Exercise 8.1

Answer 1E.

Consider the following curve:

$y = 2x - 5, -1 \le x \le 3$

Find the length of the curve by using the arc length formula.

Differentiate the equation $y = 2x - 5$ with respect to x.

$\frac{dy}{dx} = 2$

If $x = -1$, then the corresponding y coordinate is calculated as follows:

$$\begin{aligned} y &= 2(-1) - 5 \\ &= -7 \end{aligned}$$

If $x = 3$, then the corresponding y coordinate is calculated as follows:

$$\begin{aligned} y &= 2(3) - 5 \\ &= 1 \end{aligned}$$

Find the length of the curve from the point $(-1,-7)$ to $(3,1)$.

So, the variable *x* varies from -1 to 3.

Plug these values in the arc length formula:

The length of the curve from the point $(-1,-7)$ to $(3,1)$ is,

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{-1}^{3} \sqrt{1+2^2}\, dx \\ &= \sqrt{5} \int_{-1}^{3} dx \\ &= \sqrt{5}\left[x\right]_{-1}^{3} \\ &= \sqrt{5}\left[3-(-1)\right] \\ &= 4\sqrt{5} \end{aligned}$$

Therefore, the length of the curve from the point $(-1,-7)$ to $(3,1)$ is $\boxed{4\sqrt{5}}$.

Verify the length of the curve obtained above using the distance formula for the points $(-1,-7),(3,1)$.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(3+1)^2+(1+7)^2}$$
$$=\sqrt{16+64}$$
$$=\sqrt{80}$$
$$=\sqrt{16\cdot 5}$$
$$=\boxed{4\sqrt{5}}$$

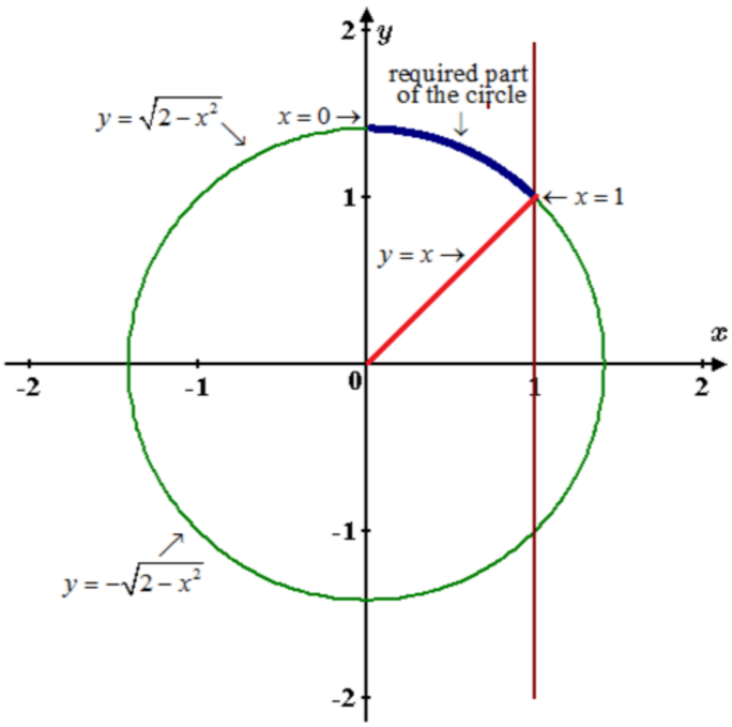
Hence, verified.

Answer 2E.

Let us consider the curve $y=\sqrt{2-x^2},0\le x\le 1$.

Need to find the length of the curve.

