AREA [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. The area bounded by the curves $y = f(x)$, the x-axis, and the ordinates $x = 1$ and $x = b$ is $(b - 1) \sin (3b + 4)$. Then $f(x)$ is
   - **a.** $(x - 1) \cos (3x + 4)$
   - **b.** $\sin (3x + 4)$
   - **c.** $\sin (3x + 4) + 3(x - 1) \cos (3x + 4)$
   - **d.** none of these **(IIT-JEE 1982)**

2. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is
   - **a.** 1 sq. units **b.** 2 sq. units
   - **c.** $2\sqrt{2}$ sq. units **d.** 4 sq. units

   (IIT-JEE 2002)

3. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$, and x-axis in the 1st quadrant is
   - **a.** 18 sq. units **b.** 27/4 sq. units
   - **c.** 36 sq. units **d.** 9 sq. units

   (IIT-JEE 2002)

4. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ (where $a > 0$) is 1 sq. unit, then the value of a is
   - **a.** $1/\sqrt{3}$ **b.** 1/2
   - **c.** 1 **d.** 1/3 **(IIT-JEE 2004)**

5. The area bounded by the parabolas $y = (x + 1)^2$ and $y = (x - 1)^2$ and the line $y = 1/4$ is
   - **a.** 4 sq. units **b.** 1/6 sq. units
   - **c.** 4/3 sq. units **d.** 1/3 sq. units

   (IIT-JEE 2005)

6. The area of the region between the curves $y = \sqrt{\dfrac{1+\sin x}{\cos x}}$ and $y = \sqrt{\dfrac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \dfrac{\pi}{4}$ is
   - **a.** $\displaystyle\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$ **b.** $\displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$
   - **c.** $\displaystyle\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$ **d.** $\displaystyle\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

   (IIT-JEE 2008)

7. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$, and $x = 0$ into two parts R_1 $(0 \le x \le b)$ and R_2 $(b \le x \le 1)$ such that $R_1 - R_2 = \dfrac{1}{4}$. Then b equals
   - **a.** 3/4 **b.** 1/2
   - **c.** 1/3 **d.** 1/4 **(IIT-JEE 2011)**

8. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $[0, \pi/2]$ is
   - **a.** $4(\sqrt{2} - 1)$ **b.** $2\sqrt{2}(\sqrt{2} - 1)$
   - **c.** $2(\sqrt{2} + 1)$ **d.** $2\sqrt{2}(\sqrt{2} + 1)$

   (JEE Advanced 2013)

Multiple Correct Answers Type

1. For which of the following values of m is the area of the regions bounded by the curve $y = x - x^2$ and the line $y = mx$ equals 9/2?
   - **a.** -4 **b.** -2
   - **c.** 2 **d.** 4 **(IIT-JEE 1999)**

2. The area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **(IIT-JEE 2009)**
   - **a.** $e - 1$ **b.** $\displaystyle\int_1^e \ln(e+1-y)\,dy$
   - **c.** $e - \displaystyle\int_0^1 e^x\,dx$ **d.** $\displaystyle\int_1^e \ln y\,dy$

Linked Comprehension Type

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$. **(IIT-JEE 2008)**

1. The area of the region bounded by the curves $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$ is
   - **a.** $\displaystyle\int_a^b \frac{x}{3((f(x))^2-1)}dx + bf(b) - af(a)$
   - **b.** $-\displaystyle\int_a^b \frac{x}{3((f(x))^2-1)}dx + bf(b) - af(a)$
   - **c.** $\displaystyle\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$
   - **d.** $-\displaystyle\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$

2. $\displaystyle\int_{-1}^{1} g'(x)dx =$
   - **a.** $2g(-1)$ **b.** 0 **c.** $-2g(1)$ **d.** $2g(1)$

Matching Column Type

1. Match the statements given in Column I with the values given in Column II.

Column I	Column I
(i) $\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \log(\sin x)^{\sin x})dx$	(a) 1
(ii) Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$	(b) 0
(iii) Cosine of the angle of intersection of curves $y = 3^{x-1}\log x$ and $y = x^x - 1$ is	(c) 6 ln 2
	(d) 4/3

(IITJEE 2006)

2. Match the statements given in Column I with the values given in Column II.

Column I	Column II
(a) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X - Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	(p) 1
(b) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	(q) 2
(c) In R^2, let $\sqrt{3}\hat{i} + \hat{j}, \hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1 - \beta)\hat{j}$ be the position vectors of X, Y and Z with respect of the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overline{OX}$ and $\overline{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	(r) 3
(d) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0, x = 2, y^2 = 4x$ and $y = \lvert\alpha x - 1\rvert + \lvert\alpha x - 2\rvert + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	(s) 5
	(t) 6

(JEE Advanced 2015)

Integer Answer Type

1. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is

(JEE Advanced 2014)

Subjective Type

1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. **(IIT-JEE 1981)**
2. For any real t, $x = \frac{1}{2}(e^t + e^{-t}), y = \frac{1}{2}(e^t - e^{-t})$ is a point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1. **(IIT-JEE 1982)**
3. Find the area bounded by the x-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$, and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, then find a. **(IIT-JEE 1983)**
4. Find the area of the region bounded by the x-axis and the curves defined by $y = \tan x$ $\left(\text{where } -\frac{\pi}{3} \le x \le \frac{\pi}{3}\right)$ and $y = \cot x$ $\left(\text{where } \frac{\pi}{6} \le x \le \frac{3\pi}{2}\right)$. **(IIT-JEE 1984)**
5. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = \lvert x - 1\rvert$ and find its area. **(IIT-JEE 1985)**
6. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}y$, and $x = y$. **(IIT-JEE 1986)**
7. Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = \lvert 4 - x^2\rvert$, and $x = 0$ above the x-axis. **(IIT-JEE 1987)**
8. Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to C at $x = \frac{\pi}{4}$, and the x-axis. **(IIT-JEE 1988)**
9. Compute the area of the region bounded by the curves $y = ex \log_e x$ and $y = \frac{\log x}{ex}$. **(IIT-JEE 1990)**
10. Sketch the curves and identify the region bounded by $x = \frac{1}{2}$, $x = 2$, $y = \ln x$, and $y = 2^x$. Find the area of this region. **(IIT-JEE 1991)**
11. Sketch the region bounded by the curves $y = x^2$ and $y = \frac{2}{1+x^2}$. Find the area. **(IIT-JEE 1992)**
12. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$? **(IIT-JEE 1993)**
13. Consider a square with vertices at (1, 1), (−1, 1), (−1, −1), and (1, −1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. **(IIT-JEE 1995)**

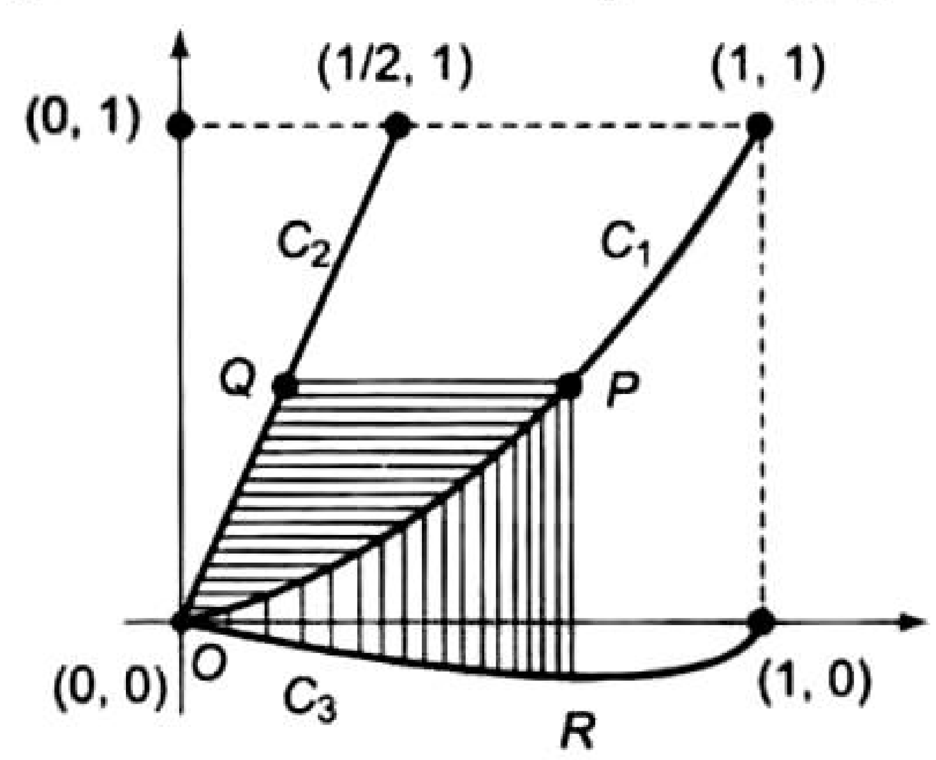
14. Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$, and $x = \frac{\pi}{4}$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$. **(IIT-JEE 1996)**

15. Find all the possible values of $b > 0$, so that the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum. **(IIT-JEE 1997)**

16. Let O (0, 0), A(2, 0), and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \le \min [d(P, OB), d(P, AB)]$, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area. **(IIT-JEE 1997)**

17. Let $f(x)$ = Maximum $\{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$, and $x = 1$. **(IIT-JEE 1997)**

18. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, respectively, where $0 \le x \le 1$. Let C_3 be the graph of a function $y = f(x)$, where $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axis, meet C_2 and C_3 at Q and R, respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$. **(IIT-JEE 1998)**



19. Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

. Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$. **(IIT-JEE 1999)**

20. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$, and $y = 2$, which lies to the right of the line $x = 1$. **(IIT-JEE 2002)**

21. Find the area bounded by the curve $x^2 = y$, $x^2 = -y$, and $y^2 = 4x - 3$. **(IIT-JEE 2005)**

22. If $f(x)$ is a differentiable function such that $f'(x) = g(x)$, $g''(x)$ exists, $|f(x)| < 1$, and $(f(0))^2 + (g(0))^2 = 9$. Prove that there is a point c where $c \in (-3, 3)$ such that $g(c).g''(c) < 0$. **(IIT-JEE 2005)**

23. $f(x)$ is a quadratic polynomial and a, b, c are three real and distinct numbers satisfying

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}.$$

Given $f(x)$ cuts the x-axis at A, and V is the point of maxima. If AB is any chord which subtends a right angle at V, find curve $f(x)$ and the area bounded by the chord AB and curve $f(x)$. **(IIT-JEE 2005)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. c. **2.** b. **3.** d. **4.** a. **5.** d.
6. b. **7.** b. **8.** b.

Multiple Correct Answers Type

1. b., d. **2.** b., c., d.

Linked Comprehension Type

1. a. **2.** d.

Matching Column Type

1. (ii) – (d) **2.** (d) – (s), (t)

Integer Answer Type

1. 6

Subjective Type

1. $\frac{9}{8}$ sq. units

3. $a = 2\sqrt{2}$

4. $\log\frac{3}{2}$ sq.units

5. $\frac{5\pi-2}{4}$ sq. units

6. $\pi+\frac{1}{3}$ sq.units

7. $4+25\sin^{-1}\frac{4}{5}$ sq. units

8. $\frac{1}{2}\left[\log 2-\frac{1}{2}\right]$ sq. units

9. $\frac{e^2-5}{4e}$ sq. units

10. $\frac{4-\sqrt{2}}{\log 2}-\frac{5}{2}\log 2+\frac{3}{2}$ sq. units

11. $\left(\pi-\frac{2}{3}\right)$ sq. units

12. $121:4$

13. $\frac{16\sqrt{2}-20}{3}$ sq. units

15. $b=1$

16. $2-\sqrt{3}$ sq. units

17. $\frac{17}{27}$ sq. units

18. $f(x)=x^3-x^2$

19. $\frac{257}{192}$ sq. units

20. $\left(\frac{20}{3}-4\sqrt{2}\right)$ sq. units

21. $\frac{1}{3}$ sq. units

23. $\frac{125}{3}$ sq. units.

Hints and Solutions

JEE Advanced

Single Correct Answer Type

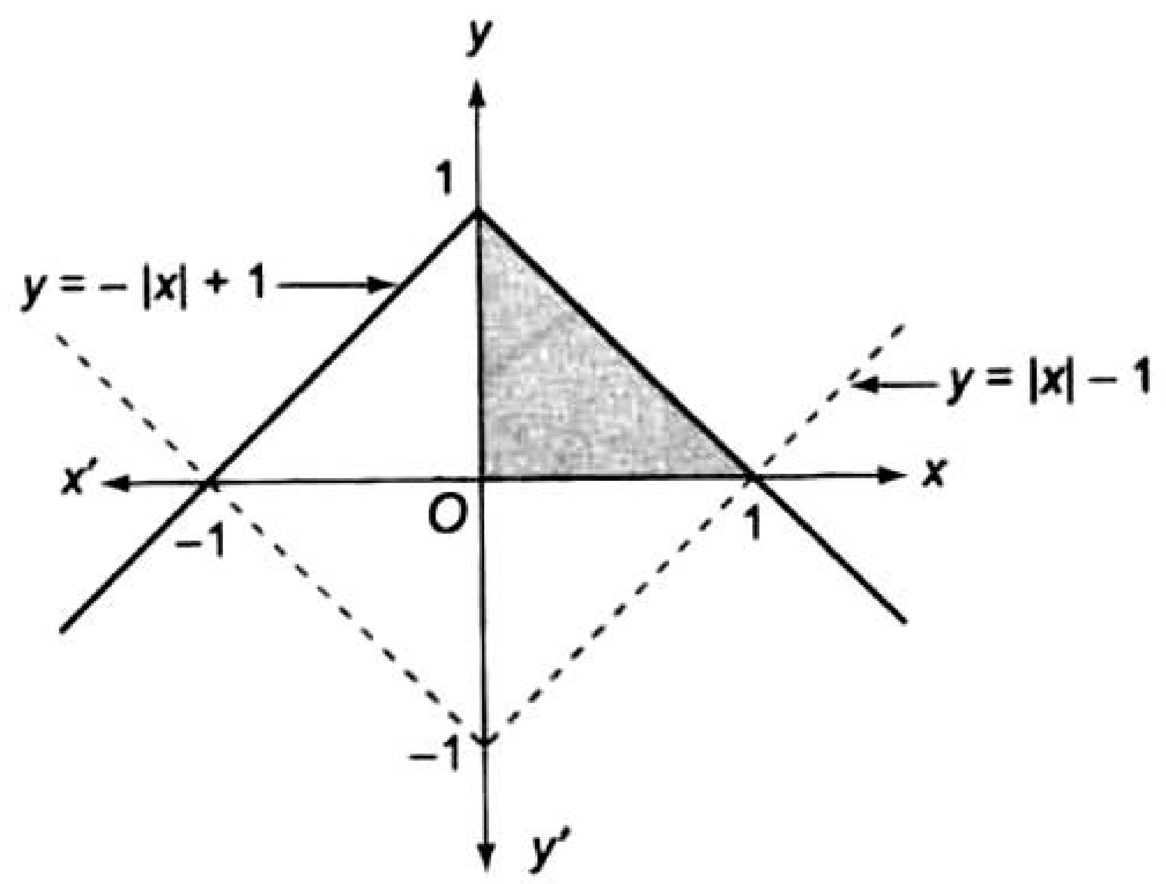
1. c. Given $\int_1^b f(x)\,dx = (b-1)\sin(3b+4)$

Differentiating both sides w.r.t. b, we get

$\Rightarrow f(b) = 3(b-1)\cos(3b+4) + \sin(3b+4)$

$\Rightarrow f(x) = \sin(3x+4) + 3(x-1)\cos(3x+4)$.

2. b.

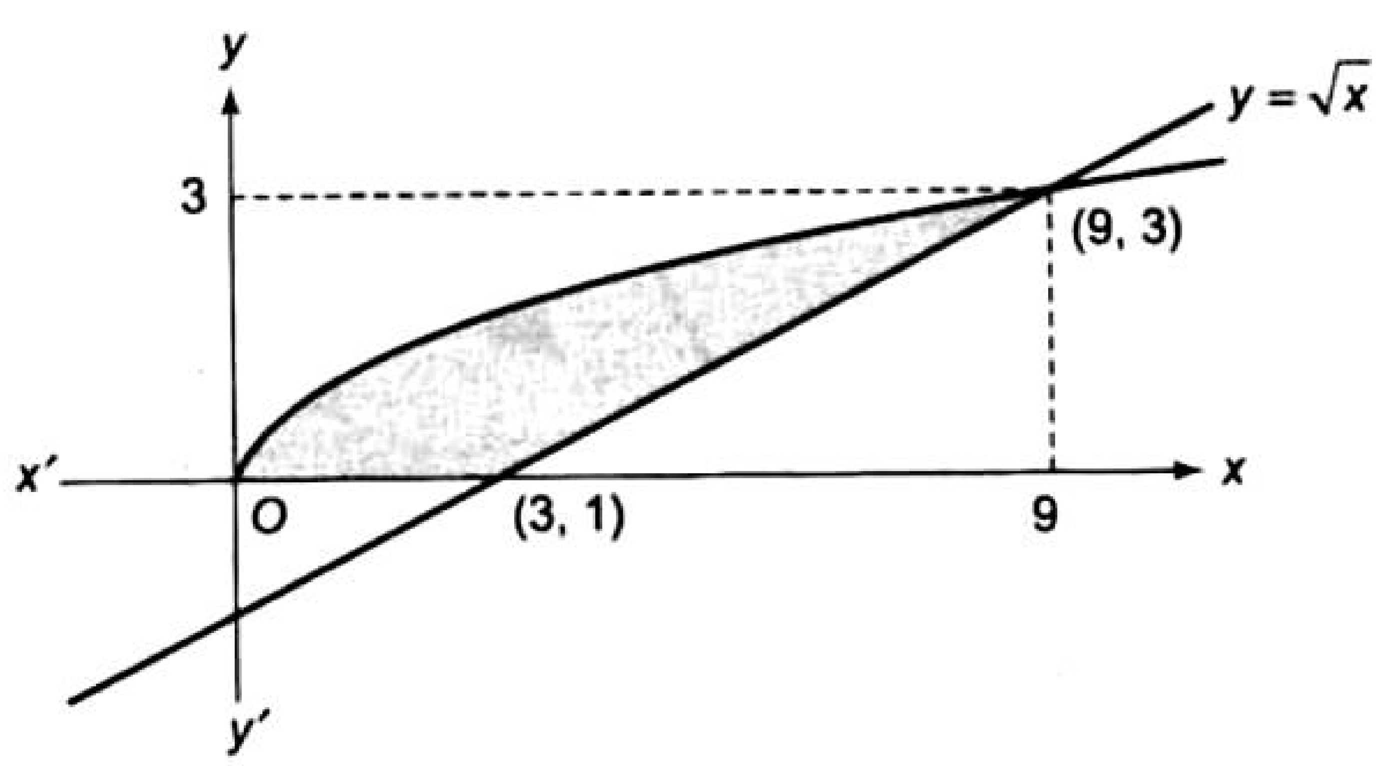


Required area = 4 × (Shaded area shown in the figure)

$= 4 \times \frac{1}{2}$

$= 2$.

