14

Trigonometric Ratios of Acute Angles

14.01 Right Angled Triangle :

In previous chapters we have studied about the triangle in which one angle is right angle, such triangles are called right triangles. Fig. 14.01 is a right triangle in which $\angle B$ is a right angle. The side opposite the right angle is called hypotenuse. Hence in right angled triangle ΔABC, AC is hypotenuse. Regarding other two angles of the right angled triangle, the side which makes angle with the hypotenuse is called base or adjacent side and the side opposite to this angle is called perpendeicular. In Fig. 14.01 for $\angle C$, side CB is base AB is perpendicular. Similarly for $\angle A$ side AB is base and side BC is perpendicular. In right angled triangle the angles other than the right angle are acute angles. The relation between the sides of a right triangle is "Square on hypotenuse is equal to the sum of squares on other two sides" known as Baudhayan Theorem. In brief the above theroem may be read as

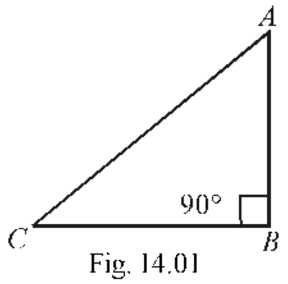


Fig. 14.01

$$AC^2 = AB^2 + BC^2$$

Clearly if out of three sides AB, BC, and AC two are given then third can easily be obtained.

Example 1. In right angled triangle ABC give the names of sides corresponding to angle θ and ϕ .

Solution : In ΔABC, $\angle B$ is a right angle, so AC is hypotenuse. Now for angle θ BC is base and AB is perpendicular. Similarly for angle ϕ side AB is base and side BC is perpendicular.

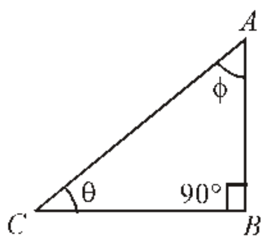


Fig. 14.02

Example 2. In triangle ABC, $\angle B$ is right angle. If $AB = 4$ cm and $AC = 5$ cm then find BC.

Solution : Draw rough figure of ΔABC as per the given specification.

$\angle B = 90°$

$AC = 5$ cm

$AB = 4$ cm

From Baudhayan theorem

$$AC^2 = AB^2 + BC^2$$

or $\quad (5)^2 = (4)^2 + BC^2$

or $\quad BC^2 = 25 - 16$

or $\quad BC^2 = 9$

Hence $\quad BC = 3$ cm.

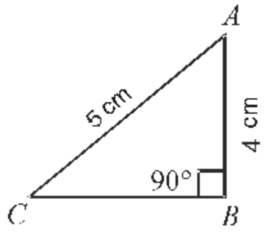


Fig. 14.03

14.02 Trigonometric Ratios of an Acutre Angle :

In right angled triangle the ratio of any two sides is called trigonometrical ratio.

Let triangle ABC be a right angled triangle in which $\angle ABC$ is a right angle and $\angle CAB = \theta$, then

(i) $\dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{BC}{AC} = \text{sine}\,\theta$ or $\sin\theta$

In brief sine θ is written as $\sin\theta$

(ii) $\dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{AB}{AC} = \text{cosine}\,\theta$ or $\cos\theta$

In brief cosine θ or is written as $\cos\theta$.

(iii) $\dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{BC}{AB} = \text{tangent}\,\theta$ or $\tan\theta$

In brief tangent θ is written as $\tan\theta$

(iv) $\dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{AB}{BC} = \text{cotangent}\,\theta$ or $\cot\theta$

In brief cotangent θ is written as $\cot\theta$.

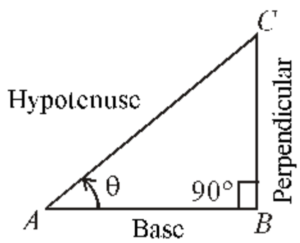


Fig. 14.04

(v) $\dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{AC}{AB} = \text{secant}\,\theta$ or $\sec\theta$

In brief secant θ is written as $\sec\theta$.

(vi) $\dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{AC}{BC} = \text{cosecant}\,\theta$ or $\text{cosec}\,\theta$

In brief cosecant θ is written as $\text{cosec}\,\theta$.

Note : (i) Suppose the revolving line AX moves in anticlock wise direction keeping the vertex A fixed and makes an acute angle $\angle XAP = \theta$. Draw perpendiculars CB, C_1B_1 and C_2B_2 from the points C, C_1, C_2 respectively on AX.

We find that the triangle CAB, C_1AB_1, C_2AB_2 etc. are similar. Therefore

$$\sin\theta = \frac{BC}{AC} = \frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} = \cdots$$

and $$\cos\theta = \frac{AB}{AC} = \frac{AB_1}{AC_1} = \frac{AB_2}{AC_2} = \cdots$$

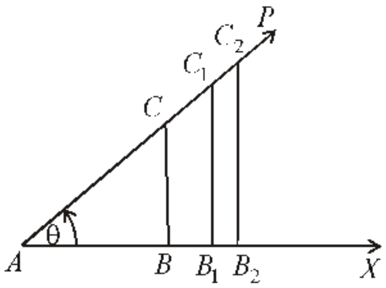


Fig. 14.05

From this we observe that the value of $\sin\theta$ or $\cos\theta$ remains unchanged in every case, i.e. these values do not depend on the position of the point P on the revolving line. Similarly the other trigonometrical ratios also do not depend on the position of point P on the revolving line.

