CHAPTER XVIII.

INTEREST AND ANNUITIES.

229. In this chapter we shall explain how the solution of questions connected with Interest and Discount may be simplified by the use of algebraical formulæ.

We shall use the terms *Interest*, *Discount*, *Present Value* in their ordinary arithmetical sense; but instead of taking as the rate of interest the interest on £100 for one year, we shall find it more convenient to take the interest on £1 for one year.

230. *To find the interest and amount of a given sum in a given time at simple interest.*

Let P be the principal in pounds, r the interest of £1 for one year, n the number of years, I the interest, and M the amount.

The interest of P for one year is Pr, and therefore for n years is Pnr; that is,

$$I = Pnr \quad \text{.............................(1).}$$

Also $$M = P + I;$$

that is, $$M = P(1 + nr) \quad \text{........................(2).}$$

From (1) and (2) we see that if of the quantities P, n, r, I, or P, n, r, M, any three be given the fourth may be found.

231. *To find the present value and discount of a given sum due in a given time, allowing simple interest.*

Let P be the given sum, V the present value, D the discount, r the interest of £1 for one year, n the number of years.

Since V is the sum which put out to interest at the present time will in n years amount to P, we have

$$P = V(1 + nr);$$

$$\therefore \quad V = \frac{P}{1 + nr}.$$

And $$D = P - V = P - \frac{P}{1 + nr};$$

$$\therefore \quad D = \frac{Pnr}{1 + nr}.$$

NOTE. The value of D given by this equation is called the *true discount.* But in practice when a sum of money is paid before it is due, it is customary to deduct the *interest* on the debt instead of the true discount, and the money so deducted is called the *banker's discount;* so that

$$\text{Banker's Discount} = Pnr.$$

$$\text{True Discount} = \frac{Pnr}{1 + nr}.$$

Example. The difference between the true discount and the banker's discount on £1900 paid 4 months before it is due is 6*s*. 8*d*.; find the rate per cent., simple interest being allowed.

Let r denote the interest on £1 for one year; then the banker's discount is $\frac{1900r}{3}$, and the true discount is $\dfrac{\frac{1900r}{3}}{1 + \frac{1}{3}r}$.

$$\therefore \quad \frac{1900r}{3} - \frac{\frac{1900r}{3}}{1 + \frac{1}{3}r} = \frac{1}{3};$$

whence $$1900r^2 = 3 + r;$$

$$\therefore \quad r = \frac{1 \pm \sqrt{1 + 22800}}{3800} = \frac{1 \pm 151}{3800}.$$

Rejecting the negative value, we have $r = \frac{152}{3800} = \frac{1}{25}$;

$$\therefore \text{ rate per cent.} = 100r = 4.$$

232. *To find the interest and amount of a given sum in a given time at compound interest.*

Let P denote the principal, R the amount of £1 in one year, n the number of years, I the interest, and M the amount.

The amount of P at the end of the first year is PR; and, since this is the principal for the second year, the amount at the end of the second year is $PR \times R$ or PR^2. Similarly the amount at the end of the third year is PR^3, and so on; hence the amount in n years is PR^n; that is,

$$M = PR^n;$$

$$\therefore \quad I = P(R^n - 1).$$

NOTE. If r denote the interest on £1 for one year, we have

$$R = 1 + r.$$

233. In business transactions when the time contains a fraction of a year it is usual to allow *simple* interest for the fraction of the year. Thus the amount of £1 in $\frac{1}{2}$ year is reckoned $1 + \frac{r}{2}$; and the amount of P in $4\frac{2}{3}$ years at compound interest is $PR^4\left(1 + \frac{2}{3}r\right)$. Similarly the amount of P in $n + \frac{1}{m}$ years is $PR^n\left(1 + \frac{r}{m}\right)$.

If the interest is payable more than once a year there is a distinction between the *nominal annual rate* of interest and that actually received, which may be called the *true annual rate*; thus if the interest is payable twice a year, and if r is the *nominal* annual rate of interest, the amount of £1 in half a year is $1 + \frac{r}{2}$, and therefore in the whole year the amount of £1 is $\left(1 + \frac{r}{2}\right)^2$, or $1 + r + \frac{r^2}{4}$; so that the *true* annual rate of interest is $r + \frac{r^2}{4}$.

234. If the interest is payable q times a year, and if r is the nominal annual rate, the interest on £1 for each interval is $\frac{r}{q}$, and therefore the amount of P in n years, or qn intervals, is $P\left(1 + \frac{r}{q}\right)^{qn}$.

