Chapter 11

Inequalities

CHAPTER HIGHLIGHTS

- *Inequalities and Modulus*
- *Symbols and Notations*
- *Absolute Value*
- *Properties of Modulus*

Inequalities and Modulus

If 'a' is any real number, then 'a' is either positive or negative or zero. When 'a' is positive, we write $a > 0$, which is read 'a is greater than zero'. When 'a' is negative, we write $a < 0$, which is read 'a is less than zero'. If 'a' is zero, we write $a = 0$ and in this case, 'a' is neither positive nor negative.

Symbols and Notations

'>' means 'greater than'
'<' means 'less than'
'≥' means 'greater than or equal to'
'≤' means 'less than or equal to'

For any two non-zero real numbers a and b,

1. a is said to be greater than b when $a - b$ is positive.
2. a is said to be less than b when $a - b$ is negative.

These two statements are written as

1. $a > b$ when $a - b > 0$ and
2. $a < b$ when $a - b < 0$.

For example,
3 is greater than 2 because $3 - 2 = 1$ and 1 is greater than zero. -3 is less than -2 because $-3 - (-2) = -1$ and -1 is less than zero.

Certain properties and useful results pertaining to inequalities are given below. A thorough understanding of these properties results is very essential for being able to solve the problems pertaining to inequalities.

[In the following list of properties and results, numbers like a, b, c, d, etc. are real numbers.]

1. For any two real numbers a and b, either $a > b$ or $a < b$ or $a = b$.
2. If $a > b$, then $b < a$.
3. If $a \nless b$, then $a \geq b$ and if $a > b$, then $a \leq b$.
4. If $a > b$ and $b > c$, then $a > c$.
5. If $a < b$ and $b < c$, then $a < c$.
6. If $a > b$, then $a \pm c > b \pm$ c.
7. If $a > b$ and $c > 0$, then $ac > bc$.
8. If $a < b$ and $c > 0$, then $ac < bc$.
9. If $a > b$ and $c < 0$, then $ac < bc$.
10. If $a < b$ and $c < 0$, then $ac > bc$.
11. If $a > b$ and $c > d$, then $a + c > b + d$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. Let A, G and H be the Arithmetic mean, Geometric mean and Harmonic mean of n positive real numbers. Then $A \geq G \geq H$, the equality occurring only when the numbers are all equal.

14. If the sum of two positive quantities is given, their product is the greatest when they are equal; if the product of two positive quantities is given, their sum is the least when they are equal.
15. If $a > b$ and $c > d$, then we cannot say anything conclusively about the relationship between $(a - b)$ and $(c - d)$; depending on the values of a, b, c, and d, it is possible to have $(a - b) > (c - d)$, $(a - b) = (c - d)$ or $(a - b) < (c - d)$.

Absolute Value

(written as $|x|$ and read as 'modulus of x')

For any real number x, the absolute value is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \text{ and} \\ -x, & \text{if } x < 0 \end{cases}$$

Properties of Modulus

For any real number x and y,

1. $x = 0 \Leftrightarrow |x| = 0$
2. $|x| \geq 0$ and $-|x| \leq 0$
3. $|x + y| \leq |x| + |y|$
4. $||x| - |y|| \leq |x - y|$
5. $-|x| \leq x \leq |x|$
6. $|x \cdot y| = |x| \cdot |y|$
7. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$; $(y \neq 0)$
8. $|x|^2 = x^2$

In inequalities, the variables generally take a range of values unlike in the case of equations where the variables in general, take one value or a discrete set of values. (In some specific cases, the variables may take only one value.)

Solved Examples

Example 1

If $13x - 19 \leq 4x + 26$, find the range of x.

Solution

$13x - 4x \leq 26 + 19$

$$x \leq 5.$$

$(-\infty, 5]$ in the interval notation.

Example 2

Solve the following inequalities:

$$5x + 21 < 46 \text{ and } 4x + 18 < 54.$$

Solution

$$5x + 21 < 46 \Rightarrow x < 5 \quad (1)$$

$$4x + 18 < 54 \Rightarrow x < 9 \quad (2)$$

The common inequality satisfying (1) and (2) is $x < 5$ or $(-\infty, 5)$ in the interval notation.

Example 3

Which of the numbers 50^{51} and 51^{50} is greater?