Hence the trigonometrical ratio depend on the acute angle θ not on the size of the right angled triangle. Since for every acute angle θ, the value of the trigonometrical ratio is unique, therefore, trigonometrical ratios are also called as trigonometrical functions.

(ii) $\sin\theta, \cos\theta, \tan\theta, \ldots$ does not mean the multiplication of $\sin$ or $\cos$ or $\tan$ or $\cdots$ by θ

i.e., $$\sin\theta \neq \sin\times\theta$$
$$\cos\theta \neq \cos\times\theta$$
$$\tan\theta \neq \tan\times\theta$$

(iii) Trigonometrical ratios of any positive acute angle are always positive.

Illustrative Examples

Example 3. In triangle ABC angle B is a right angle, find all trigonometrical ratios of the angle A.

Solution : From Fig. 14.06, side AC is hypotenuse and side opposite to $\angle A$ is BC. Therefore in ΔABC side AB is base, BC is perpendicular and AC is hypotenuse.

$$\therefore \quad \sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{BC}{AB}$$

$$\cot A = \frac{\text{Base}}{\text{Perpendicular}} = \frac{AB}{BC}$$

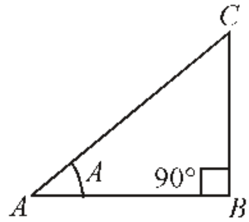


Fig. 14.06

$$\sec A = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{AC}{AB}$$

$$\operatorname{cosec} A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{AC}{BC}$$

Example 4. In triangle ABC angle C is a right angle and $AB = 25$ cm and $BC = 24$ cm then find all trigononmetric ratios of the angle A.

Solution : $\because$ ΔABC is a right triangle.

$$\therefore \quad AC = \sqrt{AB^2 - BC^2}$$

$$= \sqrt{(25)^2 - (24)^2}$$

$$= \sqrt{49}$$

$$= 7 \text{ cm}$$

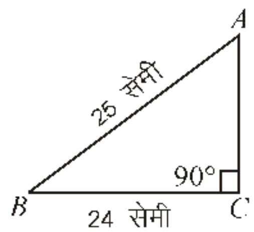


Fig. 14.07

$$\therefore \quad \sin A = \frac{BC}{AB} = \frac{24}{25}$$

$$\cos A = \frac{AC}{AB} = \frac{7}{25}$$

$$\tan A = \frac{BC}{AC} = \frac{24}{7}$$

$$\cot A = \frac{AC}{BC} = \frac{7}{24}$$

$$\sec A = \frac{AB}{AC} = \frac{25}{7}$$

$$\operatorname{cosec} A = \frac{AB}{BC} = \frac{25}{24}$$

Example 5. If $\sin\theta = \frac{3}{5}$, then find remaining trigonometrical ratios of θ .

Solution : We draw a right ΔABC where $\sin\theta = \frac{3}{5}$ (Fig. 14.08) i.e., perpendicular AB and hypotenuse AC are in the ratio 3 : 5. Let $AB = 3k$ and $AC = 5k$, where $k > 0$, which is proportionality constant.

Hence form Baudhayan theorem

$$BC^2 = AC^2 - AB^2 = (5k)^2 - (3k)^2 = 16k^2$$

$$\therefore \quad BC = \pm 4k$$

Hence angle θ is an acute angle, therefore BC will be positive.

$$BC = 4k$$

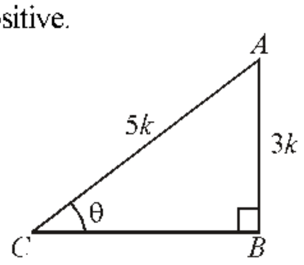


Fig. 14.08

$$\therefore \quad \cos\theta = \frac{BC}{AC} = \frac{4k}{5k} = \frac{4}{5}$$

$$\tan\theta = \frac{AB}{BC} = \frac{3k}{4k} = \frac{3}{4}$$

$$\cot\theta = \frac{BC}{AB} = \frac{4k}{3k} = \frac{4}{3}$$

$$\sec\theta = \frac{AC}{BC} = \frac{5k}{4k} = \frac{5}{4}$$

$$\text{cosec}\,\theta = \frac{AC}{AB} = \frac{5k}{3k} = \frac{5}{3}.$$

Example 6. If $\sec\theta = \frac{13}{12}$ then find the value of $\frac{1-\tan\theta}{1+\tan\theta}$.

Solution : Draw a right angled triangle ABC where $\sec\theta = \frac{13}{12}$ (Fig. 14.09)

i.e., hypotenuse AC and base BC are in the ratio of 13 : 12.
Let AC = 13k, BC = 12k, where k > 0, which is proportionality constant.
Hence form Baudhayan theorem

$$AB^2 = AC^2 - BC^2 = (13k)^2 - (12k)^2 = 25k^2$$

$$\therefore \quad AB = \pm 5k$$

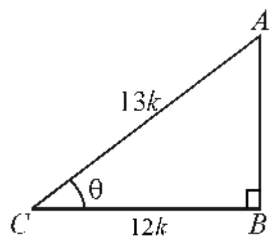


Fig. 14.09

Since angle θ is an acute angle, therefore AB will be positive.

$$\therefore \quad AB = 5k$$

$$\therefore \quad \tan\theta = \frac{AB}{BC} = \frac{5k}{12k} = \frac{5}{12}$$

Now
$$\frac{1-\tan\theta}{1+\tan\theta} = \frac{1-\frac{5}{12}}{1+\frac{5}{12}} = \frac{7}{17}$$

Example 7. If $\text{cosec}\,A = 2$, then find the value of $\cot A + \frac{\sin A}{1+\cos A}$.