Sketch the graph is shown below:



Note that, if f' is continuous on $[a,b]$, then the length of the curve $y=f(x),a\le x\le b$ is

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \cdots\cdots(1)$$

Since $y=\sqrt{2-x^2}$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{2-x^2}}(-2x)$$
$$\frac{dy}{dx}=\frac{-x}{\sqrt{2-x^2}}$$

Now

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{2-x^2}$$
$$1+\left(\frac{dy}{dx}\right)^2=\frac{2}{2-x^2}$$
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{\frac{2}{2-x^2}}$$

By the formula (1), the length of the given curve is

$$L = \int_0^1 \sqrt{\frac{2}{2-x^2}}dx$$
$$= \sqrt{2}\int_0^1 \frac{1}{\sqrt{2-x^2}}dx$$
$$= \sqrt{2}\left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$$
$$= \sqrt{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0)\right]$$
$$= \sqrt{2}\left[\frac{\pi}{4} - 0\right]$$
$$= \frac{\pi\sqrt{2}}{4}$$

Hence the length of the given curve is $\boxed{\frac{\pi\sqrt{2}}{4}}$.

The above result can be confirmed by the observation that the arc between $x = 0$ and $x = 1$ is nothing but the part of the circle in the first quadrant divided by the line $y = x$.

Further, $y = x$ halves the arc of the circle into equal parts in the 1st quadrant.

So, the required part of the circle is $\frac{1}{8}^{\text{th}}$ part of the circumference of the circle with radius $\sqrt{2}$.

Thus, the length of the curve is $\frac{1}{8}\left(2\pi\sqrt{2}\right) = \boxed{\frac{\pi\sqrt{2}}{4}}$.

Answer 3E.

Given $y = \sin x, \quad 0 \le x \le \pi$

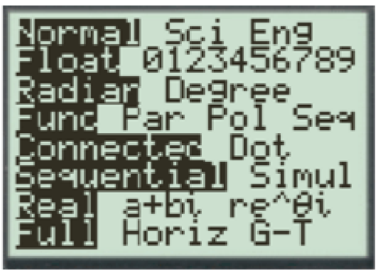
$$\frac{dy}{dx} = \cos x$$

Length of the curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^\pi \sqrt{1 + (\cos x)^2}\,dx$$

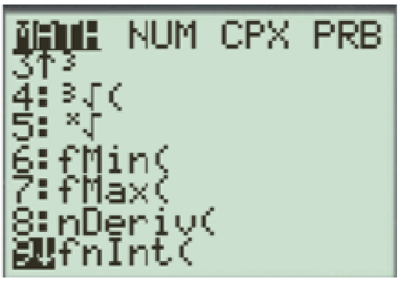
Using the graphing calculator with the following key strokes to solve the integral

This is problem involving with angles so set in the radian mode



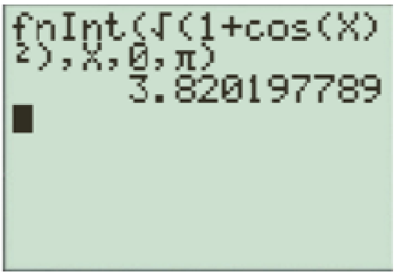
Using the graphing calculator with the following key strokes to solve the integral

Choose the key of $\boxed{MATH}$ and select the option $\boxed{fnInt}$



Instead of varible y using the varible x and eneter the integrand with the limits

$\boxed{2nd}\boxed{x^2}\boxed{1}\boxed{+}\boxed{\cos}\boxed{X,T,\theta,\eta}\boxed{)}\boxed{)}\boxed{,}\boxed{X,T,\theta,\eta}\boxed{,}\boxed{0}\boxed{,}\boxed{2nd}\boxed{\wedge}\boxed{)}\boxed{enter}$



The length of the curve is

$\boxed{L \approx 3.8202}$

Answer 4E.