3. d. To find the area between the curves $y = \sqrt{x}$ and $2y + 3 = x$ and x-axis in the 1st quadrant.



Given curves intersect when $y^2 = 2y + 3$.

$\Rightarrow y^2 - 2y - 3 = 0 \Rightarrow (y-3)(y+1) = 0 \Rightarrow y = 3, -1$

when $y = 3, x = 9$ (Ist quadrant)

$$\text{Required area} = \int_0^9 \sqrt{x}\,dx - \int_3^9 \left(\frac{x-3}{2}\right)dx$$

$$= \left|\frac{x^{3/2}}{3/2}\right|_0^9 - \frac{1}{2}\left|\frac{x^2}{2} - 3x\right|_3^9$$

$$= \frac{2}{3}(27) - \frac{1}{2}\left[\left(\frac{81}{2} - 27\right) - \left(\frac{9}{2} - 9\right)\right]$$

$= 9$ sq. units.

4. a. We have $ay^2 = x$ and $ax^2 = y$

Clearly curves intersect at (0, 0) and $(1/a, 1/a)$

$$\therefore \quad \text{Area} = \int_0^{1/a} \left(\sqrt{x/a} - ax^2\right) dx$$

$$= \left(\frac{2x^{3/2}}{3\sqrt{a}} - \frac{ax^3}{3}\right)_0^{1/a}$$

$$= \frac{2}{3a^2} - \frac{a}{3a^3}$$

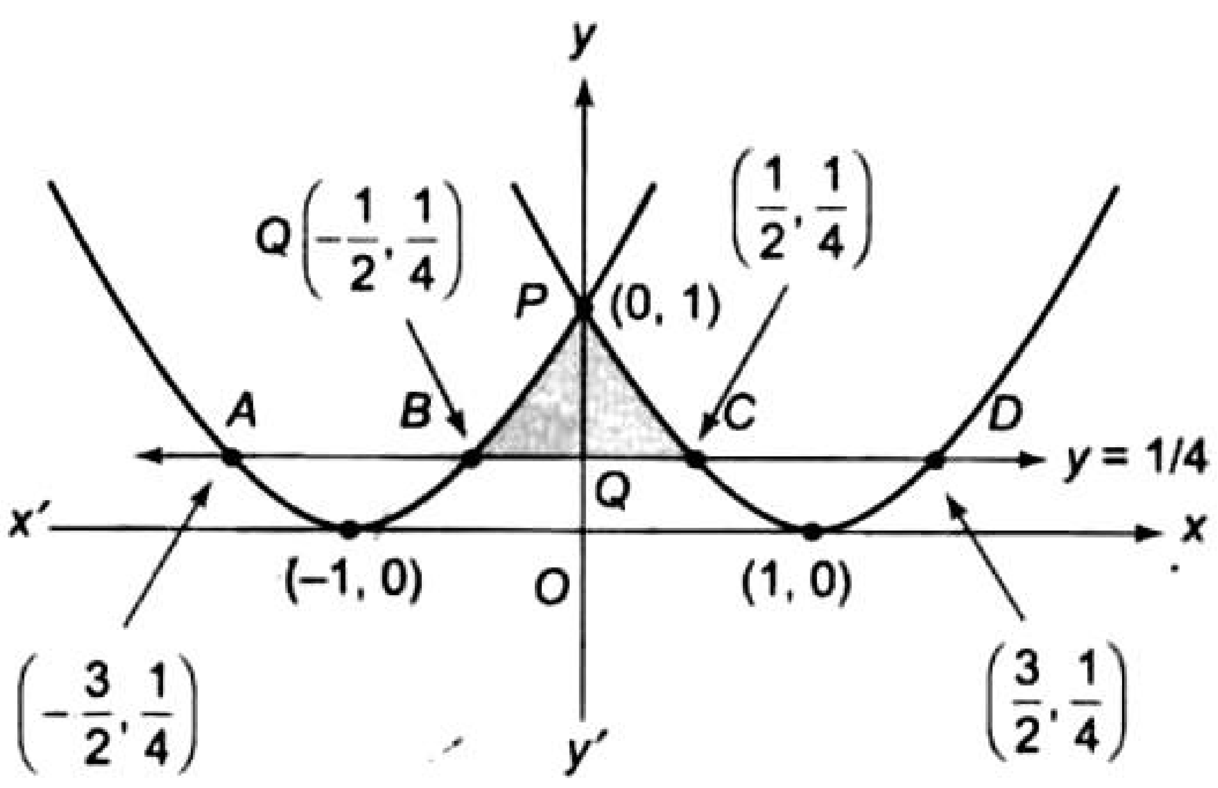
$$= \frac{1}{3a^2}$$

Given $\frac{1}{3a^2} = 1$

$$\therefore \quad a = \pm\frac{1}{\sqrt{3}}$$

5. d. The given curves are $y = (x + 1)^2$ and $y = (x - 1)^2$ and $y = 1/4$

The graphs are as shown in the following figure.



The required area is the shaded portion,
given by Ar $(BPCQB)$ = 2Ar $(PQCP)$ (by symmetry)

$$= 2\left[\int_0^{1/2}\left((x-1)^2 - \frac{1}{4}\right)dx\right] = 2\left[\left(\frac{(x-1)^3}{3} - \frac{x}{4}\right)_0^{1/2}\right]$$

$$= 2\left[\left(-\frac{1}{24} - \frac{1}{8}\right) - \left(-\frac{1}{3}\right)\right]$$

$$= \frac{1}{3} \text{ sq. units.}$$

6. b. Both curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ lie in 1st quadrant.

Also $\sqrt{\frac{1+\sin x}{\cos x}} > \sqrt{\frac{1-\sin x}{\cos x}}$

Required area, $I = \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$= \int_0^{\pi/4}\frac{\left(1+\tan\frac{x}{2}\right) - \left(1-\tan\frac{x}{2}\right)}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

$$= \int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

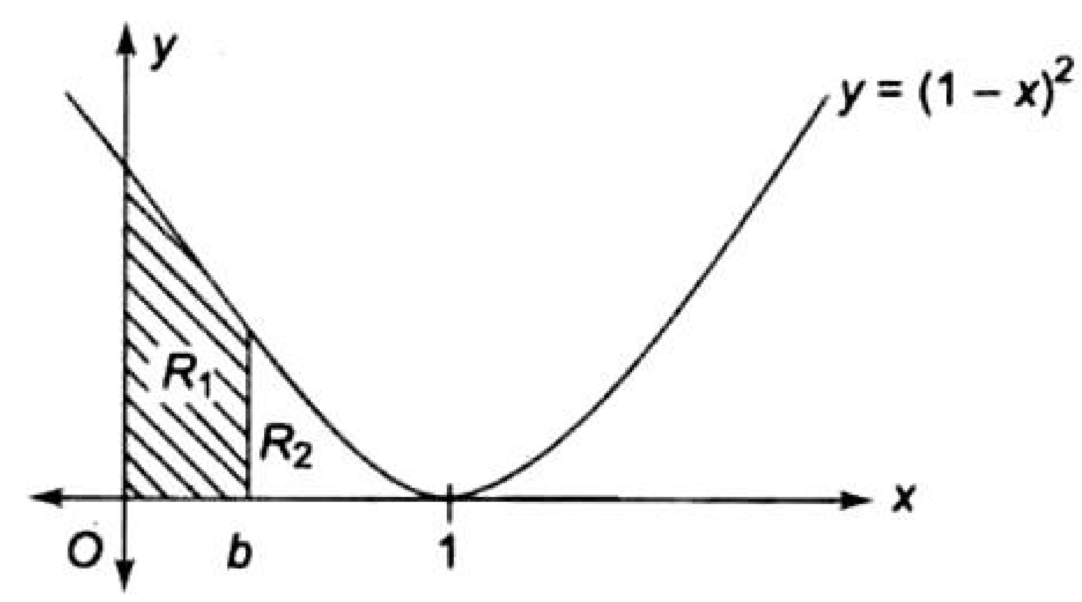
Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\therefore \quad I = \int_0^{\sqrt{2}-1}\frac{2t}{\sqrt{1-\tan^2\frac{x}{2}}}\frac{2dt}{\sec^2\frac{x}{2}}$$

$$= \int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$$

7. b. Given $R_1 - R_2 = \frac{1}{4}$

$$\therefore \quad \int_0^b (1-x)^2\,dx - \int_b^1 (1-x)^2\,dx = \frac{1}{4}$$



$$\Rightarrow \left.\frac{(x-1)^3}{3}\right|_0^b - \left.\frac{(x-1)^3}{3}\right|_b^1 = \frac{1}{4}$$

$$\Rightarrow \frac{(b-1)^3}{3} + \frac{1}{3} - \left(0 - \frac{(b-1)^3}{3}\right) = \frac{1}{4}$$

or $\frac{2(b-1)^3}{3} = -\frac{1}{12}$ or $(b-1)^3 = -\frac{1}{8}$ or $b = \frac{1}{2}$

8. b. Since $\sin x$ and $\cos x > 0$ for $x \in [0, \pi/2]$, the graph of $y = \sin x + \cos x$ always lies above the graph of $y = |\cos x - \sin x|$

Also $\cos x > \sin x$ for $x \in [0, \pi/4]$ and $\sin x > \cos x$ for $x \in [\pi/4, \pi/2]$.

$$\Rightarrow \text{Area} = \int_0^{\pi/4}((\sin x + \cos x) - (\cos x - \sin x))\,dx$$

$$+ \int_{\pi/4}^{\pi/2}((\sin x + \cos x) - (\sin x - \cos x))\,dx$$

$$= 2\int_0^{\pi/4}\sin x\,dx + 2\int_{\pi/4}^{\pi/2}\cos x\,dx$$

$$= [-2\cos x]_0^{\pi/4} + [2\sin x]_{\pi/4}^{\pi/2}$$

$$= [-\sqrt{2} + 2] + [2 - \sqrt{2}]$$

$$= 4 - 2\sqrt{2}$$

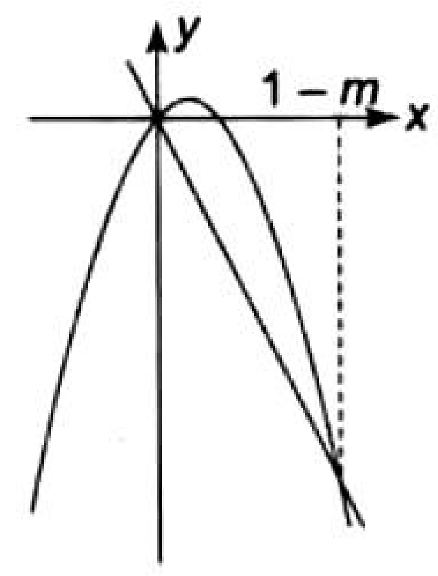
Multiple Correct Answers Type

1. b., d.

The two curves meet at $mx = x - x^2$ or $x^2 = x(1 - m)$

$\therefore x = 0, 1 - m$

If $m < 1$



$$A = \int_0^{1-m} \left(x - x^2 - mx\right) dx$$

$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m} = \frac{9}{2} \text{ if } m < 1$$

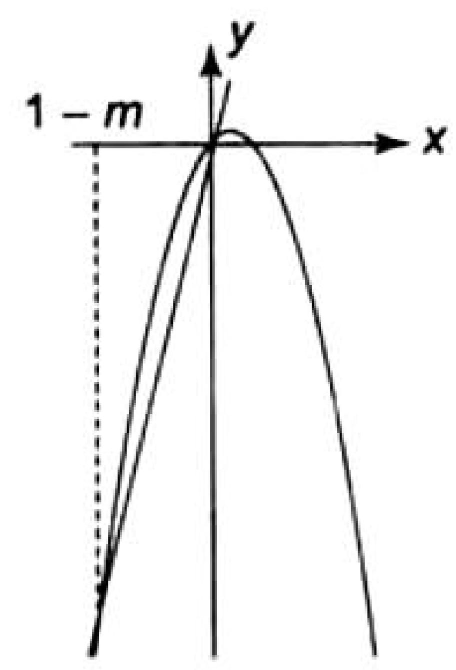
$$\Rightarrow (1-m)^3 \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{9}{2}$$

or $(1 - m)^3 = 27$

or $m = -2$.

But if $m > 1$ and $1 - m$ is – ve, then

$$\therefore A = \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0} = \frac{9}{2}$$



or $-(1-m)^3 \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{9}{2}$

or $-(1 - m)^3 = -27$.

or $m = 4$.

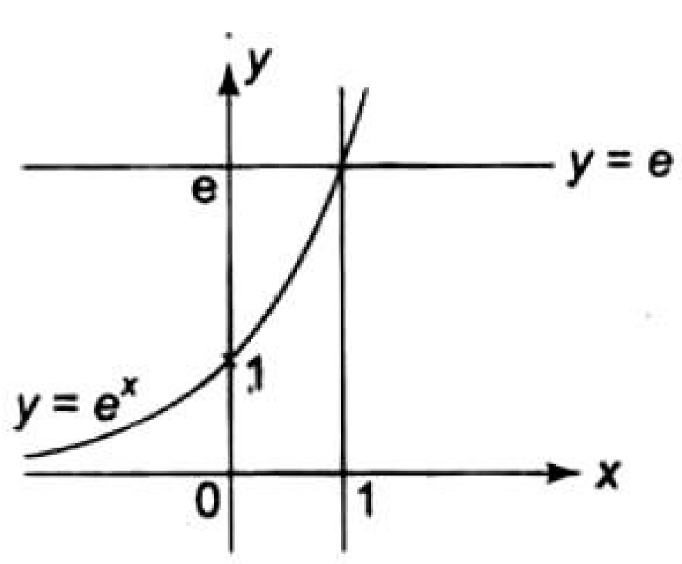
2. b., c., d.

Required area $= \int_1^e \ln y\,dy$

$$= (y \ln y - y)_1^e = (e - e) - \{-1\} = 1$$

Also, $\int_1^e \ln y\,dy = \int_1^e \ln(e + 1 - y)\,dy$

Further, required area $= e \times 1 - \int_0^1 e^x\,dx$



Linked Comprehension Type

1. a. $y^3 - 3y + x = 0$

Differentiating the given equation, we get

$3y^2 y' - 3y' + 1 = 0$

$$\Rightarrow \quad f'(x) = \frac{1}{3(1 - (f(x))^2)} \qquad (1)$$

$\therefore$ Required area

$$= \int_a^b f(x)\,dx$$

$$= [xf(x)]_a^b - \int_a^b xf'(x)\,dx$$

$$= bf(b) - af(a) + \int_a^b \frac{x}{3[(f(x)^2 - 1)]}dx \quad \text{(from (1))}$$

2. d. We have $y' = \dfrac{1}{3(1 - (f(x)^2))}$, which is even.

Hence $\int_{-1}^{1} g'(x)\,dx = g(1) - g(-1) = 2g(1)$

Matching Column Type

1. (ii) – (d)

The points of intersection of $-4y^2 = x$ and $x - 1 = -5y^2$ are $(-4, -1)$ and $(-4, 1)$.

Hence, required area

$$= 2\left|\left[\int_0^1 (1 - 5y^2)dy - \int_0^1 -4y^2 dy\right]\right| = \frac{4}{3}$$

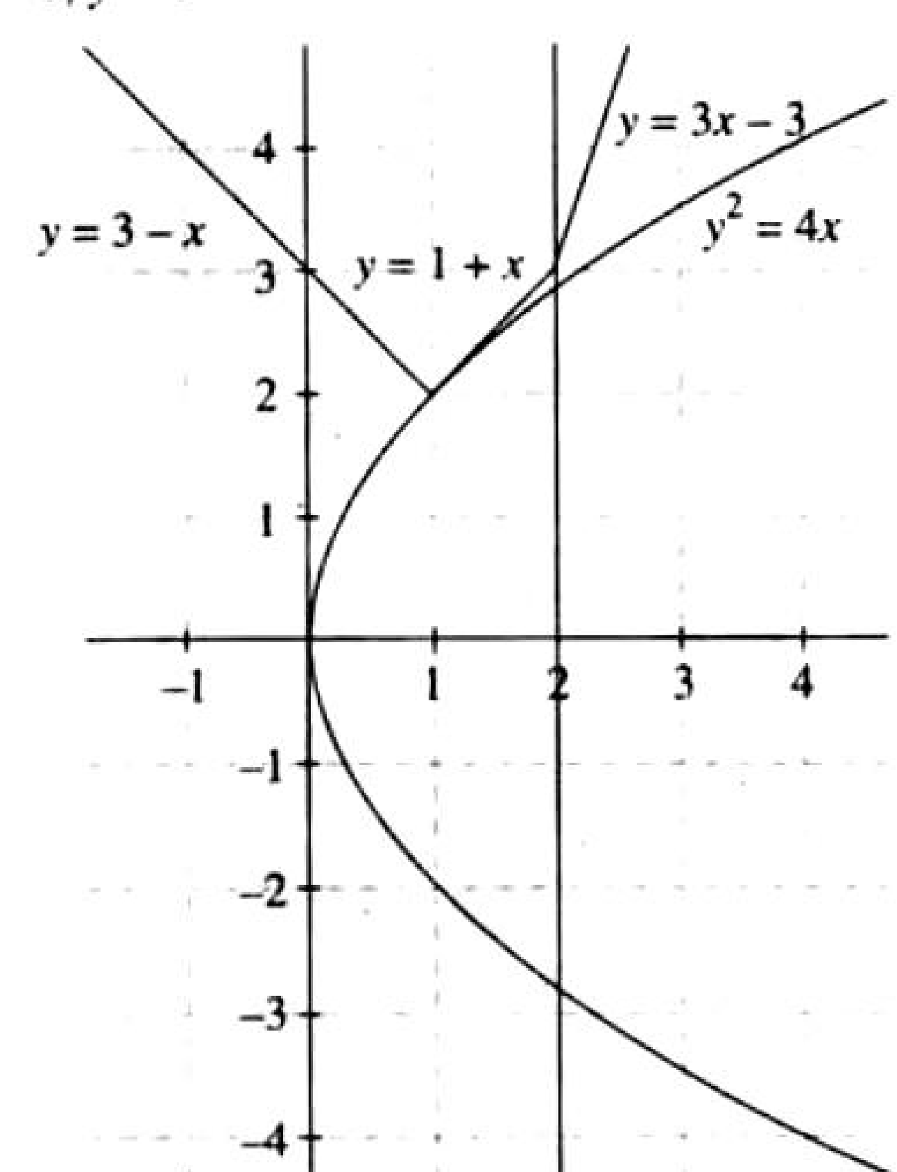
Note: Solutions of the remaining parts are given in their respective chapters.

