9.13 Surface Integral

Scalar functions: $f(x,y,z)$, $z(x,y)$
Position vectors: $\vec{r}(u,v)$, $\vec{r}(x,y,z)$
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$
Surface: S
Vector field: $\vec{F}(P,Q,R)$

Divergence of a vector field: $\text{div }\vec{F} = \nabla \cdot \vec{F}$

Curl of a vector field: $\text{curl }\vec{F} = \nabla \times \vec{F}$

Vector element of a surface: $d\vec{S}$
Normal to surface: $\vec{n}$
Surface area: A
Mass of a surface: m
Density: $\mu(x,y,z)$
Coordinates of center of mass: $\overline{x}$, $\overline{y}$, $\overline{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z
Volume of a solid: V

Force: $\vec{F}$
Gravitational constant: G
Fluid velocity: $\vec{v}(\vec{r})$
Fluid density: ρ
Pressure: $p(\vec{r})$
Mass flux, electric flux: Φ
Surface charge: Q
Charge density: $\sigma(x,y)$

Magnitude of the electric field: $\vec{E}$

1140. Surface Integral of a Scalar Function
Let a surface S be given by the position vector
$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$,
where (u,v) ranges over some domain $D(u,v)$ of the uv-

plane.

The surface integral of a scalar function $f(x,y,z)$ over the surface S is defined as

$$\iint_S f(x,y,z)dS = \iint_{D(u,v)} f(x(u,v),y(u,v),z(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dudv,$$

where the partial derivatives $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$ are given by

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u,v)\vec{i} + \frac{\partial y}{\partial u}(u,v)\vec{j} + \frac{\partial z}{\partial u}(u,v)\vec{k},$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u,v)\vec{i} + \frac{\partial y}{\partial v}(u,v)\vec{j} + \frac{\partial z}{\partial v}(u,v)\vec{k}$$

and $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$ is the cross product.

1141. If the surface S is given by the equation $z = z(x,y)$ where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

$$\iint_S f(x,y,z)dS = \iint_{D(x,y)} f(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dxdy.$$

1142. Surface Integral of the Vector Field $\vec{F}$ over the Surface S

- If S is oriented outward, then

$$\iint_S \vec{F}(x,y,z) \cdot d\vec{S} = \iint_S \vec{F}(x,y,z) \cdot \vec{n}dS$$

$$= \iint_{D(u,v)} \vec{F}(x(u,v),y(u,v),z(u,v)) \cdot \left[\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right] dudv.$$

- If S is oriented inward, then

$$\iint_S \vec{F}(x,y,z) \cdot d\vec{S} = \iint_S \vec{F}(x,y,z) \cdot \vec{n}dS$$

$$\vec{F} \quad x \quad u,v \;, y \quad u,v \;, z \quad u,v$$

$$= \iint\limits_{D(u,v)} {}^{-}((\quad)(\quad)(\quad)) \cdot \left[\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial u}\right] du dv.$$

$d\vec{S} = \vec{n}dS$ is called the **vector element of the surface**. Dot means the scalar product of the appropriate vectors. The partial derivatives $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$ are given by

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u,v) \cdot \vec{i} + \frac{\partial y}{\partial u}(u,v) \cdot \vec{j} + \frac{\partial z}{\partial u}(u,v) \cdot \vec{k},$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u,v) \cdot \vec{i} + \frac{\partial y}{\partial v}(u,v) \cdot \vec{j} + \frac{\partial z}{\partial v}(u,v) \cdot \vec{k}.$$

1143. If the surface S is given by the equation $z = z(x,y)$, where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

- If S is oriented **upward**, i.e. the k-th component of the normal vector is positive, then

$$\iint\limits_{S} \vec{F}(x,y,z) \cdot d\vec{S} = \iint\limits_{S} \vec{F}(x,y,z) \cdot \vec{n}dS$$

$$= \iint\limits_{D(x,y)} \vec{F}(x,y,z) \cdot \left(-\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}\right)dxdy,$$

- If S is oriented **downward**, i.e. the k-th component of the normal vector is negative, then

$$\iint\limits_{S} \vec{F}(x,y,z) \cdot d\vec{S} = \iint\limits_{S} \vec{F}(x,y,z) \cdot \vec{n}dS$$

$$= \iint\limits_{D(x,y)} \vec{F}(x,y,z) \cdot \left(\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} - \vec{k}\right)dxdy.$$

1144. $\iint\limits_{S} (\vec{F} \cdot \vec{n})dS = \iint\limits_{S} Pdydz + Qdzdx + Rdxdy$

$$= \iint\limits_{S} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS,$$

where $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ are the components of the vector field $\vec{F}$.

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the angles between the outer unit normal vector $\vec{n}$ and the x-axis, y-axis, and z-axis, respectively.

