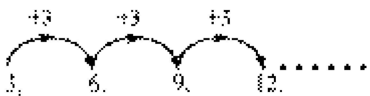
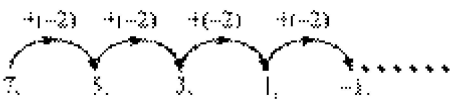
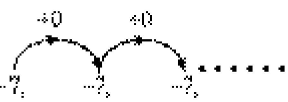
4. Sequence and Series

- **The Concept of Arithmetic Progression**
  - An arithmetic progression is a list of numbers in which the difference between any two consecutive terms is equal.
  - In an AP, each term, except the first term, is obtained by adding a fixed number called common difference to the preceding term.
  - The common difference of an AP can be positive, negative or zero.

Example 1:

1.  is an AP whose first term and common difference are 3 and 3 respectively.

2.  is an AP whose first term and common difference are 7 and –2 respectively.

3.  is an AP whose first term and common difference are –7 and 0 respectively.

  - The general form of an AP can be written as a, $a + d$, $a + 2d$, $a + 3d$ …, where a is the first term and d is the common difference.
  - A given list of numbers i.e., a_1, a_2, a_3 … forms an AP if $a_{k+1} - a_k$ is the same for all values of k.

Example 2:

Which of the following lists of numbers forms an AP? If it forms an AP, then write its next three terms.

(a) –4, 0, 4, 8, …

(b) 2, 4, 8, 16, …

Solution:

(a) –4, 0, 4, 8, …

$a_2 - a_1 = 0 - (-4) = 4$
$a_3 - a_2 = 4 - 0 = 4$
$a_4 - a_3 = 8 - 4 = 4$
$a_{n+1} - a_n = 4$; for all values of n

Therefore, the given list of numbers forms an AP with 4 being its common difference.
The next three terms of the AP are 8 + 4 = 12, 12 + 4 = 16, 16 + 4 = 20

Hence, AP: –4, 0, 4, 8, 12, 16, 20 …

(b) 2, 4, 8, 16, …

$a_2 - a_1 = 4 - 2 = 2$
$a_3 - a_2 = 8 - 4 = 4$
$a_3 - a_2 \neq a_2 - a_1$
Therefore, the given list of numbers does not form an AP.

- ***n*^th^ term of an AP**

The n^{th} term (a_n) of an AP with first term a and common difference d is given by $a_n = a + (n - 1)\, d$. Here, a_n is called the general term of the AP.

- **n^{th} term from the end of an AP**

The n^{th} term from the end of an AP with last term l and common difference d is given by $l - (n - 1)\, d$.

Example:
Find the 12^{th} term of the AP 5, 9, 13 …
Solution:
Here, $a = 5$, $d = 9 - 5 = 4$, $n = 12$

$a_{12} = a + (n - 1)\, d$
$= 5 + (12 - 1)\, 4$
$= 5 + 11 \times 4$
$= 5 + 44$
$= 49$

- **Sum of *n* terms of an AP**
  - The sum of the first *n* terms of an AP is given by Sn=n22a+n-1d , where *a* is the first term and *d* is the common difference.
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  - If there are only *n* terms in an AP, then Sn=n2a+l, where $l = a_n$ is the last term.

Example :

Find the value of 2 + 10 + 18 + …. + 802.

Solution:

2, 10, 18… 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be *n* terms in the series. Then,

$a_n = 802$

$\Rightarrow a + (n - 1)\, d = 802$

$\Rightarrow 2 + (n - 1)\ 8 = 802$

$\Rightarrow 8(n - 1) = 800$

$\Rightarrow n - 1 = 100$

$\Rightarrow n = 101$

Thus, required sum = n2a+l = 10122+802 = 40602

- **Properties of an Arithmetic progression**
  - If a constant is added or subtracted or multiplied to each term of an A.P. then the resulting sequence is also an A.P.
  - If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

- **Arithmetic mean**
  - For any two numbers a and b, we can insert a number A between them such that a, A, b is an A.P. Such a number i.e., A is called the arithmetic mean (A.M) of numbers a and b and it is given by $A = \frac{a+b}{2}$.
  - For any two given numbers a and b, we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

Let $A_1, A_2 \ldots A_n$ be n numbers between a and b such that $a, A_1, A_2 \ldots A_n, b$ is an A.P.