In this case the interest is said to be "converted into principal" q times a year.

If the interest is convertible into principal every moment, then q becomes infinitely great. To find the value of the amount, put $\frac{r}{q} = \frac{1}{x}$, so that $q = rx$; thus

$$\text{the amount} = P\left(1+\frac{r}{q}\right)^{qn} = P\left(1+\frac{1}{x}\right)^{xnr} = P\left\{\left(1+\frac{1}{x}\right)^{x}\right\}^{nr}$$

$$= Pe^{nr}, \text{ [Art. 220, Cor.,]}$$

since x is infinite when q is infinite.

235. *To find the present value and discount of a given sum due in a given time, allowing compound interest.*

Let P be the given sum, V the present value, D the discount, R the amount of £1 for one year, n the number of years.

Since V is the sum which, put out to interest at the present time, will in n years amount to P, we have

$$P = VR^n;$$

$$\therefore \quad V = \frac{P}{R^n} = PR^{-n},$$

and

$$D = P(1 - R^{-n}).$$

Example. The present value of £672 due in a certain time is £126; if compound interest at $4\frac{1}{6}$ per cent. be allowed, find the time; having given

$$\log 2 = \cdot 30103, \ \log 3 = \cdot 47712.$$

Here

$$r = \frac{4\frac{1}{6}}{100} = \frac{1}{24}, \text{ and } R = \frac{25}{24}.$$

Let n be the number of years; then

$$672 = 126\left(\frac{25}{24}\right)^n;$$

$$\therefore \ n \log \frac{25}{24} = \log \frac{672}{126},$$

or

$$n \log \frac{100}{96} = \log \frac{16}{3};$$

$$\therefore \ n(\log 100 - \log 96) = \log 16 - \log 3,$$

$$n = \frac{4\log 2 - \log 3}{2 - 5\log 2 - \log 3}$$

$$n = \frac{\cdot 72700}{\cdot 01773} = 41, \text{ very nearly};$$

thus the time is very nearly 41 years.

EXAMPLES. XVIII. a.

When required the following logarithms may be used.

$$\log 2 = \cdot 3010300, \quad \log 3 = \cdot 4771213,$$
$$\log 7 = \cdot 8450980, \quad \log 11 = 1 \cdot 0413927.$$

1. Find the amount of £100 in 50 years, at 5 per cent. compound interest; given $\log 114 \cdot 674 = 2 \cdot 0594650$.

2. At simple interest the interest on a certain sum of money is £90, and the discount on the same sum for the same time and at the same rate is £80; find the sum.

3. In how many years will a sum of money double itself at 5 per cent. compound interest?

4. Find, correct to a farthing, the present value of £10000 due 8 years hence at 5 per cent. compound interest; given

$$\log 67683 \cdot 94 = 4 \cdot 8304856.$$

5. In how many years will £1000 become £2500 at 10 per cent. compound interest?

6. Shew that at simple interest the discount is half the harmonic mean between the sum due and the interest on it.

7. Shew that money will increase more than a hundredfold in a century at 5 per cent. compound interest.

8. What sum of money at 6 per cent. compound interest will amount to £1000 in 12 years? Given

$$\log 106 = 2 \cdot 0253059, \quad \log 49697 = 4 \cdot 6963292.$$

9. A man borrows £600 from a money-lender, and the bill is renewed every half-year at an increase of 18 per cent.: what time will elapse before it reaches £6000? Given $\log 118 = 2 \cdot 071882$.

10. What is the amount of a farthing in 200 years at 6 per cent. compound interest? Given $\log 106 = 2 \cdot 0253059$, $\log 115 \cdot 0270 = 2 \cdot 0611800$.

Annuities.

236. An **annuity** is a fixed sum paid periodically under certain stated conditions; the payment may be made either once a year or at more frequent intervals. Unless it is otherwise stated we shall suppose the payments annual.

An **annuity certain** is an annuity payable for a fixed term of years independent of any contingency; a **life annuity** is an annuity which is payable during the lifetime of a person, or of the survivor of a number of persons.

A **deferred annuity,** or **reversion,** is an annuity which does not begin until after the lapse of a certain number of years; and when the annuity is deferred for n years, it is said to commence after n years, and the first payment is made at the end of $n+1$ years.

If the annuity is to continue for ever it is called a **perpetuity**; if it does not commence at once it is called a **deferred perpetuity.**

An annuity left unpaid for a certain number of years is said to be **forborne** for that number of years.

237. *To find the amount of an annuity left unpaid for a given number of years, allowing simple interest.*

Let A be the annuity, r the interest of £1 for one year, n the number of years, M the amount.