Consider the function:

$$y = f(x), \qquad a \le x \le b$$

The derivative of the above function f' that continuous over the closed interval $[a,b]$, so the formula used to determine the arc length of the curve is:

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Consider the equation:

$$y = xe^{-x}, \qquad 0 \le x \le 2$$

Differentiate the above equation with respect to *x*:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(xe^{-x}\right)\\ &= e^{-x}\frac{d}{dx}(x) + x\frac{d}{dx}\left(e^{-x}\right)\\ &= \left(e^{-x}\times 1\right) + \left[x\times\left(-e^{-x}\right)\right]\\ &= e^{-x}(1-x)\end{aligned}$$

Determine the equation for the length of the curve:

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^2 \sqrt{1+\left(e^{-x}(1-x)\right)^2}\,dx \\ &= \int_0^2 \sqrt{1+e^{-2x}(1-x)^2}\,dx \\ &= 2.1024 \end{aligned}$$

So, the integral that represents the length of the curve is $\boxed{\int_0^2 \sqrt{1+e^{-2x}(1-x)^2}\,dx}$.

Hence, length of the curve calculated by the use of a calculator is $\boxed{2.1024}$.

Answer 5E.

Consider the curve $x=\sqrt{y}-y, 1\le y\le 4$.

The objective is to set up an integral that represents the length of the above curve.

Use the following formula to solve the problem.

It a curve has the equation $x=g(y), c\le y\le d$, and $g'(y)$ is continuous then the length of the curve is $L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$.

Now find $\frac{dx}{dy}$ from $x=\sqrt{y}-y$.

Differentiate $x=\sqrt{y}-y$ with respect to y.

$$\begin{aligned} \frac{dx}{dy} &= \frac{d}{dy}\left(y^{1/2}\right)-\frac{d}{dy}(y) \\ &= \frac{1}{2}y^{(1/2)-1}-1 \\ &= \frac{1}{2}y^{-1/2}-1 \\ &= \frac{1}{2\sqrt{y}}-1 \end{aligned}$$

Now substitute in the length of the curve formula.

Thus, the length of the given curve is, $L=\int_1^4 \sqrt{1+\left(\frac{1}{2\sqrt{y}}-1\right)^2}\,dy$.

Now use a calculator to compute the length of the curve.

$$\int_1^4 \sqrt{1+\left(\frac{1}{2\sqrt{y}}-1\right)^2}\,dy = 3.6094$$

Therefore, the length of the given curve is $\boxed{3.6094}$.

Answer 6E.

Consider the curve $x = y^2 - 2y$ in the interval $0 \le y \le 2$.

Differentiate the curve.

$$x = y^2 - 2y$$

$$\frac{dx}{dy} = 2y - 2$$

Recall that, the length of the curve $x = f(y)$ in the interval $a \le y \le d$ is as follows:

$$L = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy$$

Calculate the required length of the curve.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^2 \sqrt{1 + (2y-2)^2}\, dy$$

$$= \int_0^2 \sqrt{1 + 4y^2 + 4 - 8y}\, dy$$

$$= \int_0^2 \sqrt{4y^2 - 8y + 5}\, dy$$

Now evaluate this integral using a calculator.

Maple software can be used to find the definite integral.

Key board strokes of the command are as follows:

Maple command:

int ((4y^2-8y+5)^1/2,y=0..2,numeric);

Maple command and output:

$$> \; int\left(\left(4y^2 - 8y + 5\right)^{\frac{1}{2}}, y = 0..2, numeric\right);$$

2.957885715

Hence, the required length of the curve is

$$L = \int_0^2 \sqrt{4y^2 - 8y + 5}\, dy$$

$$= \boxed{2.9578857}$$

Answer 7E.

$$y = 1 + 6x^{3/2},\ 0 \le x \le 1$$
$$\frac{dy}{dx} = 0 + 6.\frac{3}{2}x^{3/2-1} = 9x^{1/2}$$

The length of the curve is

$$= \int_0^1 \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$
$$= \int_0^1 \sqrt{1 + \left(9\sqrt{x}\right)^2}\, dx$$
$$= \int_0^1 \sqrt{1 + 81x}dx$$
$$= \left[\frac{(1+81x)^{3/2}}{(3/2).81}\right]_0^1$$
$$= \frac{2}{243}\left[82^{3/2} - 1\right] = \boxed{\frac{2}{243}\left[82\sqrt{82} - 1\right]}$$

Answer 8E.