2.(d) – (s), (t)

For $\alpha = 1$

$$y = |x - 1| + |x - 2| + x = \begin{cases} 3 - x; & x < 1 \\ 1 + x; & 1 \le x < 2 \\ 3x - 3; & x \ge 2 \end{cases}$$

For $\alpha = 0$, $y = 3$

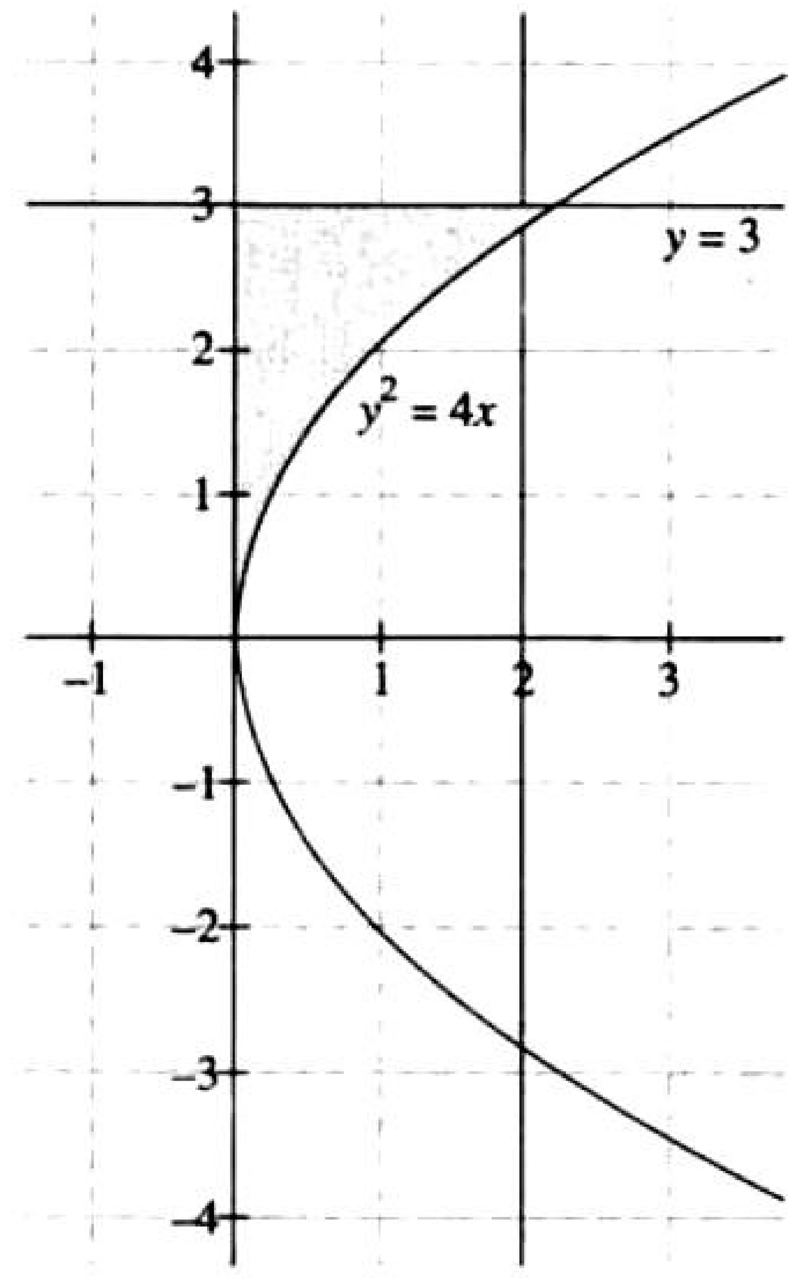


$$A = \frac{1}{2}(2+3) \times 1 + \frac{1}{2}(2+3) \times 1 - \int_0^2 2\sqrt{x}\,dx$$

$$\Rightarrow \quad A = 5 - \frac{8}{3}\sqrt{2}$$

$$\therefore \quad F(1) + \frac{8}{3}\sqrt{2} = 5$$

For $\alpha = 0$, $y = |-1| + |-2| = 3$



$$A = 6 - \int_0^2 2\sqrt{x}\,dx$$

$$\Rightarrow \quad A = 6 - \frac{8}{3}\sqrt{2}$$

$$\therefore \quad F(0) + \frac{8}{3}\sqrt{2} = 6$$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (6) For $P(x, y)$, we have

$$2 \le d_1(P) + d_2(P) \le 4$$

$$\Rightarrow \quad 2 \le \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \le 4$$

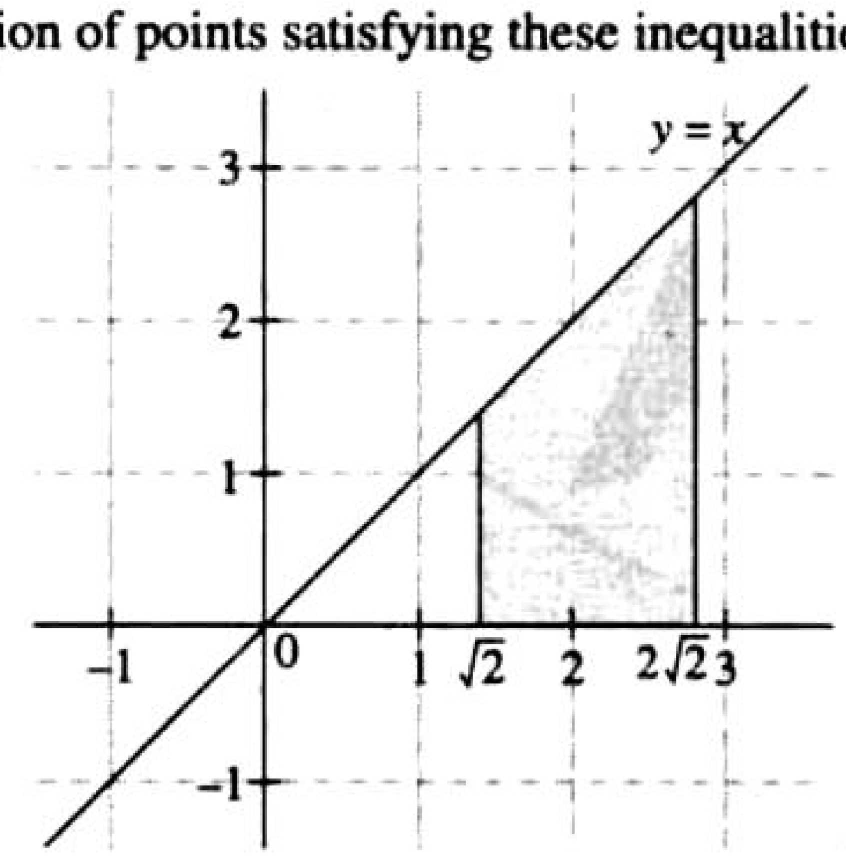
$$\Rightarrow \quad 2\sqrt{2} \le |x-y| + |x+y| \le 4\sqrt{2}$$

In first quadrant if $x > y$, we have

$$2\sqrt{2} \le x - y + x + y \le 4\sqrt{2}$$

or $\quad \sqrt{2} \le x \le 2\sqrt{2}$

The region of points satisfying these inequalities is

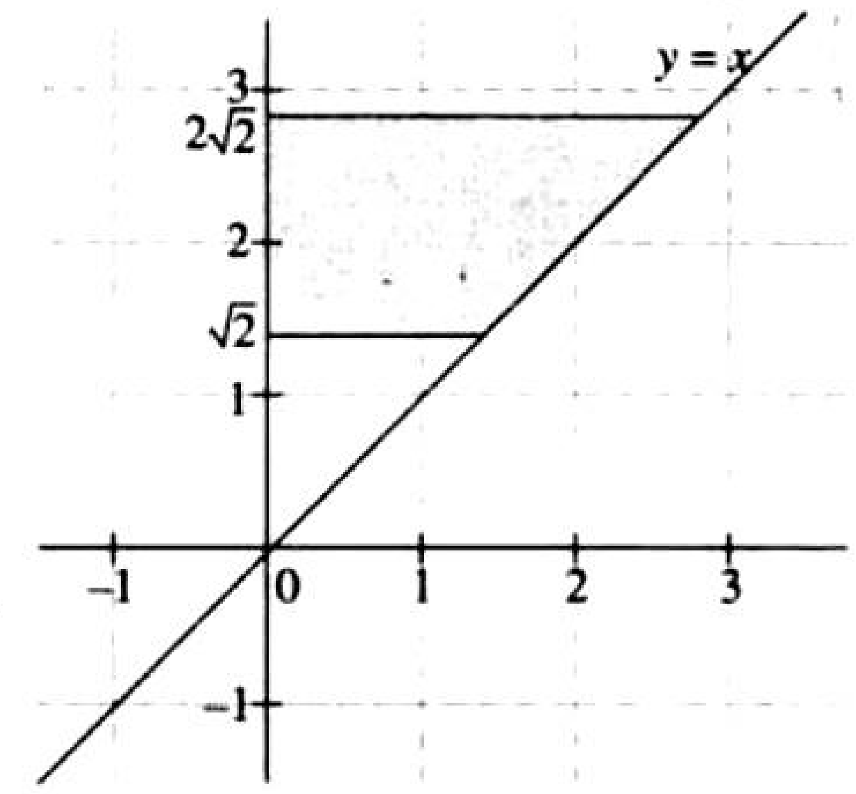


In first quadrant if $x < y$, we have

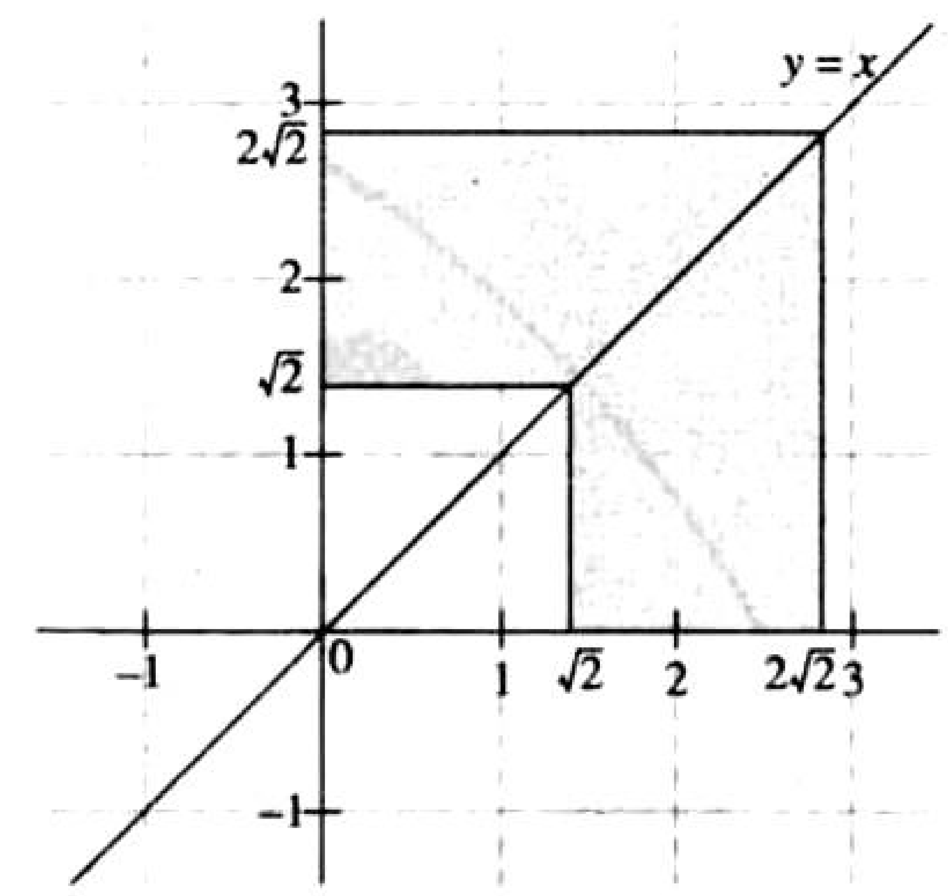
$$2\sqrt{2} \le y - x + x + y \le 4\sqrt{2}$$

or $\quad \sqrt{2} \le y \le 2\sqrt{2}$

The region of points satisfying these inequalities is



Combining above two regions we have



Area of the shaded region $= ((2\sqrt{2})^2 - (\sqrt{2})^2)$

$= 8 - 2 = 6$ sq. units

Subjective Type

1. Given curves $x^2 = 4y$ and $x = 4y - 2$ intersect, when

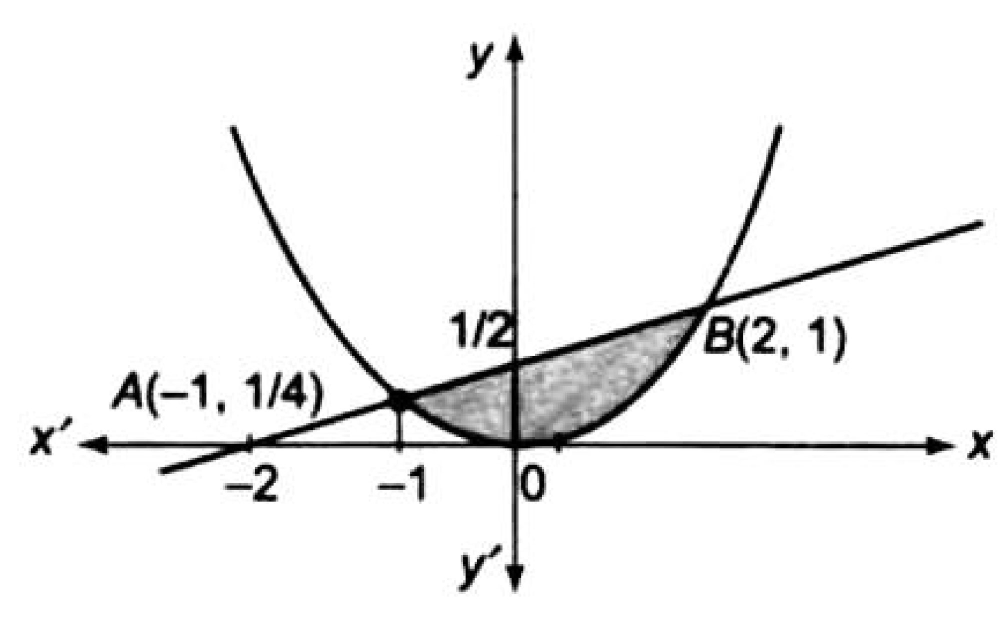
$x^2 = x + 2$

or $x^2 - x - 2 = 0$

or $x = 2, -1$

$\Rightarrow y = 1, 1/4$

Hence, points of intersection are $A(-1, 1/4)$, $B(2, 1)$.



Required area = Area of shaded region

$$= \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$$

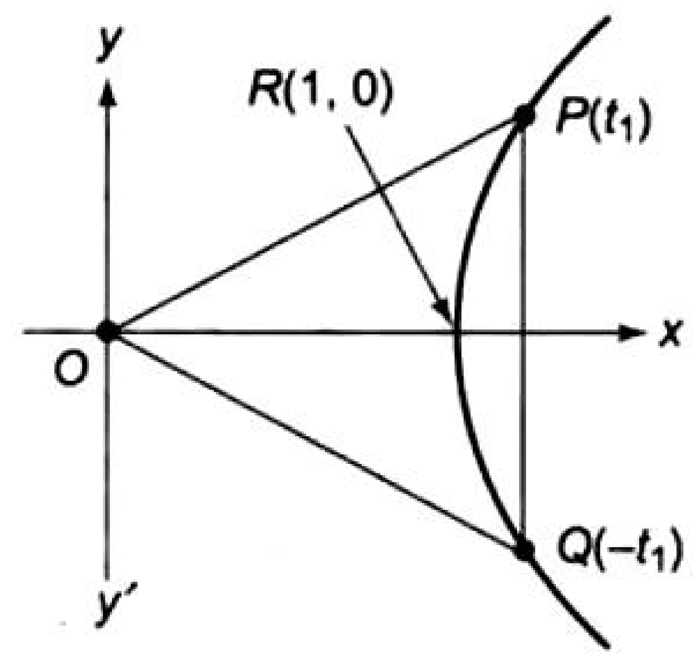
$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{10}{3} - \left(\frac{-7}{6}\right)\right]$$

$$= \frac{1}{4}\left[\frac{27}{6}\right] = 9/8 \text{ sq. units.}$$

2.



$$x = \frac{e^t + e^{-t}}{2};\ y = \frac{e^t - e^{-t}}{2}$$

It is a point on hyperbola $x^2 - y^2 = 1$.