Here, common difference (d) is given by $\frac{b-a}{n+1}$.

Example:
Insert three numbers between –2 and 18 such that the resulting sequence is an A.P.
Solution:
Let A_1, A_2, and A_3 be three numbers between – 2 and 18 such that – 2, A_1, A_2, A_3, 18 are in an A.P.
Here, $a = -2$, $b = 18$, $n = 5$
$\therefore 18 = -2 + (5 - 1)\ d$
$\Rightarrow 20 = 4\ d$
$\Rightarrow d = 5$
Thus, $A_1 = a + d = -2 + 5 = 3$
$A_2 = a + 2d = -2 + 10 = 8$
$A_3 = a + 3d = -2 + 15 = 13$
Hence, the required three numbers between –2 and 18 are 3, 8, and 13.

- **Geometric Progression:** A sequence is said to be a geometric progression (G.P.) if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by r.

- In standard form, the G.P. is written as a, ar, ar^2 ... where, a is the first term and r is the common ratio.

- **General Term of a G.P.:** The n^{th} term (or general term) of a G.P. is given by $a_n = ar^{n-1}$

Example: Find the number of terms in G.P. 5, 20, 80 ... 5120.

Solution: Let the number of terms be n.

Here $a = 5$, $r = 4$ and $t_n = 5120$

n^{th} term of G.P. = ar^{n-1}

$$\therefore 5(4)^{n-1} = 5120$$

$$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$$

$$\Rightarrow (2)^{2n-2} = (2)^{10}$$

$$\Rightarrow 2n - 2 = 10$$

$$\Rightarrow 2n = 12$$

$$\therefore n = 6$$

- **Sum of n Term of a G.P.:** The sum of n terms (S_n) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, \quad \text{if } r < 1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, \text{ if } r > 1 \\ na, \text{ if } r = 1 \end{cases}$$

Example: Find the sum of the series 1 + 3 + 9 + 27 + ... to 10 terms.

Solution: The sequence 1, 3, 9, 27, ... is a G.P.

Here, $a = 1$, $r = 3$.

Sum of n terms of G.P. = $\frac{a(r^n-1)}{r-1}$ $[r > 1]$

S_{10} = 1 + 3 + 9 + 27 + ... to 10 terms

$$= \frac{1 \times \left[(3)^{10} - 1\right]}{(3-1)}$$

$$= \frac{59049-1}{2}$$

$$= \frac{59048}{2}$$

$$= 29524$$

- Three consecutive terms can be taken as ar, a, ar. Here, common ratio is r.
- Four consecutive terms can be taken as ar3, ar, ar, ar3. Here, common ratio is $r2$.

- **Geometric Mean:** For any two positive numbers a and b, we can insert a number G between them such that a, G, b is a G.P. Such a number i.e., G is called a geometric mean (G.M.) and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then $G_1, G_2, \ldots, G_n$ are given by

$G_1 = ar, G_2 = ar^2, \ldots, G_n = ar^n$

Where, r is calculated from the relation $b = ar^{n+1}$, that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

Example: Insert three geometric means between 2 and 162.
Solution:
Let G_1, G_2, G_3 be 3 G.M.'s between 2 and 162.
Therefor 2, G_1, G_2, G_3, 162 are in G.P.
Let r be the common ratio of G.P.
Here, $a = 2$, $b = 162$ and $n = 3$

$$r = \left(\frac{162}{2}\right)^{\frac{1}{3+1}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$$

$G_1 = ar = 2 \times 3 = 6$
$G_2 = ar^2 = 2 \times (3)^2 = 2 \times 9 = 18$
$G_3 = ar^3 = 2 \times (3)^3 = 2 \times 27 = 54$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let A and G be the respective A.M. and G.M. of two given positive real numbers a and b. Accordingly, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$

• If the reciprocals of the terms of a sequence form an arithmetic progression, then the sequence is said to be in harmonic progression.

Therefore, the sequence $a_1, a_2, a_3, \ldots, a_n$ will be in H.P. if and only if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in A.P.

• **General term or n^{th} term of H.P.:**

• If a_n is the n^{th} term of a H.P., then $\frac{1}{a_n}$ will be the n^{th} term of the A.P. obtained by the reciprocals of the terms in H.P.

The general term of a H.P. is given by $\frac{1}{a+(n-1)d}$, where a is the first term and d is the common difference of the A.P.