Solution

Let $a = 50^{51}$ and $b = 51^{50}$.

$$\frac{b}{a} = \frac{51^{50}}{50^{51}} = \left(\frac{51}{50}\right)^{50}\left(\frac{1}{50}\right) = \left(1+\frac{1}{50}\right)^{50}\left(\frac{1}{50}\right)$$

$\left(1+\dfrac{1}{x}\right)^x$ where $x > 0$ always lies between 2 and 2.8.

$\therefore$ $\dfrac{b}{a}$ lies between

$$\frac{2}{50} = 0.04 \text{ and } \frac{2.8}{50} = 0.056$$

$\therefore$ $\dfrac{b}{a} < 1$

$\therefore$ $a > b$.

Example 4

Solve for x if $4x^2 - 21x + 20 > 0$

Solution

$$4x^2 - 21x + 20 > 0 \Rightarrow (4x - 5)(x - 4) > 0$$

Both factors are positive (i.e. the smaller is positive) or both are negative (i.e. the greater is negative), i.e. $x > 4$ or $x < \dfrac{5}{4}$ or it can be expressed in the interval notation as $(4, \infty) \cup \left(-\infty, \dfrac{5}{4}\right)$

Example 5

Solve for x, if $\dfrac{x^2 + 5x - 24}{2x^2 - 5x - 3} < 0$

Solution

$$x^2 + 5x - 24 = (x + 8)(x - 3)$$

Similarly $2x^2 - 5x - 3 = (2x + 1)(x - 3)$

Given: $\dfrac{x^2 + 5x - 24}{2x^2 - 5x - 3} < 0$

$$\Rightarrow \frac{(x+8)(x-3)}{(2x+1)(x-3)} < 0$$

$$\Rightarrow \frac{x+8}{2x+1} < 0$$

$$\frac{(x+8)(2x+1)}{(2x+1)^2} < 0$$

$$\Rightarrow \quad (x+8)\left(x+\frac{1}{2}\right) < 0$$

$$\therefore \quad -8 < x < -\frac{1}{2}.$$

Example 6

Solve the inequality $|3x + 6| > -12$.

Solution

The modulus of any number is always non-negative.

$\therefore \quad |3x + 6| \geq 0.$

$\therefore$ The given inequality is always satisfied.

$\therefore \quad -\infty < x < \infty$

Example 7

Solve the inequality $|2x + 4| < -6$.

Solution

The modulus of any number is always non- negative.

$\therefore \quad |2x + 4| \geq 0$

$\therefore$ The given inequality will never satisfy.
The solution is null set.

Example 8

Solve for x: $|2x - 3| = 5$

Solution

$2x - 3 = 5$ or $2x - 3 = -5$

$$(\text{If } |y| = a, y = \pm a) \quad \Rightarrow \quad x = 4 \text{ or } x = -1.$$

Example 9

Find the maximum value of

$$g(x) = 16 - |-x - 6|;\ x \in R.$$

Solution

$g(x)$ is maximum when $|-x - 6|$ is minimum.
The minimum value of the modulus of all numbers is 0.
$\therefore$ The maximum value of $g(x) = 16 - 0 = 16$.

Exercises

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. If $a < b$ and $c < 0$, then which of the following is true?
(A) $ac < bc$ (B) $\frac{a}{c} < \frac{b}{c}$
(C) $ac > bc$ (D) None of these

2. If p and q are two real numbers, then which of the following statements is always true?
(A) $\frac{p}{q} < 1 \Rightarrow p < q$
(B) $p > 0$, $q > 0$ and $\frac{p}{q} > 1 \Rightarrow p > q$
(C) $\frac{p}{q} > 1 \Rightarrow p > q$
(D) All the above

3. If $5x - 8 < 2x + 9$ and $4x + 7 > 7x - 8$, then the range of the values of x that satisfies the inequalities is
(A) $(5, \infty)$ (B) $(-\infty, 5)$
(C) $\left(5, \frac{17}{3}\right)$ (D) $\left(-\infty, \frac{17}{3}\right)$

4. Solve for real values of x; $5x^2 - 3x - 2 \geq 0$.
(A) $\left[\frac{-2}{5}, 1\right]$ (B) $R - \left(\frac{-2}{5}, 1\right)$
(C) $[1, \infty)$ (D) $R - (0, 1)$

5. If $x^2 - 9x - 36$ is negative, then find the range of x.
(A) $(-3, 12)$ (B) $[-3, 12]$
(C) $(-12, 3)$ (D) $[-12, 3]$

6. Which of the following is true?
(A) $|x + y| \leq |x| + |y|$ (B) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$
(C) $|x - y| \geq ||x| - |y||$ (D) All the above