At the end of the first year A is due, and the amount of this sum in the remaining $n-1$ years is $A+(n-1)rA$; at the end of the second year another A is due, and the amount of this sum in the remaining $(n-2)$ years is $A+(n-2)rA$; and so on. Now M is the sum of all these amounts;

$$\therefore\ M=\{A+(n-1)rA\}+\{A+(n-2)rA\}+\ldots\ldots+(A+rA)+A,$$

the series consisting of n terms;

$$\therefore\ M=nA+(1+2+3+\ldots\ldots+\overline{n-1})rA$$

$$=nA+\frac{n(n-1)}{2}rA.$$

238. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

Let A be the annuity, R the amount of £1 for one year, n the number of years, M the amount.

At the end of the first year A is due, and the amount of this sum in the remaining $n-1$ years is AR^{n-1}; at the end of the second year another A is due, and the amount of this sum in the remaining $n-2$ years is AR^{n-2}; and so on.

$$\therefore\ M=AR^{n-1}+AR^{n-2}+\ldots\ldots+AR^2+AR+A$$

$$=A(1+R+R^2+\ldots\ldots\text{ to }n\text{ terms})$$

$$=A\,\frac{R^n-1}{R-1}.$$

239. In finding the present value of annuities it is always customary to reckon compound interest; the results obtained when simple interest is reckoned being contradictory and untrustworthy. On this point and for further information on the subject of annuities the reader may consult Jones on the *Value of Annuities and Reversionary Payments*, and the article *Annuities* in the *Encyclopædia Britannica*.

240. *To find the present value of an annuity to continue for a given number of years, allowing compound interest.*

Let A be the annuity, R the amount of £1 in one year, n the number of years, V the required present value.

The present value of A due in 1 year is AR^{-1};

the present value of A due in 2 years is AR^{-2};

the present value of A due in 3 years is AR^{-3};

and so on. [Art. 235.]

Now V is the sum of the present values of the different payments;

$$\therefore\ V = AR^{-1} + AR^{-2} + AR^{-3} + \ldots\ldots \text{to } n \text{ terms}$$

$$= AR^{-1}\,\frac{1-R^{-n}}{1-R^{-1}}$$

$$= A\,\frac{1-R^{-n}}{R-1}.$$

NOTE. This result may also be obtained by dividing the value of M, given in Art. 238, by R^n. [Art. 232.]

COR. If we make n infinite we obtain for the present value of a *perpetuity*

$$V = \frac{A}{R-1} = \frac{A}{r}.$$

241. If mA is the present value of an annuity A, the annuity is said to be worth m years' purchase.

In the case of a perpetual annuity $mA = \dfrac{A}{r}$; hence

$$m = \frac{1}{r} = \frac{100}{\text{rate per cent.}};$$

that is, the number of years' purchase of a perpetual annuity is obtained by dividing 100 by the rate per cent.

As instances of perpetual annuities we may mention the income arising from investments in irredeemable Stocks such as many Government Securities, Corporation Stocks, and Railway Debentures. A good test of the credit of a Government is furnished by the number of years' purchase of its Stocks; thus the $2\frac{3}{4}$ p. c. Consols at $96\frac{1}{4}$ are worth 35 years' purchase; Egyptian 4 p. c. Stock at 96 is worth 24 years' purchase; while Austrian 5 p. c. Stock at 80 is only worth 16 years' purchase.

242. *To find the present value of a deferred annuity to commence at the end of* p *years and to continue for* n *years, allowing compound interest.*

Let A be the annuity, R the amount of £1 in one year, V the present value.

The first payment is made at the end of $(p+1)$ years. [Art. 236.]

Hence the present values of the first, second, third ... payments are respectively

$$AR^{-(p+1)},\ AR^{-(p+2)},\ AR^{-(p+3)},\ \ldots$$

$$\therefore\ V = AR^{-(p+1)} + AR^{-(p+2)} + AR^{-(p+3)} + \ldots\ldots \text{ to } n \text{ terms}$$

$$= AR^{-(p+1)}\frac{1-R^{-n}}{1-R^{-1}}$$

$$= \frac{AR^{-p}}{R-1} - \frac{AR^{-p-n}}{R-1}.$$

Cor. The present value of a *deferred perpetuity* to commence after p years is given by the formula

$$V = \frac{AR^{-p}}{R-1}.$$

243. A **freehold estate** is an estate which yields a perpetual annuity called the *rent;* and thus the value of the estate is equal to the present value of a perpetuity equal to the rent.