Then, the equation of line joining $P(t_1)$ and $Q(-t_1)$, or

$$\left(\frac{e^{t_1} + e^{-t_1}}{2}, \frac{e^{t_1} - e^{-t_1}}{2}\right) \text{ and } \left(\frac{e^{-t_1} + e^{t_1}}{2}, \frac{e^{-t_1} - e^{t_1}}{2}\right)$$

is $x = \dfrac{e^{t_1} + e^{-t_1}}{2}$.

$$\therefore \text{ Area } (PQRP) = 2\int_1^{\frac{e^{t_1} + e^{-t_1}}{2}} y\,dx$$

$$= 2\int_1^{\frac{e^{t_1} + e^{-t_1}}{2}} \sqrt{x^2 - 1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2 - 1} - \frac{1}{2}\log|x + \sqrt{x^2 - 1}|\right]_1^{\frac{e^{t_1} + e^{-t_1}}{2}}$$

$$= \left(\frac{e^{t_1} + e^{-t_1}}{2}\right)\left(\frac{e^{t_1} - e^{-t_1}}{2}\right) - \log\left|\frac{e^{t_1} + e^{-t_1}}{2} + \frac{e^{t_1} - e^{-t_1}}{2}\right|$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \log e^{t_1}$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - t_1 \quad (1)$$

$$\text{Area of } \Delta OPQ = 2 \times \frac{1}{2}\left(\frac{e^{t_1} + e^{-t_1}}{2}\right)\left(\frac{e^{t_1} - e^{-t_1}}{2}\right)$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} \quad (2)$$

$\therefore$ Required area = Ar ΔOPQ – Ar$(PQRP)$

$= t_1$ (Using (1) and (2))

3. Given $y = 1 + \dfrac{8}{x^2}$.

Here, y is always positive; hence, curve is lying above the x-axis.

$$\therefore \text{ Required area} = \int_2^4 y\,dx = \int_2^4\left(1 + \frac{8}{x^2}\right)dx$$

$$= \left[x - \frac{8}{x}\right]_2^4 = 4.$$

If $x = a$ bisects the area, then we have

$$\int_2^a\left(1 + \frac{8}{x^2}\right)dx = \left[x - \frac{8}{x}\right]_2^a = \left[a - \frac{8}{a} - 2 + 4\right] = \frac{4}{2}.$$

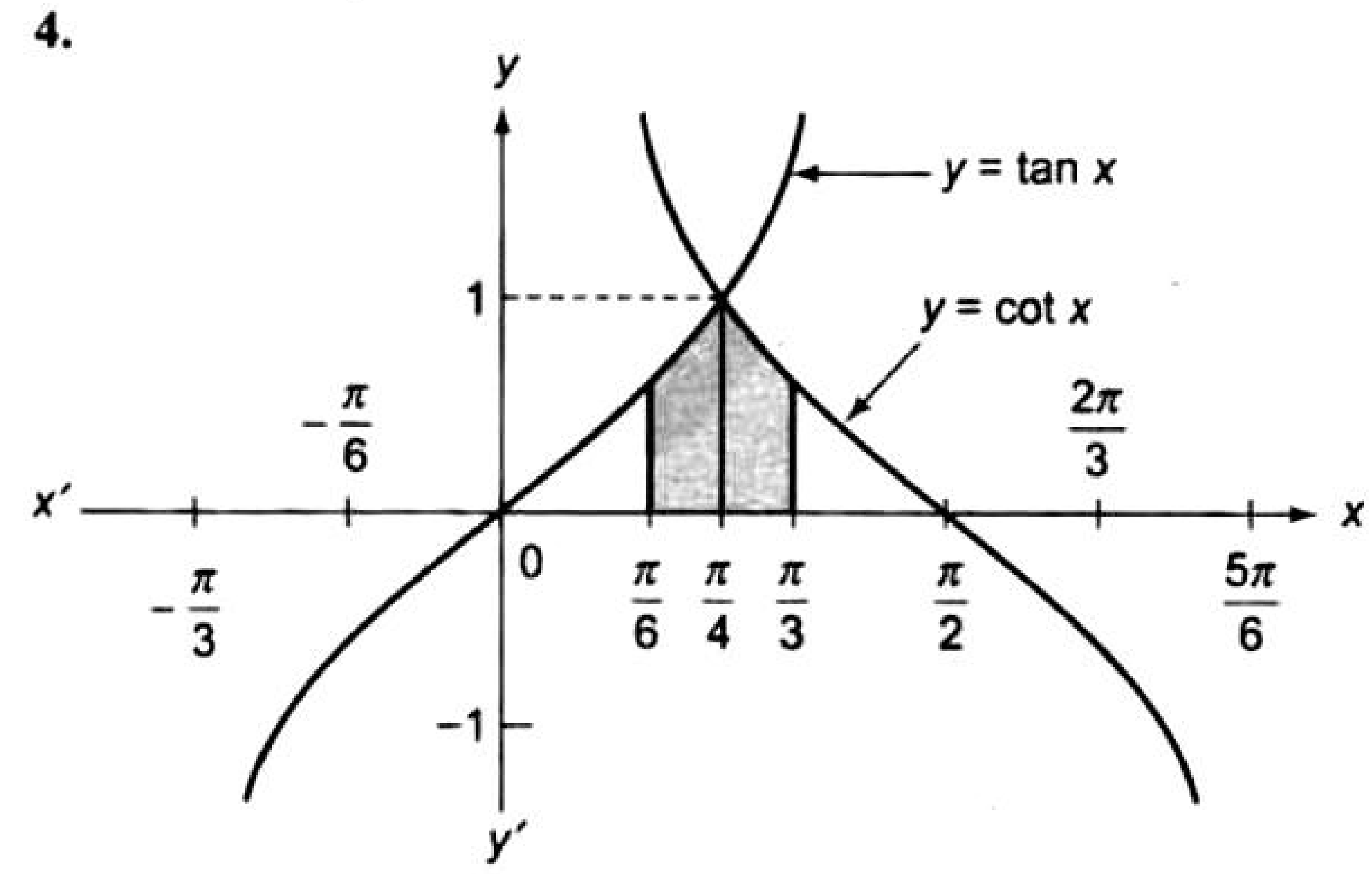
or $a - \dfrac{8}{a} = 0$

or $a^2 = 8$

or $a = \pm 2\sqrt{2}$

Since $a > 2$, $a = 2\sqrt{2}$.

4.



The two curves are

$y = \tan x$, where $-\pi/3 \le x \le \pi/3$ (1)

$y = \cot x$, where $\pi/6 \le x \le 3\pi/2$ (2)

At the point of intersection of the two curves,

$\tan x = \cot x$ or $\tan^2 x = 1$ or $\tan x = \pm 1$, $x = \pm\pi/4$

Thus, the curves intersect at $x = \pi/4$

The required area is the shaded area. Therefore,

$$A = \int_{\pi/6}^{\pi/4}\tan x\,dx + \int_{\pi/4}^{\pi/3}\cot x\,dx$$

$$= [\log\ \sec x]_{\pi/6}^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/3}$$

$$= \left(\log\sqrt{2} - \log\frac{2}{\sqrt{3}}\right) + \left(\log\frac{\sqrt{3}}{2} - \log\frac{1}{\sqrt{2}}\right)$$

$$= \log\sqrt{2} + \log\frac{\sqrt{3}}{2} + \log\frac{\sqrt{3}}{2} + \log\sqrt{2}$$

$$= 2\left(\log\sqrt{2}\,\frac{\sqrt{3}}{2}\right)$$

$$= 2\log\sqrt{\frac{3}{2}} = \log 3/2 \text{ sq. units.}$$

5. The given curves are

$y = \sqrt{5 - x^2}$ (1)

$y = |x - 1|$ (2)

We can clearly see that on squaring the both sides of (1), equation (1) represents a circle.

But as y is + ve square root, (1) represents the upper-half of the circle with centre (0, 0) and radius $\sqrt{5}$.

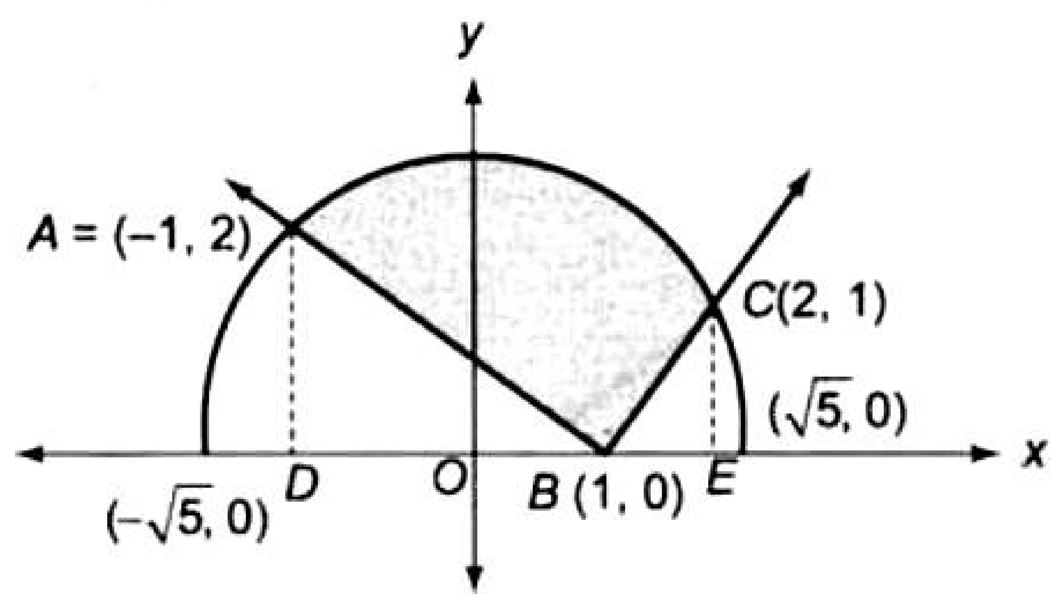
Equation (2) represents the curve

$$y=\begin{cases}-x+1 \text{ if } x<1\\ x-1 \quad \text{if } x\geq 1\end{cases}$$

Graphs of these curves are as shown in the following figure with point of

intersection of $y=\sqrt{5-x^2}$ and $y=-x+1$ as $A(-1, 2)$

and of $y=\sqrt{5-x^2}$ and $y=x-1$ as $C(2, 1)$.



$\therefore$ Required area = Area of shaded region

$$=\int_{-1}^{2}\sqrt{5-x^2}\,dx-\int_{-1}^{2}|x-1|\,dx$$

$$=\left[\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2}$$

– Area of ΔADB – Area of ΔBEC

$$=\left(\frac{2}{2}\sqrt{5-4}+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)-\left(\frac{-1}{2}\sqrt{5-1}+\frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$$

$$-\frac{1}{2}(2)(2)-\frac{1}{2}(1)(1)$$

$$=1+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}+1+\frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$=2+\frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right]-2-\frac{1}{2}$$

$$=\frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}\frac{2}{\sqrt{5}}\right]-\frac{1}{2}=\frac{5}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}$$

$$=\frac{5\pi-2}{4}\text{ sq. units.}$$

6. The given curves are

$x^2+y^2=4$ (circle) (1)

$x^2=-\sqrt{2}y$ (parabola, concave downward) (2)

$x=y$ (straight line through origin) (3)

Solving equations (1) and (2), we get

$$y^2-\sqrt{2}y-4=0$$

$$\Rightarrow (y-2\sqrt{2})(y+\sqrt{2})=0$$

$$\Rightarrow y=2\sqrt{2}\text{ or }-\sqrt{2}$$

$\Rightarrow x^2=2$ (rejecting $y=2\sqrt{2}$ as x^2 is positive)

or $x=\pm\sqrt{2}$.

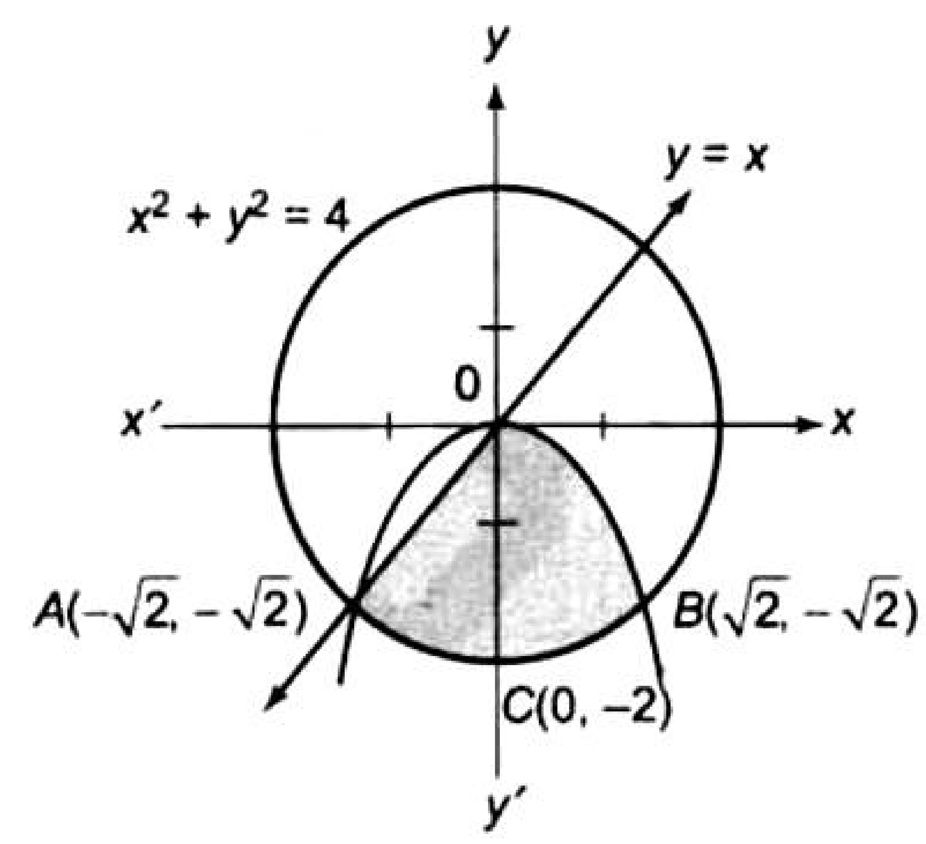
Therefore, points of intersection of (1) and (2) are $B\left(\sqrt{2}, -\sqrt{2}\right)$, $A\left(-\sqrt{2}, -\sqrt{2}\right)$.

Solving (1) and (3), we get

$2x^2=4$ or $x^2=2$ or $x=\pm\sqrt{2}$ or $y=\pm\sqrt{2}$.

Therefore, points of intersection are $(-\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, \sqrt{2})$.

Thus, all the three curves pass through the same point $A(-\sqrt{2}, -\sqrt{2})$.



Now, required area = Area of shaded region

$$=\int_{-\sqrt{2}}^{0}\left(x-\left(-\sqrt{4-x^2}\right)\right)dx+\int_{0}^{\sqrt{2}}\left(-\frac{x^2}{\sqrt{2}}-\left(-\sqrt{4-x^2}\right)\right)dx$$

$$=2\int_{0}^{\sqrt{2}}\sqrt{4-x^2}\,dx+\int_{-\sqrt{2}}^{0}x\,dx-\int_{0}^{\sqrt{2}}\frac{x^2}{\sqrt{2}}dx$$

$$=2\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{0}^{\sqrt{2}}+\left[\frac{x^2}{2}\right]_{-\sqrt{2}}^{0}-\left[\frac{x^3}{3\sqrt{2}}\right]_{0}^{\sqrt{2}}$$

$$=2\left[\frac{\sqrt{2}}{2}\sqrt{4-2}+2\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right]+\left[\frac{-2}{2}\right]-\left[\frac{2\sqrt{2}}{3\sqrt{2}}\right]$$

$$=2\left[1+2\frac{\pi}{4}\right]-1-\frac{2}{3}=\pi+\frac{1}{3}\text{ sq. units.}$$

7. Given curves are

$x^2+y^2=25$ (1)

$4y=|4-x^2|$ (2)

$x=0$ (3)

Solving (1) and (2), we get

$4y+4+y^2=25$

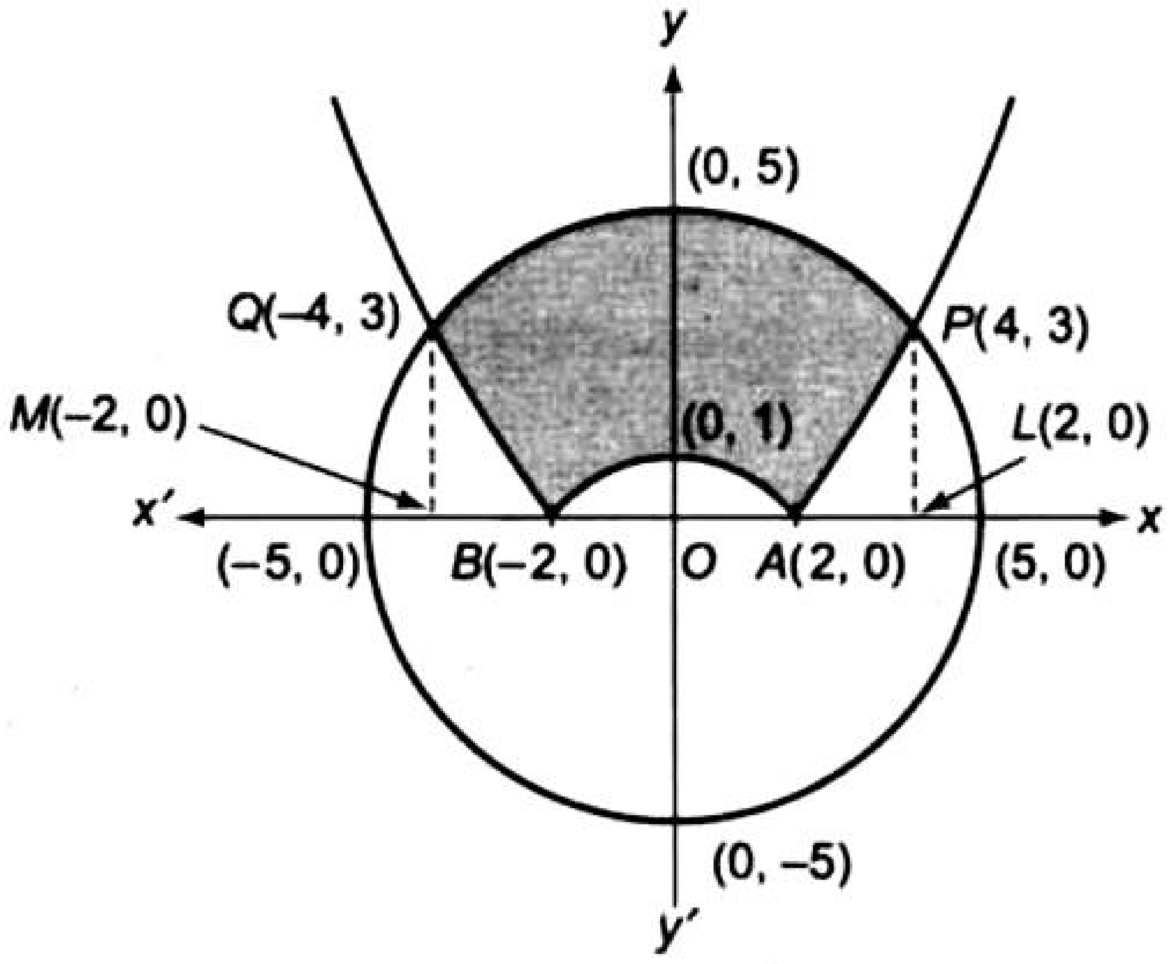
or $(y+2)^2=5^2$

or $y=3, -7$

$y=-7$ is rejected, $y=3$ gives the points above x-axis.

