − 2C = 0
− A + B + C = 1

हल करने पर, $A = -\frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{4}$

$$\int \frac{1}{(x^3 - x^2 - x + 1)} dx$$

$$= -\frac{1}{4} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{(x-1)^2} dx + \frac{1}{4} \int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{4} \log |x-1| + \frac{1}{2} \left[-\frac{1}{(x-1)}\right] \frac{1}{4} \log |x+1| + C$$

$$= \frac{1}{4} \log \frac{x+1}{x-1} - \frac{1}{2(x-1)} + C$$

प्रश्न 8.

$$\int \frac{x^2}{(x+1)(x-2)} dx$$

हल:

$$\int \frac{x^2}{(x+1)(x-2)} dx$$

$$\frac{x^2}{(x+1)(x-2)} = \frac{(x^2 - x - 2) + (x + 2)}{(x+1)(x-2)}$$

$$\therefore (x + 1)(x - 2) = x^2 - x - 2$$

तथा अंश की बात हर को घात से बड़ी या बराबर नहीं होनी चाहिएः

∴ घात का संयोजन किया गया है।

$$= 1 + \frac{x+2}{(x+1)(x-2)}$$

अब माना

$$\frac{x+2}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$\Rightarrow \quad x + 2 = A(x - 2) + B(x + 1)$

$\Rightarrow \quad x + 2 = (A + B)x - 2A + B$

तुलना से, $A + B = 1, \quad -2A + B = 2$

हल करने पर, $A = -\frac{1}{3}, B = \frac{4}{3}$

$$\therefore \int \frac{x^2}{(x+1)(x-2)} dx$$

$$= \int dx + \left(-\frac{1}{3}\right) \int \frac{1}{x+1} dx + \frac{4}{3} \int \frac{1}{x-2} dx$$

$$= x - \frac{1}{3} \log|x+1| + \frac{4}{3} \log|x-2| + C$$

$$= x + \frac{4}{3} \log|x-2| - \frac{1}{3} \log(x+1) + C$$

$$= x + \frac{1}{3} \log \frac{(x-2)^4}{|x+1|} + C$$

प्रश्न 9.

$$\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

हल:

$$\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

माना $x^2 = y$

$$\frac{x^2}{(x^2+d^2)(x^2+b^2)} = \frac{y}{(y+a^2)(y+b^2)}$$

$$\Rightarrow \frac{y}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$$

$$\Rightarrow y = A(y+b^2) + B(y+a^2)$$

$$\Rightarrow y = (A+B)y + (Ab^2 + Ba^2)$$

तुलना से, $A + B = 1, Ab^2 + Ba^2 = 0$

हल करने पर, $A = \frac{a^2}{a^2-b^2}$ तथा $B = \frac{b^2}{a^2-b^2}$

$$\Rightarrow \int \frac{y}{(y+a^2)(y+b^2)} dy$$

$$= \frac{a^2}{a^2-b^2} \int \frac{dy}{(y+a^2)} - \frac{b^2}{a^2-b^2} \int \frac{dy}{y+b^2}$$

$$\Rightarrow \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

$$= \frac{a^2}{a^2-b^2} \int \frac{1}{x^2+a^2} dx - \frac{b^2}{a^2-b^2} \int \frac{1}{x^2+b^2} dx$$

$$= \frac{a^2}{a^2-b^2} \left(\frac{1}{a} \tan^{-1} \frac{x}{a}\right) - \frac{b^2}{a^2-b^2} \times \frac{1}{b} \tan^{-1} \frac{x}{b} + C$$

$$= \frac{a}{a^2-b^2} \tan^{-1} \frac{x}{a} - \frac{b}{a^2-b^2} \tan^{-1} \frac{x}{b} + C$$

$$= \frac{1}{a^2-b^2} \left[a \tan^{-1} \frac{x}{a} - b \tan^{-1} \frac{x}{b}\right] + C$$

प्रश्न 10.

$$\int \frac{x+1}{x^3+x^2-6x} dx$$

हल:

$$\int \frac{x+1}{x^3+x^2-6x} dx$$

माना $\frac{x+1}{x^3+x^2-6x} = \frac{x+1}{x(x^2+x-6)}$

$$= \frac{x+1}{x(x^2+3x-2x-6)}$$

$$= \frac{x+1}{x(x+3)(x-2)}$$

अब $\frac{x+1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x-2)}$

$x + 1 = A(x + 3)(x - 2) + Bx(x - 2) + Cx(x + 3)$

$x + 1 = A(x^2 + 3x - 2x - 6) + B(x^2 - 2x) + C(x^2 + 3x)$

$x + 1 = A(x^2 + x - 6) + B(x^2 - 2x) + (Cx^2 + 3Cx)$

$x + 1 = (A + B + C)x^2 + (A - 2B + 3C)x - 6A$

तुलना करने पर,

A + B + C = 0, A − 2B + 3C = 1, − 6A = 1
हल करने पर,

$$A = -\frac{1}{6}, B = -\frac{2}{15}, C = \frac{3}{10}$$

$$\Rightarrow \int \frac{x+1}{x^3+x^2-6x} dx = -\frac{1}{6} \int \frac{1}{x} dx - \frac{2}{15} \int \frac{1}{x+3} dx]$$

$$+ \frac{3}{10} \int \frac{1}{x-2} dx$$

$$= -\frac{1}{6} \log|x| - \frac{2}{15} \log|x+3| + \frac{3}{10} \log|x-2| + C$$

प्रश्न 11.

$$\int \frac{x^2-8x+4}{x^3-4x} dx$$

हल:

$$\int \frac{x^2-8x+4}{x^3-4x} dx$$

माना $\frac{x^2-8x+4}{x^3-4x} = \frac{x^2-8x+4}{x(x^2-4)}$

$$= \frac{x^2-8x+4}{x(x-2)(x+2)}$$

$$\Rightarrow \frac{x^2-8x+4}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

⇒ $x^2 - 8x + 4 = A(x^2 - 4) = A(x^2 - 4) + B(x^2 + 2x) + C(x^2 - 2x)$
⇒ $x^2 - 8x + 4 = (A + B + C)x^2 + (2B - 2C)x - 4A$
तुलना करने पर
A + B + C = 1, 2B − 2 C = − 8, − 4 A = 4
हल करने पर, A = -1, B = − 1, C = 3

$$\Rightarrow \int \frac{x^2+8x+4}{x^3-4x} dx = -1 \int \frac{1}{x} dx - \int \frac{1}{(x-2)} dx + 3$$

$$\int \frac{1}{(x+2)} dx$$

= − log | x | − log | x − 2 | + 3 log | x + 2 | + C

प्रश्न 12.

$$\int \frac{1}{(x-1)^2(x+2)}\,dx$$

हल:

$$\int \frac{1}{(x-1)^2(x+2)}\,dx$$

माना

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$$

$\Rightarrow 1 = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$
$\Rightarrow 1 = A(x^2 - x + 2x - 2) + B(x + 2) + C(x^2 - 2x + 1)$
$\Rightarrow 1 = A(x^2 + x - 2) + B(x + 2) + C(x^2 - 2x + 1)$
$\Rightarrow 1 = (A + C)x^2 + (A + B - 2C)x - 2A + 2B + C$
तुलना करने पर,
$A + C = 0,\ A + B - 2C = 0,\ -2A + 2B + C = 1$
हल करने पर,

$$A = \frac{-1}{9},\ B = \frac{1}{3},\ C = \frac{1}{9}$$

$$\int \frac{1}{(x-1)^2(x+2)}\,dx$$

$$= \frac{-1}{9}\int\frac{1}{x-1}\,dx+\frac{1}{3}\int\frac{1}{(x-1)^2}\,dx+\frac{1}{9}\int\frac{1}{x+2}\,dx$$

$$= \frac{-1}{9}\log|x-1|-\frac{1}{3}\frac{1}{(x-1)}+\frac{1}{9}\log|x+2|+C$$

$$= \frac{1}{9}\log\frac{|x+2|}{|x-1|}-\frac{1}{3(x-1)}+C$$

प्रश्न 13.

$$\int \frac{1-3x}{1+x+x^2+x^3}\,dx$$

हल:

$$\int \frac{1-3x}{1+x+x^2+x^3} dx$$

$$\frac{1-3x}{1+x+x^2+x^3} = \frac{1-3x}{(1+x)+x^2(1+x)}$$

$$= \frac{1-3x}{(1+x)(1+x^2)}$$

माना $$\frac{1-3x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$\Rightarrow 1 - 3x = A(1 + x^2) + (Bx + C)(1 + x)$
$\Rightarrow 1 - 3x = A + Ax^2 + Bx + C + Bx^2 + Cx$
$\Rightarrow 1 - 3x = Ax^2 + Bx^2 + Bx + Cx + A + C$
$\Rightarrow 1 - 3x = (A + B)x^2 + (B + C)x + A + C$

तुलना करने पर,

$A + B = 0, B + C = -3, A + C = 1$

हल करने पर

$A = 2, B = -2, C = -1$

$$\therefore \int \frac{1-3x}{1+x+x^2+x^3} dx$$

$$= 2\int \frac{1}{(x+1)} dx + \int \frac{-2x+(-1)}{1+x^2} dx$$

$$= 2\int \frac{1}{1+x} dx - \int \frac{2x+1}{1+x^2} dx$$

$$= 2\int \frac{1}{1+x} dx - \int \frac{2x}{1+x^2} dx - \int \frac{1}{1+x^2} dx$$

$$= 2 \log|1+x| - \log|1+x^2| - \tan^{-1} x + C$$

$$= \log \frac{|1+x|^2}{|1+x^2|} - \tan^{-1} x + C$$

प्रश्न 14.

$$\int \frac{1+x^2}{x^5-x} dx$$

हल:

$$\int \frac{1+x^2}{x^5-x}\,dx$$

$$\frac{1+x^2}{x^5-x} = \frac{1+x^2}{x(x^4-1)} = \frac{1+x^2}{x(x^2-1)(x^2+1)}$$

$$= \frac{1+x^2}{x(x-1)(x+1)(x^2+1)} = \frac{1}{x(x-1)(x+1)}$$

अब $\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$\Rightarrow \quad \frac{1}{x(x-1)(x+1)} = \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x-1)(x+1)}$

$\Rightarrow \quad 1 = (A+B+C)x^2 + (B-C)x - A$

तुलना करने पर,

$$A+B+C=0,\ B-C=0,\ -A=1$$

हल करने पर,

$$A=-1,\ B=1/2\ C=1/2$$

$\therefore \int \frac{1+x^2}{x^5-x}\,dx = -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-1} + \frac{1}{2}\int \frac{1}{x+1}\,dx$

$$= -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1|\,C$$

$$= \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| - \log|x| + C$$

$$= \frac{1}{2}\log\left|\frac{(x-1)(x+1)}{x^2}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| + C$$

प्रश्न 15.

$$\int \frac{x^2+5x+3}{x^2+3x+2}\,dx$$

हल:

$$\int \frac{x^2+5x+3}{x^2+3x+2}\,dx$$

$$\frac{x^2+5x+3}{x^2+3x+2} = \frac{(x^2+3x+2)+(2x+1)}{(x^2+3x+2)}$$

$$= 1 + \frac{2x+1}{x^2+3x+2}$$

$$= 1 + \frac{2x+1}{(x+1)(x+2)}$$

माना $\frac{2x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

⇒ $2x + 1 = Ax + Bx + 2A + B$

⇒ $2x + 1 = (A + B)x + (2A + B)$

तुलना करने पर,

$A + B = 2, 2A + B = 1$

हल करने पर,

$A = -1, B = 3$

$$\therefore \int \frac{x^2+5x+3}{x^2+3x+2}\,dx$$

$$= \int 1\,dx + (-1)\int \frac{1}{x+1}\,dx + 3\int \frac{1}{(x+2)}\,dx$$

$= x - \log|x + 1| + 3 \log|x + 2|$

प्रश्न 16.

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx$$

हल:

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx$$

माना $\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$(x - 1) = A(x^2 + 1) + (Bx + C)(x + 1)$

$x - 1 = Ax^2 + A + Bx^2 + Cx + Bx + C$

$x - 1 = (A + B)x^2 + (B + C)x + A + C$

तुलना करने पर,

$A + B = 0, B + C = 1, A + C = -1$

हल करने पर,
A = − 1, B = 1, C = 0

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx = -\int \frac{1}{x+1}\,dx + \int \frac{x}{x^2+1}\,dx$$

$$= -\log|x+1| + \frac{1}{2}\log|x^2+1| + C$$

$$= -\log|x+1| + \log\sqrt{x^2+1} + C$$

$$= \log\frac{\sqrt{x^2+1}}{|x+1|} + C$$

प्रश्न 17.

$$\int \frac{1}{(1+e^x)(1-e^{-x})}\,dx$$

हल :

$$\int \frac{1}{(1+e^x)(1-e^{-x})}\,dx = \int \frac{e^x}{(e^x+1)(e^x-1)}\,dx$$

माना $e^x = t \Rightarrow e^x\,dx = dt$

$$= \int \frac{dt}{(t+1)(t-1)}$$

माना $\frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1}$

$\Rightarrow \quad 1 = A(t-1) + B(t+1)$

$\Rightarrow \quad 1 = (A+B)t + (-A+B)$

तुलना से, $A + B = 0, \ -A + B = 1$

$$2B = 1 \Rightarrow B = \frac{1}{2}, \ A = -\frac{1}{2}$$

$$\therefore \int \frac{1}{(1+e^x)(1-e^{-x})}\,dx = \int \frac{-1/2}{(t+1)}\,dt + \int \frac{1}{t-1}\,dt$$

$$= -\frac{1}{2}\int \frac{1}{t+1}\,dt + \frac{1}{2}\int \frac{1/2}{t-1}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{2}\log|t-1| + C$$

$$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{e^x - 1}{e^x + 1} \right| + C$$

प्रश्न 18.

$$\int \frac{1}{(e^x - 1)^2} dx$$

हल :

$$\int \frac{1}{(e^x - 1)^2} dx$$

(e^x से अंश व हर में गुणा करने पर)

माना $e^x - 1 = t$, तो $e^x = t + 1$

$e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{e^x (e^x - 1)^2} dx = \int \frac{dt}{(t+1)t^2}$$

अब $$\frac{1}{(t+1)t^2} = \frac{A}{t+1} + \frac{B}{t} + \frac{C}{t^2}$$

$1 = At^2 + Bt(t - 1) + C(t + 1)$

$1 = At^2 + Bt^2 + Bt + Ct + C$

$1 = (A + B)t^2 - (B + C)t + C$

तुलना से, $A + B = 0, B + C = 0, C = 1$

हल करने पर,

$C = 1, B = -1, A = 1$

अतः $$\int \frac{1}{(e^x - 1)^2} dx = \int \frac{1}{t+1} dt - \int \frac{1}{t} dt + \int \frac{1}{t^2} dt$$

$$= \log|t + 1| - \log|t| - \frac{1}{t} + C$$

$$= \log \left| \frac{t+1}{t} \right| - \frac{1}{t} + C$$

$$= \log \left| \frac{e^x - 1 + 1}{e^x - 1} \right| - \frac{1}{e^x - 1} + C$$

$$= \log \left| \frac{e^x}{e^x - 1} \right| - \frac{1}{e^x - 1} + C$$

प्रश्न 19.