$$y^2 = 4(x+4)^3,\ 0 \le x \le 2,\ y > 0$$
$$\Rightarrow y = 2(x+4)^{3/2}$$
$$\Rightarrow \frac{dy}{dx} = 2.\frac{3}{2}(x+4)^{1/2}$$
$$= 3(x+4)^{1/2}$$

The length of the curve is

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_0^2 \sqrt{1 + 9(x+4)}dx$$
$$= \int_0^2 \sqrt{9x + 37}dx$$
$$= \left[\frac{2(9x+37)^{3/2}}{3 \times 9}\right]_0^2$$
$$= \frac{2}{27}\left[55^{3/2} - 37^{3/2}\right]$$
$$= \boxed{\frac{2}{27}\left[55\sqrt{55} - 37\sqrt{37}\right]}$$

Answer 9E.

Consider the curve,

$$y=\frac{x^3}{3}+\frac{1}{4x}, \qquad 1\le x\le 2 \quad \text{......} \ (1)$$

The objective is to find the arc length of the curve by using arc length formula.

The Arc length formula:

If f' is continuous on $[a,b]$, then the length of the curve, $y=f(x)$, $a\le x\le b$ is,

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dy}\right)^2}\,dx \quad \text{......} \ (2)$$

From equation (1),

$$y=\frac{x^3}{3}+\frac{1}{4x}, \qquad 1\le x\le 2$$

Differentiate on both sides with respect to *x*.

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^3}{3}+\frac{1}{4x}\right)$$

$$\frac{dy}{dx}=x^2-\frac{1}{4x^2}$$

The limits are $1\le x\le 2.$

Substitute these values in equation (2) to find the arc length.

$$L=\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=\int_1^2 \sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}\,dx$$

$$=\int_1^2 \sqrt{1+x^4+\frac{1}{16x^4}-\frac{1}{2}}dx$$

$$=\int_1^2 \sqrt{x^4+\frac{1}{16x^4}+\frac{1}{2}}dx$$

$$=\int_1^2 \sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}\,dx$$

$$=\int_1^2 \left(x^2+\frac{1}{4x^2}\right)dx$$

$$=\left[\frac{x^{2+1}}{(2+1)}+\frac{1}{4}\frac{x^{-2+1}}{(-2+1)}\right]_1^2$$

$$=\left[\frac{x^3}{3}-\frac{1}{4}\cdot\frac{1}{x}\right]_1^2$$

$$=\frac{2^3-1^3}{3}-\frac{1}{4}\left(\frac{1}{2}-\frac{1}{1}\right)$$

$$=\frac{7}{3}+\frac{1}{8}$$

$$=\boxed{\frac{59}{24}}$$

Therefore, the arc length of the curve is $L=\boxed{\frac{59}{24}}$

Answer 10E.

Consider the curve.

$$x=\frac{y^4}{8}+\frac{1}{4y^2}, 1\le y\le 2 \;\ldots\ldots\; (1)$$

The objective is to find the arc length of the curve by using arc length formula.

The Arc length formula:

If f' is continuous on $[a,b]$, then the length of the curve, $y=f(x)$, $a\le x\le b$ is,

$$L=\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \;\ldots\ldots\; (2)$$

From equation (1).

$$x=\frac{y^4}{8}+\frac{1}{4y^2}, 1\le y\le 2$$

Find the value of $\frac{dy}{dx}$ as follows:

Differentiate on both sides with respect to y.

$$\begin{aligned}\frac{dx}{dy}&=\frac{d}{dx}\left(\frac{y^4}{8}+\frac{1}{4y^2}\right)\\ &=\frac{4y^3}{8}+\frac{1}{4}\left(-\frac{2}{y^3}\right)\\ &=\frac{y^3}{2}-\frac{1}{2y^3}\end{aligned}$$

The limits are $1\le y\le 2.$

Substitute these values in equation (2).