When $y=3$, $x=\pm 4$.

Hence, the points of intersection are $P(4, 3)$ and $Q(-4, 3)$.



Required area

$$= 2\left[\int_0^4 \sqrt{25-x^2}\,dx - \frac{1}{4}\int_0^2 \left(4-x^2\right)dx - \frac{1}{4}\int_2^4 \left(x^2-4\right)dx\right]$$

$$= 2\left[\left(\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right)_0^4 - \frac{1}{4}\left(4x - \frac{x^3}{3}\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3} - 4x\right)_2^4\right]$$

$$= 2\left[6 + \frac{25}{2}\sin^{-1}\frac{4}{5}\right] - \frac{1}{2}\left[8 - \frac{8}{3}\right] - \frac{1}{2}\left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right]$$

$= 4 + 25\sin^{-1}\dfrac{4}{5}$ sq. units.

8. The given curve is $y = \tan x$ (1)

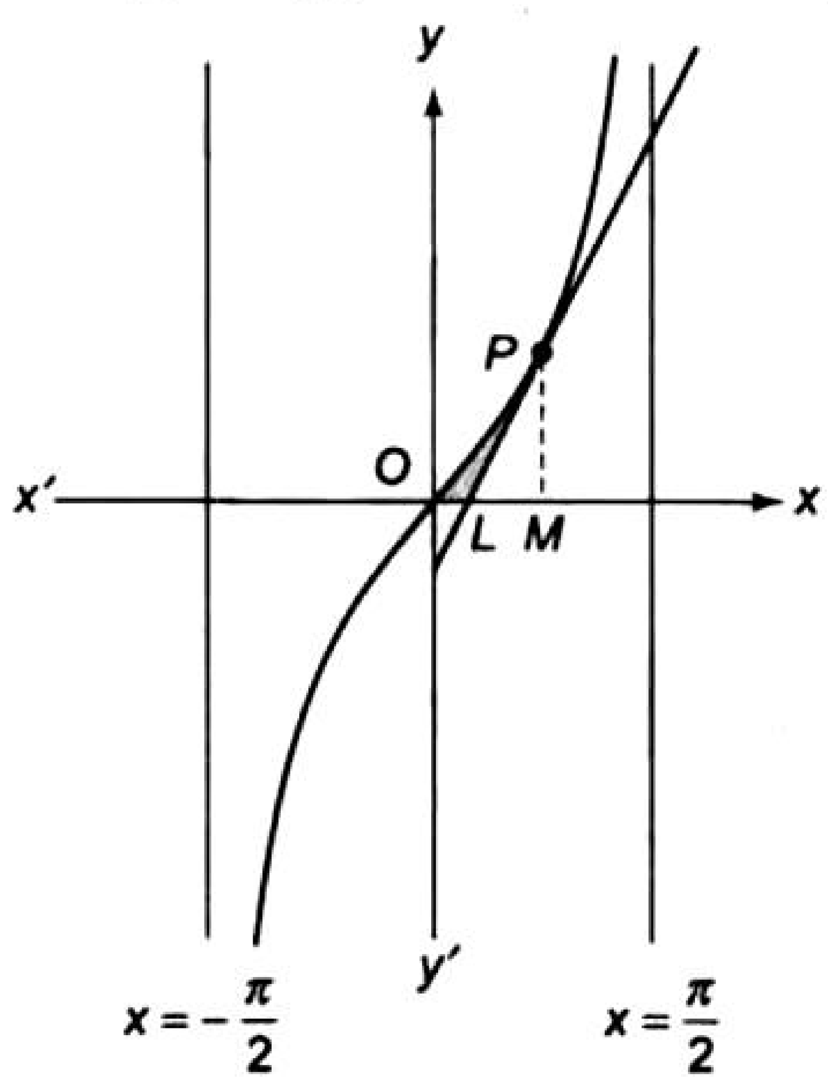
When $x = \pi/4$, $y = 1$

i.e., co-ordinates of P are $(\pi/4, 1)$

$\therefore$ equation of tangent at P is $y - 1 = \left(\sec^2\dfrac{\pi}{4}\right)(x - \pi/4)$

or $y = 2x + 1 - \pi/2$ (2)

The graphs of (1) and (2) are as shown in the figure.



Tangent (2) meets x-axis at $L\left(\dfrac{\pi-2}{4}, 0\right)$

Now, required area = Area of shaded region

= Area $OPMO$ – Ar(ΔPLM)

$$= \int_0^{\pi/4} \tan x\,dx - \frac{1}{2}(OM - OL)PM$$

$$= \left[\log \sec x\right]_0^{\pi/4} - \frac{1}{2}\left\{\frac{\pi}{4} - \frac{\pi-2}{4}\right\} 1$$

$$= \frac{1}{2}\left[\log 2 - \frac{1}{2}\right] \text{sq. units.}$$

9. The given curves are

$y = ex\log_e x$ (1)

$y = \dfrac{\log x}{ex}$ (2)

The two curves intersect where $ex\log x = \dfrac{\log x}{ex}$

$\Rightarrow \left(ex - \dfrac{1}{ex}\right)\log x = 0$

$\Rightarrow x = \dfrac{1}{e}$ or $x = 1$

At $x = 1/e$, $y = -1$ (from (1))

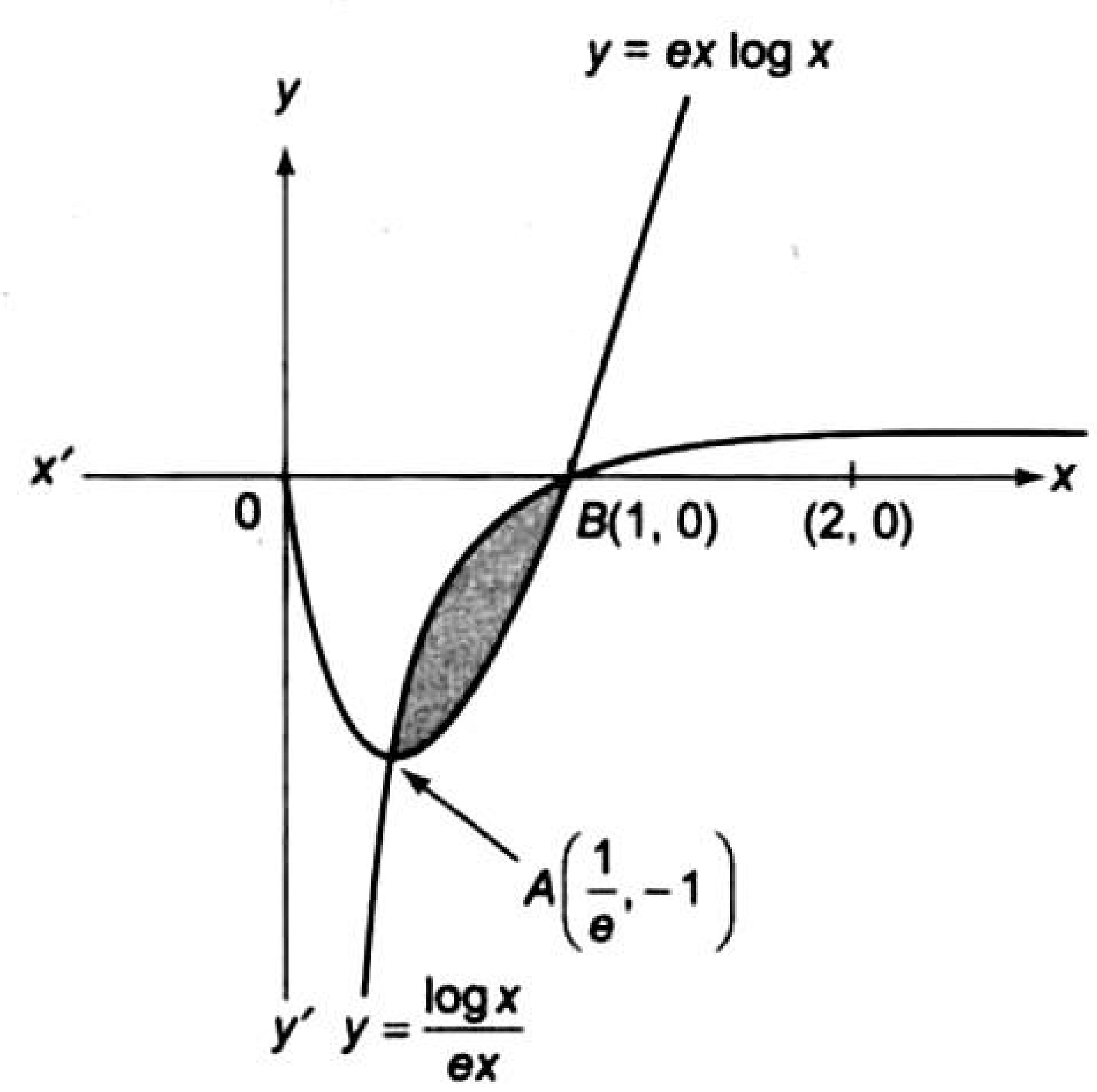
At $x = 1$, $y = 0$ (from (1))

So, points of intersection are $\left(\dfrac{1}{e}, -1\right)$ and $(1, 0)$.

Curve Tracing:

Curve 1	Curve 2
For $0 < x < 1, y < 0$	For $0 < x < 1, y < 0$
For $x > 1, y > 0$	For $x > 1, y > 0$
When $x \to 0, y \to 0$	When $x \to 0, y \to -\infty$
When $x \to \infty, y \to \infty$	When $x \to \infty, y \to 0$
$\dfrac{dy}{dx} = e(\log x + 1)$	$\dfrac{dy}{dx} = \dfrac{(1 - \log x)}{ex^2}$
$x = \dfrac{1}{e}$ is a point of min.	$x = e$ is a point of max.

From the above information, the rough sketch of two curves is as shown in the figure and the shaded area is the required area.



$\therefore$ Required area = Area of shaded region

$$= \left| \int_{1/e}^{1} \left[ex \log x - \frac{\log x}{ex} \right] dx \right|$$

$$= \left| e\left[\frac{x^2}{2}\log x - \frac{x^2}{4} \right]_{1/e}^{1} - \frac{1}{e}\left[\frac{(\log x)^2}{2} \right]_{1/e}^{1} \right|$$

$$= \left| e\left[\left(-\frac{1}{4}\right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2}\right) \right] - \frac{1}{e}\left[0 - \frac{1}{2} \right] \right|$$

$$= \left| e\left[-\frac{1}{4} + \frac{3}{4e^2} \right] + \frac{1}{2e} \right|$$

$$= \left| \frac{5 - e^2}{4e} \right|$$

$$= \frac{e^2 - 5}{4e} \text{ sq. units.}$$

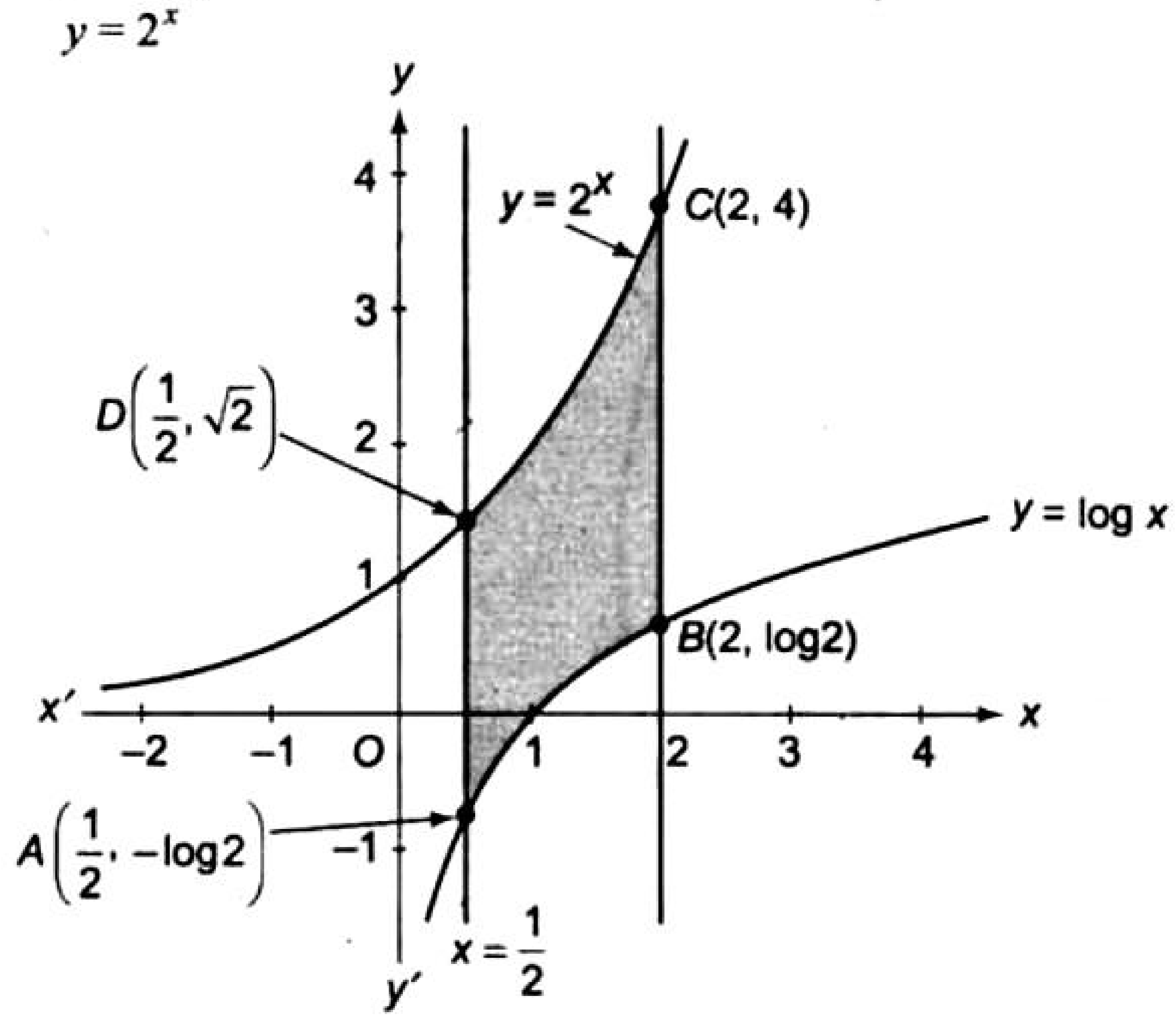
10. The given curves are

$x = \frac{1}{2}$ (1)

$x = 2$ (2)

$y = \log_e x$ (3)

$y = 2^x$ (4)



Required area = $ABCDA$

$$= \int_{1/2}^{2} \left(2^x - \log x\right) dx$$

$$= \left[\frac{2^x}{\log 2} - (x \log x - x) \right]_{1/2}^{2}$$

$$= \left(\frac{4}{\log 2} - 2\log 2 + 2 \right) - \left(\frac{\sqrt{2}}{\log 2} - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2} \right)$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2} \right) \text{ sq. units.}$$

11. The given curves are $y = x^2$ (1)

and $y = \frac{2}{1 + x^2}$ (2)

Solving (1) and (2), we have

$$x^2 = \frac{2}{1 + x^2}$$

or $x^4 + x^2 - 2 = 0$

or $(x^2 - 1)(x^2 + 2) = 0$

or $x = \pm 1$

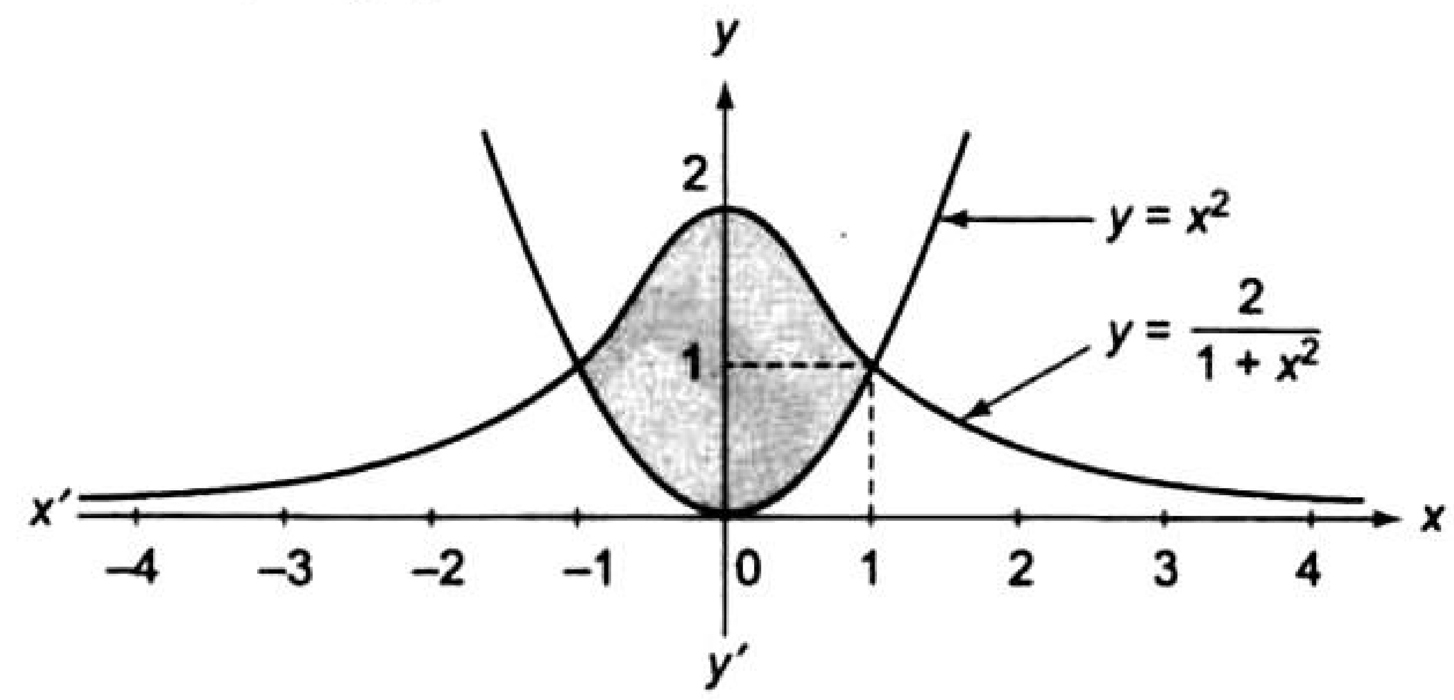
Also, $y = \frac{2}{1 + x^2}$ is an even function.