$$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$$

हल :

$$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$$

माना
$e^x = t$
$e^x\,dx = dt$

$$\int \frac{e^x}{(e^x)^2+5e^x+6}\,dx = \int \frac{dt}{t^2+5t+6}$$

$$= \int \frac{1}{(t+3)(t+2)}\,dt$$

पुनः माना $\frac{1}{(t+3)(t+2)} = \frac{A}{t+3} + \frac{B}{t+2}$

⇒ $1 = At + 2A + Bt + 3B$
⇒ $1 = (A + B)t + (2A + 3B)$
तुलना करने पर,
$A + B = 0,\ 2A + 3B = 1$
हल करने पर,
$A = -1,\ B = 1$

$$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$$

$$= \int \frac{dt}{t^2+5t+6} = \int \frac{dt}{(t+3)(t+2)}$$

$$= \int \frac{(-1)}{t+3}\,dt + \int \frac{1}{t+2}\,dt$$

$$= -\log|t+3| + \log|t+2| + C$$

$$= \log\left|\frac{t+2}{t+3}\right| + C$$

$$= \log\left|\frac{e^x+2}{e^x+3}\right| + C$$

प्रश्न 20.

$$\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)} dx$$

हल :

$$\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)} dx$$

माना

tan x = t

sec² x dx = dt

$$\Rightarrow \int \frac{\sec^2 x\, dx}{(2+\tan x)(3+\tan x)}$$

$$= \int \frac{dt}{(2+t)(3+t)}$$

माना $\frac{1}{(2+t)(3+t)} = \frac{A}{2+t} + \frac{B}{3+t}$

⇒ 1 = 3A + At + 2B + Bt

⇒ 1 = (A + B)t + 3A + 2B

तुलना करने पर,

A + B = 0

तथा 3A + 2B = 1

हल करने पर, A = 1, B = -1

$$\int \frac{\sec^2 x\, dx}{(2+\tan x)(3+\tan x)} = \int \frac{dt}{(2+t)(3+t)}$$

$$= \int \frac{1}{2+t} dt - \int \frac{1}{3+t} dt$$

$$= \log|2+t| - \log|3+t| + C$$

$$= \log\left|\frac{2+\tan x}{3+\tan x}\right| + C$$

प्रश्न 21.

$$\int \frac{1}{x(x^5+1)} dx$$

हल:

$$\int \frac{1}{x(x^5+1)} dx$$

(अंश व हर में x^4 से गुणा करने पर)

$$= \int \frac{x^4}{x^5(x^5+1)} dx$$

माना $x^5 = t$

$5x^4 dx = dt$ या $x^4 dx = dt$

$$= \frac{1}{5} \int \frac{dt}{t(t+1)}$$

माना $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$

$\Rightarrow 1 = At + A + Bt$

$\Rightarrow 1 = (A + B)t + A$

तुलना करने पर

$A + B = 0, A = 1$

हल करने पर,

$A = 1, B = -1$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\int \frac{1}{t(t+1)} dt = \int \frac{1}{t} dx - \int \frac{1}{t+1} dt$$

$$= \log|t| - \log|t+1| + C$$

अतः $\int \frac{1}{x(x^5+1)} dx$

$$= \frac{1}{5}\log|x^5| - \frac{1}{5}\log|x^5+1| + C$$

प्रश्न 22.

$$\int \frac{1}{x(a+bx^n)} dx$$

हल:

$$\int \frac{1}{x(a+bx^n)}\,dx = \int \frac{x^{n-1}}{x.x^{n-1}(a+bx^n)}\,dx$$

$$= \int \frac{x^{n-1}}{x^n(a+bx^n)}\,dx$$

$$a + bx^n = t$$

$$bn\, x^{n-1}\, dx = dt$$

$$\Rightarrow \quad x^{n-1}\, dx = \frac{1}{bn}\, dt$$

$$\Rightarrow \quad \int \frac{1}{x(a+bx^n)}\,dx = \int \frac{1}{\left(\frac{t-a}{b}\right)} \cdot \frac{1}{bn}\, dt$$

$$= \frac{1}{n} \int \frac{1}{t(t-a)}\, dt$$

$$= \frac{1}{n}\left[\int \frac{-1}{at}\, dt + \int \frac{1}{a(t-a)}\, dt\right]$$

$$= \frac{1}{n}\left[-\frac{1}{a}\log t + \frac{1}{a}\log|t-a|\right] + C$$

$$= \frac{1}{an}\log\left|\frac{t-a}{t}\right| + C$$

$$= \frac{1}{an}\log\left|\frac{a+bx^n-a}{a+bx^n}\right| + C$$

$$= \frac{1}{an}\log\left|\frac{bx^n}{a+bx^n}\right| + C$$

प्रश्न 23.

$$\int \frac{8}{(x+2)(x^2+4)}\,dx$$

हल:

$$\int \frac{8}{(x+2)(x^2+4)}\,dx$$

माना $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$\Rightarrow 8 = (x^2 + 4) = (Bx + C)(x + 2)$

$= A(x^2 + 4) + Bx^2 + Cx + 2Bx + 2C$
$= (A + B)x^2 + (2B + C)x + 4A + 2$

तुलना करने पर,

$A + B = 0, 2B + C = 0, 4A + 2C = 8$

हल करने पर,

$A = 1, B = -1, C = 2$

$$\Rightarrow \int \frac{8}{(x+2)(x^2+4)} dx$$

$$= \int \frac{1}{x+2} dx + \int \frac{-x+2}{x^2+4} dx$$

$$= \int \frac{1}{x+2} dx - \int \frac{x}{x^2+4} dx + 2\int \frac{1}{x^2+4} dx$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2\times\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\frac{x}{2} + C$$

प्रश्न 24.

$$\int \frac{1-\cos x}{\cos x(1+\cos x)} dx$$

हल : $\int \frac{1-\cos x}{\cos x(1+\cos x)} dx$

$$= \int \frac{1}{\cos x(1+\cos x)} dx - \int \frac{1}{1+\cos x} dx$$

$$= \int \frac{1}{\cos x} dx - \int \frac{1}{1+\cos x} dx - \int \frac{1}{1+\cos x} dx$$

$$= \int \sec x \, dx - \int \sec^2 \frac{x}{2} dx$$

$$= \log|\sec x + \tan x| - 2\tan\left(\frac{x}{2}\right) + C$$

Ex 9.5

निम्न फलनों का x के सापेक्ष समाकलन कीजिए-

प्रश्न 1.

$$\frac{1}{x^2+2x+10}$$

हल :

$$\int \frac{1}{x^2+2x+10}\,dx$$

$$= \int \frac{1}{x^2+2\cdot1\cdot x+1^2+9}\,dx$$

$$= \int \frac{1}{(x+1)^2+3^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right)+C$$

प्रश्न 2.

$$\frac{1}{2x^2+x-1}$$

हल :

$$\int \frac{1}{2x^2+x-1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}x-\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x^2+2\cdot\frac{1}{4}\cdot x+\frac{1}{16}\right)-\frac{1}{16}-\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}\,dx$$

$$= \frac{1}{2} \times \frac{1}{2 \times \left(\frac{3}{4}\right)} \log \left| \frac{\left(x + \frac{1}{4}\right) - \frac{3}{4}}{x + \frac{1}{4} + \frac{3}{4}} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{4x - 2}{4x + 4} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{2x - 1}{2x + 2} \right| + C$$

प्रश्न 3.

$$\int \frac{1}{9x^2 - 12x + 8} dx$$

हल :

$$\int \frac{1}{9x^2 - 12x + 8} dx$$

$$= \frac{1}{9} \int \frac{1}{x^2 - \frac{12}{9}x + \frac{8}{9}} dx$$

$$= \frac{1}{9} \int \frac{1}{x^2 - \frac{4}{3}x + \frac{8}{9}} dx$$

$$= \frac{1}{9} \int \frac{1}{\left(x^2 - 2 \times \frac{2}{3} \times x + \frac{4}{9}\right) + \frac{4}{9}} dx$$

$$= \frac{1}{9} \int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} dx$$

$$= \frac{1}{9} \times \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1} \left(\frac{x - \frac{2}{3}}{\frac{2}{3}} \right) + C$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3x - 2}{2} \right) + C$$

प्रश्न 4.

$$\int \frac{1}{3+2x-x^2}$$

हल :

$$\int \frac{1}{3+2x-x^2}\,dx$$

$$= \int \frac{1}{4-1+2x-x^2}\,dx$$

$$= \int \frac{1}{2^2-(x-1)^2}\,dx$$

$$= \frac{1}{2\times 2}\log\left|\frac{2+x-1}{2-x+1}\right|+C$$

$$= \frac{1}{4}\log\left|\frac{x+1}{3-x}\right|+C$$

प्रश्न 5.

$$\int \frac{x}{x^4+x^2+1}\,dx$$

हल :

$$\int \frac{x}{x^4+x^2+1}\,dx$$

माना $x^2 = t$, तो $2x\,dx = dt$

$$x\,dx = \frac{1}{2}\,dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2+t+1}$$

$$= \frac{1}{2}\int \frac{1}{\left(t^2+2\times\frac{1}{2}+t+\frac{1}{4}\right)+\frac{3}{4}}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= \frac{1}{2} \times \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \left| \frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right| + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left| \frac{2t+1}{\sqrt{3}} \right| + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left| \frac{2x^2+1}{\sqrt{3}} \right| + C$$

प्रश्न 6.

$$\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

हल :

$$\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

sin x = t

cos x dx = dt

$$= \int \frac{dt}{t^2 + 4t + 5}$$

$$= \int \frac{dt}{(t^2 + 2 \cdot 2t + 4) + 1}$$

$$= \int \frac{dt}{(t+2)^2 + 1}$$

$$= \frac{1}{1} \tan^{-1} \frac{t+2}{1} + C$$

$= \tan^{-1}(t+2)+c$

$= \tan^{-1}(\sin x+2)+c$

प्रश्न 7.

$$\int \frac{x-3}{x^2 + 2x - 4} dx$$

हल :

$$\int \frac{x-3}{x^2 + 2x - 4} dx$$

$x - 3 = A \frac{d}{dx}(x^2 + 2x - 4) + B$
$\Rightarrow x - 3 = A(2x - 2) + B$
$\Rightarrow x - 3 = 2Ax + (2A + B)$
तुलना करने पर,

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$2A + B = -3$$
$$2 \times \frac{1}{2} + B = -3 \Rightarrow B = -4$$
$$x - 3 = \frac{1}{2}(2x + 2) - 4$$

$$\therefore \int \frac{x-3}{x^2+2x-4}\,dx = \int \frac{\frac{1}{2}(2x+2)-4}{x^2+2x-4}\,dx$$
$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}\,dx - \int \frac{4}{x^2+2x-4}\,dx$$
$$= \frac{1}{2}\log|x^2+2x-4| - 4\int \frac{1}{(x^2+2\cdot 1\cdot x+1)-5}\,dx$$
$$= \frac{1}{2}\log|x^2+2x-4| - 4\int \frac{1}{(x+1)^2-(\sqrt{5})^2}\,dx$$
$$= \frac{1}{2}\log|x^2+2x-4| - \frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$$

प्रश्न 8.

$$\int \frac{3x+1}{2x^2-2x+3}$$

हल :

$$\int \frac{3x+1}{2x^2-2x+3}\,dx$$

$3x + 1 = A \frac{d}{dx}(2x^2 - 2x + 3) + B$
$\Rightarrow 3x + 1 = A(4x - 2) + B$
$\Rightarrow 3x + 1 = 4Ax - 2A + B$
तुलना करने पर,
$4A = 3 \Rightarrow A = \frac{3}{4}$

$$-2A + B = 1 \Rightarrow -2 \times \frac{3}{4} + B = 1$$

$$B = 1 + \frac{3}{2} = \frac{5}{2}$$

$$3x + 1 = \frac{3}{4}(4x-2) + \frac{5}{2}$$

$$\int \frac{3x+1}{2x^2-2x+3}\,dx = \int \frac{\frac{3}{4}(4x-2)+\frac{5}{2}}{2x^2-2x+3}\,dx$$

$$= \frac{3}{4}\int \frac{4x-2}{2x^2-2x+3}\,dx + \frac{5}{2}\int \frac{1}{2x^2-2x+3}\,dx$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{5}{4}\int \frac{1}{x^2-x+\frac{3}{2}}\,dx + C_1 \quad \text{...(i)}$$

$$\int \frac{1}{x^2-x+\frac{3}{2}}\,dx = \int \frac{1}{x^2-2\times\frac{1}{2}\times x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{3}{2}}\,dx$$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{5}}{2}\right)^2}\,dx$$

$$= \frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + C_2$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_2 \quad \text{...(ii)}$$

समीकरण (i) व (ii) से

$$\int \frac{3x+1}{2x^2-2x+3}\,dx$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{5}{4}\times\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_1 + C_2$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C$$

(जहाँ $C = C_1 + C_2$)

प्रश्न 9.

$$\int \frac{x+1}{x^2+4x+5}\,dx$$

हल :

$$\int \frac{x+1}{x^2+4x+5}$$

माना $x + 1 = A\frac{d}{dx}(x^2 + 4x + 5) + B$

$\Rightarrow x + 1 = A(2x + 4) + B$

$\Rightarrow 2A = 1 \Rightarrow A = 1/2$

$\Rightarrow 4A + B = 1 \Rightarrow B = -1$

$\therefore x + 1 = \frac{1}{2}(2x + 4) - 1$

$$= \int \frac{\frac{1}{2}(2x+4)-1}{x^2+4x+5}$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,dx - \int \frac{1}{x^2+4x+5}\,dx$$

$$= \frac{1}{2}\log|x^2+4x+5| - \int \frac{1}{(x+2)^2+1}\,dx$$

$$= \frac{1}{2}\log|x^2+4x+5| - \tan^{-1}(x+2) + C$$

प्रश्न 10.

$$\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\,dx$$

हल :

$$\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\,dx$$

माना $\sin x = t$

cos x dx = dt

$$= \int \frac{(3t-2)}{4+t^2-4t}\,dt$$

$$= \int \frac{(3t-2)}{t^2-4t+4}\,dt$$

$$= \int \frac{3t-2}{(t-2)^2}\,dt$$

$$= \int \frac{(3t-6)+4}{(t-2)^2}\,dt$$

$$= \int \frac{3(t-2)}{(t-2)^2}\,dt + \int \frac{4}{(t-2)^2}$$

$$= 3\int \frac{1}{(t-2)}\,dt + 4\int (t-2)^{-2}\,dt$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + C$$

$$= 3\log(2-t| + \frac{4}{(t-2)} + C$$

$$= 3\log|2-\sin x| + \frac{4}{(2-\sin x)} + C$$

$$(\because\ 0 \le \sin x \le 1)$$

प्रश्न 11.

$$\int \frac{1}{2e^{2x}+3e^{x}+1}\,dx$$

हल :

माना $I = \int \frac{1}{2e^{2x}+3e^{x}+1}\,dx$

$$= \int \frac{e^{-x}\cdot e^{-x}\,dx}{2+3e^{-x}+e^{-2x}}$$

(e^{-2x} से हर व अंश में गुणा करने पर)