Solution : $\because \quad \text{cosec}\,A = \frac{2}{1}$

Draw a right angled triangle ABC in which hypotenuse AC and perpendicular BC are in the ratio 2 : 1 (Fig. 14.10).

Let AC = 2k, BC = k

where k > 0, is the proportionality constant.

Hence from Baudhayan theroem

$$AB^2 = AC^2 - BC^2$$

or $$AB^2 = (2k)^2 - k^2 = 3k^2$$

or $$AB = \pm\sqrt{3}k$$

Since angle A is an acute angle, therefore AB will be positive.

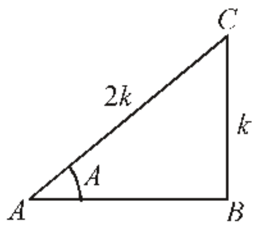


Fig. 14.10

$$\therefore \quad AB = \sqrt{3}k$$

$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{k}{2k} = \frac{1}{2}$$

$$\cos A = \frac{AB}{AC} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$$

$$\cot A = \frac{AB}{BC} = \frac{\sqrt{3}k}{k} = \sqrt{3}$$

Now

$$\cot A + \frac{\sin A}{1+\cos A} = \sqrt{3} + \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}$$

$$= \sqrt{3} + \frac{1}{2+\sqrt{3}} \times \frac{(2-\sqrt{3})}{(2-\sqrt{3})}$$

$$= \sqrt{3} + \frac{2-\sqrt{3}}{(2)^2 - (\sqrt{3})^2} = \sqrt{3} + 2 - \sqrt{3} = 2$$

Exercise 14.1

1. If in ΔABC, $\angle A = 90°$, $a = 25$ cm, $b = 7$ cm then find all the trigonometric ratio of $\angle B$ and $\angle C$.

2. If in ΔABC, $\angle B = 90°$, $a = 12$ cm, $b = 13$ cm then find all the trigonometrical ratios of $\angle A$ and $\angle C$.

3. If $\tan A = \sqrt{2} - 1$ then prove that $\sin A \cos A = \frac{1}{2\sqrt{2}}$.

4. If $\sin A = \frac{1}{3}$ then find the value of $\cos A \operatorname{cosec} A + \tan A \sec A$.

5. If $\cos\theta = \frac{8}{17}$ then find all the remaining trigonometrical ratios.

6. If $\cos A = \frac{5}{13}$ then find the value of $\frac{\operatorname{cosec} A}{\cos A + \operatorname{cosec} A}$.

7. If $5\tan\theta = 4$ then find the value of $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}$.

8. In ΔABC, $\angle C = 90°$ and if $\cot A = \sqrt{3}$ and $\cot B = \frac{1}{\sqrt{3}}$ then prove that $\sin A \cos B + \cos A \sin B = 1$.

9. If $16\cot A = 12$ then find the value of $\frac{\sin A + \cos A}{\sin A - \cos A}$.

10. In Fig. 14.13, $AD = DB$ and $\angle B = 90°$ than find the value of the following :

    (i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$

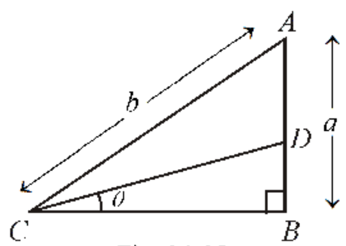


Fig. 14.13

14.03 Relation between Trigonometic Ratios :

In any right angle triangle OMP, PM is perpendicular, OM is base and OP is hypotenuse for the $\angle\theta$.

(i) $\sin\theta \operatorname{cosec}\theta = 1$

$$\sin\theta = \frac{PM}{OP} \qquad \cdots (1)$$

and $$\operatorname{cosec}\theta = \frac{OP}{PM} \qquad \cdots (2)$$

Multiplying (1) and (2)

$$\sin\theta \cdot \operatorname{cosec}\theta = \frac{PM}{OP} \times \frac{OP}{PM} = 1$$

i.e., $\sin\theta \operatorname{cosec}\theta = 1$

$$\Rightarrow \quad \sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

or $$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

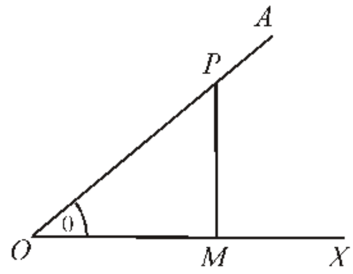


Fig. 14.14

Hence $\sin\theta$ and $\operatorname{cosec}\theta$ are reciprocal of each other.

(ii) $\cos\theta \cdot \sec\theta = 1$

From Fig. (14.14)

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{OM}{OP} \qquad \cdots (3)$$

and $$\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{OP}{OM} \qquad \cdots (4)$$

Multiplying (3) and (4)

$$\cos\theta \cdot \sec\theta = \frac{OM}{OP} \cdot \frac{OP}{OM} = 1$$

i.e., $\cos\theta \cdot \sec\theta = 1$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sec\theta} \quad \text{या} \quad \sec\theta = \frac{1}{\cos\theta}$$

Hence $\cos\theta$ and $\sec\theta$ are reciprocal of each other.