Hence, its graph is symmetrical about y-axis.

At $x = 0, y = 2$, by increasing the values of x, y decreases and when $x \to \infty, y \to 0$.

Therefore, $y = 0$ is an asymptote of the given curve.

Thus, the graphs of the two curves are as follows:



$$\text{Required area} = 2\int_{0}^{1} \left(\frac{2}{1 + x^2} - x^2 \right) dx$$

$$= \left(4\tan^{-1} x - \frac{2x^3}{3} \right)_0^1$$

$$= \pi - \frac{2}{3} \text{ sq.units.}$$

12. Both the given curves are parabola.

$y = 4x - x^2$ (1)

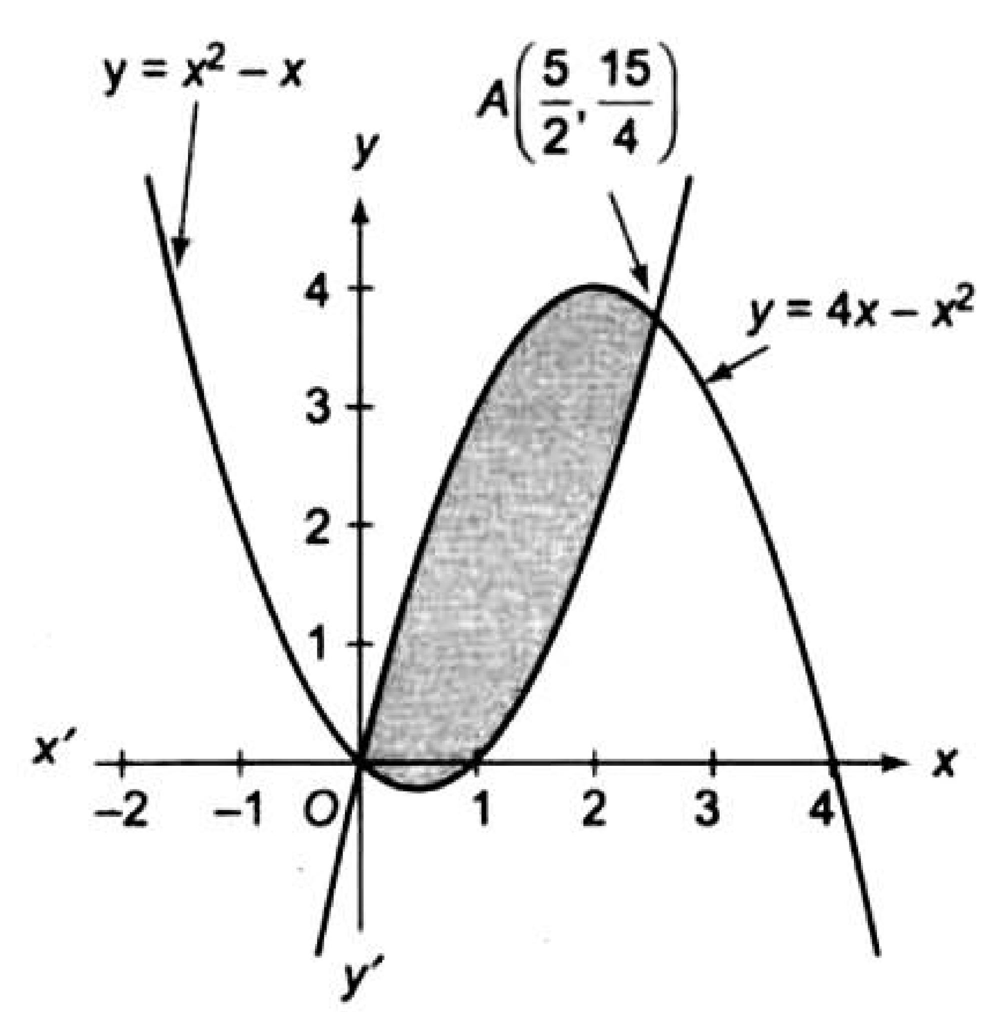
and $y = x^2 - x$ (2)

Solving (1) and (2), we get

$4x - x^2 = x^2 - x$

or $x = 0, x = \frac{5}{2}$.

Thus, two curves intersect at $O(0, 0)$ and $A\left(\frac{5}{2}, \frac{15}{4}\right)$.



Area below x-axis,

$$A_1 = \int_0^1 \left(x - x^2\right) dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units}$$

Area above x-axis,

$$A_2 = \int_0^{5/2} (4x - x^2)dx - \int_1^{5/2} (x^2 - x)dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^{5/2}$$

$$= \left(\frac{25}{2} - \frac{125}{24}\right) - \left[\left(\frac{125}{24} - \frac{25}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

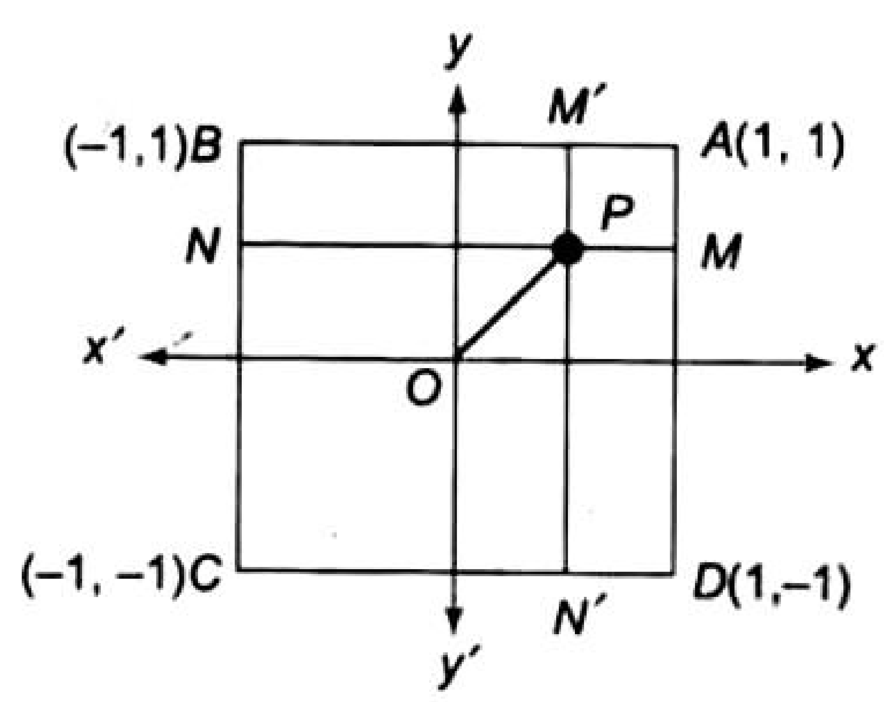
$$= \frac{25}{2} - \frac{125}{12} + \frac{25}{8} - \frac{1}{6}$$

$$= \frac{300 - 250 + 75 - 4}{24} = \frac{121}{24} \text{ sq. units.}$$

Thus, the ratio of area above x-axis to area below x-axis is

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4.$$

13.



Let us consider any point (x, y) inside the square such that its distance from origin $\leq$ its distance from any of the edges, say AD. Therefore,

$OP < PM$

$$\Rightarrow \sqrt{\left(x^2 + y^2\right)} < 1 - x \text{ or } y^2 \leq -2\left(x - \frac{1}{2}\right) \qquad (1)$$

Above represents all points within the parabola P_1.

If we consider the edge BC, then $OP < PN$ will imply

$$y^2 < 2\left(x + \frac{1}{2}\right) \qquad (2)$$

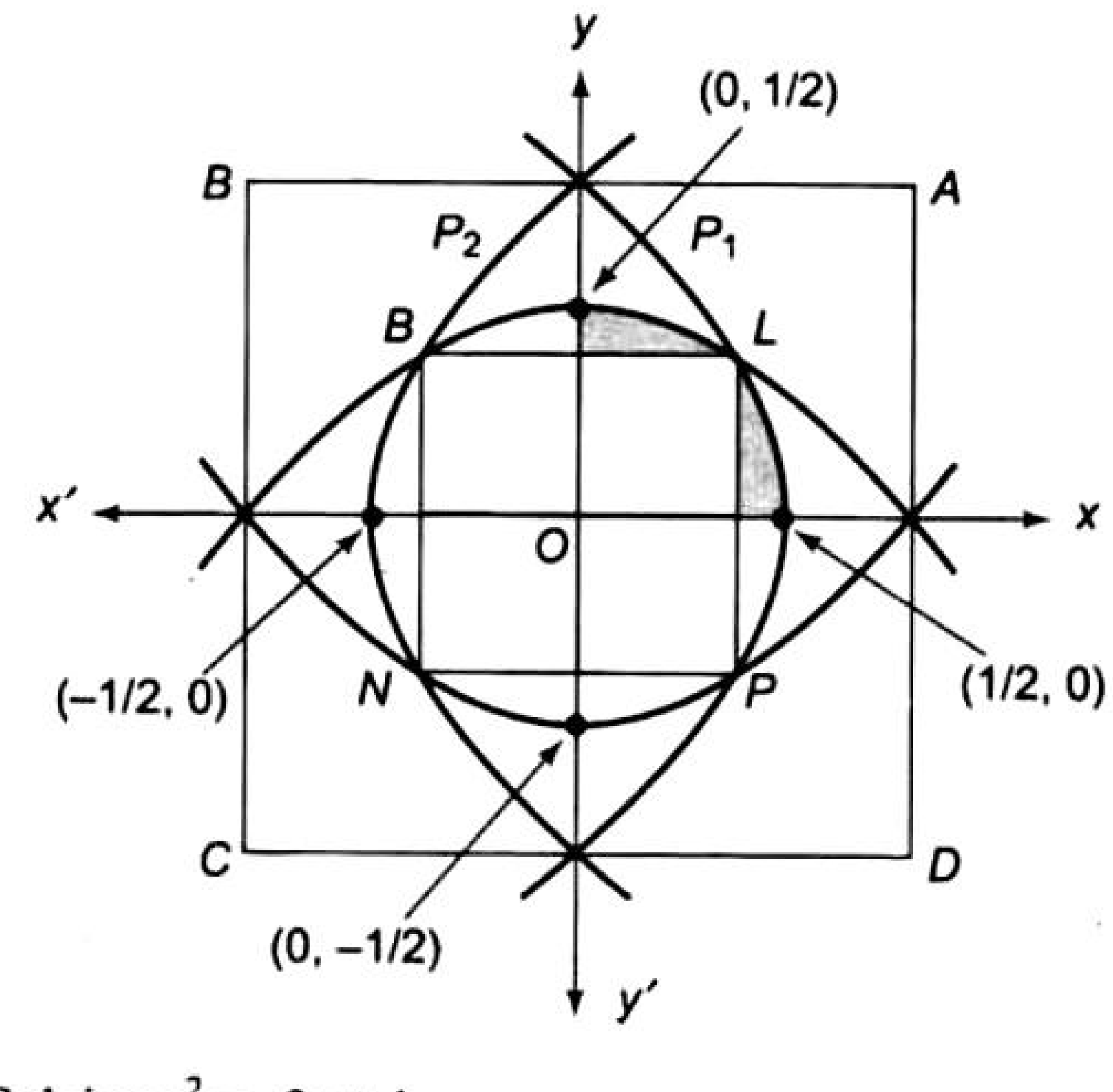
Similarly, if we consider the edges AB and CD, we will have

$$x^2 < -2\left(y - \frac{1}{2}\right) \qquad (3)$$

$$x^2 < 2\left(y + \frac{1}{2}\right) \qquad (4)$$

Hence, S consists of the region bounded by four parabolas meeting the axes at $\left(\pm\frac{1}{2}, 0\right)$ and $\left(0, \pm\frac{1}{2}\right)$.

The point L is intersection of P_1 and P_3 given by (1) and (3):



Solving $y^2 = -2x + 1$

and $x^2 = -2y + 1$

$y^2 - x^2 = -2(x - y) = 2(y - x)$

or $y - x = 0$

or $y = x$

$\Rightarrow x^2 + 2x - 1 = 0$

or $(x + 1)^2 = 2$

or $x = \sqrt{2} - 1$ as x is +ve

$\therefore L$ is $\left(\sqrt{2} - 1, \sqrt{2} - 1\right)$.

$$\therefore \text{ Total area} = 4\left[\text{Area of square of side } \left(\sqrt{2} - 1\right) + 2\int_{\sqrt{2}-1}^{1/2} \sqrt{1 - 2x}\,dx\right]$$

$$= 4\left\{\left(\sqrt{2} - 1\right)^2 + 2\int_{\sqrt{2}-1}^{1/2} \sqrt{(1 - 2x)}\,dx\right\}$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\left\{(1 - 2x)^{3/2}\right\}_{\sqrt{2}-1}^{1/2}\right]$$

$$= 4\left[3 - 2\sqrt{2} - \frac{2}{3}\left\{0 - \left(1 - 2\sqrt{2} + 2\right)^{3/2}\right\}\right]$$

$$= 4\left[3 - 2\sqrt{2} + \frac{2}{3}\left(3 - 2\sqrt{2}\right)^{3/2}\right]$$

$$= 4\left(3 - 2\sqrt{2}\right)\left[1 + \frac{2}{3}\sqrt{\left(3 - 2\sqrt{2}\right)}\right]$$

$$= 4\left(3 - 2\sqrt{2}\right)\left[1 + \frac{2}{3}\left(\sqrt{2} - 1\right)\right]$$

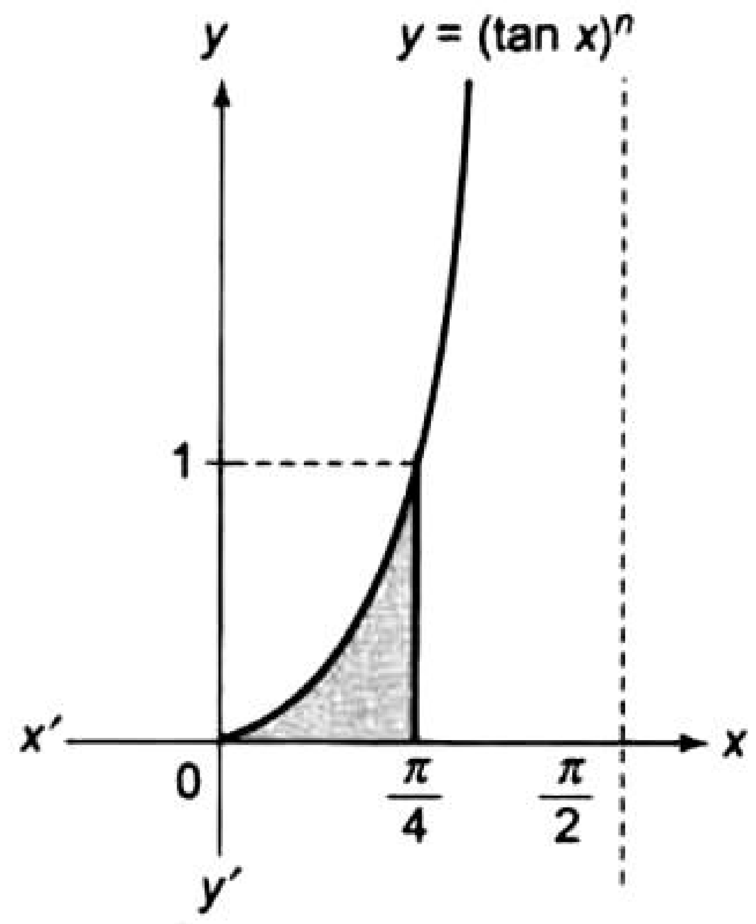
$$= \frac{4}{3}\left(3 - 2\sqrt{2}\right)\left(1 + 2\sqrt{2}\right) = \frac{4}{3}\left[\left(4\sqrt{2} - 5\right)\right]$$

$$= \frac{16\sqrt{2} - 20}{3} \text{ sq. units.}$$

14. We have $A_n = \int_0^{\pi/4} (\tan x)^n\,dx$

Since $0 < \tan x < 1$, when $0 < x < \pi/4$, we have

$0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$



$$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1}\, dx < \int_0^{\pi/4} (\tan x)^n\, dx$$

$$\Rightarrow A_{n+1} < A_n$$

Now, for $n > 2$,

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}]\, dx$$

$$= \int_0^{\pi/4} (\tan x)^n (1 + \tan^2 x)\, dx$$

$$= \int_0^{\pi/4} (\tan x)^n (\sec^2 x)\, dx$$

$$= \left[\frac{1}{(n+1)} (\tan x)^{n+1}\right]_0^{\pi/4}$$

$$\left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{(n+1)} (1-0)$$

Since $A_{n+2} < A_{n+1} < A_n$, we get

$A_n + A_{n+2} < 2A_n$

$$\Rightarrow \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n \qquad (1)$$

Also for $n > 2$, $A_n + A_n < A_n + A_{n-2} = \dfrac{1}{n-1}$

or $2A_n < \dfrac{1}{n-1}$

$$\text{or } A_n < \frac{1}{2n-2} \qquad (2)$$

Combining (1) and (2), we get $\dfrac{1}{2n+2} < A_n < \dfrac{1}{2n-2}$.

15. The given curves are $y = x - bx^2$ (1)

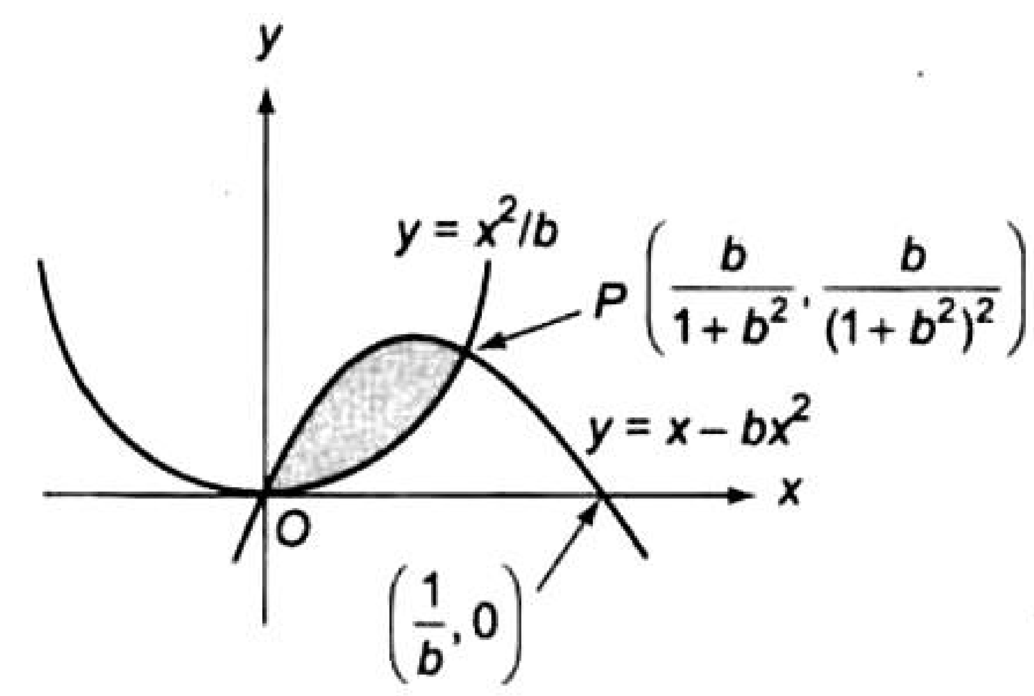
and $y = x^2/b$ (2)

$$\Rightarrow \left(y - \frac{1}{4b}\right) = -b\left(x - \frac{1}{2b}\right)^2 \text{ and } x^2 = by$$

Here, clearly the first curve is a downward parabola which meets x-axis at (0, 0) and $(1/b, 0)$, while the second is an upward parabola with vertex at (0, 0).