$$L=\int_1^2 \sqrt{1+\left(\frac{y^3}{2}-\frac{1}{2y^3}\right)^2}\,dy$$

Simplify the value of $1+\left(\frac{y^3}{2}-\frac{1}{2y^3}\right)^2$ as follows:

$$\begin{aligned}1+\left(\frac{y^3}{2}-\frac{1}{2y^3}\right)^2&=1+\left(\frac{y^3}{2}-\frac{1}{2y^3}\right)^2\\ &=1+\frac{1}{4}\left(y^3-y^{-3}\right)^2\end{aligned}$$

$$\begin{aligned}&=1+\frac{1}{4}\left(y^6-2+y^{-6}\right)\\ &=\frac{1}{4}y^6+\frac{1}{4}y^{-6}+\frac{1}{2}\\ &=\frac{1}{4}\left(y^3+y^{-3}\right)^2\end{aligned}$$

Plug the value of $1+\left(\frac{y^3}{2}-\frac{1}{2y^3}\right)^2=\frac{1}{4}\left(y^3+y^{-3}\right)^2$ in L.

$$L=\int_1^2\sqrt{\frac{1}{4}\left(y^3+y^{-3}\right)^2}\,dy$$

$$=\frac{1}{2}\int_1^2\sqrt{\left(y^3+y^{-3}\right)^2}\,dy$$

$$=\frac{1}{2}\int_1^2\left(y^3+y^{-3}\right)dy$$

$$=\frac{1}{2}\left(\frac{y^4}{4}-\frac{1}{2}y^{-2}\right)\Big]_1^2$$

$$=\frac{1}{2}\left[\left(\frac{2^4}{4}-\frac{2^{-2}}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left[4-\frac{1}{8}+\frac{1}{4}\right]$$

$$=\frac{1}{2}\left(\frac{32-1+2}{8}\right)$$

$$=\frac{33}{16}$$

Therefore, the arc length of the curve is,

$$L=\boxed{\frac{33}{16}}.$$

Answer 11E.

Consider the following curve in the interval $1\le y\le 9$:

$$x=\frac{1}{3}\sqrt{y}\,(y-3)$$

Differentiate the equation with respect to *y*.

$$\frac{dx}{dy}=\frac{1}{3}\left[\sqrt{y}\cdot 1+(y-3)\cdot\frac{1}{2\sqrt{y}}\right] \qquad \text{Use Product Rule.}$$

$$=\frac{1}{3}\left[\frac{2y+y-3}{2\sqrt{y}}\right]$$

$$=\frac{1}{3}\left[\frac{3(y-1)}{2\sqrt{y}}\right]$$

$$=\frac{y-1}{2\sqrt{y}}$$

Use the arc length formula to find the length of a curve from one point to another.

$$\text{Length } L=\int_{y=a}^{b}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad \ldots\ldots()$$

Now use the formula to calculate L.

$$L=\int_1^9\sqrt{1+\left(\frac{y-1}{2\sqrt{y}}\right)^2}\,dy$$

$$=\int_1^9\sqrt{1+\frac{(y-1)^2}{4y}}\,dy$$

$$=\int_1^9\sqrt{\frac{4y+y^2-2y+1}{4y}}\,dy$$

$$=\int_1^9\sqrt{\frac{y^2+2y+1}{4y}}\,dy$$

$$=\int_1^9\sqrt{\frac{(y+1)^2}{\left(2\sqrt{y}\right)^2}}\,dy$$

Solve the equation further to find L.

$$L=\int_1^9\frac{y+1}{2\sqrt{y}}\,dy$$

$$=\frac{1}{2}\int_1^9\left(y^{\frac{1}{2}}+y^{-\frac{1}{2}}\right)dy$$

$$=\frac{1}{2}\left[\frac{2}{3}y^{\frac{3}{2}}+2y^{\frac{1}{2}}\right]_1^9$$

$$=\frac{1}{2}\left[\left\{\frac{2}{3}(9)^{\frac{3}{2}}+2(9)^{\frac{1}{2}}\right\}-\left\{\frac{2}{3}(1)^{\frac{3}{2}}+2(1)^{\frac{1}{2}}\right\}\right]$$

$$=\frac{1}{2}\left(24-\frac{8}{3}\right)$$

$$=\frac{1}{2}\left(\frac{72-8}{3}\right)$$

$$=\frac{1}{2}\left(\frac{64}{3}\right)$$

$$=\frac{32}{3}$$

Thus, the length of the given curve is $\boxed{\frac{32}{3}}$.