(iii) $\tan\theta \cdot \cot\theta = 1$

From Fig. 14.14

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{PM}{OM} \qquad \cdots (5)$$

and $$\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{OM}{PM} \qquad \cdots (6)$$

Multiplying (5) and (6)

$$\tan\theta \cdot \cot\theta = \frac{PM}{OM} \times \frac{OM}{PM} = 1$$

i.e., $\tan\theta \cdot \cot\theta = 1$

$$\Rightarrow \quad \tan\theta = \frac{1}{\cot\theta} \text{ or } \cot\theta = \frac{1}{\tan\theta}$$

Hence $\tan\theta$ and $\cot\theta$ are reciprocal of each other.

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

From equation (1) and (3)

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{PM}{OP}}{\frac{OM}{OP}} = \frac{PM}{OP} \times \frac{OP}{OM} = \frac{PM}{OM} = \frac{\text{Perpendicular}}{\text{Base}} = \tan\theta$$

i.e., $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

From equation (1) and (3)

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{OM}{OP}}{\frac{PM}{OP}} = \frac{OM}{OP} \times \frac{OP}{PM} = \frac{OM}{PM} = \frac{\text{Base}}{\text{Perpendicular}} = \cot\theta$$

i.e., $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(vi) $\sin^2\theta + \cos^2\theta = 1$

From equation (1) and (3)

$$\sin\theta = \frac{PM}{OP} \text{ एवं } \cos\theta = \frac{OM}{OP}$$

Squaring and adding

$$\sin^2\theta + \cos^2\theta = \left(\frac{PM}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2$$

$$= \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1$$ [from Baudhayan theorem (Fig. 14.14)]

(vii) $1 + \tan^2\theta = \sec^2\theta$

From Fig. 14.14

$$\tan\theta = \frac{PM}{OM}$$

$$\therefore \quad 1+\tan^2\theta = 1+\frac{PM^2}{OM^2} = \frac{OM^2+PM^2}{OM^2}$$

$$= \frac{OP^2}{OM^2} \quad \text{[From Baudhayab theroem]}$$

$$[\because \sec\theta = \frac{OP}{OM},\ \text{Fig. (14.14)}]$$

or $\quad 1+\tan^2\theta = \sec^2\theta$

Aliter :

We know that

$$\sin^2\theta + \cos^2\theta = 1$$

Dividing both sides by $\cos^2\theta$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad \tan^2\theta + 1 = \sec^2\theta \qquad \left(\because \frac{\sin\theta}{\cos\theta} = \tan\theta, \frac{1}{\cos\theta} = \sec\theta\right)$$

(viii) $1+\cot^2\theta = \operatorname{cosec}^2\theta$

From Fig. 14.14

$$\cot\theta = \frac{OM}{PM}$$

$$\therefore \quad 1+\cot^2\theta = 1+\left(\frac{OM}{PM}\right)^2 = \frac{PM^2+OM^2}{PM^2}$$

$$= \frac{OP^2}{PM^2} \qquad \text{(from Baudhayan theorem)}$$

$$= \operatorname{cosec}^2\theta \quad \left(\because \operatorname{cosec}\theta = \frac{OP}{PM} \text{ from fig. 14.14}\right)$$

Aliter :

We know that

$$\cos^2\theta + \sin^2\theta = 1$$

Dividing both sides by $\sin^2\theta$

$$\left(\frac{\cos\theta}{\sin\theta}\right)^2 + 1 = \frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad \cot^2\theta + 1 = \csc^2\theta \quad \left(\because \frac{\cos\theta}{\sin\theta} = \cot\theta, \frac{1}{\sin\theta} = \csc\theta\right)$$

Note : $(\sin\theta)^2$ is always written as $\sin^2\theta$ and same is read as 'sign square theta'

i.e., $(\sin\theta)^2 = \sin^2\theta \neq \sin\theta^2$

Other trigonometrical ratios are also to be treated in the same manner

Illustrative Examples

Example 8. If $\cos\theta = \frac{5}{13}$ then find the value of $\sin\theta$, $\tan\theta$ with the help of relations between the trigonometrical ratios when θ is an acute angle.

Solution : We know that

$$\sin^2\theta + \cos^2\theta = 1$$

Putting $\cos\theta = \frac{5}{13}$

$$\sin^2\theta + \left(\frac{5}{13}\right)^2 = 1$$

$$\Rightarrow \quad \sin^2\theta = 1 - \frac{25}{169}$$

$$\Rightarrow \quad \sin^2\theta = \frac{144}{169}$$

$$\Rightarrow \quad \sin\theta = \pm\frac{12}{13}$$

$$\Rightarrow \quad \sin\theta = \frac{12}{13} \qquad (\because \theta \text{ is an acute angle})$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{12/13}{5/13} = \frac{12}{5}$$

Example 9. If $\tan\theta = \sqrt{3}$ then find all the trigonometric ratios using relations between the trigonometric ratios, when θ is an acute angle.

Solution : we know that

$$1 + \tan^2\theta = \sec^2\theta$$

or

$$\sec^2\theta = 1 + \left(\sqrt{3}\right)^2$$
$$= 1 + 3 = 4$$

$\Rightarrow \quad \sec\theta = \pm 2$

or $\quad \sec\theta = 2 \quad (\because\ \theta$ is an acute angle)

$\because \quad \cos\theta = \frac{1}{\sec\theta} = \frac{1}{2}$

Now $\sin\theta = \tan\theta \cdot \cos\theta$

$\Rightarrow \quad \sin\theta = \sqrt{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2},$

$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\sqrt{3}/2\right)} = \frac{2}{\sqrt{3}},$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}.$

Example 10. If $\text{cosec}\,A = \sqrt{10}$ then find $\cot A, \sin A, \cos A$ using the relations between the trigonometrical ratios, when θ is an acute angle.