Solving (1) and (2), we get the intersection points of two curves as (0, 0) and $\left(\dfrac{b}{1+b^2}, \dfrac{b}{(1+b^2)^2}\right)$.

Hence, the graph of given curves is as shown here.



Shaded region represents the required area

$$\therefore A = \int_0^{\frac{b}{1+b^2}} \left(x - bx^2 - \frac{x^2}{b}\right) dx$$

$$= \left(\frac{x^2}{2} - \frac{bx^3}{3} - \frac{x^3}{3b}\right)_0^{\frac{b}{1+b^2}}$$

$$= \frac{b^2}{2(1+b^2)^2} - \frac{b^4}{3(1+b^2)^3} - \frac{b^2}{3(1+b^2)^3}$$

$$= \frac{b^4 + b^2}{6(1+b^2)^3} = \frac{b^2}{6(1+b^2)^2}$$

$$= \frac{1}{6\left(\frac{1}{b} + b\right)^2} \text{ sq. units.}$$

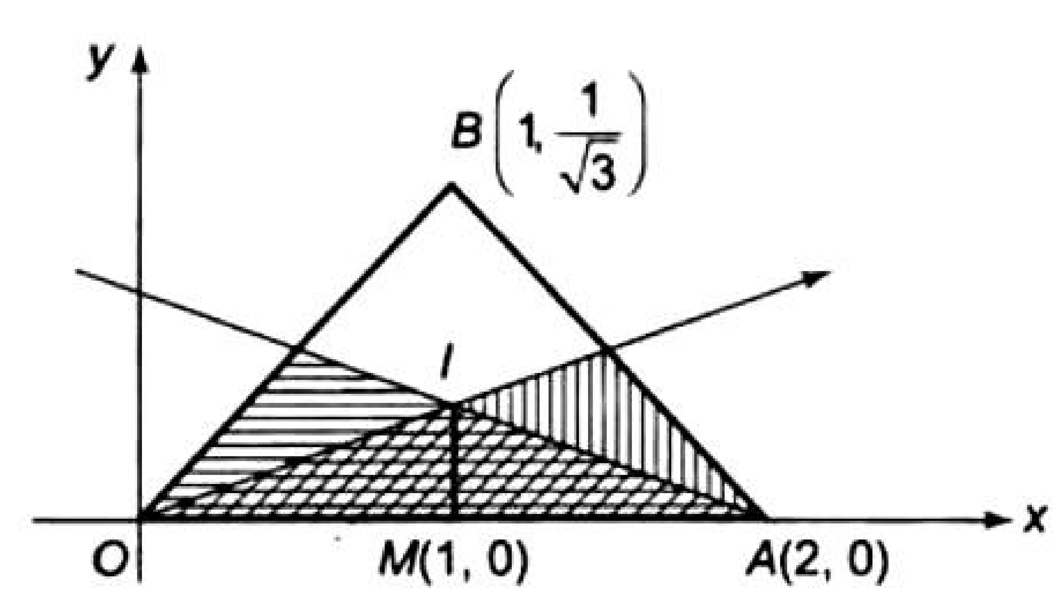
Now, $\left(\dfrac{1}{b} + b\right) \ge 2$ or $\le -2 \Rightarrow \left(\dfrac{1}{b} + b\right)^2 \ge 4$.

Hence, area is max. when $\left(\dfrac{1}{b} + b\right)^2_{\min} = 4$, for which $b = \pm 1$

but given that $b > 0$

$\therefore b = 1$.

16.



$$d(P, OA) \le \min [d(P, OB), d(P, AB)] \qquad (1)$$

$\Rightarrow d(P, OA) \le d(P, OB)$ and $d(P, OA) \le d(P, AB)$

When $d(P, OA) = d(P, OB)$, P is equidistant from OA and OB, or P lies on the angular bisector of lines OA and OB.

Hence, when $d(P, OA) \le d(P, OB)$, point P is nearer to OA than OB or lies on or below the bisector of OA and OB.

Similarly, when $d(P, OA) \le d(P, AB)$, P is nearer to OA than OB, or lies on or below the bisector of OA and AB.

From (1), point P lies in the region ΔOIA.

$\therefore$ Required area = Area of ΔOIA.

Now, $\tan\angle BOA = \dfrac{1/\sqrt{3}}{1} = \dfrac{1}{\sqrt{3}}$

or $\angle BOA = 30° \Rightarrow \angle IOA = 15°$

$\Rightarrow IM = \tan 15° = 2 - \sqrt{3}$.

Hence, Area of $\Delta OIA = \dfrac{1}{2} OA \times IM = \dfrac{1}{2} \times 2 \times \left(2 - \sqrt{3}\right)$

$= 2 - \sqrt{3}$ sq. units

17. $f(x) = \text{Maximum } \{x^2, (1-x)^2, 2x(1-x)\}$

We draw the graphs of

$y = x^2$ (1)

$y = (1-x)^2$ (2)

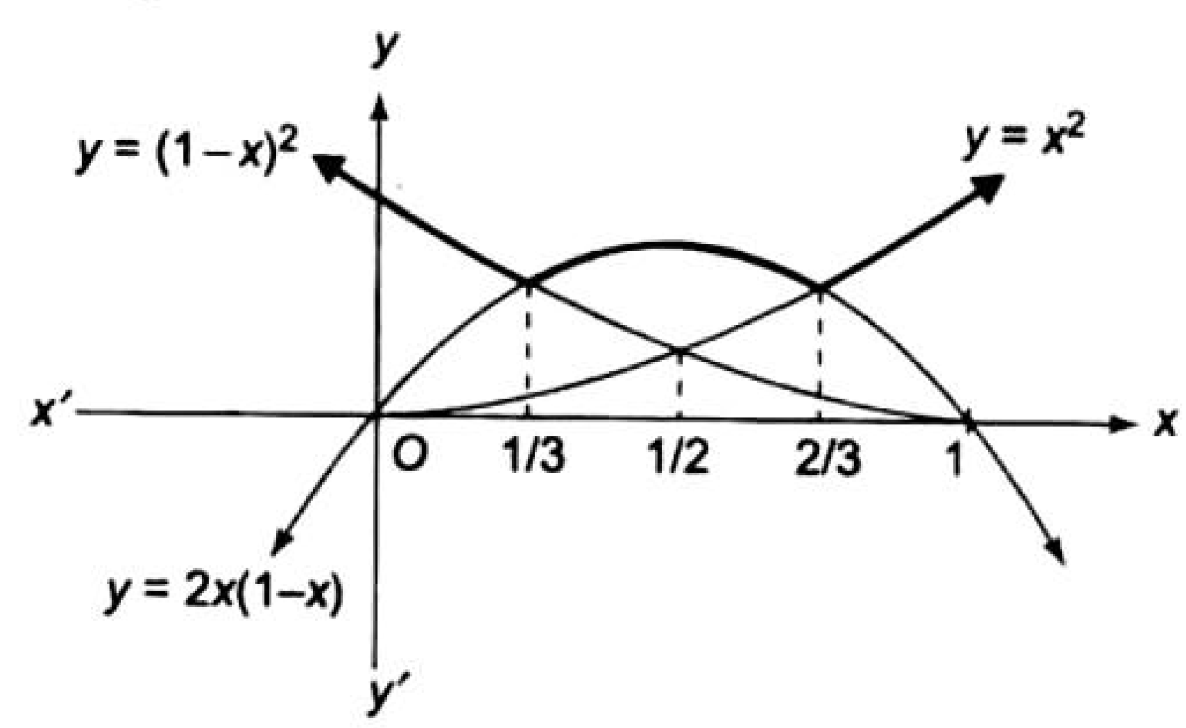
$y = 2x(1-x)$ (3)

Solving (1) and (3), we get $x^2 = 2x(1-x)$

or $3x^2 = 2x \Rightarrow x = 0$ or $x = 2/3$.

Solving (2) and (3) we get $(1-x)^2 = 2x(1-x)$

$\Rightarrow x = 1/3$ or $x = 1$,



From the figure, it is clear that

$$f(x) = \begin{cases} (1-x)^2 \text{ for } 0 \le x \le 1/3 \\ 2x(1-x) \text{ for } 1/3 \le x \le 2/3 \\ x^2 \text{ for } 2/3 < x \le 1 \end{cases}$$

The required area A is given by

$$A = \int_0^1 f(x)\,dx$$

$$= \int_0^{1/3} (1-x)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} x^2 dx$$

$$= \left[-\frac{1}{3}(1-x)^3\right]_0^{1/3} + \left[\left(x^2 - \frac{2x^3}{3}\right)\right]_{1/3}^{2/3} + \left[\frac{x^3}{3}\right]_{2/3}^{1}$$

$$= -\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3} + \left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^3$$

$= \dfrac{17}{27}$ sq. units.

18. Let P be on C_1, $y = x^2$ be (t, t^2)

$\therefore$ y co-ordinate of Q is also t^2

Now, Q on $y = 2x$, $y = t^2$

$\therefore x = t^2/2$

$\therefore Q\left(\dfrac{t^2}{2}, t^2\right)$

For point R, $x = t$ and it is on $y = f(x)$

$\therefore R(t, f(t))$

Given that,

Area OPQ = Area OPR

$$\Rightarrow \int_0^{t^2} \left(\sqrt{y} - \frac{y}{2}\right) dy = \int_0^t \left(x^2 - f(x)\right) dx$$

Diff. both sides w.r.t. t, we get

$$\left(\sqrt{t^2} - \frac{t^2}{2}\right)(2t) = t^2 - f(t)$$

$\Rightarrow f(t) = t^3 - t^2 \Rightarrow f(x) = x^3 - x^2$

19. $f(x) = \begin{cases} x^2 + ax + b;\ x < -1 \\ 2x;\ -1 \le x \le 1 \\ x^2 + ax + b;\ x > 1 \end{cases}$

$\because f(x)$ is continuous at $x = -1$ and $x = 1$

$\therefore (-1)^2 + a(-1) + b = -2 \Rightarrow b - a = -3$ (1)

and $2 = (1)^2 + a.1 + b \Rightarrow a + b = 1$ (2)

On solving, we get $a = 2$, $b = -1$

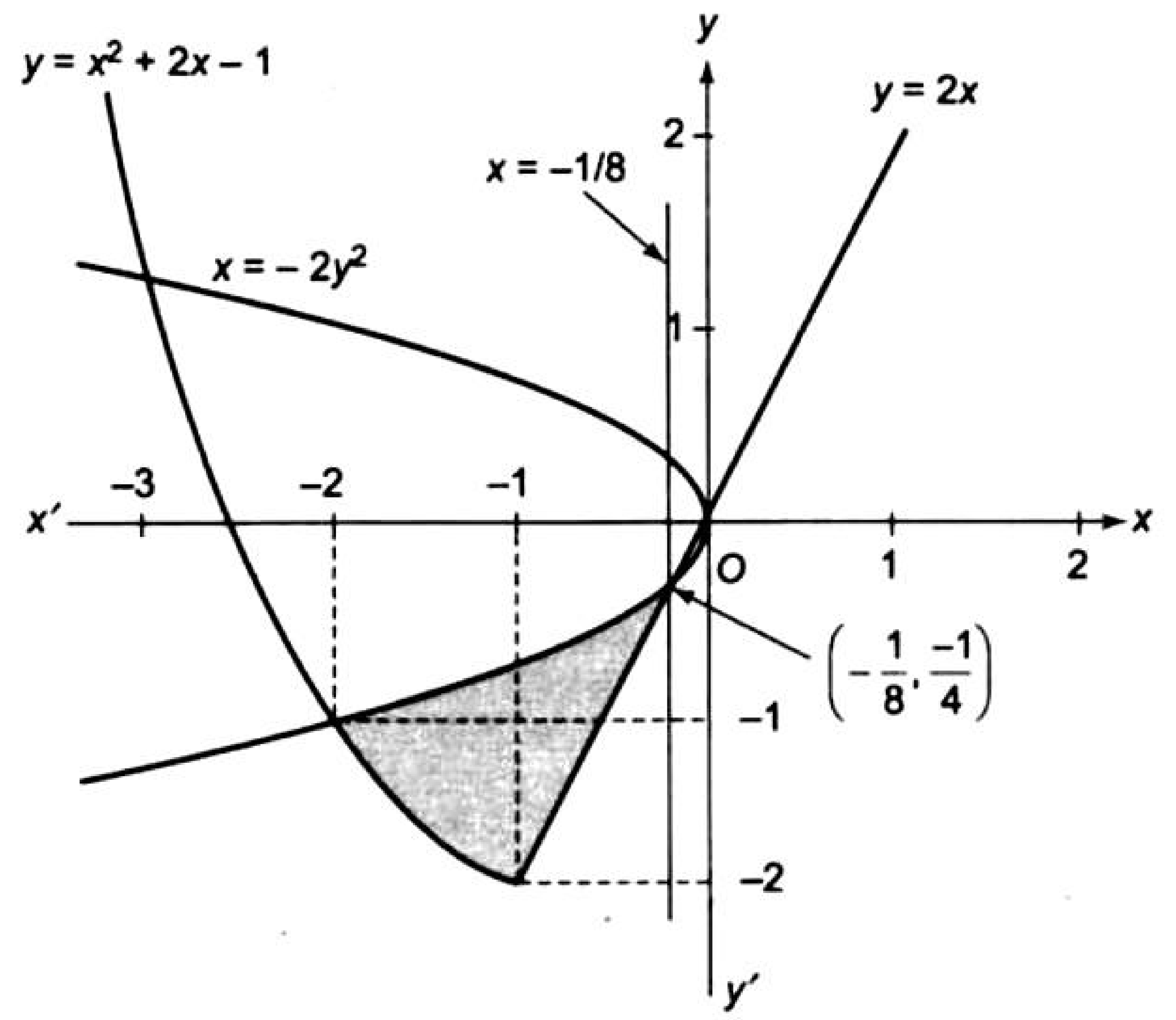
$\therefore f(x) = \begin{cases} x^2 + 2x - 1;\ x < -1 \\ 2x;\ -1 \le x \le 1 \\ x^2 + 2x - 1;\ x > 1 \end{cases}$

Given curves are $y = f(x)$, $x = -2y^2$ and $8x + 1 = 0$

Solving $x = -2y^2$, $y = x^2 + 2x - 1$ (where $x < -1$), we get $x = -2$.

Also, $y = 2x$, $x = -2y^2$ meet at (0, 0).

$y = 2x$ and $x = -1/8$ meet at $\left(-\dfrac{1}{8}, \dfrac{-1}{4}\right)$.



The required area is the shaded region in the figure.

∴ Required area

$$= \int_{-2}^{-1}\left[-\sqrt{\frac{-x}{2}} - \left(x^2 + 2x - 1\right) dx\right] + \int_{-1}^{-1/8}\left[-\sqrt{\frac{-x}{2}} - 2x\right] dx$$

$$= \left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3} - \frac{x^3}{3} - x^2 + x\right]_{-2}^{-1} + \left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3} - x^2\right]_{-1}^{-1/8}$$

$$= \left(\frac{\sqrt{2}}{3} + \frac{1}{3} - 1 - 1\right) - \left(\frac{4}{3} + \frac{8}{3} - 4 - 2\right)$$

$$+ \left(\frac{\sqrt{2}}{3} \times \frac{1}{16\sqrt{2}} - \frac{1}{64}\right) - \left(\frac{\sqrt{2}}{3} - 1\right)$$

$$= \left(\frac{\sqrt{2} - 5}{3}\right) - \left(\frac{4 + 8 - 18}{3}\right) + \left(\frac{4 - 3}{192}\right) - \left(\frac{\sqrt{2} - 3}{3}\right)$$

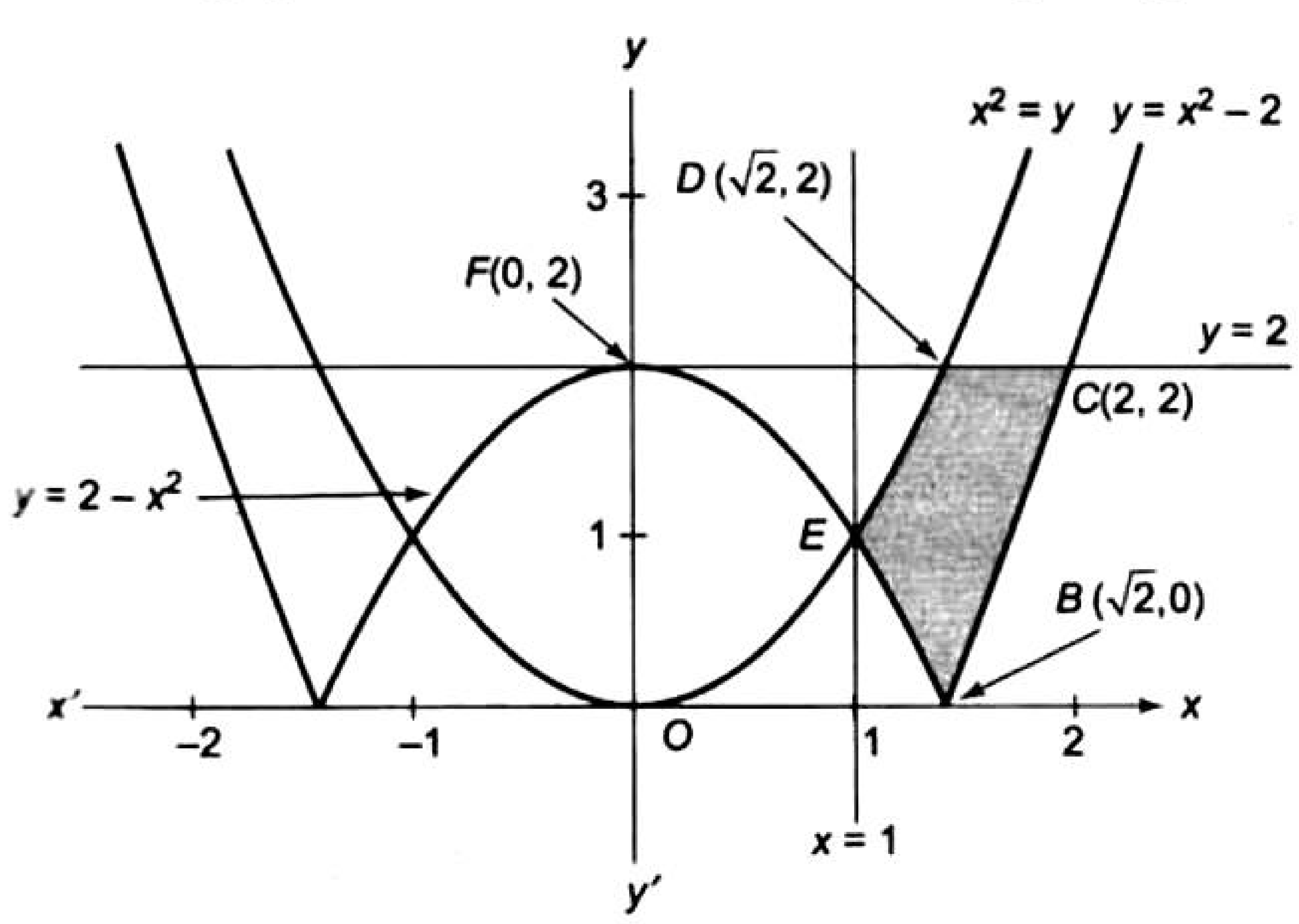
$$= \frac{257}{192} \text{ sq. units.}$$

20. The given curves are

$y = x^2$ (1)

$y = |2 - x^2|$ (2)

The graphs of these curves are as shown in the given figure.