Answer 12E.

Consider the function $y = \ln(\cos x)$

The arc length function:

If a smooth curve C has the equation $y = f(x)$, $a \le x \le b$, let $s(x)$ be the distance along C from the initial point $p_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the arc length function,

That is, $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$

Rewrite the equation as $y = \ln(\cos x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x)$$

Or $\frac{dy}{dx} = -\tan x$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

The arc length along the curve from 0 to $\frac{\pi}{3}$ is

$$s(x) = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/3}$$

$$= \ln|\sec(\pi/3) + \tan(\pi/3)| - \ln|\sec 0 + \tan 0|$$

$$= \ln|2 + \sqrt{3}| - \ln|1 + 0|$$

$$= \boxed{\ln(2 + \sqrt{3})}$$

Answer 13E.

$y = \ln(\sec x)$

Then $\frac{dy}{dx} = \frac{1}{\sec x}.\sec x.\tan x$

$= \tan x$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x$$

$= \sec^2 x$

We know that the arc lengh formula is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The length of the curve is

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sec x\, dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$$

$$= \left[\ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec 0 + \tan 0|\right]$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

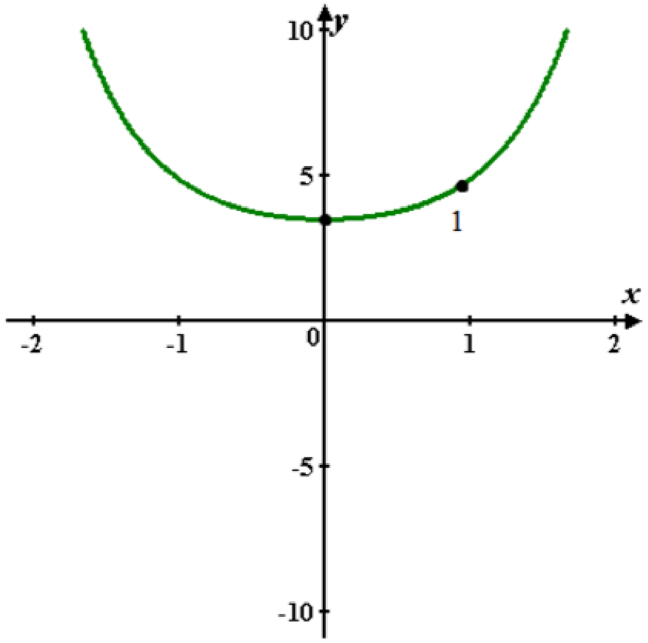
Thus $\boxed{L = \ln(\sqrt{2} + 1)}$

Answer 14E.

Consider the following curve:

$$y = 3 + \frac{1}{2}\cosh 2x, 0 \le x \le 1$$

Draw this curve as shown below:



Find the length of the arc between the two points.

Use the arc length formula to compute the length of the curve of this function.

Write the formula for arc length.

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Find the derivative of the given function.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(3 + \frac{1}{2}\cosh 2x\right)\\ &= \frac{1}{2}\cdot 2\sinh 2x\\ &= \sinh 2x\end{aligned}$$

Plug in to obtain the value of L.

Recall the hyperbolic sine and cosine identity that is used.

$$\begin{aligned}L &= \int_0^1 \sqrt{1 + (\sinh 2x)^2}\, dx\\ &= \int_0^1 \sqrt{\cosh^2 2x}\, dx\\ &= \int_0^1 \cosh 2x\, dx\\ &= \frac{1}{2}\sinh 2x\Big]_0^1\\ &= \frac{1}{2}\sinh 2(1) - \frac{1}{2}\sinh 2(0)\end{aligned}$$

Therefore, $L = \boxed{\frac{1}{2}\sinh 2}$.