It follows from Art. 241 that if we know the number of years' purchase that a tenant pays in order to buy his farm, we obtain the rate per cent. at which interest is reckoned by dividing 100 by the number of years' purchase.

Example. The reversion after 6 years of a freehold estate is bought for £20000; what rent ought the purchaser to receive, reckoning compound interest at 5 per cent.? Given log 1·05 = ·0211893, log 1·340096 = ·1271358.

The rent is equal to the annual value of the perpetuity, deferred for 6 years, which may be purchased for £20000.

Let £A be the value of the annuity; then since $R = 1{\cdot}05$, we have

$$20000 = \frac{A \times (1{\cdot}05)^{-6}}{\cdot 05};$$

$$\therefore\ A \times (1{\cdot}05)^{-6} = 1000;$$

$$\log A - 6 \log 1{\cdot}05 = 3,$$

$$\log A = 3{\cdot}1271358 = \log 1340{\cdot}096.$$

$$\therefore\ A = 1340{\cdot}096, \text{ and the rent is £1340. 1}s. \text{ 11}d.$$

244. Suppose that a tenant by paying down a certain sum has obtained a lease of an estate for $p + q$ years, and that when q years have elapsed he wishes to renew the lease for a term $p + n$ years; the sum that he must pay is called the **fine** for renewing n years of the lease.

Let A be the annual value of the estate; then since the tenant has paid for p of the $p + n$ years, the fine must be equal to the present value of a deferred annuity A, to commence after p years and to continue for n years; that is,

$$\text{the fine} = \frac{AR^{-p}}{R-1} - \frac{AR^{-p-n}}{R-1}. \qquad \text{[Art. 242.]}$$

EXAMPLES. XVIII. b.

The interest is supposed compound unless the contrary is stated.

1. A person borrows £672 to be repaid in 5 years by annual instalments of £120; find the rate of interest, reckoning simple interest.

2. Find the amount of an annuity of £100 in 20 years, allowing compound interest at $4\frac{1}{2}$ per cent. Given

$$\log 1{\cdot}045 = {\cdot}0191163, \quad \log 24{\cdot}117 = 1{\cdot}3823260.$$

3. A freehold estate is bought for £2750; at what rent should it be let so that the owner may receive 4 per cent. on the purchase money?

4. A freehold estate worth £120 a year is sold for £4000; find the rate of interest.

5. How many years' purchase should be given for a freehold estate, interest being calculated at $3\frac{1}{2}$ per cent.?

6. If a perpetual annuity is worth 25 years' purchase, find the amount of an annuity of £625 to continue for 2 years.

7. If a perpetual annuity is worth 20 years' purchase, find the annuity to continue for 3 years which can be purchased for £2522.

8. When the rate of interest is 4 per cent., find what sum must be paid now to receive a freehold estate of £400 a year 10 years hence; having given $\log 104 = 2{\cdot}0170333$, $\log 6{\cdot}75565 = {\cdot}8296670$.

9. Find what sum will amount to £500 in 50 years at 2 per cent., interest being payable every moment; given $e^{-1} = {\cdot}3678$.

10. If 25 years' purchase must be paid for an annuity to continue n years, and 30 years' purchase for an annuity to continue $2n$ years, find the rate per cent.

11. A man borrows £5000 at 4 per cent. compound interest; if the principal and interest are to be repaid by 10 equal annual instalments, find the amount of each instalment; having given

$$\log 1{\cdot}04 = {\cdot}0170333 \text{ and } \log 675565 = 5{\cdot}829667.$$

12. A man has a capital of £20000 for which he receives interest at 5 per cent.; if he spends £1800 every year, shew that he will be ruined before the end of the 17th year; having given

$$\log 2 = {\cdot}3010300, \ \log 3 = {\cdot}4771213, \ \log 7 = {\cdot}8450980.$$

13. The annual rent of an estate is £500; if it is let on a lease of 20 years, calculate the fine to be paid to renew the lease when 7 years have elapsed allowing interest at 6 per cent.; having given

$$\log 106 = 2{\cdot}0253059, \quad \log 4{\cdot}688385 = {\cdot}6710233, \quad \log 3{\cdot}118042 = {\cdot}4938820.$$

14. If a, b, c years' purchase must be paid for an annuity to continue n, $2n$, $3n$ years respectively; shew that

$$a^2 - ab + b^2 = ac.$$

15. What is the present worth of a perpetual annuity of £10 payable at the end of the first year, £20 at the end of the second, £30 at the end of the third, and so on, increasing £10 each year; interest being taken at 5 per cent. per annum?