माना $I = \int \frac{1}{2e^{2x}+3e^{x}+1}\,dx$

$$= \int \frac{e^{-x}\cdot e^{-x}\,dx}{2+3e^{-x}+e^{-2x}}$$

$$= \int \frac{e^{-x} \cdot e^{-x}\, dx}{e^{-2x} + 3e^{-x} + 2}$$

माना $e^{-x} = t$

$-e^{-x}\, dx = dt$

$e^{-x}\, dx = -dt$

t = At + 2A + Bt + B

t = (A +B)t + (2A + B)

तुलना से, A + B = 1

2A + B = 0

B = -2A

A + (-2A) = 1

A − 2A = 1

− 2A = 1

− A = 1 या A = -1

B = -2 x (- 1) = 2 या B = 2

$$\Rightarrow \quad I = -\int \frac{-1}{t+1}\, dt - \int \frac{2}{t+2}\, dt$$

$$= \log|t+1| - 2\log|t+2| + C$$

$$= \log \frac{|t+1|}{|t+2|^2} + C$$

$$= \log \left| \frac{t+1}{t^2 + 4t + 4} \right| + C$$

$$= \log \left| \frac{e^{-x} + 1}{e^{-2x} + 4e^{-x} + 4} \right| + C$$

प्रश्न 12.

$$\int \frac{1}{\sqrt{4x^2 - 5x + 1}}\, dx$$

हल :

$$\int \frac{1}{\sqrt{4x^2 - 5x + 1}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2 - \frac{5}{4}x + \frac{1}{4}}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2 - 2 \times \frac{5}{8} \times x + \frac{25}{64} + \frac{1}{4} - \frac{25}{64}}} dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\left(x - \frac{5}{8}\right)^2 - \frac{9}{64}}} dx$$

$$= \frac{1}{2} \log \left| \left(x - \frac{5}{8}\right) + \sqrt{\left(x - \frac{5}{8}\right)^2 - \frac{9}{64}} \right| + C$$

$$= \frac{1}{2} \log \left| \left(x - \frac{5}{8}\right) + \sqrt{x^2 - \frac{5}{4}x + \frac{1}{4}} \right| + C$$

प्रश्न 13.

$$\int \frac{1}{\sqrt{5x - 6 - x^2}}$$

हल :

$$\int \frac{1}{\sqrt{5x - 6 - x^2}} dx$$

$$= \int \frac{1}{\sqrt{\frac{25}{4} - \frac{24}{4} - \left(x^2 + \frac{25}{4} - 5x\right)}} dx$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} - \left(x - \frac{5}{2}\right)^2}} dx$$

$$= \sin^{-1} \left(\frac{x - 5/2}{1/2}\right) + C$$

$$= \sin^{-1} \left(\frac{(2x - 5)/2}{1/2}\right)$$

$$= \sin^{-1} (2x - 5) + C$$

प्रश्न 14.

$$\int \frac{1}{\sqrt{1 - x - x^2}}$$

हल :

$$\int \frac{1}{\sqrt{1-x-x^2}}\,dx$$

$$1-x-x^2 = 1+\frac{1}{4}-\left(\frac{1}{4}+x+x^2\right)$$

$$= \frac{5}{4}-\left(\frac{1}{2}+x\right)^2$$

$$\int \frac{1}{\sqrt{1-x-x^2}}\,dx = \int \frac{1}{(\sqrt{5}/2)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{5}/2}\right)$$

$$= \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$$

प्रश्न 15.

$$\int \frac{1}{\sqrt{4+3x-2x^2}}$$

हल :

$$\int \frac{1}{\sqrt{4+3x-2x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{4-(2x^2-3x)}}\,dx$$

$$= \int \frac{1}{\sqrt{4-2\left(x^2-\frac{2\times 3}{4}x+\frac{9}{16}-\frac{9}{16}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{4+\frac{9}{8}-2\left(x-\frac{3}{4}\right)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{41}{8} - 2\left(x - \frac{3}{4}\right)^2}} \, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2}} \, dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x - \frac{3}{4}}{\sqrt{41/4}} \right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4x - 3}{\sqrt{41}} \right) + C$$

प्रश्न 16.

$$\int \frac{x + 2}{\sqrt{x^2 - 2x + 4}} \, dx$$

हल :

$$\int \frac{x + 2}{\sqrt{x^2 - 2x + 4}} \, dx$$

माना

$x^2 - 2x + 4 = t$

$(2x - 2) = t$

$2(x - 1)dx = dt$

$$\Rightarrow \quad (x - 1) \, dx = \frac{dt}{2}$$

$$= \int \frac{x - 1 + 3}{\sqrt{x^2 - 2x + 4}} \, dx$$

$$= \int \frac{x - 1}{\sqrt{x^2 - 2x + 4}} \, dx + \int \frac{3}{\sqrt{x^2 - 2x + 4}} \, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{t}} + \int \frac{3}{\sqrt{(x - 1)^2 + (\sqrt{3})^2}} \, dx$$

$$= \frac{1}{2} \frac{t^{-1/2 + 1}}{-1/2 + 1} + 3 \log |(x - 1) + \sqrt{(x - 1)^2 + (\sqrt{3})^2}| + C$$

$$= \frac{1}{2} \frac{t^{1/2}}{1/2} + 3 \log |(x - 1) + \sqrt{x^2 - 2x + 4}| + C$$

$$= \sqrt{(x^2 - 2x + 4)} + 3 \log |(x - 1) + \sqrt{x^2 - 2x + 4}| + C$$

प्रश्न 17.

$$\int \frac{x+1}{\sqrt{x^2-x+1}}\, dx$$

हल :

$$\int \frac{x+1}{\sqrt{x^2-x+1}}$$

$x + 1 = A\frac{d}{dx}(x^2 - x + 1) + B$
$x + 1 = A(2x - 1) + B$
$x + 1 = 2Ax - A + B$

तुलना करने पर,

$2A = 1, A = 1/2$
$-A + B = 1$
$\Rightarrow B = \frac{3}{4}$

$$\int \frac{x+1}{\sqrt{x^2-x+1}}\, dx = \frac{1}{2}\int \frac{(2x-1)}{\sqrt{x^2-x+1}}\, dx + \frac{3}{2}\int \frac{1}{\sqrt{x^2-x+1}}\, dx$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}\, dx$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\log\left|\left(x-\frac{1}{2}\right) + \sqrt{x^2-x+1}\right| + C$$

प्रश्न 18.

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\, dx$$

हल :

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\, dx$$

$x + 3 = A\frac{d}{dx}(x^2 + 2x + 2) + B$
$\Rightarrow x + 3 = A(2x + 2) + B$
$\Rightarrow x + 3 = 24x + 2A + B$

तुलना करने पर,

2A = 1, A = $\frac{1}{2}$

2A + B = 3, B = 2

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+2)}{\sqrt{x^2+2x+2}}\,dx + 2\int \frac{1}{\sqrt{x^2+2x+2}}\,dx$$

$$= \sqrt{(x^2+2x+2)} + 2\int \frac{1}{\sqrt{(x+1)^2+1}}$$

$$= \sqrt{(x^2+2x+2)} + 2\log|(x+1)+\sqrt{x^2+2x+2}| + C$$

प्रश्न 19.

$$\int \sqrt{\sec x - 1}\;dx$$

हल :

$$\int \sqrt{\sec x - 1}\,dx$$

$$= \int \sqrt{\frac{1}{\cos x} - 1}\,dx$$

$$= \int \sqrt{\frac{1-\cos x}{\cos x} \times \frac{1+\cos x}{1+\cos x}}\,dx$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x\,(1+\cos x)}}\,dx$$

$$= \int \frac{\sin x}{\sqrt{\cos x\,(1+\cos x)}}\,dx$$

माना $\cos = t$

$-\sin x\,dx = dt$

$\sin x\,dx = -\,dt$

$$= -\int \frac{1}{\sqrt{t(1+t)}}\,dx$$

$$= -\int \frac{1}{\sqrt{t^2+t+1/4-1/4}}\,dt$$

$$= -\int \frac{1}{\sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dt$$

$$= -\log\left|\left(t+\frac{1}{2}\right)+\sqrt{t^2+t}\right|+C$$

$$= -\log\left|\left(\cos x+\frac{1}{2}\right)+\sqrt{\cos^2 x+\cos x}\right|+C$$

प्रश्न 20.

$$\int\sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}$$

हल :

माना $I = \int\sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

$$= \int\sqrt{\frac{\sin(x-\alpha).\sin(x-\alpha)}{\sin(x+\alpha).\sin(x-\alpha)}}\,dx$$

$$= \int\sqrt{\frac{\sin^2(x-\alpha)}{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= \int\frac{\sin(x-\alpha)}{\sqrt{\sin^2 x - \sin^2\alpha}}\,dx$$

$$= \int\frac{\sin x\cos\alpha - \cos x\sin\alpha}{\sqrt{\cos^2\alpha - \cos^2 x}}\,dx$$

$$= \cos\alpha\underbrace{\int\frac{\sin x}{\sqrt{\cos^2\alpha-\cos^2 x}}\,dx}_{I_1}$$

$$- \sin\alpha\underbrace{\int\frac{\cos x}{\sqrt{\cos^2\alpha-\cos^2 x}}\,dx}_{I_2}$$

$$I_1 = \int\frac{\sin x}{\sqrt{\cos^2\alpha-\cos^2 x}}\,dx$$

माना $\cos x = t \Rightarrow \sin x\,dx = -\,dt$

$$I_1 = \int \frac{-dt}{\sqrt{\cos^2 \alpha - t^2}}$$

$$I_1 = -\sin^{-1}\left(\frac{t}{\cos \alpha}\right)$$

$$I_1 = -\sin^{-1}\left(\frac{\cos x}{\cos \alpha}\right)$$

तथा $$I_2 = \int \frac{\cos x \, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

माना $\sin x = y$

$\cos x \, dx = dy$

$$I_2 = \int \frac{dy}{\sqrt{y^2 - \sin^2 \alpha}}$$

$$= \log\left[y + \sqrt{(y^2 - \sin^2 \alpha)}\right]$$

$$I_2 = \log\left[\sin x + \sqrt{\sin^2 x - \sin^2 \alpha}\right]$$

$\Rightarrow$ $$I = -\cos \alpha \sin^{-1}\left(\frac{\cos x}{\cos \alpha}\right) - \sin \alpha \log\left[\sin x + \sqrt{\sin^2 x - \sin^2 \alpha}\right] + C$$

प्रश्न 21.

$$\int \frac{x^3}{x^2 + x + 1} dx$$

हल :

$$\int \frac{x^3}{x^2 + x + 1} dx$$

$$= \int \frac{x^3 + x^2 - x^2 - x + x}{x^2 + x + 1} dx$$

$$= \int \frac{x^3 + x^2 + x - x^2 - x}{x^2 + x + 1} dx$$

$$= \int \left[\frac{x(x^2 + x + 1)}{(x^2 + x + 1)} - \frac{(x^2 + x + 1) - 1}{x^2 + x + 1}\right] dx$$

$$= \int x\,dx - \int dx + \int \frac{1}{x^2 + x + 1}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{1}{x^2 + x + \frac{1}{4} - \frac{1}{4} + 1}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{x^2}{2} - x + \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1}\left(\frac{x + \frac{1}{2}}{\sqrt{\frac{3}{2}}}\right) + C$$

$$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

प्रश्न 22.

$$\int \frac{e^x}{e^{2x} + 6e^x + 5}\,dx$$

हल :

$$\int \frac{e^x}{e^{2x} + 6e^x + 5}\,dx$$

माना $e^x = t$

$e^x\,dx = dt$

$$= \int \frac{dt}{t^2 + 6t + 5}$$

$$= \int \frac{dt}{t^2 + 2 \cdot 3 \cdot t + 9 - 4}$$

$$= \int \frac{1}{(t+3)^2 - 2^2}\,dt$$

$$= \frac{1}{2 \times 2} \log\left|\frac{t + 3 - 2}{t + 3 + 2}\right| + C$$

$$= \frac{1}{4} \log\left|\frac{e^x + 1}{e^x + 5}\right| + C$$

Ex 9.6

निम्न फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.
(a) ∫x cos
(b) ∫x sec^2x

हल : (a) ∫x cos dx

$$= x\int \cos x\, dx - \int\left[\frac{d}{dx}(x).\int \cos x\, dx\right]dx$$

= x sin x – ∫1.sin x dx
= x sin x – (– cos x) + C
= x sin x + cos x + C

(b) ∫x sec^2x dx

$$= x\int \sec^2 x\, dx - \int\left[\frac{d}{dx}(x)\int \sec^2 x\, dx\right]dx$$

= x tan x – ∫1.tan x dx
= x tan x + log |cos x| + C
= x tan x – log |sec x| + C

प्रश्न 2.
(a) x^3e^{-x}
(b) x^3 sinx

हल : (a) ∫x^3e^{-x}

$$= x^3\int e^{-x}\, dx - \int\left[\frac{d}{dx}x^3.\int e^{-x}\, dx\right]dx$$

$$= x^3\,\frac{e^{-x}}{-1} - \int 3x^2\,\frac{e^{-x}}{-1}\,dx$$

= – x^3e^{-x} + 3∫x^2e^{-x} dx ….(i)
∫x^2 e^{-x} dx

$$= x^2\int e^{-x}\, dx - \int\left[\frac{d}{dx}(x^2)\int e^{-x}\, dx\right]dx$$

= – x^2e^{-x} + ∫2xe^{-x} dx
= – x^2e^{-x} + 2∫xe^{-x} dx ……(ii)
∫x e^{-x} dx

$= x\int e^{-x}\,dx - \int \frac{d}{dx}(x)\int e^{-x}\,dx$

= − xe^{-x} + $\int e^{-x}$ dx

= − xe^{-x} − e^{-x} + C …..(iii)

समी. (i), (ii) व (iii) को हल करने पर

$\int x^3e^{-x}$ dx = − x^3e^{-x} + 3[- x^2e^{-x} + 2 $\int xe^{-x}$ dx]

= -x^3e^{-x} + 3(-x^2e^{-x} + 2(-xe^{-x} − e^{-x} + C1))

= -x^3e^{-x} − $3x^2e^{-x}$ + 6(-xe^{-x} − e^{-x} + C1)

= -x^3e^{-x} − $3x^2e^{-x}$ − $6xe^{-x}$ − $6e^{-x}$ + 6C1

= -x^3e^{-x} − $3x^2e^{-x}$ − $6xe^{-x}$ − $6e^{-x}$ + C

= -e [x^3 + $3x^2$ + 6x + 6] + C

(b) $\int x^3$ sinx

$$= x^3\int \sin x\,dx - \int\left(\frac{d}{dx}(x^3)\int \sin x\,dx\right)dx$$

$$= x^3(-\cos x) - 3\int x^2(-\cos x)\,dx$$

$$= -x^3\cos x + 3\int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\cos x}\,dx$$

$$= -x^3\cos x + 3\left[x^2\int \cos x\,dx - \int\left\{\frac{d}{dx}(x^2)\int \cos x\,dx\right\}dx\right]$$

$$= -x^3\cos x + 3\left[x^2\sin x - 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sin x}\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x - 6\left[x\int \sin x\,dx - \left\{\int \frac{d}{dx}(x)\int \sin x\,dx\right\}dx\right]$$

= -x^3 cos x + $3x^2$ sin x − 6(-x cos x + $\int$cos x dx)

= -x^3 cosx + $3x^2$ sin x + 6x cos x − 6 sin x + C

प्रश्न 3.