Solution : We know that

$1 + \cot^2 A = \text{cosec}^2 A$

$\Rightarrow \quad \cot^2 A = \text{cosec}^2 A - 1$

$\Rightarrow \quad \cot^2 A = \left(\sqrt{10}\right)^2 - 1 = 10 - 1 = 9$

$\Rightarrow \quad \cot A = 3 \quad (\because\ \theta$ is an acute angle)

Now $\sin A = \frac{1}{\text{cosec}\,A} = \frac{1}{\sqrt{10}}$

and $\cos A = \cot A \cdot \sin A$

$= 3 \cdot \frac{1}{\sqrt{10}} = \frac{3}{\sqrt{10}}$

Example 11. If $\sec\theta = \frac{17}{8}$ then find all the trigonometrical ratios with the help of relation between the trigonometrical ratios, when θ is an acute angle.

Solution : We know that

$1 + \tan^2\theta = \sec^2\theta$

or $\quad \tan^2\theta = \sec^2\theta - 1$

or $\tan^2\theta = \left(\frac{17}{8}\right)^2 - 1$

$$= \frac{289-64}{64} = \frac{225}{64}$$

$\Rightarrow$ $\tan\theta = \frac{15}{8}$ $(\because\ \theta$ is an acute angle)

$\therefore$ $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{(15/8)} = \frac{8}{15}$

and $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{17/8} = \frac{8}{17}.$

Now $\sin\theta = \tan\theta\cos\theta$

$$= \frac{15}{8}\cdot\frac{8}{17}$$

$$= \frac{15}{17}$$

and $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{15}{17}\right)} = \frac{17}{15}.$

Example 12. If $\sin\theta = \frac{a^2-b^2}{a^2+b^2}$ then find the value of $\cos\theta$ and $\tan\theta$ using the relation between the trigonometrical ratios, when θ is an acute angle.

Solution : $\sin\theta = \frac{a^2-b^2}{a^2+b^2}$

Now $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow$ $\cos^2\theta = 1 - \sin^2\theta$

or $\cos^2\theta = 1 - \frac{\left(a^2-b^2\right)^2}{\left(a^2+b^2\right)^2}$

$$= \frac{\left(a^2+b^2\right)^2 - \left(a^2-b^2\right)^2}{\left(a^2+b^2\right)^2}$$

$$= \frac{4a^2b^2}{\left(a^2+b^2\right)^2}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{2ab}{\left(a^2+b^2\right)}$$

or $$\cos\theta = \frac{2ab}{\left(a^2+b^2\right)} \quad (\because \ \theta \text{ is an acute angle})$$

Now $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\dfrac{a^2-b^2}{a^2+b^2}\right)}{\left(\dfrac{2ab}{a^2+b^2}\right)} = \frac{a^2-b^2}{2ab}$$

Exercise 14.2

Solve with the help of relation between trigonometric ratio [Q. 1 to 10]

1. If $\operatorname{cosec} A = \frac{5}{4}$ then find the value $\cot A, \sin A, \cos A$.

2. If $\tan A = \frac{20}{21}$ then find the value $\cos A$ and $\sin A$.

3. If $\sin A = \frac{3}{5}$ then find the value $\cos A$ and $\tan A$.

4. If $\cos B = \frac{1}{3}$ then find remaining trigonometric ratios.

5. If $\sin A = \frac{5}{13}$ then find the value $\cos A$ and $\tan A$.

6. If $\tan A = \sqrt{2}-1$ then prove that $\sin A \cos A = \frac{1}{2\sqrt{2}}$.

7. If $\tan A = 2$ then find the value $\sec A \sin A + \tan^2 A - \operatorname{cosec} A$.

8. If $\sin\theta = \frac{4}{5}$ then find the value $\frac{4\tan\theta - 5\cos\theta}{\sec\theta + 4\cot\theta}$.

9. If $\cos\theta = \frac{1}{\sqrt{2}}$ then find the value $\sin\theta$ and $\cot\theta$.

10. If $\sec\theta = 2$ then evaluate $\tan\theta, \cos\theta$ and $\sin\theta$.

14.04 Trigonometric Identities :

Such trigonometric relations which are always true for the angles involved and for those angles for which trigonometrical ratios are defined are called trigonometric identities.

The relations defined in article 14.02 and 14.03 are the true for all values of angle θ . Thus these relations are called basic identities.

In trigonometry all relations are not identities, for example $\sin\theta = \cos\theta$ is an equation because this is not true for all values of θ .

To prove the trigonometrical identities one should take care of the following points :

(i) Always start form the difficult side of the identity and making use of basic identities and find second side of the identity.

(ii) If the identity contains the trigonometrical ratios then it is always better to convert these ratios in term of sines and cosines

(iii) If there exists any radical sign then it should be removed.

(iv) In some problems we may use rationalisation.

(v) If it is not possible to obtain one side form the other then simplify both the sides as far as possible and prove them identically equal.