1145. If the surface S is given in parametric form by the vector $\vec{r}(x(u,v),y(u,v),z(u,v))$, then the latter formula can be written as

$$\iint_S (\vec{F}\cdot\vec{n})dS = \iint_S Pdydz + Qdzdx + Rdxdy = \iint_{D(u,v)} \begin{vmatrix} P & Q & R \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} dudv,$$

where (u,v) ranges over some domain $D(u,v)$ of the uv-plane.

1146. Divergence Theorem

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_G (\nabla\cdot\vec{F})dV,$$

where

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of $\vec{F}$, also denoted $\text{div}\vec{F}$. The symbol $\oiint$ indicates that the surface integral is taken over a closed surface.

1147. Divergence Theorem in Coordinate Form

$$\oiint_S Pdydz + Qdxdz + Rdxdy = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz.$$

1148. Stoke's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S},$$

where

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

is the curl of $\vec{F}$, also denoted $\operatorname{curl} \vec{F}$.

The symbol $\oint$ indicates that the line integral is taken over a closed curve.

1149. Stoke's Theorem in Coordinate Form

$$\oint_C P dx + Q dy + R dz$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

1150. Surface Area

$$A = \iint_S dS$$

1151. If the surface S is parameterized by the vector

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k},$$

then the surface area is

$$A = \iint_{D(u,v)} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv,$$

where $D(u,v)$ is the domain where the surface $\vec{r}(u,v)$ is defined.

1152. If S is given explicitly by the function $z(x,y)$, then the surface area is

$$A = \iint\limits_{D(x,y)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dxdy,$$

where $D(x,y)$ is the projection of the surface S onto the xy-plane.

1153. Mass of a Surface

$$m = \iint\limits_{S} \mu(x,y,z)dS,$$

where $\mu(x,y,z)$ is the mass per unit area (density function).

1154. Center of Mass of a Shell

$$\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iint\limits_{S} x\mu(x,y,z)dS,$$

$$M_{xz} = \iint\limits_{S} y\mu(x,y,z)dS,$$

$$M_{xy} = \iint\limits_{S} z\mu(x,y,z)dS$$

are the first moments about the coordinate planes $x = 0$, $y = 0$, $z = 0$, respectively. $\mu(x,y,z)$ is the density function.

1155. Moments of Inertia about the xy-plane (or $z = 0$), yz-plane ($x = 0$), and xz-plane ($y = 0$)

$$I_{xy} = \iint\limits_{S} z^2\mu(x,y,z)dS,$$

$$I_{yz} = \iint\limits_{S} x^2\mu(x,y,z)dS,$$

$$I_{xz} = \iint\limits_S y^2 \mu(x,y,z) dS.$$

1156. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = \iint\limits_S (y^2 + z^2) \mu(x,y,z) dS,$$

$$I_y = \iint\limits_S (x^2 + z^2) \mu(x,y,z) dS,$$

$$I_z = \iint\limits_S (x^2 + y^2) \mu(x,y,z) dS.$$

1157. Volume of a Solid Bounded by a Closed Surface

$$V = \frac{1}{3} \left| \iint\limits_S x\,dydz + y\,dxdz + z\,dxdy \right|$$

1158. Gravitational Force

$$\vec{F} = Gm \iint\limits_S \mu(x,y,z) \frac{\vec{r}}{r^3} dS,$$

where m is a mass at a point $\langle x_0, y_0, z_0 \rangle$ outside the surface,
$\vec{r} = \langle x - x_0, y - y_0, z - z_0 \rangle$,
$\mu(x,y,z)$ is the density function,
and G is gravitational constant.

1159. Pressure Force

$$\vec{F} = \iint\limits_S p(\vec{r}) d\vec{S},$$

where the pressure $p(\vec{r})$ acts on the surface S given by the position vector $\vec{r}$.

1160. Fluid Flux (across the surface S)

$$\Phi = \iint\limits_S \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{v}(\vec{r})$ is the fluid velocity.

1161. Mass Flux (across the surface S)
$$\Phi = \oiint_S \rho \vec{v}(\vec{r}) \cdot d\vec{S},$$
where $\vec{F} = \rho \vec{v}$ is the vector field, ρ is the fluid density.

1162. Surface Charge
$$Q = \iint_S \sigma(x,y) dS,$$
where $\sigma(x,y)$ is the surface charge density.

1163. Gauss' Law
The electric flux through any closed surface is proportional to the charge Q enclosed by the surface
$$\Phi = \oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0},$$
where
Φ is the electric flux,
$\vec{E}$ is the magnitude of the electric field strength,
$\varepsilon_0 = 8{,}85 \times 10^{-12} \dfrac{F}{m}$ is permittivity of free space.