(a) $\int x^3 (\log x)^2$ dx

(b) $\int x^3 e_x 2$ dx.

हल :

(a) $\int x^3 (\log x)^2\, dx$

$$= (\log x)^2 \int x^3\, dx - \int \left[\frac{d}{dx} (\log x)^2 \int x^3\, dx \right] dx$$

$$= \frac{(\log x)^2\ x^4}{4} - \int \left[2 \log x \cdot \frac{1}{x} \times \frac{x^4}{4} \right] dx$$

$$= \frac{(\log x)^2\ x^4}{4} - \frac{1}{2} \int \underset{\text{I}}{\log x} . \underset{\text{II}}{x^3}\, dx$$

$$= \frac{1}{4} (\log x)^2\ x^4$$

$$- \frac{1}{2} \left[\log x \int x^3\, dx - \int \left\{ \frac{d}{dx} \log x \int x^3\, dx \right\} dx \right]$$

$$= \frac{1}{4} (\log x)^2\ x^4 - \frac{1}{2} \left[\log x\ \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx \right]$$

$$= \frac{1}{4} (\log x)^2 x^4 - \frac{1}{2} \left[\frac{1}{4} \log x \cdot x^4 - \frac{1}{4} \int x^3\, dx \right]$$

$$= \frac{1}{4} (\log x)^2 x^4 - \frac{1}{2} \left[\frac{1}{4} \log x \cdot x^4 - \frac{1}{4} \times \frac{x^4}{4} \right] + C$$

$$= \frac{1}{4} (\log x)^2 x^4 - \frac{1}{8} \log x \cdot x^4 + \frac{1}{32} x^4 + C$$

$$= \frac{x^4}{4} \left[(\log x)^2 - \frac{1}{2} \log x + \frac{1}{8} \right] + C$$

(b) $\int x^3 e_x 2\, dx$.

माना $x^2 = t$

$\Rightarrow 2x\, dx = dt$

$\Rightarrow x\, dx = \frac{dt}{2}$

$$= \int te^t \frac{dt}{2}$$

$$= \frac{1}{2}\int \underset{I}{t} . \underset{II}{e^t} \, dt$$

$$= \frac{1}{2}\left[t\int e^t \, dt - \int\left[\frac{d}{dt}(t)\int e^t \, dt\right]dt\right]$$

$$= \frac{1}{2}\left[te^t - \int e^t \, dt\right]$$

$$= \frac{1}{2}\,[te^t - e^t]$$

$$= \frac{e^t}{2}\,[t-1] + C$$

$$= \frac{1}{2}e^{x^2}\,(x^2-1) + C$$

प्रश्न 4. (a) $\int e^{2x} e_e x \, dx$
(b) $\int (\log x)^2 \, dx$

हल : (a) $\int e^{2x} e_e x \, dx$
माना $e^x = t$
$e^x \, dx = dt$

$$= \int e^x . e^{e^x} . e^x \, dx$$

$$= \int \underset{I}{t} \, \underset{II}{e^t} \, dt$$

$$= t\int e^t \, dt - \int\left\{\frac{d}{dt}(t)\int e^t \, dt\right\}dt$$

$= t\, e^t - \int 1\, e^t \, dt = te^t - e^t + c$
$= e^t\,(t-1) + c$
$= e_e x\,(e^x - 1) + c$

(b) $I = \int (\log x)^2 \, dx = \int (\log x)^2 . 1 \, dx$
[$(\log x)^2$ को प्रथम फलन तथा 1 को द्वितीय फलन लेने पर]

$$= (\log x)^2 \int 1\, dx - \int \left[\frac{d}{dx} (\log x)^2 \int 1\, dx \right] dx$$

$$= (\log x)^2\, x - \int \left[(2 \log x). \frac{1}{x} \times x\, dx \right] dx$$

$$= x(\log x)^2 - 2 \int \underset{\text{I}}{\log x}.\underset{\text{II}}{1}\, dx$$

(log x को प्रथम फलन तथा 1 को द्‌वितीय फलन लेने पर)

$$= x (\log x)^2 - 2\left[\log x \int 1\, dx - \int \frac{d}{dx} \log x \int 1\, dx \right] dx$$

$$= x (\log x)^2 - 2\left[\log x \int dx - \int \left\{ \frac{d}{dx} (\log x). \int dx \right\} dx \right]$$

$$= x (\log x)^2 - 2\left[x \log x - \int \frac{1}{x} x\, dx \right]$$

$= x (\log x)^2 - 2x \log x + 2x + C$

प्रश्न 5.

(a) $\int \cos^{-1} x\, dx$ **(b)** $\int \text{cosec}^{-1} \sqrt{\frac{x+a}{x}}\, dx$

हल :

(a) $I = \int \cos^{-1}x\, dx$

माना $\cos^{-1} x = t$

तब $x = \cos t$

$dx = - \sin t\, dt$

$\therefore I = - \int t \sin t\, dt$

(t को प्रथम फलन तथा sin t को द्‌वितीय फलन लेने पर)

$$= - \left[t \int \sin t\, dt - \int \left\{ \frac{d}{dt} (t) \int \sin t\, dt \right\} dt \right]$$

$$= - t (- \cos t) + \int 1.(- \cos t)\, dt$$

$$= t \cos t - \int \cos t\, dt = t \cos t - \sin t + C$$

$$= x \cos^{-1} x - \sqrt{1 - x^2} + C$$

(b) $\int \text{cosec}^{-1} \sqrt{\frac{x+a}{x}}\, dx$

माना $I = \int \text{cosec}^{-1} \sqrt{\frac{x+a}{x}}\, dx$

$$= \sin^{-1} \sqrt{\frac{x}{x+a}}\, dx$$

माना $x = a \tan^2\theta$

$\therefore dx = 2a \tan\theta \sec^2\theta \, d\theta$

अत: $$I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a(1+\tan^2\theta)}}\,2a\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\int \theta(\tan\theta\sec^2\theta)\,d\theta$$

$$= a\int \underset{\text{I}}{\theta}\underset{\text{II}}{(2\tan\theta\sec^2\theta)}\,d\theta$$

$$= a[\theta\tan^2\theta) - \int 1\cdot\tan^2\theta\,d\theta]$$

(θ को पहला एवं $\tan\theta\sec^2\theta$ को दूसरा फलन मानने पर)

खण्डशः समाकलन से,

$$= a\left[\theta\tan^2\theta - \int(\sec^2\theta - 1)\right]d\theta$$

$$= d\theta\tan^2\theta - \tan\theta + \theta$$

$$= a[\theta(1+\tan^2\theta) - \tan\theta)] + C$$

$$= a\left[\tan^{-1}\sqrt{\frac{x}{a}}\left(1+\frac{x}{a}\right) - \sqrt{\frac{x}{a}}\right] + C$$

$$= \left[(a+x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax}\right] + C$$

प्रश्न 6.

(a) $\int \sin^{-1}(3x - 4x^3)\,dx$ **(b)** $\int \frac{x}{1+\cos x}\,dx$

हल : (a) $I = \int \sin^{-1}(3x - 4x^3)\,dx$

माना $x = \sin t$, तब $dx = \cos t\,dt$

$\therefore I = \int \sin^{-1}(3\sin t - 4\sin^3 t).\cos t\,dt$

$= \int \sin^{-1}(\sin 3t)\cos t\,dt$

$= \int 3t\cos t\,dt$

$= 3\int t \cos t\, dt$

(t को प्रथम फलन तथा cos t को द्वितीय फलन लेने पर)

$$= 3\left[t\int \cos t\, dt - \int\left\{\frac{d}{dt}(t)\int \cos t\, dt\right\}dt\right]$$

$$= 3t \sin t - 3\int 1.\sin t\, dt$$

$$= 3t \sin t - 3(-\cos t) + C$$

$$= 3t \sin t + 3\cos t + C$$

$$= 3(\sin^{-1} x)(x) + 3\sqrt{1-\sin^2 t} + C$$

$$= 3x \sin^{-1} x + 3\sqrt{1-x^2} + C$$

(b) $\int \frac{x}{1+\cos x}dx$

$$I = \int \frac{x}{1+\cos x}dx = \int \frac{x}{2\cos^2 \frac{x}{2}}dx$$

$$= \frac{1}{2}\int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2 \frac{x}{2}}\, dx$$

(x को प्रथम फलन तथा $\sec^2 \frac{x}{2}$ को द्वितीय फलन लेने पर)।

$$= \frac{1}{2}\left[x\int \sec^2 \frac{x}{2}dx - \int\left\{\frac{d}{dx}(x)\int \sec^2 \frac{x}{2}dx\right\}\right]dx$$

$$= \frac{1}{2}\left[x \times 2\tan \frac{x}{2} - \int 1\cdot 2\tan \frac{x}{2}dx\right]$$

$$= x\tan \frac{x}{2} - \int \tan \frac{x}{2}dx$$

$$= x\tan \frac{x}{2} - 2\log\left|\sec \frac{x}{2}\right| + C$$

प्रश्न 7.

(a) $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$ **(b)** $\int \cos\sqrt{x}\,dx$

हल : (a) माना $x = \cos\theta$

$dx = -\sin\theta\, d\theta$,

$\therefore \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

$$= \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-\sin\theta\, d\theta)$$

$$= -\int \tan^{-1}\sqrt{\frac{1-1+2\sin^2\theta/2}{1+2\cos^2\theta/2-1}}\,(\sin\theta\, d\theta)$$

$$= -\int \tan^{-1}\sqrt{\tan^2\theta/2}\,\sin\theta\, d\theta$$

$$= -\int \tan^{-1}(\tan\theta/2)\sin\theta\, d\theta$$

$$= -\int \frac{\theta}{2}\sin\theta d\theta = -\frac{1}{2}\int \theta\sin\theta\, d\theta$$

$$= -\frac{1}{2}\left[\theta\int(\sin\theta) - \int\left[\frac{d}{dx}\theta\int\sin\theta\cdot\theta\right]\right]d\theta$$

$$-\frac{1}{2}\left[-\theta\cos\theta + \int 1.\cos\theta\, d\theta\right] + C$$

$$= -\frac{1}{2}[-\theta\cos\theta + \sin\theta] + C$$

$$= -\frac{1}{2}[-\theta\cos\theta + \sqrt{1-\cos^2\theta}] + C$$

$$= -\frac{1}{2}[-x\cos^{-1}x + \sqrt{1-x^2}] + C$$

$$= \frac{1}{2}[x\cos^{-1}x - \sqrt{1-x^2}] + C$$

(b) $\int \cos\sqrt{x}\, dx$

माना $\sqrt{x} = t, x = t^2$

$dx = 2t\, dt$

$\int \cos\sqrt{x}\, dx = \int \cos t.2t\, dt$

$$= 2\int \underset{\text{I}}{t}\cdot\underset{\text{II}}{\cos t}\, dt$$

$$= 2\left[t\int\cos t\, dt - \int\left\{\frac{d}{dt}(t).\int\cos t\, dt\right\}dt\right]$$

$$= 2\left[t\sin t - \int\sin t\, dx\right]$$

$$= 2[t\sin t + \cos t] + C$$

$$= 2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + C$$

प्रश्न 8.

(a) $\int \frac{x}{1+\sin x} dx$ (b) $\int x^2 \tan^{-1} x \, dx$

हल :

(a) $\int \frac{x}{1+\sin x} dx$

$$= \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= \int \frac{x(1-\sin x)}{(1-\sin^2 x)} dx$$

$$= \int \frac{x(1-\sin x)}{\cos^2 x} dx$$

$$= \int \frac{x}{\cos^2 x} dx - \int x \tan x \sec x \, dx$$

= $\int x \sec^2 x \, dx - \int x \tan x \sec x \, dx$
= $I_1 - I_2$ …….(i)
$I_1 = \int x \sec^2 x \, dx$

$$I_1 = x \int \sec^2 x \, dx - \int \left\{ \frac{d}{dx} x \int \sec^2 x \, dx \right\} dx$$

= $x \tan x - \int 1.\tan x \, dx$
= $x \tan x - \int \tan x \, dx$
= $x \tan x - (-\log|\cos x|) + C_1$
= $x \tan x + \log|\cos x| + C_1$ ….(ii)
$I_2 = \int x.\tan x \sec x \, dx$

$$= x \int \tan x \sec x \, dx - \int \left(\frac{d}{dx} x \cdot \int \tan x \sec x \, dx \right) dx$$

= $x \sec x - \int 1.\sec dx$
= $x \sec x - \int \sec x \, dx$

= x sec x – log | sec x + tan x | + C2 …(iii)

समीकरण (i), (ii) व (iii) से,

$$\int \frac{x}{1+\sin x}\,dx$$

$$= [x \tan x + \log|\cos x| + C_1] - [x \sec x - \log|\sec x + \tan x| + C_2]$$

$$= x \tan x + \log|\cos x| - x \sec x + \log|\sec x + \tan x| + (C_1 - C_2)$$

$$= x(\tan x - \sec x) + \log\{\cos x(\sec x + \tan x)\} + C$$

$$= x(\tan x - \sec x) + \log(1 + \sin x) + C,$$

(जहाँ $C = C_1 + C_2$)

$$= x\left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right) + \log(1+\sin) + C$$

$$= \frac{-x(1-\sin x)}{\cos x} + \log(1+\sin x) + C$$

(b) $\int \underset{\text{II}}{x^2}\ \underset{\text{I}}{\tan^{-1} x}\,dx$

माना प्रथम फलन $\tan^{-1} x$ तथा द्वितीय फलन x^2 है।

$$\therefore \int x^2 \tan^{-1} x\,dx$$

$$= (\tan^{-1} x)\cdot\frac{x^3}{3} - \int \frac{1}{1+x^2}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int\left[x - \frac{x}{x^2+1}\right]dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\left(\frac{1}{2}x^2\right) + \frac{1}{3}\left[\frac{1}{2}\log(x^2+1)\right] + C$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\log(x^2+1) + C$$

प्रश्न 9.