Illustrative Examples

Example 13. Prove the identity :

$$(\sec\theta+\cos\theta)(\sec\theta-\cos\theta)=\tan^2\theta+\sin^2\theta.$$

Solution : L.H.S $= \sec^2\theta - \cos^2\theta$

$$= 1+\tan^2\theta-(1-\sin^2\theta) \quad \left[\because \sec^2\theta = 1+\tan^2\theta \text{ and } \cos^2\theta = 1-\sin^2\theta\right]$$

$$= \tan^2\theta + \sin^2\theta$$

= R.H.S

Example 14. Prove the identity :

$$(\text{cosec}\,\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1.$$

Solution : L.H.S $= (\text{cosec}\,\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)$

$$= \left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

(converting all ratios into sine and cosine θ

$$= \left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \frac{\cos^2\theta}{\sin\theta}\cdot\frac{\sin^2\theta}{\cos\theta}\cdot\frac{1}{\sin\theta\cos\theta} \quad \text{(Using basic identities)}$$

$= 1$

= R.H.S

Example 15. Prove the identity :

$$\sqrt{\sec^2\theta + \text{cosec}^2\theta} = \tan\theta + \cot\theta \cdot$$

Solution : L.H.S $= \sqrt{\sec^2\theta + \text{cosec}^2\theta}$

$= \sqrt{\left(1+\tan^2\theta\right)+\left(1+\cot^2\theta\right)}$ [use of trigonometric identities]

$= \sqrt{\tan^2\theta + 2 + \cot^2\theta}$

$= \sqrt{\tan^2\theta + 2\tan\theta\cot\theta + \cot^2}$ $[\because \tan\theta\cot\theta = 1]$

$= \sqrt{(\tan\theta + \cot\theta)^2}$

$= \tan\theta + \cot\theta$

= R.H.S

Example 16. Prove the identity :

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\,\theta + \cot\theta \cdot$$

Solution : To remove radical sign multiplying numerator and denominator by $\sqrt{1+\cos\theta}$ in the L.H.S.

L.H.S. $= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \times \sqrt{\frac{1+\cos\theta}{1+\cos\theta}}$

$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$

$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$ $\left[\because 1-\cos^2\theta = \sin^2\theta\right]$

$= \frac{1+\cos\theta}{\sin\theta}$

$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$

$= \text{cosec}\,\theta + \cot\theta$

= R.H.S.

Example 17. Prove the identity :

$$\frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta}=1-2\sec\theta\tan\theta+2\tan^2\theta\cdot$$

Solution : L.H.S. $=\dfrac{\dfrac{1}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}}$

$$=\frac{1-\sin\theta}{\cos\theta}\times\frac{\cos\theta}{1+\sin\theta}$$

$$=\frac{1-\sin\theta}{1+\sin\theta} \qquad \cdots(1)$$

$$\text{R.H.S.} =1-2\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}+2\cdot\frac{\sin^2\theta}{\cos^2\theta}$$

$$=\frac{\cos^2\theta-2\sin\theta+2\sin^2\theta}{\cos^2\theta}$$

$$=\frac{\left(\cos^2\theta+\sin^2\theta\right)-2\sin\theta+\sin^2\theta}{\cos^2\theta}$$

$$=\frac{1-2\sin\theta+\sin^2\theta}{1-\sin^2\theta}$$

$$=\frac{(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$

$$=\frac{1-\sin\theta}{1+\sin\theta}$$

From equation (1) and (2)

$\therefore$ L.H.S. = R.H.S.

Example 18. Prove the identity :

$$\frac{1}{\sec\theta-\tan\theta}-\frac{1}{\cos\theta}=\frac{1}{\cos\theta}-\frac{1}{\sec\theta+\tan\theta}\cdot$$

Solution : On rearranging both the sides

$$\frac{1}{\sec\theta-\tan\theta}+\frac{1}{\sec\theta+\tan\theta}=\frac{2}{\cos\theta}=2\sec\theta$$

Now, L.H.S. $= \dfrac{\sec\theta + \tan\theta + \sec\theta - \tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}$

$= \dfrac{2\sec\theta}{\sec^2\theta - \tan^2\theta}$

$= \dfrac{2\sec\theta}{1 + \tan^2\theta - \tan^2\theta}$

$= 2\sec\theta$

= R.H.S.

Example 19. Prove the identity :

$\sec^6\theta - \tan^6\theta = 1 + 3\tan^2\theta + 3\tan^4\theta$.

Solution : We know that

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Now, L.H.S. $= (\sec^2\theta)^3 - (\tan^2\theta)^3$

$= (\sec^2\theta - \tan^2\theta)(\sec^4\theta + \sec^2\theta\tan^2\theta + \tan^4\theta)$

$= (1 + \tan^2\theta - \tan^2\theta)\{\sec^2\theta(\sec^2\theta + \tan^2\theta) + \tan^4\theta\}$

$= 1 \cdot \{(1 + \tan^2\theta)(1 + \tan^2\theta + \tan^2\theta) + \tan^4\theta\}$

$= (1 + \tan^2\theta)(1 + 2\tan^2\theta) + \tan^4\theta$

$= 1 + 3\tan^2\theta + 3\tan^4\theta$

= R.H.S.

Exercise 14.3

Prove the following identities :

1. $\cos\theta \cdot \tan\theta = \sin\theta$
2. $(1 - \sin^2\theta)\tan^2\theta = \sin^2\theta$
3. $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta = \operatorname{cosec}\theta$
4. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$
5. $\operatorname{cosec}^6\theta - \cot^6\theta = 1 + 3\operatorname{cosec}^2\theta\cot^2\theta$

6. $\sin^2\theta\cos\theta+\tan\theta\sin\theta+\cos^3\theta=\sec\theta$

7. $\dfrac{\cos\theta}{1-\tan\theta}+\dfrac{\sin\theta}{1-\cot\theta}=\sin\theta+\cos\theta$

8. $\dfrac{\text{cosec}\theta}{\text{cosec}-1}+\dfrac{\text{cosec}\theta}{\text{cosec}\theta+1}=2\sec^2\theta$