• $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}$ is the standard form of a H.P.

• Like A.P. and G.P., there is no formula to find the sum of n terms of a H. P.

Arithmetic Mean (A.M.):

• If A is the A.M. of the numbers a and b, then A is given by A = $\frac{a+b}{2}$.

• For any two given numbers, a and b, we can insert as many numbers between them as we want in such a way that the resulting sequence becomes an A.P.

Geometric Mean (G.M.):

• The geometric mean (G.M.) of two numbers a and b is $\sqrt{ab}$ and is denoted by G.

• G is a real number if and only if both a and b have the same sign.

• For any two numbers, a and b, there exist two values of G, $\sqrt{ab}$ and $-\sqrt{ab}$.

• If $a = b$, then G = $\pm\sqrt{aa} = \pm a$.

• If we are given two numbers, a and b, then we can insert a number G between these two numbers so that the sequence a, G, b becomes a G.P. Here, G is the **geometric mean (G.M.)** of the numbers a and b.

• For any two positive numbers, we can insert as many numbers between them as we want in such a way that the resulting sequence becomes a G.P.

Harmonic Mean (H.M.):

The harmonic mean (H.M.) of two numbers a and b is $\frac{2ab}{a+b}$ and is denoted by H.

Relation between A.M., G.M. and H.M. of two distinct real numbers:

Let a and b be two distinct positive numbers.

If A, G and H are their A.M., G.M. and H.M. respectively, then G is the geometric mean of A and H.

∴ A, G and H are in G.P.

Also, **A.M. ≥ G.M. ≥ H.M.**

Arithmetico-Geometric Progression

The series whose each term is formed by multiplying the corresponding terms of an A.P. and a G.P. is called an arithmetico-geometric series. Let a, $(a + d)$, $(a + 2d)$, $(a + 3d)$, ... ,$[a + (n - 1)d]$, ... be an A.P. with first term a and common difference d ($d \neq 0$); $1, r, r^2, r^3, \ldots, r^{n-1}$, ...be a G.P. with first term 1 and common ratio r ($r \neq 1$). Then,
$a, (a + d) r, (a + 2d) r^2, (a + 3d) r^3, \ldots, [a + (n - 1)d] r^{n-1}$,... is an arithmetico-geometric progression.

***n* th term** of an arithmetico-geometric progression is $t_n = [a + (n - 1)d] r^{n-1}$.

Sum of *n* terms of an arithmetico-geometric progression is given by

Sn=a1-r+dr1-rn-11-r2-[a+(n-1)d]rn1-r,   r≠1

If $|r| < 1$, then the sum of an infinite arithmetico-geometric progression denoted by S∞is given by S∞=a1-r+rd1-r2.

- **Sum of *n*-terms of some special series:**
  - Sum of first *n* natural numbers

    $$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

  - Sum of squares of the first *n* natural numbers

    $$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

  - Sum of cubes of the first *n* natural numbers

    $$1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2$$

Example: Find the sum of *n* terms of the series whose n^{th} term is $n(n+1)(n-2)$.

Solution: It is given that

$$\begin{aligned} a_n &= n(n+1)(n-2) \\ &= n(n^2+n-2n-2) \\ &= n(n^2-n-2) \\ &= n^3-n^2-2n \end{aligned}$$

Thus, the sum of *n* terms is given by

$$S_n = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3} - 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)-2(2n+1)-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n-4n-2-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2-n-14}{6}\right]$$

$$= \frac{n(n+1)(3n^2-n-14)}{12}$$

$$= \frac{n(n+1)(3n^2-7n+6n-14)}{12}$$

$$= \frac{n(n+1)[n(3n-7)+2(3n-7)]}{12}$$

$$= \frac{n(n+1)(n+2)(3n-7)}{12}$$

Exponential Series

- ex=1+x1!+x22!+x33!+...
- e-x=1-x1!+x22!-x33!+...
- ex+e-x=21+x22!+x44!+...
- ex-e-x=2x+x33!+x55!+...
- For *a* > 0, ax = 1 + xlogea + x22!logea2 + x33!logea3 + ...

Note: The value of *e* lies between 2 and 3.

Logarithmic Series

For x<1,

- loge1+x=x-x22+x33-...
- loge1-x=-x-x22-x33-...
- loge1+x-loge1-x=loge1+x1-x=2x+x33+x55+...