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

हल :

$$I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

$\sin^{-1} x = \theta$ रखने पर

$$\frac{1}{\sqrt{1-x^2}} = d\theta$$

$$\therefore \quad I = \int \theta \cdot \sin\theta \, d\theta \quad (\because \sin^{-1} x = \theta \text{ से } x = \sin\theta)$$

$$= \theta \cdot \int \sin\theta \, d\theta - \int \frac{d}{d\theta} \cdot \theta \left[\int \sin\theta \, d\theta \right] d\theta$$

$$= \theta(-\cos\theta) - \int 1 \cdot (-\cos\theta) \, d\theta$$

$$= -\theta \cos\theta + \sin\theta + C$$

$$= -\sin^{-1} x \sqrt{1-\sin^2\theta} + x + C$$

$$= -\sin^{-1} \sqrt{1-x^2\theta} + x + C$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + C$$

प्रश्न 10.

$$\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$$

हल :

$$\int \frac{x \tan^{-1} x \, dx}{(1+x^2)^{3/2}}$$

माना $x = \tan\theta$

$$= \int \frac{\tan\theta . \tan^{-1}(\tan\theta)}{(1+\tan^2\theta)^{3/2}} \cdot \sec^2\theta \, d\theta$$

$$= \int \frac{\tan\theta \cdot \theta \sec^2\theta \, d\theta}{\sec^3\theta} = \int \frac{\theta \tan\theta}{\sec\theta}$$

$$= \int \theta \sin\theta \, d\theta = \theta \int \sin\theta \, d\theta - \int \left[\frac{d}{d\theta}(\theta) \int \sin\theta \right] d\theta$$

$= \theta(-\cos\theta) + \int \cos\theta \, d\theta$

$= -\theta\cos\theta + \sin\theta + c$

$= \sin\theta - \theta\cos\theta + c$

= cos θ [tanθ – θ] + c

$$= \frac{\tan\theta - \theta}{\sec\theta} + C = \frac{-\theta + \tan\theta}{\sqrt{1+\tan^2\theta}} + C$$

$$= \frac{-\tan^{-x}}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$$

प्रश्न 11. ∫ e^x (cot x + log sin x) dx

हल : ∫ e^x (cot x + log sin x) dx

माना I = ∫e^x [log |sin x| + cot x]dx

= ∫e^x log | sinx | dx + ∫e^x cot x dx

= ∫log | sin x | e^x dx + ∫e^x cot x dx

अब ∫log |sin x| e^x dx

$$= \log|\sin x| \cdot \int e^x\, dx - \int\left[\frac{d}{dx}|\sin x| \cdot \int e^x\, dx\right] dx$$

$$= \log|\sin x| \cdot e^x - \int \frac{1 \cdot \cos x}{\sin x} e^x\, dx$$

$$= \log|\sin x| \cdot e^x - \int \frac{\cot x}{\sin x} e^x\, dx$$

∴ I = e^x log | sin x |e^x – ∫e^x cotx dx + ∫e^x cotx dx

= log | sin x |e^x + C

= e^x log | sin x | + C

प्रश्न 12.

$$\int \frac{2x + \sin 2x}{1 + \cos 2x} dx$$

हल :

$$\int \frac{2x + \sin 2x}{1 + \cos 2x} dx$$

$$= \int \frac{2x + 2\sin x \cos x}{2\cos^2 x} dx$$

= ∫x $\sec^2$ x dx + ∫tan x dx

= x tan x – ∫tan x dx + ∫tan x dx

= x tan + C

प्रश्न 13.

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx$$

हल :

माना $I = \int e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx$

$$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$$

$$= \int e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right) dx$$

$$I = \int e^x\left\{\left(-\cot\frac{x}{2}\right) + \frac{1}{2}\text{cosec}^2\frac{x}{2}\right\} dx$$

$$= -\int e^x \cot\frac{x}{2}\, dx + \frac{1}{2}\int e^x \text{cosec}^2\frac{x}{2}\, dx$$

$$= -\left\{\cot\frac{x}{2}e^x - \int -\text{cosec}^2\frac{x}{2}\cdot\frac{1}{2}e^x\, dx\right\} + \frac{1}{2}\int e^x \text{cosec}^2\frac{x}{2}\, dx$$

(केवल पहले भाग का खण्डशः समाकलन करने पर)

$$= -e^x\cot\frac{x}{2} - \frac{1}{2}\int e^x \text{cosec}^2\frac{x}{2}\, dx + \frac{1}{2}\int e^x \text{cosec}^2\frac{x}{2}\, dx + C$$

$$= -e^x\cot\frac{x}{2} + C$$

प्रश्न 14.

$$\int e^x\left[\log x + \frac{1}{x^2}\right] dx$$

हल :

$$\int e^x \left[\log x + \frac{1}{x^2}\right] dx$$

$$= \int e^x . \log x \, dx + \int e^x . \frac{1}{x^2} \, dx$$

$$= \int \log x \cdot e^x \, dx + \int x^{-2} \cdot e^x \, dx \quad \ldots(i)$$

$$= I_1 + I_2 \quad \text{माना}$$

$$I_1 = \int \log \underset{I}{x} . \underset{II}{e^x} \, dx$$

$$= \log x \int e^x \, dx - \int \left[\frac{d}{dx}(\log x) . \int e^x \, dx\right] dx$$

$$= \log x . e^x - \int \underset{I}{\frac{1}{x}} . \underset{II}{e^x} \, dx$$

$$= \log x . e^x - \frac{1}{x} . \int e^x \, dx + \int \left[\frac{d}{dx}\left(\frac{1}{x}\right) . \int e^x \, dx\right] dx$$

$$= e^x . \log x - \frac{1}{x} e^x + \int -\frac{1}{x^2} e^x \, dx$$

$$= e^x . \log x - \frac{1}{x} e^x - \int x^{-2} e^x \, dx \quad \ldots(ii)$$

समीकरण (i) व (ii) से,

$$\int e^x \left(\log x + \frac{1}{x^2}\right) dx$$

$$= e^x \log x - \frac{1}{x} e^x - \int x^{-2} e^x \, dx + \int x^{-2} e^x \, dx + C$$

$$= e^x \log x - \frac{1}{x} e^x + C$$

$$= e^x \left[\log x - \frac{1}{x}\right] + C$$

प्रश्न 15. ∫e^x [log (sec x + tan x) + sec x) dx

हल :

= ∫e^x [log (sec x + tan x) + sec x) dx

= ∫e^x log (sec x + tan x) dx + ∫e^x sec x dx

$= \int \log(\sec x + \tan x)\, e^x\, dx + \int \sec x e^x\, dx$

$$= \log(\sec x + \tan x) \int e^x\, dx - \int \left[\frac{d}{dx} \log(\sec x + \tan x) \int e^x\, dx\right] dx + \int \sec x . e^x\, dx$$

$$= \log|\sec x + \tan x| . e^x - \int \frac{1}{(\sec x + \tan x)} \times (\sec x \tan x) + \sec^2 x |\, e^x\, dx + \int \sec x . e^x\, dx + C$$

$$= e^x \log|\sec x + \tan x| - \int \frac{\sec x (\tan x + \sec x)}{(\sec x + \tan x)} e^x\, dx + \int \sec x\, e^x\, dx + C$$

$= e^x \log|\sec x + \tan x| - \int \sec x e^x\, dx + \int \sec x e^x\, dx + C$

$= e^x \log|\sec x + \tan x| + C$

प्रश्न 16. $\int e^x (\sin x + \cos x) \sec^2 x\, dx$

हल : $\int e^x (\sin x + \cos x) \sec^2 x\, dx$

$$= e^x \left(\frac{\cos x}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$= \int e^x (\sec x + \sec x \tan)\, dx$

$= e^x \sec x + C$ [$\because e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$, $f(x) = \sec x$, $f'(x) = \sec x \tan x$]

प्रश्न 17.

$$\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$$

हल :

$$\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$$

$$= \int e^x \cdot \frac{1}{x^2}\, dx - 2\int \frac{e^x}{x^3}\, dx$$

$$= \int e^x x^{-2}\, dx - 2\int \frac{e^x}{x^3}\, dx$$

$$= \int \underset{\text{I}}{x^{-2}}\, \underset{\text{II}}{e^x}\, dx - 2\int \frac{e^x}{x^3}\, dx$$

$$= x^{-2}\int e^x\,dx - \int\left[\frac{d}{dx}(x^{-2})\int e^x\,dx\right]dx$$

$$-2\int\frac{e^x}{x^3}\,dx + C$$

$$= x^{-2}e^x - \int -2x^{-3}e^x\,dx - 2\int\frac{e^x}{x^3}\,dx + C$$

$$= x^{-2}e^x + 2\int\frac{e^x}{x^3}\,dx - 2\int\frac{e^x}{x^3}\,dx + C$$

$$= x^{-2}e^x + C$$

प्रश्न 18.

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$$

हल :

माना $I = \int e^x\,\frac{(1-x)^2}{(1+x^2)^2}\,dx = \int e^x\left\{\frac{1-2x+x^2}{(1+x^2)^2}\right\}dx$

$$= \int e^x\left\{\frac{1+x^2+(-2x)}{(1+x^2)^2}\right\}dx$$

$$= \int e^x\left[\frac{1}{1+x^2}+\frac{(-2x)}{(1+x^2)^2}\right]dx$$

$$= \int e^x\frac{1}{1+x^2}\,dx + \int e^x\,\frac{(-2x)}{(1+x^2)^2}\,dx$$

$$= \frac{1}{1+x^2}e^x - \int\frac{(-2x)}{(1+x^2)^2}\,e^x\,dx + \int e^x\,\frac{(-2x)}{(1+x^2)^2}\,dx$$

$$= \frac{e^x}{1+x^2} + C$$

प्रश्न 19.

$$\int \cos 2\theta.\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$$

हल :

$$I = \int \cos 2\theta . \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}\right) d\theta$$

$\log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}\right)$ प्रथम फलन मानकर, खण्डशः समाकलन पर,

$$= \frac{\sin 2\theta}{2} \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}\right)$$

$$- \frac{1}{2} \int \frac{d}{d\theta} \left[\log \left(\frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}\right) \sin 2\theta\right] \quad ...(i)$$

लेकिन

$$= \frac{d}{d\theta} [\log (\cos \theta + \sin \theta) - \log (\cos \theta - \cos \theta)]$$

$$= \frac{1}{(\cos\theta + \sin\theta)} (-\sin\theta + \cos\theta) - \frac{(-\sin\theta \cos\theta)}{\cos\theta - \sin\theta}$$

$$= \frac{(\cos\theta - \sin\theta)(\cos\theta - \sin\theta) - (\cos\theta + \sin\theta)(-\sin\theta - \cos\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}$$

$$= \frac{(\cos^2 \theta - \cos \theta \sin \theta - \sin \theta \cos \theta + \sin^2 \theta + \cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta + \cos \theta \sin \theta)}{(\cos^2 \theta - \sin^2 \theta)}$$

$$= \frac{2(\cos^2 \theta + \sin^2 \theta)}{\cos 2\theta} = \frac{2}{\cos 2\theta}$$

समी. (i) में उपरोक्त मान रखने पर,

$$I = \frac{1}{2} \sin 2\theta \log \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \frac{1}{2} \int \sin 2\theta \cdot \frac{2}{\cos 2\theta}$$

$$= \frac{1}{2} \sin 2\theta \log \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) + \frac{1}{2} \log (\cos 2\theta) + C$$

प्रश्न 20.

$$\int \frac{x^2}{(x \cos x - \sin x)^2} dx$$

हल :

$$\int \frac{x^2}{(x\cos x - \sin x)^2}\,dx$$

$$= x \text{ cosec } x \cdot \frac{x\sin x}{(x\cos x - \sin x)^2}$$

I II

$$= x \text{ cosec } x \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx$$

$$- \int \frac{d}{dx}(x \text{ cosec } x) \int \left[\frac{x\sin x}{(x\cos x - \sin x)^2}\,dx\right]$$

$$= x \text{ cosec } x \cdot \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx$$

$$- \int (\text{cosec } x - \cot x . \text{cosec } x \cdot \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx \quad ...(i)$$

माना $\qquad I_1 = \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx$

तथा $\qquad x \cos x - \sin x = u$

$(x (-\sin x) + \cos x - \cos x)\,dx = du$

$\Rightarrow \qquad -x \sin dx = du$

$\Rightarrow \qquad x \sin dx = -du$

$\Rightarrow \qquad I_1 = \int \frac{-du}{u^2}$

$$= -\int u^2\,du = 1 \times \left(-\frac{1}{u}\right)$$

$\Rightarrow \qquad I_1 = \frac{1}{u} + C_1$

$\Rightarrow \qquad I_1 = \frac{1}{x\cos x - \sin x} \qquad ...(ii)$

समीकरण (i) व (ii) से,

$$\int \frac{x^2}{(x\cos x - \cos x)^2}\,dx$$

$$= x\ \text{cosec}\ x \frac{1}{(x\cos x - \sin x)}$$

$$- \int (\text{cosec}\ x - \cot x \text{cosec}\ x) \cdot \frac{1}{(x\cos x - \sin x)} dx$$

$$= \frac{x\ \text{cosec}\ x}{(x\cos x - \sin x)} - \int \frac{\text{cosec}\ x(1 - x\cot x)}{(x\cos x - \sin x)} dx$$

$$= \frac{x\ \text{cosec}\ x}{(x\cos x - \sin x)} - \int \frac{\text{cosec} \cdot (x\cot x - 1)}{(x\cos x - \sin x)} dx$$

$$= \frac{x\ \text{cosec}\ x}{(x\cos x - \sin x)} + \int \frac{\text{cosec}\ x \cdot (x\cos x - \sin x)}{\sin x(x\cos x - \sin x)} dx$$

$$= \frac{x\ \text{cosec}\ x}{x\cos x - \sin x} + \int \text{cosec}^2\ x\, dx$$

$$= \frac{x\ \text{cosec}\ x}{x\cos x - \sin x} - \cot x + C$$

$$= \frac{x}{\sin x(x\cos x - \sin x)} - \frac{\cos x}{\sin x} + C$$

$$= \frac{x - \cos x\ (x\cos x - \sin x)}{\sin x(x\cos x - \sin x)} + C$$

$$= \frac{x - x\cos^2 x + \sin x \cos x}{\sin x(x\cos x - \sin x)}$$

$$= \frac{x(1 - \cos^2 x) + \sin x \cos x}{\sin x(x\cos x - \sin x)} + C$$

$$= \frac{\sin x(\sin x + \cos x)}{\sin x(x\cos x - \sin x)} + C$$

$$= \frac{x\sin x + \cos x}{x\cos x - \sin x} + C$$

प्रश्न 21.

$$\int \cos^{-1}\left(\frac{1}{x}\right) dx$$

हल :

माना $I = \int \cos^{-1} \frac{1}{x} dx$

$= \int \sec^{-1} x \, dx \quad \left(\because \cos^{-1} \frac{1}{x} = \sec^{-1} x\right)$

$= \int \sec^{-1} x \cdot 1 \, dx$

Sec^{-1} x को प्रथम तथा 1 को द्वितीय फलन लेकर खण्डशः माकलन करने पर,

$I = \sec^{-1} x \int dx - \int \left[\frac{d}{dx} (\sec^{-1} x) \cdot \int dx\right] dx$

$= x \sec^{-1} x - \int \frac{1}{x\sqrt{x^2 - 1}} x \, dx$

$= x \sec^{-1} x - \int \frac{1}{\sqrt{x^2 - 1}}$

$= x \sec^{-1} x - \log |x + \sqrt{x^2 - 1}| + C$

प्रश्न 22. $\int (\sin^{-1} x)^2 dx$

हल : माना $I = \int(\sin^{-1} x)^2 dx$

माना $\sin^{-1} x = \theta$

$x = \sin\theta$

$\therefore dx = c0s \theta d\theta$

$\therefore I = \int\theta^2 \cdot \cos \theta a\theta$

(θ^2 को प्रथम फलन एवं $\cos \theta$ को द्वितीय फलन लेने पर)