9. $\dfrac{\sin\theta}{1-\cos\theta}=\dfrac{1+\cos\theta}{\sin\theta}$

10. $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\,\text{cosec}\theta$

11. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=\dfrac{1-\sin\theta}{\cos\theta}$

12. $\sqrt{\dfrac{\sec\theta+1}{\sec\theta-1}}=\cot\theta+\text{cosec}\theta$

13. $\dfrac{\sqrt{\left(\text{cosec}^2\theta-1\right)}}{\text{cosec}\theta}=\cos\theta$

14. $(1+\cot\theta-\text{cosec}\theta)(1+\tan\theta+\sec\theta)=2$

Importants Points

1. In right angled triangle

(i) $\sin\theta=\dfrac{\text{Perpendicular}}{\text{Base}}$ (ii) $\cos\theta=\dfrac{\text{Base}}{\text{Hypotenuse}}$

(iii) $\tan\theta=\dfrac{\text{Perpendicular}}{\text{Base}}$ (iv) $\cot\theta=\dfrac{\text{Base}}{\text{Perpendicular}}$

(v) $\sec\theta=\dfrac{\text{Hypotenuse}}{\text{Base}}$ (vi) $\text{cosec}\,\theta=\dfrac{\text{Hypotenuse}}{\text{Perpendicular}}$

2. (i) $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

(iii) $\text{cosec}\theta=\dfrac{1}{\sin\theta}$ (iv) $\sec\theta=\dfrac{1}{\cos\theta}$

3. $\sin^2\theta+\cos^2\theta=1$

4. $1+\tan^2\theta=\sec^2\theta$

5. $1+\cot^2\theta=\text{cosec}^2\theta$

Miscellaneous Exercise 14

1. If $\tan\theta = \sqrt{3}$ then the value of $\sin\theta$ is :

   (A) $\frac{1}{\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{2}{\sqrt{3}}$ (D) 1

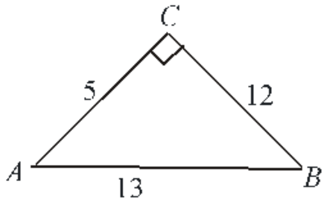
2. If $\sin\theta = \frac{5}{13}$ then the value of $\tan\theta$ is :

   (A) $\frac{5}{12}$ (B) $\frac{12}{13}$ (C) $\frac{13}{12}$ (D) $\frac{12}{5}$

3. If $\sqrt{3}\cos A = \sin A$ then the value of $\cot A$ is :

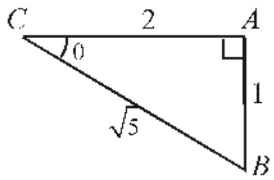
   (A) $\sqrt{3}$ (B) 1 (C) $\frac{1}{\sqrt{3}}$ (D) 2

4. In given ΔABC the value of $\cot A$ is :



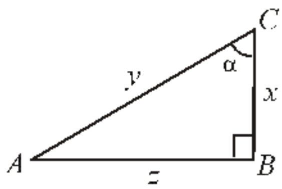
   (A) $\frac{12}{13}$ (B) $\frac{5}{12}$ (C) $\frac{5}{13}$ (D) $\frac{13}{5}$

5. In given ΔABC the value of $\tan\theta$ is :



   (A) 2 (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{1}{2}$

6. In given ΔABC the value of $\operatorname{cosec}\alpha$ is :



   (A) $\frac{y}{x}$ (B) $\frac{y}{z}$ (C) $\frac{x}{z}$ (D) $\frac{x}{y}$

7. The value of $\sin^2 30° + \cos^2 30°$ is :

   (A) 0 (B) 2 (C) 3 (D) 1

8. The value of $\text{cosec}^2 55° - \cot^2 55°$ is :

(A) 1 (B) 2 (C) 3 (D) 0

9. If $\cot\phi = \frac{20}{21}$ then the value of $\text{cosec}\,\phi$ is :

(A) $\frac{21}{20}$ (B) $\frac{20}{29}$ (C) $\frac{29}{21}$ (D) $\frac{21}{29}$

10. If in ΔABC , $\angle B = 90°$, $c = 12$ cm $a = 9$ cm then the value of $\cos C$ is :

(A) $\frac{3}{5}$ (B) $\frac{3}{4}$ (C) $\frac{5}{3}$ (D) $\frac{4}{5}$

11. The value of $(\sec 40° + \tan 40°)(\sec 40° - \tan 40°)$ is :

(A) -1 (B) 1 (C) $\cos 40°$ (D) $\sin 40°$

12. The value of $\frac{1}{\sin\theta - \tan\theta}$ is :

(A) $\frac{\cot\theta}{\cos\theta - 1}$ (B) $\frac{\cot\theta}{\cot\theta - \text{cosec}\,\theta}$

(C) $\text{cosec}\,\theta - \cot\theta$ (D) $\cot\theta$

13. The value of $\frac{\sec A - 1}{\sec A + 1}$ is:

(A) $\frac{1+\cos A}{1-\cos A}$ (B) $\frac{\cos A - 1}{1+\cos A}$ (C) $\frac{1-\cos A}{1+\cos A}$ (D) $\frac{\cos A - 1}{1 - \cos A}$

14. The value of $\cot^2\theta - \frac{1}{\sin^2\theta}$ is :

(A) 2 (B) 1 (C) 0 (D) -1

15. If $\text{cosec}\,\theta = \frac{41}{40}$ then find the value of $\tan\theta$ and $\cos\theta$.

....