7. If $6x + 8 > 7x - 9$ and $4x - 7 < 6x - 3$, then the values of x is
(A) $(-17, 2)$ (B) $(2, 17)$
(C) $(-2, 17)$ (D) $(-\infty, 17)$

8. The solution set of the inequality $|x - 5| < 9$ is
(A) $(0, 14)$ (B) $(-4, 14)$ (C) $(-4, 0)$ (D) $(9, 14)$

9. The number of integral values of x that do not satisfy the inequation $\frac{x+5}{x-2} \geq 0$ is
(A) 7 (B) 5 (C) 6 (D) 4

10. If $(x + 5)(x + 9)(x + 3)^2 < 0$, then the solution set for the inequality is
(A) $(-9, -3)$ (B) $(-9, -5)$
(C) $(-3, \infty)$ (D) $(-9, \infty)$

11. Find the range of the real values of x satisfying $8 - 3x \leq 5$ and $4x + 5 \leq -7$.
(A) $[-3, 1]$ (B) $(-\infty, -3] \cup [1, \infty)$
(C) $(-3, 1)$ (D) ϕ

12. Which of the following is true?
(A) $30^{31} < 31^{30}$ (B) $71^{69} > 70^{70}$
(C) $(155)^{29} < (150)^{30}$ (D) Both (B) and (C)

13. At what value of x is $-|x-3| + \frac{21}{2}$ maximum?

(A) -3 (B) $\frac{21}{2}$ (C) 0 (D) 3

14. Find the range of all real values of x if $|3x+5| < 5x - 11$.

(A) $(8, \infty)$ (B) $(-\infty, -5/3) \cup (8, \infty)$
(C) $(-5/3, 8)$ (D) $(-5/3, \infty)$

15. If $ac = bd = 2$, then the minimum value of $a^2 + b^2 + c^2 + d^2$ is

(A) 4 (B) 6 (C) 8 (D) 16

16. If $x, y > 0$ and $x + y = 3$ then

(A) $xy \le 0.72$ (B) $xy \le 1.8$
(C) $xy \le 2.25$ (D) $xy \le 1.25$

17. Find the complete range of values of x that satisfies $|x - 16| > x^2 - 7x + 24$.

(A) $(0, 2)$ (B) $\left(\frac{3}{2}, \frac{5}{2}\right)$
(C) $(1, 3)$ (D) $(2, 4)$

18. For which of the following range of values of x is $x^2 + x$ less than $x^3 + 1$?

(A) $(-\infty, -1)$ (B) $(1, \infty)$
(C) $(-1, 1) \cup (1, \infty)$ (D) $[-1, 1]$

19. If x, y, z are positive, then the value of

$A=\dfrac{(4x^2+x+4)(5y^2+y+5)(7z^2+z+7)}{xyz}$ can be

(A) 400 (B) 500 (C) 1000 (D) 1500

20. The range of x for which $2x^2 - 5x - 8 \le |2x^2 + x|$ is

(A) $\left[-\frac{4}{3}, \infty\right)$ (B) $\left(-\frac{4}{3}, -1\right)$
(C) $[-1, \infty)$ (D) $[-1, 2]$

21. For how many integral values of x, is the inequation $\frac{x-5}{x+7} > 4$ satisfied?

(A) 5 (B) 4 (C) 2 (D) 3

22. If $1 \le x \le 3$ and $2 \le y \le 5$, then the minimum value of $\frac{x+y}{y}$ is

(A) $\frac{3}{5}$ (B) $\frac{1}{5}$ (C) $\frac{6}{5}$ (D) $\frac{5}{6}$

23. If $|b| \ge 5$ and $x = |a|\, b$, which of the following is true?

(A) $a - xb > 0$ (B) $a + xb < 0$
(C) $a + xb > 0$ (D) $a - xb \le 0$

24. Find the number of solutions of the equation $\big|x - |x-2|\big| = 6$.

(A) 2 (B) 1 (C) 3 (D) 4

25. If x, y and z are positive real numbers, then the minimum value of

$\dfrac{x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2}{xyz}$ is

(A) 6 (B) 9 (C) 12 (D) 14

Answer Keys

1. C	**2.** B	**3.** B	**4.** B	**5.** A	**6.** D	**7.** C	**8.** B	**9.** A	**10.** B
11. D	**12.** C	**13.** D	**14.** A	**15.** C	**16.** C	**17.** D	**18.** C	**19.** D	**20.** A
21. D	**22.** C	**23.** D	**24.** B	**25.** A					