$= \theta^2 \sin \theta - \int 2\theta \sin \theta d\theta + C$

$= \theta^2 \sin \theta - 2\int \theta \sin \theta + C$

$= \theta^2 \sin \theta - 2[\theta - \cos \theta) - \int 1.(-\cos \theta) d\theta]+C$

(पुन: θ को प्रथम तथा $\sin \theta$ को द्वितीय फलन लेने पर)।

$= \theta^2 \sin \theta + 2\theta \cos \theta - 2\int \cos \theta d\theta+C$

$= \theta^2 \sin \theta + 2\theta \cos \theta - 2 \sin \theta + C$

$= \theta^2 \sin \theta + 2\theta\sqrt{1 - \sin^2 \theta} - 2 \sin \theta + C$

$= x(\sin^{-1} x)^2 + 2 \sin^{-1} x \cdot \sqrt{1 - x^2} - 2x + C$

Ex 9.7

निम्नलिखित फलनों के समाकलन कीजिए :

प्रश्न 1. $\int e^{2x} \cos x \, dx$

हल : $\int e^{2x} \cos x \, dx$

माना $I = \int e^{2x} . \cos x \, dx$

$$\Rightarrow \quad I = e^{2x} \int \cos x \, dx - \int \left[\frac{d}{dx} e^{2x} \int \cos x \, dx \right] dx$$

$$\Rightarrow \quad I = e^{2x} \sin x - 2 \int \underset{I}{e^{2x}} \underset{II}{\sin x} \, dx$$

$$\Rightarrow \quad I = e^{2x} \sin x$$
$$- 2 \left[e^{2x} \int \sin x \, dx - \int \left\{ \frac{d}{dx} e^{2x} \int \sin 2x \, dx \right\} dx \right]$$

$$\Rightarrow \quad I = e^{2x} \sin x$$
$$- 2 \left[e^{2x} (- \cos x) - 2 \int e^{x} (- \cos x) \, dx \right] + C$$

$$\Rightarrow \quad I = e^{2x} \sin x$$
$$- 2 \left[- e^{2x} \cos 2x + 2 \int e^{x} \cos x \, dx \right] + C$$

$$\Rightarrow \quad I = e^{2x} \sin x + 2 e^{2x} \cos 2x - 4 \int e^{x} \cos x \, dx + C$$

$$\Rightarrow \quad I = e^{2x} \sin x + 2e^{2x} \cos x - 4I + C$$

$$\Rightarrow \quad 5I = e^{2x} \sin x + 2e^{2x} \cos x + C$$

$$\Rightarrow \quad I = \frac{1}{5} e^{2x} (\sin 2x + 2 \cos x) + C$$

$$\therefore \int e^{2x} \cos x \, dx = \frac{1}{5} e^{2x} (2 \cos x + \sin 2x) + C$$

प्रश्न 2.

$\int \sin (\log x) \, dx$

हल :

माना $I = \int \sin (\log x) \, dx$

माना $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$\therefore \quad I = \int \underset{II}{\sin t}\ \underset{I}{e^t}\,dt$$

$$= -e^t \cos t - \int e^t(-\cos t)\,dt$$

(खंडशः समाकलन से)

$$= -e^t \cos t + \int e^t \cos t\,dt$$

$$= -e^t \cos t + \left\{e^t \sin t - \int e^t \sin t\,dt\right\}$$

$$= -e^t \cos t + e^t \sin t - I$$

$$\Rightarrow \quad 2I = e^t(\sin t - \cos t)$$

$$\Rightarrow \quad I = \frac{e^t}{2}(\sin t - \cos t) + C$$

$$= \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$$

प्रश्न 3.

$$\int \frac{e^{a\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}}\,dx$$

हल :

$$\int \frac{e^{a\tan^{-1}x}}{(1+x^2)^{3/2}}\,dx$$

माना $\tan^{-1} x = t$

$x = \tan t$

$$\frac{1}{1+x^2}\,dx = dt$$

$$\text{अत:} \quad I = \int \frac{e^{at}}{\sqrt{1+\tan^2 t}}\,dt$$

$$\Rightarrow \quad I = \int e^{at} \cos t\,dt$$

$$\Rightarrow \quad I = e^{at}\int \cos t\,dt - \int\left[\frac{d}{dx}(e^{at})\int \cos t\,dt\right]dt$$

$$\Rightarrow \quad I = e^{at} \sin t - \int a e^{at} \sin t\,dt$$

$\Rightarrow \quad I = e^{at} \sin t$

$$- a\left[e^{at} \int \sin t \, dt - \int\left[\frac{d}{dt}(e^{at}) \int \sin t \, dt\right] dt\right]$$

$\Rightarrow \quad I = e^{at} \sin t + ae^{at} \cos t - a \int (ae^{at} \cos t) dt$

$\Rightarrow \quad I = e^{at} \sin t + ae^{at} \cos t - a^2 \int e^{at} \cos t \, dt$

$\Rightarrow \quad I = e^{at} \sin t + ae^{at} \cos t - a^2 I$

$\Rightarrow \quad I(1 + a^2) = e^{at} \sin + ae^{at} \cos t$

$$\Rightarrow \quad I = \frac{e^{at}}{(1+a^2)}[\sin t + a \cos t]$$

यहाँ $\tan x = x$, $\sin t = \dfrac{x}{\sqrt{1+x^2}}$, $\cos t = \dfrac{1}{\sqrt{1+x^2}}$

$$\Rightarrow \quad I = \frac{e^{at}}{1+a^2}\left[\frac{x}{\sqrt{1+x^2}} + \frac{a}{\sqrt{1+x^2}}\right] + C$$

$$\Rightarrow \quad I = \frac{e^{at}}{1+a^2}\left[\frac{x+a}{\sqrt{x^2+1}}\right] + C$$

यही अभीष्ट हल है|

प्रश्न 4.

$\int e^{\frac{x}{\sqrt{2}}} \cos (x + \alpha)$

हल :

$$\int e^{x/\sqrt{2}} \cos (x + \alpha) \, dx \qquad \text{(माना)}$$

$$I = \int \underset{\text{I}}{\cos} (x + \alpha) \underset{\text{II}}{e}^{x/\sqrt{2}} \, dx$$

$$= \cos (x + \alpha) \int e^{x/\sqrt{2}} \, dx - \int\left[\frac{d}{dx}[\cos (x + \alpha) \int e^{x/\sqrt{2}} \, dx\right] dx$$

$$= \cos (x + \alpha) \frac{e^{x/\sqrt{2}}}{1/\sqrt{2}} - \int\left[\{-\sin (x + \alpha)\} \frac{e^{x/\sqrt{2}}}{1/\sqrt{2}}\right] dx$$

$$= \sqrt{2} \cos (x + \alpha) e^{x/\sqrt{2}} + \sqrt{2} \int\left[\underset{\text{I}}{\sin} (x + \alpha) \underset{\text{II}}{e}^{x/\sqrt{2}}\right] dx$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos(x+\alpha) + \sqrt{2}\Big[\sin(x+\alpha) \int e^{x/\sqrt{2}}\, dx$$

$$- \int \frac{d}{dx}[\sin(x+\alpha) \int e^{x/\sqrt{2}}\, dx\Big] dx$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos(x+\alpha) + \sqrt{2}\Big[\sin(x+\alpha) \frac{e^{x/\sqrt{2}}}{1/\sqrt{2}}$$

$$- \int \cos(x+\alpha) \frac{e^{x/\sqrt{2}}}{1/\sqrt{2}}\, dx\Big]$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos(x+\alpha) + \sqrt{2}\Big[\sqrt{2} \sin(x+\alpha)e^{x/\sqrt{2}}$$

$$- \sqrt{2} \int \cos(x+\alpha)e^{x/\sqrt{2}}\, dx\Big]$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos(x+\alpha) + 2\sin(x+\alpha)e^{x/\sqrt{2}}$$

$$- 2\int \cos(x+\alpha)e^{x/\sqrt{2}}\, dx + C$$

$$3I = 2e^{x/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\cos(x+\alpha) + \sin(x+\alpha)\right] + C$$

$$\Rightarrow \int e^{x/\sqrt{2}} \cos(x+\alpha)\, dx$$

$$= \frac{2}{3} e^{x/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\cos(x+a) + \sin(x+\alpha)\right] + C$$

प्रश्न 5.

$\int e^x \sin^2 x\, dx$

हल : माना $\int e^x \sin^2 x\, dx = I$
$I = \int e^x \sin^2 x\, dx$

$$= \int e^x \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int e^x \, dx - \frac{1}{2}\int \underset{\text{I}}{e^x} \underset{\text{II}}{\cos} 2x\,dx \qquad \ldots\text{(i)}$$

पुनः माना $I_1 = \int e^x \cos 2x \, dx$

$$I = \frac{1}{2}\int e^x - \frac{1}{2} I_1 \qquad \ldots\text{(ii)}$$

$$I_1 = \left[e^x \frac{\sin 2x}{2} - \int e^x \frac{\sin 2x}{2} dx\right]$$

$$= \left[\frac{1}{2} e^x \sin 2x - \frac{1}{2}\int \underset{\text{I}}{e^x} \underset{\text{II}}{\sin} 2x\,dx\right]$$

$$= \frac{1}{2}\left[e^x \sin 2x - \left\{\frac{-e^x \cos 2x}{2} - \int \frac{e^x(-\cos 2x)}{2}\right\} dx\right]$$

$$= \frac{1}{2}\left[e^x \sin 2x + \frac{1}{2} e^x \cos 2x - \frac{1}{2}\int e^x \cos 2x\right] dx$$

$$= \frac{1}{2}\left[e^x \sin 2x + \frac{1}{2} e^x \cos 2x - \frac{1}{2} I_1\right] + C$$

$$\Rightarrow \qquad I_1 = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} I_1 + C$$

$$\Rightarrow \left(1 + \frac{1}{4}\right) I_1 = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x + C$$

$$\Rightarrow \qquad I_1 = \frac{4}{5\times 2} e^x \sin 2x + \frac{4}{5\times 4} e^x \cos 2x + C$$

$$\Rightarrow \qquad I_1 = \frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x + C$$

I_1 का मान समीकरण (ii) में रखने पर,

$$I = \frac{1}{2}\int e^x\, dx - \frac{1}{2}\left[\frac{2}{5}e^x \sin 2x + \frac{1}{5}e^x \cos 2x + C\right]$$

$$= \frac{1}{2}e^x - \frac{1}{5}e^x \sin 2x - \frac{1}{10}e^x \cos 2x + C$$

$$\Rightarrow \quad I = \frac{1}{2}e^x - \frac{1}{5}e^x \sin 2x - \frac{1}{10}e^x \cos 2x + C$$

$$\Rightarrow \int e^x \sin^2 x\, dx$$

$$= \frac{1}{2}e^x - \frac{1}{5}e^x \sin 2x - \frac{1}{10}e^x \cos 2x\, dx + C$$

$$= \frac{e^x}{2} - \frac{e^x}{10}[\cos 2x + 2\sin 2x] + C$$

प्रश्न 6. $\int e^a \sin^{-1x} dx$

हल : $\int e^a \sin^{-1x} dx$

माना $a \sin^{-1} x = t$

$$x = \sin\left(\frac{t}{a}\right)$$

$$dx = \cos\left(\frac{t}{a}\right)\cdot\frac{1}{a}\, dt$$

$$\Rightarrow \quad dx = \frac{1}{a}\cos\left(\frac{t}{a}\right) dt$$

$$I = \int e^{a \sin^{-1} x}\, dx$$

$$= \int e^t \cdot \left(\frac{1}{a}\right) \cos\frac{t}{a}\, dt$$

$$= \frac{1}{a}\int e^t \cos\frac{t}{a}\, dt$$

सूत्र $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos ax + b\sin bx) + C$ से

$$= \frac{1}{a}\cdot\frac{e^t}{\left(1+\frac{1}{a^2}\right)}\left[\cos t + \frac{1}{a}\sin b\left(\frac{t}{a}\right)\right] + C$$

$$= \frac{1}{a} \times \frac{e^t}{\left(\frac{1+a^2}{a^2}\right)} \left[\cos t + \frac{1}{a} \sin b \left(\frac{t}{a}\right)\right] + C$$

$$= \frac{e^t . a^2}{a^2(1+a^2)} \left[a \cos t + \sin \frac{t}{a}\right] + C$$

$$= \frac{e^t}{(1+a^2)} \left[a \cos t + \sin \frac{t}{a}\right] + C$$

$$= \frac{e^{a \sin^{-1} x}}{(1+a^2)} [a\sqrt{1-x^2} + x] + C$$

प्रश्न 7.

$$\int \cos\left(b \cos \frac{x}{a}\right) dx$$

हल :

$$\int \cos\left(b \log \frac{x}{a}\right) dx$$

माना $b \log \frac{x}{a} = t$

$\Rightarrow \quad \log \frac{x}{a} = \frac{t}{b}$

$\Rightarrow \quad \frac{x}{a} = e^{t/b}$

$\Rightarrow \quad x = ae^{t/b}$

$$dx = \frac{a}{b} e^{t/b}\, dt$$

$$\int \cos\left(b \cos \frac{x}{a}\right) dx = \int \cos t \frac{a}{b} e^{t/b}\, dt$$

$$= \frac{a}{b} \int e^{t/b} \cos t\, dt$$

$$= \frac{a}{b} \frac{e^{t/b}}{\left(\frac{1}{b^2}+1\right)} (\cos t + \sin t) + C$$

$$= \frac{a \times b^2}{b(1+b^2)} e^{\frac{b \log}{b}\left(\frac{x}{a}\right)} \left[\cos\left(b \log \frac{x}{a}\right) + \sin\left(b \log \frac{x}{a}\right)\right] + C$$

$$= \frac{a}{1+b^2} e^{\log \frac{x}{a}} \left[\cos\left(b \cos \frac{x}{a}\right) + \sin\left(b \log \frac{x}{a}\right)\right] + C$$

$$= \frac{x}{1+b^2} \left[\cos\left(b \cos \frac{x}{a}\right) + \sin\left(b \log \frac{x}{a}\right)\right] + C$$

प्रश्न 8. $\int e^{4x} \cos 4x \cos 2x \, dx$

हल : $\int e^{4x} \cos 4x \cos 2x \, dx$

$$= \int e^{4x} \frac{1}{2}\{\cos(4x+2x) + \cos(4x-2x)\} \, dx$$

$$= \frac{1}{2}\int e^{4x} \cos 6x \, dx + \frac{1}{2}\int e^{4x} \cos 2x \, dx$$

$$= \frac{1}{2} I_1 + \frac{1}{2} I_2 \quad \ldots(i)$$

$$I_1 = \int \underset{II}{e^{4x}} \underset{I}{\cos 6x} \, dx$$

$$= \frac{e^{4x}}{(4^2+6^2)} [4\cos 6x + 6\sin 6x] + C_1$$

$$\left[\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C\right]$$

$$= \frac{e^{4x}}{52} [4\cos 6x + \sin 6x] + C_1 \quad \ldots(ii)$$

$$I_2 = \int e^{4x} \cos 2x \, dx$$

$$= \frac{e^{4x}}{4^2+2^2} [4\cos 2x + 2\sin 2x] + C_2$$

$$= \frac{e^{4x}}{20} [4\cos 2x + 2\sin 2x] + C_2 \quad \ldots(iii)$$

$$I=\frac{1}{2}\times\frac{e^{4x}}{52}[4\cos 6x+6\sin 6x]$$

$$+\frac{1}{2}\frac{e^{4x}}{20}[4\cos 2x+2\sin 2x]+C_1+C_2$$

$$=\frac{e^{4x}}{8}\left[\frac{1}{13}(4\cos 6x+6\sin 6x)\right.$$

$$\left.+\frac{1}{5}(4\cos 2x+2\sin 2x)\right]+C$$

(जहाँ $C_1+C_2=C$)

प्रश्न 9.