∴ Required area = $BCDEB$

$$= \int_1^{\sqrt{2}}\left[x^2 - \left(2 - x^2\right)\right]dx + \int_{\sqrt{2}}^{2}[2 - (x^2 - 2)]\,dx$$

$$= \int_1^{\sqrt{2}}\left(2x^2 - 2\right)dx + \int_{\sqrt{2}}^{2}\left(4 - x^2\right)dx$$

$$= \left[\frac{2x^3}{3} - 2x\right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2\right) + \left(8 - \frac{8}{3} - 4\sqrt{2} + \frac{2\sqrt{2}}{3}\right)$$

$$= \left(\frac{20}{3} - 4\sqrt{2}\right) \text{ sq. units.}$$

21. The given curves are

$x^2 = y$ (1)

$x^2 = -y$ (2)

$y^2 = 4x - 3$ (3)

Clearly (1) and (2) meet at (0, 0).

Solving (1) and (3), we get $x^4 - 4x + 3 = 0$

or $(x - 1)(x^3 + x^2 + x - 3) = 0$

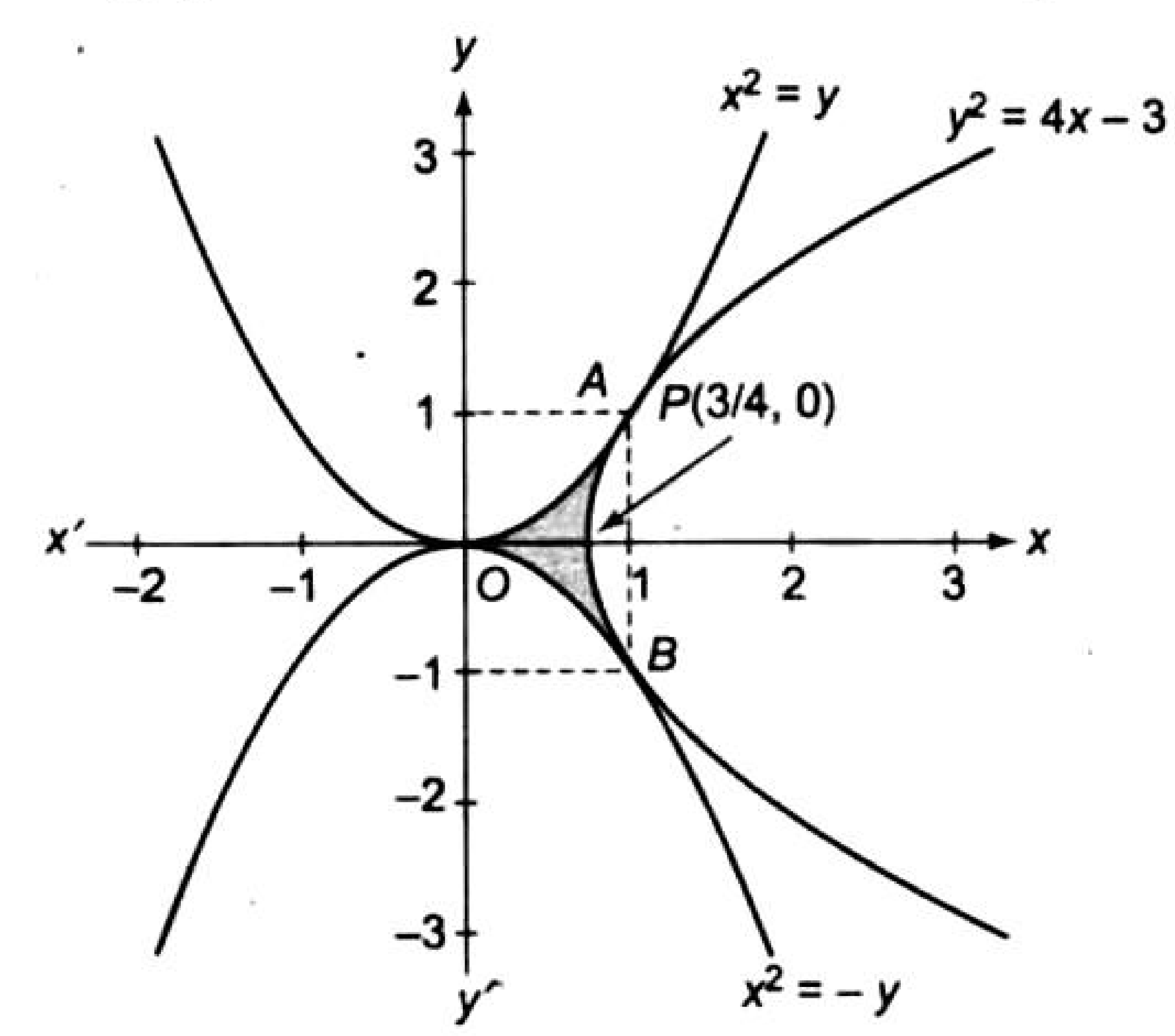
or $(x - 1)^2(x^2 + 2x + 3) = 0$

$\Rightarrow x = 1 \Rightarrow y = 1$

Thus, point of intersection is (1, 1).

Similarly, point of intersection of (2) and (3) is (1, – 1).

The graphs of three curves are as shown in the figure.



We also observe that at $x = 1$ and $y = 1$, $\frac{dy}{dx}$ for (1) and (3) is same and hence the two curves touch each other at $A(1, 1)$.

Same is the case with (2) and (3) at $B(1, -1)$.

Required area = Area of shaded region

= 2 (Ar OPA)

$$= 2\left[\int_0^1 x^2\,dx - \int_{3/4}^{1}\sqrt{4x - 3}\,dx\right]$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2(4x-3)^{3/2}}{4\times 3}\right)_{3/4}^1\right] = 2\left[\frac{1}{3}-\frac{1}{6}\right]$$

$= \frac{1}{3}$ sq. units.

22. $f'(x) = g(x)$

$$\int_0^3 g(x)dx = \int_0^3 f'(x)dx = [f(x)]_0^3 = [f(3) - f(0)] \in (-2, 2)$$

$$\int_{-3}^0 g(x)dx = \int_{-3}^0 f'(x)dx = [f(x)]_{-3}^0$$

$$= [f(0) - f(-3)] \in (-2, 2)$$

$$\Rightarrow (f(0))^2 + (g(0))^2 = 9$$

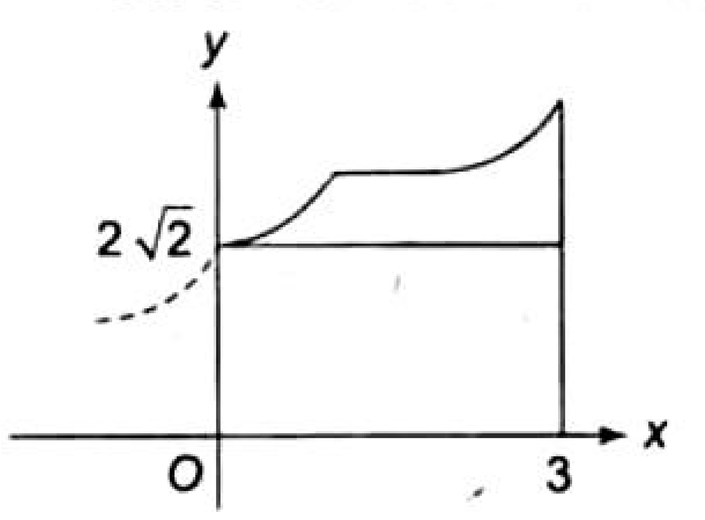
Now $|g(0)| > 2\sqrt{2}$ $\quad (\because |f(0)| < 1)$

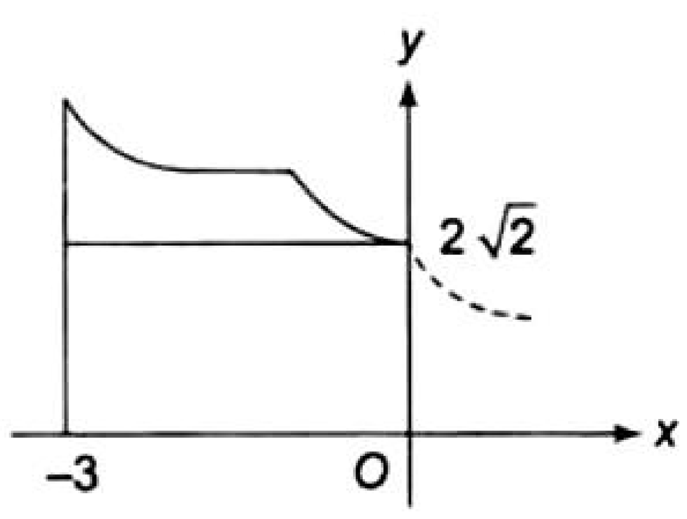
Case I:

$g(0) > 2\sqrt{2}$

Let $g''(x) \geq 0$ in $(-3, 3)$

One of the two situations is possible.





$\int_0^3 g(x)dx > 6\sqrt{2} > 2$

So contradiction arises

So $g''(x)$ has to be negative somewhere in (0, 3) while $g(x) > 0$ in (0, 3)

$\int_{-3}^0 g(x)dx > 6\sqrt{2} > 2$

So contradiction arises

So $g''(x)$ has to be negative somewhere in (–3, 0) while $g(x) > 0$ in (–3, 0)

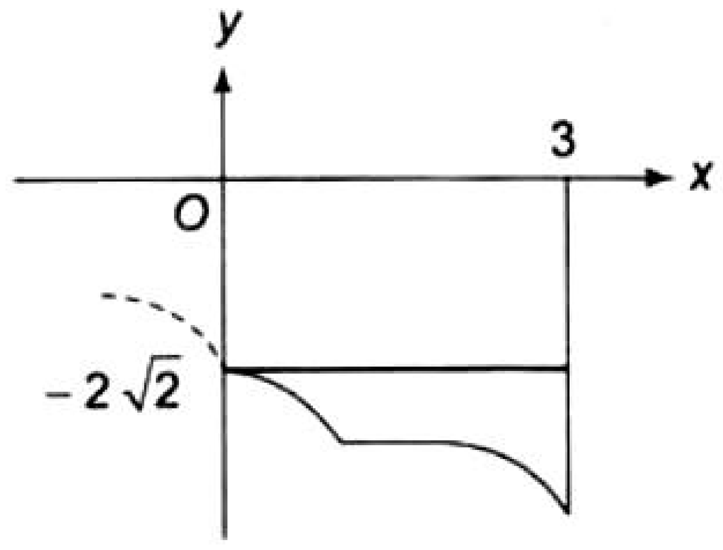
So at least somewhere $g''(x) < 0$, while $g(x) > 0$ in (–3, 3).

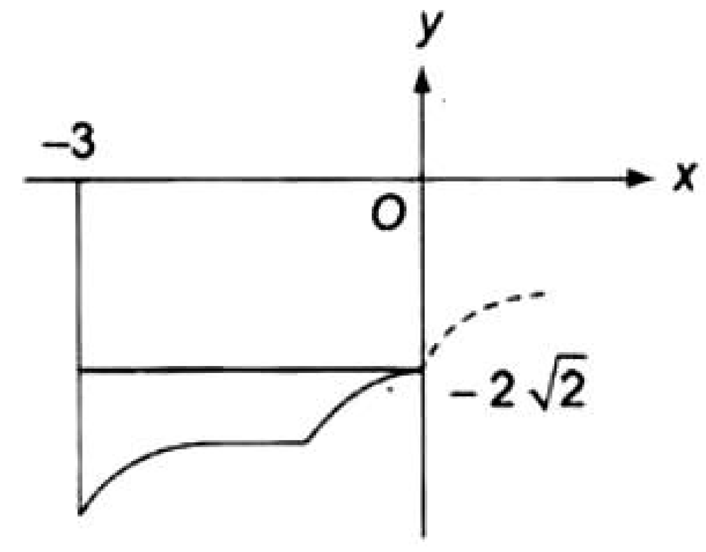
Case II:

$g(0) < -2\sqrt{2}$

Let $g''(x) \leq 0$ in $(-3, 3)$

One of the two situations is possible.





$\int_0^3 g(x)dx < -6\sqrt{2} < -2$

So contradiction arises

So $g''(x)$ has to be positive somewhere in (0, 3) while $g(x) < 0$ in (0, 3)

$\int_{-3}^0 g(x)dx < -6\sqrt{2} < -2$

So contradiction arises

So $g''(x)$ has to be positive somewhere in (–3, 0) while $g(x) < 0$ in (–3, 0)

So at least somewhere $g''(x) > 0$ while $g(x) < 0$ in (–3, 3).

So at least at one point in (–3, 3).

23. $4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$

$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3b$

$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$

Comparing coefficient of a^2, a and constant term on both sides, we get

$f(-1) = \frac{3}{4} = f(1)$ and $f(2) = 0$ (1)

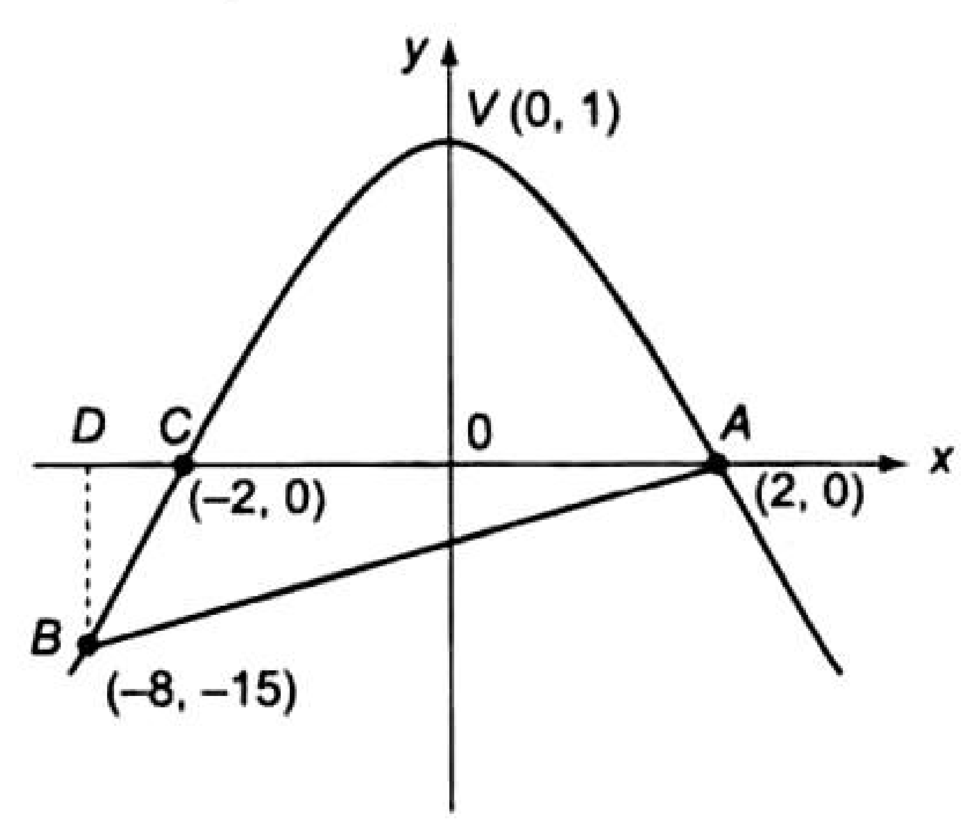
Let $f(x) = Ax^2 + Bx + C$ (2)

From (1) and (2), $A = -\frac{1}{4}$, $B = 0$, $C = 1$.

$\therefore f(x) = -\frac{1}{4}x^2 + 1$

Let $B\left(t, 1-\frac{t^2}{4}\right)$ be any point on the parabola

$f(x) = y = -\frac{x^2}{4} + 1$



As AB chord subtends right angle at V

$$\Rightarrow \left(-\frac{1}{2}\right) \times \left(\frac{\frac{t^2}{4}}{-t}\right) = -1 \quad \Rightarrow t = -8$$

$\Rightarrow B \equiv (-8, -15)$

$\therefore$ Area $(BCVAB)$

$$= 2 \times \int_0^2 \left(1 - \frac{x^2}{4}\right) dx + \frac{1}{2} \times 10 \times 15 - \left|\int_{-8}^{-2} \left(1 - \frac{x^2}{4}\right) dx\right|$$

$= \frac{125}{3}$ sq. units.