16. If in ΔABC , $\angle B$ is right angle and $AB = 12$ cm and $BC = 5$ cm then find the value of $\sin A, \tan A, \sin C$ and $\cot C$.

....

17. If $\cos\theta = \frac{3}{5}$ then evaluate $\frac{\sin\theta - \cot\theta}{2\tan\theta}$

....

18. If $\cos\theta = \frac{21}{29}$ then evaluate $\frac{\sec\theta}{\tan\theta - \sin\theta}$

....

19. If $\cot A = \sqrt{3}$ then prove that $\sin A\cos B + \cos A\sin B = 1$.

....

Using relation between trigonometric ratios prove the following [Q. 20-24].

20. If $\tan\theta = \frac{4}{3}$ then evaluate $\frac{3\sin\theta + 2\cos\theta}{3\sin\theta - 2\cos\theta}$.

21. If $\cot\theta = \frac{b}{a}$ then evaluate $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

22. If $\text{cosec}\, A = 2$ then evaluate $\cot A + \frac{\sin A}{1 + \cos A}$.

23. If $\cot\theta = \frac{1}{\sqrt{3}}$ then evaluate $\frac{1 - \cos^2\theta}{2 - \sin^2\theta} = \frac{3}{5}$.

24. If $\sin A = \frac{1}{3}$ then evaluate $\cos A\,\text{cosec}\, A + \tan A\sec A$.

Prove the following [Q. 25-27]

25. $\sqrt{\sec^2 A + \text{cosec}^2 A} = \tan A + \cot A$

26. $\frac{1 + \sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1 - \cos\theta}$

27. $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \tan A + \sec A$

Prove the following identities [Q. 28-29]

28. $\frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\tan\beta$

29. $\sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}$

Prove the following [Q. 30 - 35]

30. $\cos^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$

31. $\sec^2\theta - \text{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

32. $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$

33. $(\sin A + \text{cosec}\, A)^2 + (\cos A + \sec A)^2 = \tan^2 A + \cot^2 A + 7$

34. $\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}$

Answers

Exercise 14.1

1. $\sin B=\frac{7}{25}$, $\cos B=\frac{24}{25}$, $\tan B=\frac{7}{24}$

$\operatorname{cosec} B=\frac{25}{7}$, $\sec B=\frac{25}{24}$, $\cot B=\frac{24}{7}$

$\sin C=\frac{24}{25}$, $\cos C=\frac{7}{25}$, $\tan C=\frac{24}{7}$

$\operatorname{cosec} C=\frac{25}{24}$, $\sec C=\frac{25}{7}$, $\cot C=\frac{7}{24}$

2. $\sin A=\frac{12}{13}$, $\cos A=\frac{5}{13}$, $\tan A=\frac{12}{5}$

$\operatorname{cosec} A=\frac{13}{12}$, $\sec A=\frac{13}{5}$, $\cot A=\frac{5}{12}$

$\sin C=\frac{5}{13}$, $\cos C=\frac{12}{13}$, $\tan C=\frac{5}{12}$

$\operatorname{cosec} C=\frac{13}{5}$, $\sec C=\frac{13}{12}$, $\cot C=\frac{12}{5}$

4. $2\sqrt{2}+\frac{3}{8}$

5. $\sin\theta=\frac{15}{17}$, $\tan\theta=\frac{15}{8}$, $\operatorname{cosec}\theta=\frac{17}{15}$,

$\sec\theta=\frac{17}{8}$, $\cot\theta=\frac{8}{15}$

6. $\frac{169}{229}$

7. $\frac{5}{14}$

9. 7

10. (i) $\frac{a}{\sqrt{4b^2-3a^2}}$ (ii) $\frac{2\sqrt{b^2-a^2}}{\sqrt{4b^2-3a^2}}$ (iii) $\frac{a}{2\sqrt{b^2-a^2}}$

Exercise 14.2

1. $\cot A = \frac{3}{4}, \ \sin A = \frac{4}{5}, \cos A = \frac{3}{5}$

2. $\cos A = \frac{21}{29}, \sin A = \frac{20}{29}$

3. $\cos A = \frac{4}{5}, \tan A = \frac{3}{4}$

4. $\sin B = \frac{2\sqrt{2}}{3}, \tan B = 2\sqrt{2}, \ \cot B = \frac{1}{2\sqrt{2}},$
   $\sec B = 3, \ \operatorname{cosec} B = \frac{3}{2\sqrt{2}}$

5. $\cos A = \frac{12}{13}, \ \tan A = \frac{5}{12}$

7. $\frac{12 - \sqrt{5}}{2}$

8. $\frac{1}{2}$

9. $\sin \theta = \frac{1}{\sqrt{2}}, \cot \theta = 1$

10. $\cos\theta = \frac{1}{2}, \sin \theta = \frac{\sqrt{3}}{2}, \tan \theta = \sqrt{3}$

Miscellaneous Exercise 14

1. (B) 2. (A) 3. (C) 4. (B) 5. (D) 6. (B)
7. (D) 8. (A) 9. (C) 10. (A) 11. (B) 12. (A)
13. (C) 14. (D)

15. $\tan\theta = \frac{40}{9}, \ \cos\theta = \frac{9}{41}$ 16. $\frac{5}{13}, \frac{5}{12}, \frac{12}{13}, \frac{5}{12}$ 17. $\frac{3}{160}$

18. $\frac{841}{160}$ 20. 3 21. $\frac{b+a}{b-a}$ 22. 2 24. $\frac{16\sqrt{2}+3}{8}$

❑