$\sqrt{2x-x^2}$

हल :

$$\int\sqrt{2x-x^2}\,dx=\int\sqrt{-(x^2-2x)}\,dx$$

$$=\int\sqrt{-(x^2-2x+1-1)}\,dx$$

$$=\int\sqrt{-(x^2-2x+1)+1}\,dx$$

$$=\int\sqrt{1^2-(x-1)^2}\,dx$$

$$=\frac{1}{2}(x-1)\sqrt{2x-x^2}+\frac{1}{2}\times 1^2\sin^{-1}\left(\frac{x-1}{1}\right)+C$$

$$=\frac{1}{2}(x-1)\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+C$$

प्रश्न 10.

$\int\sqrt{x^2+4x+6}\,dx$

हल :

$$\int \sqrt{x^2+4x+6}\,dx = \int \sqrt{x^2+2.2x+2^2+2}\,dx$$

$$= \int \sqrt{(x+2)^2+(\sqrt{2})^2}\,dx$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+6}+\frac{1}{2}$$

$$+\frac{1}{2}\times(\sqrt{2})^2 \log\left[(x+2)+\sqrt{x^2+4x+6}\right]+C$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+6}$$

$$+\log\left[(x+2)+\sqrt{x^2+4x+6}\right]+C$$

प्रश्न 11.

$$\int \sqrt{x^2+6x+4}\,dx$$

हल :

$$\int \sqrt{x^2+6x+4}\,dx$$

$$= \int \sqrt{x^2+2.3.x+3^2-(\sqrt{5})^2}\,dx$$

$$= \int \sqrt{(x+3)^2-(\sqrt{5})^2}\,dx$$

$$= \frac{1}{2}(x+3)\sqrt{x^2+6x+4}$$

$$-\frac{1}{2}(\sqrt{5})^2 \log\left[(x+3)+\sqrt{x^2+6x+4}\right]$$

$$= \frac{1}{2}(x+3)\sqrt{x^2+6x+4}$$

$$-\frac{5}{2}\log\left[(x+3)+\sqrt{x^2+6x+4}\right]+C$$

प्रश्न 12.

$$\int \sqrt{2x^2+3x+4}\,dx$$

हल :

$$\int \sqrt{2x^2+3x+4}\, dx$$

$$= \sqrt{2} \int \sqrt{x^2+\frac{3}{2}x+2}\, dx$$

$$= \sqrt{2} \int \sqrt{x^2+2.\frac{3}{4}x+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+2}\, dx$$

$$= \sqrt{2} \int \sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\, dx$$

$$= \frac{1}{2}\frac{4x+3}{4}\sqrt{2x^2+3x+4}$$

$$+\frac{23}{32}\log\left(\frac{4x+3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right)+C$$

$$= \frac{4x+3}{4}\sqrt{2x^2+3x+4}$$

$$+\frac{23}{16\sqrt{2}}\log\left(\frac{4x+3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right)+C$$

प्रश्न 13.

$$\int x^2\sqrt{(a^6-x^6)}\, dx$$

हल :

$$\int x^2\sqrt{(a^6-x^6)}\, dx$$

$$= \int \sqrt{(a^3)^2-(x^3)^2}\, x^2\, dx$$

माना $x^3 = t,\ 3x^2\, dx = dt$

$$x^2\, dx = \frac{dt}{3}$$

$$= \int \sqrt{(a^3)^2-t^2}\, \frac{dt}{3}$$

$$= \frac{1}{3}\int \sqrt{(a^3)^2-t^2}\, dt$$

$$= \frac{1}{3}\left[\frac{1}{2}t\sqrt{(a^3)^2 - t^2} + \frac{1}{2}(a^3)^2 \sin^{-1}\left(\frac{t}{a^3}\right)\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{2}t\sqrt{(a^6 - t^2)} + \frac{a^6}{2} \sin^{-1}\left(\frac{t}{a^3}\right)\right] + C$$

$$= \frac{1}{6}\left[x^3\sqrt{(a^6 - x^6)} + a^6 \sin^{-1}\left(\frac{x^3}{a^3}\right)\right] + C$$

प्रश्न 14.

$$\int (x+1)\sqrt{x^2+1}\,dx$$

हल :

$$\int (x+1)\sqrt{x^2+1}\,dx$$

$$= \int x\sqrt{x^2+1}\,dx + \int \sqrt{x^2+1}\,dx$$

$$= I_1 + I_2 \qquad \text{(माना)} \quad \ldots\text{(i)}$$

$$I_1 = \int x\sqrt{(x^2+1)}\,dx$$

माना $x^2 + 1 = t$

$$2x\,dx = dt$$

$$x\,dx = \frac{dt}{2}$$

$$= \int \sqrt{(x^2+1)}.x\,dx$$

$$= \int \sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int t^{1/2}\,dt$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} + C_1 = \frac{1}{3}t^{3/2} = \frac{1}{3}(x^2+1)^{3/2} + C_1 \qquad \ldots\text{(ii)}$$

$$I_2 = \int \sqrt{x^2+1}\,dx$$

$$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log\left(x + \sqrt{x^2+1}\right) + C_2 \qquad \ldots\text{(iii)}$$

$$\int (x+1)\sqrt{x^2+1}\,dx$$

$$= \frac{1}{3}(x^2+1)^{3/2} + \frac{1}{2}x\sqrt{x^2+1)} + \frac{1}{2}\log\left(x + \sqrt{x^2+1}\right) + C$$

(जहाँ $C = C_1 + C_2$)

प्रश्न 15.

$$\int\sqrt{1-4x-x^2}\,dx$$

हल : माना

$$I=\int\sqrt{1-4x-x^2}\,dx$$

$$=\int\sqrt{1-(x^2+4x)}\,dx$$

$$=\int\sqrt{1-(x^2+4x+4-4)}\,dx$$

$$=\int\sqrt{1-(x+2)^2+4}\,dx$$

$$=\int\sqrt{5-(x+2)^2}\,dx$$

$$=\int\sqrt{(\sqrt{5})^2-(x+2)^2}\,dx$$

$$\therefore I=\frac{(x+2)}{2}\sqrt{5-(x+2)^2}+\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$$

$$I=\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+\frac{x+2}{2}\sqrt{1-4x-x^2}+C$$

$$\left[\because\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C\right]$$

प्रश्न 16.

$$\int\sqrt{4-3x-2x^2}\,dx$$

हल :

$$\int\sqrt{4-(2x^2+3x)}\,dx$$

$$=\int\sqrt{4-(2x^2+3x)}\,dx$$

$$=\int\sqrt{4-2\left(x^2+\frac{3}{2}x\right)}\,dx$$

$$=\sqrt{2}\int\sqrt{2-\left(x^2+\frac{3}{2}x\right)}\,dx$$

$$=\sqrt{2}\int\sqrt{2-\left[x^2+2.\frac{3}{4}x+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}\,dx$$

$$= \sqrt{2}\int\sqrt{\left(2+\frac{9}{16}\right)-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= \sqrt{2}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= \frac{1}{\sqrt{2}}\left(x+\frac{3}{4}\right)\sqrt{4-3x-2x^2}$$

$$+ \sqrt{2}\times\frac{1}{2}\times\left(\frac{\sqrt{41}}{4}\right)^2\sin^{-1}\left[\frac{\left(x+\frac{3}{4}\right)}{\left(\frac{\sqrt{41}}{4}\right)}\right]+C$$

$$= \sqrt{2}\left[\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{4-3x-2x^2}\right.$$

$$\left.+\frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right]+C$$

Miscellaneous Exercise

निम्नलिखित के मान ज्ञात कीजिए

प्रश्न 1.
∫[1 + 2 tan x (tan x + see x) dx

हल :
$\int[1 + 2 \tan x (\tan x + \sec x)]\, dx$
$= \int(1 + 2 \tan^2 x + 2 \tan x \sec x]\, dx$
$= \int[2(1 + \tan^2 x) + 2 \sec x \tan x - 1]\, dx$
$= 2\int(\sec^2 x + \sec x \tan x)\, dx - \int dx$
$= 2(\tan x + \sec x) - x + C$

प्रश्न 2.
$\int e^x \sin^3 x\, dx$

हल :
$\int e^x \sin^3 x\, dx$

$$= \int e^x \left[\frac{1}{4}(3 \sin x - \sin 3x)\right] dx$$

$$= \frac{3}{4}\int e^x \sin x\, dx - \frac{1}{4}\int e^x \sin 3x\, dx \quad \ldots(i)$$

$$= \frac{3}{4} \cdot \frac{e^x}{2}(\sin x - \cos x) - \frac{1}{4} \cdot \frac{e^x}{1+3^2}[\sin 3x - 3\cos 3x] + C$$

$$= \frac{3}{8}\, e^x(\sin x - \cos x) - \frac{e^x}{40}[\sin 3x - 36\cos 3x] + C$$

$$\left(\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C\right)$$

$$= \frac{e^x}{40}[(15 \sin x - 15 \cos x) - (\sin 3x - \cos 3x)] + C$$

प्रश्न 3.
$\int x^2 \log (1 - x^2) dx$

हल :

$\int x^2 \log(1-x^2)dx$

$$= \log(1-x^2) \int x^2\, dx - \int\left\{\frac{d}{dx}\log(1-x^2)\int x^2\, dx\right\} dx$$

$$= \log(1-x^2)\cdot\frac{x^3}{3} - \int\frac{1}{1-x^2}\cdot(-2x)\cdot\frac{x^3}{3}\, dx$$

$$= \frac{x^3}{3}\log(1-x^2) + \frac{2}{3}\int\frac{x^4}{1-x^2}\, dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int\frac{x^4}{x^2-1}\, dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int\left(x^2+1+\frac{1}{x^2-1}\right) dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int x^2\, dx - \frac{2}{3}\int 1\, dx - \frac{2}{3}\int\frac{1}{x^2-1}\, dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\cdot\frac{x^3}{3} - \frac{2}{3}x - \frac{2}{3}\cdot\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\left(x+\frac{x^3}{3}\right) + \frac{1}{3}\log\left|\frac{1+x}{1-x}\right| + c$$

प्रश्न 4.

$$\int\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x+a}}\, dx$$

हल :

माना, $I = \int\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x+a}}\, dx$

$$= \int\frac{\sqrt{x}}{\sqrt{x+a}}\, dx - \sqrt{a}\int\frac{1}{\sqrt{x+a}}\, dx$$

$$= I_1 - \sqrt{a}I_2$$

$$I_1 = \int\frac{\sqrt{x}}{\sqrt{x+a}}\, dx$$

माना $\sqrt{x+a} = t$

$\Rightarrow \quad x + a = t^2$

$x = t^2 - a$

$$\Rightarrow \quad I_1 = \int \frac{\sqrt{t^2 - a}}{t} 2t\, dt$$

$$= 2\int \sqrt{t^2 - (\sqrt{a})^2}\, dt$$

$$= 2\left[\frac{1}{2}\sqrt{t^2 - a} - \frac{a}{2} \log\, [t + \sqrt{t^2 - a}]\right] + c_1$$

$$= \sqrt{(x+a)}\, \sqrt{x} - a \log \sqrt{x+a} + \sqrt{x} + c_1$$

$$I_2 = \int \frac{1}{\sqrt{x+a}}\, dx = \int (x+a)^{-1/2}\, dx$$

$$= \frac{(x+a)^{-1/2\,+1}}{-1/2+1} + c_2$$

$$= \frac{(x+a)^{1/2}}{1/2} + c_2$$

$$= 2\sqrt{(x+a)} + c_2$$

अत: $\quad I = I_1 - \sqrt{a} I_2$

$$= \sqrt{(x+a)}\, \sqrt{x} - a \log\, (\sqrt{x+a}) + \sqrt{x}$$

$$- 2\sqrt{a}\, \sqrt{x+a} + C_1 - \sqrt{a}\, C_2$$

$$= \sqrt{x+a}\, \sqrt{x} - a \log \sqrt{x+a} + \sqrt{x} - 2\sqrt{a}\, \sqrt{x+a} + C$$

यही अभीष्ट हल है।

प्रश्न 5.

$$\int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x}\, dx$$

हल :

$$\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x).(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x).(\sin^2 x - \cos^2 x)}{\sin^4 x + \cos^4 x} dx$$

$$= \int (\sin^2 x - \cos^2 x)\, dx$$

$$= -\int (\cos^2 x - \sin^2 x)\, dx = -\int \cos 2x\, dx$$

$$= -\frac{\sin 2x}{2} + C = -\frac{1}{2}\sin 2x + C$$

प्रश्न 6.

$$\int \frac{x}{1 + \sin x} dx$$

हल :

$$\int \frac{x}{1 + \sin x} dx = \int \frac{x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$= \int \frac{x(1 - \sin x)}{1 - \sin^2 x} dx$$

$$= \int \frac{x - x\sin x}{\cos^2 x} dx$$

$$= \int x\sec^2 x\, dx - \int x\tan x \sec x\, dx$$

$$= \left[x\int \sec^2 x\, dx - \int\left[\frac{d}{dx}(x)\int \tan x \sec x\, dx\right] dx\right]$$

$$-\left[x\int \tan x \sec x\, dx - \int\left[\frac{d}{dx}(x)\int \tan x \sec x\, dx\right] dx\right]$$

= [x tan x − ∫tan x dx] − [xsec x − ∫sec x dx]

= [x tan x − log| sec x |] − [xsec x − log| sec x + tan x |] + C

= x tan x − log| sec x | − x sec x + log x + tan x + C

= x | tan x − sec x | − log | sec x | + log | sec x + tan x | + C

प्रश्न 7.

$$\int \frac{1}{x+\sqrt{a^2-x^2}}\,dx$$

हल :

$$\int \frac{1}{x+\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{1}{(x+\sqrt{a^2-x^2})} \times \frac{(x-\sqrt{a^2-x^2})}{(x-\sqrt{a^2-x^2})}$$

$$= \int \frac{x-\sqrt{a^2-x^2}}{x^2+(a^2-x^2)}\,dx$$

$$= \frac{1}{a^2}\int [x-\sqrt{a^2-x^2}]\,dx$$

$$= \frac{1}{a^2}\left[\frac{x^2}{2}-\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}\right]+C$$

$$= \frac{x^2}{2a^2}-\frac{x}{2a^2}\sqrt{(a^2-x^2)}+\frac{1}{2}\sin^{-1}\frac{x}{a}+C$$

प्रश्न 8.

$$\int \frac{2x-1}{(1+x)^2}\,dx$$

हल :

$$\int \frac{2x-1}{(1+x)^2}\,dx$$

$$= \int \frac{(2x+2)-3}{(x+1)^2}\,dx$$

$$= 2\int \frac{1}{(x+1)}\,dx - 3\int \frac{1}{(x+1)^2}\,dx$$

$$= 2\log|x+1| + 3\frac{1}{(x+1)} + C$$

$$= 2\log|x+1| + \frac{3}{(x+1)} + C$$

प्रश्न 9.

$$\int \frac{1}{\cos 2x + \cos 2\alpha}\, dx$$

हल :

$$\int \frac{1}{\cos 2x + \cos 2\alpha}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\cos (x+\alpha) \cos (x-\alpha)}\, dx$$

$$= \frac{1}{2\sin 2\alpha} \int \frac{\sin 2\alpha}{\cos(x+\alpha) \cos (x-\alpha)}\, dx$$

$$= \frac{1}{2\sin 2\alpha} \int \left[\frac{\sin (x+\alpha) - (x-\alpha)}{\cos(x+\alpha) \cos (x-\alpha)}\right] dx$$

$$= \frac{1}{2\sin 2\alpha} \int \frac{\sin (x+\alpha) \cos (x-\alpha) - \cos(x+\alpha) \sin(x-\alpha)}{\cos(x+\alpha) \cos (x-\alpha)}\, dx$$

$$= \frac{1}{2} \operatorname{cosec} 2a \int [\tan(x+\alpha) - \tan(x-\alpha)]\, dx$$

$$= \frac{1}{2} \operatorname{cosec} 2\alpha \cdot \log \left|\frac{\sec (x+\alpha)}{\sec(x-\alpha)}\right| + C$$

प्रश्न 10.

$$\int \sin^{-1} \left(\frac{2x}{1+x^2}\right) dx$$

हल :

$$\int \sin^{-1} \left(\frac{2x}{1+x^2}\right) dx$$

माना $\quad x = \tan\theta$

$$dx = \sec^2\theta\, d\theta$$

$$= \int \sin^{-1} \left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \sec^2\theta\, d\theta$$

$$= \int \sin^{-1}(\sin 2\theta) \sec^2\theta \, d\theta$$

$$= 2\int \underset{\text{I}}{\theta} \, \underset{\text{II}}{\sec^2\theta} \, d\theta$$

$$= 2\left[\theta \int \sec^2\theta d\theta - \int\left\{\frac{d}{d\theta}\theta \int \sec^2\theta \, d\theta\right\} d\theta\right]$$

$$= 2\left[\theta.\tan\theta - \int 1.\tan\theta \, d\theta\right]$$

$$= 2[\theta \tan\theta - \log\sec\theta]$$

$$= 2x\tan^{-1}x - 2\log\sqrt{1+x^2} + C$$

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

$$\therefore \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = 2x\tan^{-1}x - \log(1+x^2) + C$$

प्रश्न 11.

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} \, dx$$

हल :

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} \, dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x - 1}} \, dx$$

$$= \int \frac{(\sin x - \cos x)}{\sqrt{(\sin x + \cos x)^2 - 1}} \, dx$$

माना $\sin x + \cos x = t$

$$(\cos x - \sin x) \, dx = dt$$

$$(\sin x - \cos x) \, dx = -dt$$

$$= -\int \frac{dt}{\sqrt{t^2 - 1}}$$

$$= -\log(t + \sqrt{t^2 - 1}) + C$$

$$= -\log(\sin x + \cos x + \sqrt{\sin 2x}) + C$$

प्रश्न 12.

$$\int \frac{\sin 2x}{\sin 4x + \cos 4x} dx$$

हल :

$$\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \int \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \int \frac{2 \tan x \sec^2 x}{\tan^4 x + 1} dx$$

(अंश व हर में $\cos^4 x$ से भाग करने पर)

अब $\tan x = t \Rightarrow \sec^2 x \, dx = dt$ रखने पर

$$= 2 \int \frac{t}{t^4 + 1} dt$$

पुन: $t^2 = u$ रखने पर, $2t \, dt = du$

$$= 2 \int \frac{t}{u^2 + 1} \cdot \frac{du}{2} = \int \frac{1}{u^2 + 1} du$$

$= \tan^{-1} u + c = \tan^{-1} t^2 + C$
$= \tan^{-1} (\tan^2 t) + C$

प्रश्न 13.

$$\int \frac{1 + x}{(2 + x)^2} dx$$

हल :

$$\int \frac{1 + x}{(2 + x)^2} dx$$

$$= \int \frac{(x + 2) - 1}{(x + 2)^2} dx$$

$$= \int \frac{x + 2}{(x + 2)^2} dx - \int \frac{1}{(x + 2)^2} dx$$

$$= \int \frac{1}{x + 2} dx - \int (x + 2)^{-2} dx$$

$$= \log | x + 2 | + \frac{1}{(x + 2)} + C$$

प्रश्न 14.

$$\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

हल :

$$\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}$$

$$(\because \ a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

$$= \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \left(\frac{1}{\sin^2 x \cos^2 x} - 3 \right) dx$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3 \right) dx$$

$= \int secx^2 \, dx + \int cosec^2 \, xdu - 3 \int dx$

$= \tan x - cotx - 3x + C$

प्रश्न 15.

$$\int \frac{\tan^{-1} x}{x^2} dx$$

हल :

$$\int \frac{\tan^{-1} x}{x^2} dx$$

$$= \int \underset{\text{I}}{\tan^{-1} x} . \underset{\text{II}}{x^{-2}} \, dx$$

$$= \tan^{-1} x \int x^{-2} \, dx - \int \left[\frac{d}{dx} (\tan^{-1} x) \int x^{-2} \, dx \right] dx$$

$$= \tan^{-1} x \cdot \frac{x^{-2+1}}{-2+1} - \int \frac{1}{1+x^2} \cdot \frac{x^{-2+1}}{-2+1} dx$$

$$= -\frac{1}{x} \tan^{-1} x + \int \frac{1}{x(1+x^2)} dx \qquad ...(i)$$

माना $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$

$\Rightarrow \quad 1 = A(1+x^2) + (Bx+C)x$

$\Rightarrow \quad 1 = A + Ax^2 + Bx^2 + Cx$

$\Rightarrow \quad 1 = (A+B)x^2 + Cx + A.$

तुलना करने पर,

$$A + B = 0, C = 0, A = 1$$

हल करने पर

$$A = 1, B = -1, C = 0$$

$\therefore \quad \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

$$\int \frac{1}{x(1+x^2)} dx = \int \frac{1}{x} dx - \int \frac{x}{1+x^2} dx$$

माना $1 + x^2 = t$

$2x\, dx = dt$

$x\, dx = \frac{dt}{2}$

$$= \log x - \frac{1}{2} \int \frac{1}{t} dt$$

$$= \log x - \frac{1}{2} \log |t| + C$$

$$= \log x - \log |\sqrt{t}| + C$$

$$= \log x - \log |\sqrt{1+x^2}| + C$$

$$= \log \left| \frac{x}{\sqrt{1+x^2}} \right| + C$$

$$\therefore \int \frac{\tan^{-1} x}{x^2} dx = -\frac{\tan^{-1} x}{x} + \log \left| \frac{x}{\sqrt{1+x^2}} \right| + C$$

प्रश्न 16.

$$\int \frac{1}{\sin^2 x + \sin 2x} dx$$

हल :

$$\int \frac{1}{\sin^2 x + \sin 2x}\, dx$$

$$= \int \frac{1}{\sin x(\sin x + 2\cos x)}\, dx$$

$$= \int \frac{\text{cosec } x}{\cos x\, (\tan x + 2)}\, dx$$

$$= \int \frac{\text{cosec } x . \sec x}{(\tan x + 2)}\, dx$$

$$= \int \frac{\cot x \sec^2 x}{(\tan x + 2)}\, dx$$

माना $\tan x + 2 = y$

$\sec^2 x\, dx = dy$

$\tan x + 2$

$\tan x = y - 2$

$$\Rightarrow \qquad \cot x = \frac{1}{y-2}$$

$$I = \int \frac{1}{y(y-2)}\, dy$$

अब $\quad \dfrac{1}{y(y-2)} = \dfrac{A}{y} + \dfrac{B}{y-2}$

$$\Rightarrow \qquad 1 = A(y-2) + By$$

$$\Rightarrow \qquad 1 = (A+B)y - 2A$$

तुलना से, $\quad A + B = 0, \; -2A = 1$

$$A = -\frac{1}{2} \text{ तथा } B = \frac{1}{2}$$

$$I = \int \frac{-1/2}{y}\, dy + \frac{1}{2} \int \frac{1}{y-2}\, dy$$

$$= \frac{1}{2}\log(y-2) - \frac{1}{2}\log y + C$$

$$= \frac{1}{2}\log \frac{y-2}{y} + C$$

$$= \frac{1}{2}\log\left(\frac{\tan x}{\tan x + 2}\right)$$

प्रश्न 17.

$$\int \frac{1}{4x^2 - 4x + 3}\, dx$$

हल :

$$\int \frac{1}{4x^2 - 4x + 3}\, dx$$

$$= \int \frac{1}{(2x)^2 - 2 \cdot 1 \cdot 2x + 1 + 2}\, dx$$

$$= \int \frac{1}{(2x-1)^2 + (\sqrt{2})^2}\, dx$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

प्रश्न 18.

$$\int \frac{1}{x[6(\log x)^2 + 7(\log x) + 2}\, dx$$

हल :

माना log x = t

$\frac{1}{x}$dx = dt

$$= \int \frac{dt}{6t^2 + 7t + 2}$$

$$= \int \frac{dt}{6t^2 + 3t + 4t + 2}$$

$$= \int \frac{dt}{3t(2t+1) + 2(2t+1)}$$

$$= \int \frac{dt}{(2t+1)\,(3t+2)}$$

माना $\frac{1}{(2t+1)\,(3t+2)} = \frac{A}{2t+1} + \frac{B}{3t+2}$

1 = A(3t + 2) + B(2t + 1)

1 = (3A + 2B)t + (2A + B)

तुलना करने पर,

3A + 2B = 0 तथा 2A + B = 1

हल करने पर,
A = 2, B = − 3

$$\Rightarrow \int \frac{dt}{(2t+1)(3t+2)}$$

$$= 2\int \frac{1}{(2t+1)} dt - 3\int \frac{1}{(3t+2)} dt$$

$$= \frac{2 \log|2t+1|}{2} - \frac{3 \log|3t+2|}{3} + C$$

$$= \log|2t+1| - \log|3t+2| + C$$

$$= \log\left|\frac{2t+1}{3t+2}\right| + C$$

$$= \log\left|\frac{2 \log x+1}{3 \log x+2}\right| + C$$

$$\therefore \int \frac{1}{x[6(\log x)^2 + 7(\log x) + 2} dx$$

$$= \log\left|\frac{2 \log x+1}{3 \log x+2}\right| + C$$

प्रश्न 19.

$$\int \frac{\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}} dx$$

हल :

$$\int \frac{\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}} dx$$

माना $sec^2 2x = t$
$4 \sin 2x.\cos 2x\, dx = dt$

$$= \frac{1}{4} \int \frac{dt}{\sqrt{2^2 - t^2}}$$

$$= \frac{1}{4} \sin^{-1} \frac{t}{2} + C$$

$$= \frac{1}{4} \sin^{-1} \left(\frac{\sin^2 2x}{2}\right) + C$$

प्रश्न 20.

$$\int \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

हल :

$$\int \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

माना sin − cos x = t

(sin x + cos x) dx = dt

⇒ $(\sin x - \cos x)^2 = t^2$

⇒ $\sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

⇒ $1 - \sin 2x = t^2$

⇒ $\sin 2x = 1 - t^2$

$$= \int \frac{dt}{9 + 16(1 - t^2)}$$

$$= \int \frac{dt}{25 + 16t^2}$$

$$= \frac{1}{16} \int \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{16} \times \frac{1}{2 \times \left(\frac{5}{4}\right)} \log \left| \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right| + C$$

$$= \frac{1}{40} \log \left| \frac{5 + 4t}{5 - 4t} \right| + C$$

$$= \frac{1}{40} \log \left| \frac{5 + 4(\sin x - \cos x)}{5 - 4(\sin x - \cos x)} \right| + C$$

प्रश्न 21.

$$\int \frac{3x - 1}{(x - 2)^2} dx$$

हल :

$$\int \frac{3x-1}{(x-2)^2}\,dx$$

$$= \int \frac{(3x-6)+5}{(x-2)^2}\,dx$$

$$= \int \frac{3(x-2)}{(x-2)^2}\,dx + 5\int \frac{1}{(x-2)^2}\,dx$$

$$= 3\int \frac{1}{(x-2)}\,dx + 5\int \frac{1}{(x-2)^2}\,dx$$

$$= 3\log|x-2| - 5\frac{1}{(x-2)} + C$$

$$\therefore \quad \int \frac{3x-1}{(x-2)^2}\,dx = 3\log|x-2| - \frac{5}{(x-2)} + C$$

प्रश्न 22.

$$\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$$

का मान है

(a) tan x + x + C
(b) cot x + x + C
(c) tan x − x + C
(d) cot x − x + C

हल :

$$\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$$

$$= \int \frac{1-1+2\sin^2 x}{1+2\cos^2 x-1}\,dx$$

$$= \int \tan^2 x\,dx$$

$$= \int (\sec^2 x - 1)\,dx$$

$= \int \sec^2 x\,dx - \int 1\,dx$

= tan x − x + C

अत: विकल्प (c) सही है।

प्रश्न 23.

$$\int \frac{1}{\sqrt{32-2x^2}}\,dx$$

का मान है

(a) $\sin^{-1}\left(\frac{x}{4}\right)+C$ (b) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x}{4}\right)+C$

(c) $\sin^{-1}\left(\frac{\sqrt{2}x}{4}\right)+C$ (d) $\cos^{-1}\left(\frac{x}{4}\right)+C$

हल :

$$\int \frac{1}{\sqrt{32-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(4)^2-x^2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{4}+C$$

अतः विकल्प (b) सही है।

प्रश्न 24.

∫log x dx बराबर है

(a) $x\log(xe)+C$ (b) $x\log x+C$

(c) $x\log\left(\frac{x}{e}\right)+C$ (d) $\log\frac{x}{e}$

हल :

∫log x dx

= ∫log x.1 dx

$$= \log x\int 1\,dx - \int\left[\frac{d}{dx}\log x.\int 1\,dx\right]dx$$

$$= \log x\cdot x - \int\frac{1}{x}\cdot x\,dx$$

= x log x − x loge + C

= x $\log\frac{x}{e}$ + c

अत: विकल्प (c) सही है।

प्रश्न 25.

$$\int \frac{dx}{x(x+1)}$$

बराबर है

(a) $\log\left(\frac{x}{x+1}\right) + C$ (b) $\log\left(\frac{x+1}{x}\right) + C$

(c) $\frac{1}{2}\log\left(\frac{x}{x+1}\right) + C$ (d) $\frac{1}{2}\log\left(\frac{x+1}{x}\right) + C$

हल :

माना है।

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$\Rightarrow 1 = A(x + 1) + B(x)$

$\Rightarrow 1 = (A + B)x + A$

तुलना से, $A = 1, A + B = 0$

हल करने पर,

$A = 1, B = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \log|x| - \log|x+1| + C$$

$$= \log\left|\frac{x}{x+1}\right| + C$$

अत: विकल्प (a) सही है।