Answer 15E.

Consider $y=\frac{x^2}{4}-\frac{1}{2}\ln x, \quad 1\le x\le 2$

$$\frac{dy}{dx}=\frac{x}{2}-\frac{1}{2x} \quad \left[\because \frac{d}{dx}(x^n)=nx^{n-1}\right]$$

Length of the curve

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_1^2\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx$$

$$=\int_1^2\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,dx$$

$$=\int_1^2\sqrt{1+\frac{x^2}{4}+\frac{1}{4x^2}-\frac{1}{2}}\,dx$$

$$=\int_1^2\sqrt{\frac{x^2}{4}+\frac{1}{4x^2}+\frac{1}{2}}\,dx$$

To find the integral using perfect square

$$=\int_1^2\sqrt{\left(\frac{x}{2}\right)^2+\left(\frac{1}{2x}\right)^2+2\cdot\left(\frac{x}{2}\right)\cdot\left(\frac{1}{2x}\right)}\,dx$$

$$=\int_1^2\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx$$

$$=\int_1^2\left(\frac{x}{2}+\frac{1}{2x}\right)dx$$

$$=\left[\frac{x^2}{4}+\frac{1}{2}\ln|x|\right]_1^2$$

$$=\frac{2^2-1^2}{4}+\frac{1}{2}\left(\ln|2|-\ln|1|\right)$$

$$=\boxed{\frac{3}{4}+\frac{1}{2}\ln 2}$$

Answer 16E.

Consider the following function:

$$y=\sqrt{x-x^2}+\sin^{-1}\left(\sqrt{x}\right).$$

The objective is to find the exact length of the curve.

First we consider about the interval of x:

$$\begin{cases} x\ge 0 \\ x-x^2\ge 0 \end{cases}$$

$$\Rightarrow 0\le x\le 1$$

To find the integral of the length of the curve, use the following Arc Length Formula:

If f' is continuous on $[a,b]$, then the length of the curve $y=f(x)$, $a \le x \le b$, is

$$L=\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$

Re-write the function as follows:

$$f(x)=\sqrt{x-x^2}+\sin^{-1}\left(\sqrt{x}\right)$$

Compute the derivative of the function as follows:

$$\begin{aligned}
f'(x) &= \frac{1-2x}{2\sqrt{x-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\cdot\frac{1}{2\sqrt{x}} \\
&= \frac{1-2x}{2\sqrt{x-x^2}}+\frac{1}{2\sqrt{x}\sqrt{1-x}} \\
&= \frac{1-2x}{2\sqrt{x-x^2}}+\frac{1}{2\sqrt{x-x^2}} \\
&= \frac{1-x}{\sqrt{x-x^2}}
\end{aligned}$$

The arc length of the function is as follows:

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+\left(\frac{1-x}{\sqrt{x-x^2}}\right)^2}\,dx \\
&= \int_0^1 \sqrt{1+\frac{(1-x)^2}{x-x^2}}\,dx \\
&= \int_0^1 \sqrt{\frac{x-x^2+1+x^2-2x}{x-x^2}}\,dx \\
&= \int_0^1 \sqrt{\frac{1-x}{x(1-x)}}\,dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \sqrt{\frac{1}{x}}\,dx \\
&= \int_0^1 x^{-1/2}\,dx \\
&= \left[2x^{\frac{1}{2}}\right]_0^1 \\
&= \boxed{2}
\end{aligned}$$

Therefore, the arc length of the function is $\boxed{2}$.

Answer 17E.

Consider the function.

$$y = \ln\left(1-x^2\right), 0 \le x \le \frac{1}{2} \quad \ldots\ldots (1)$$

The objective is to find the arc length of the curve by using arc length formula.

The Arc length formula:

If f' is continuous on $[a,b]$, then the length of the curve, $y = f(x)$, $a \le x \le b$ is,

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \ldots\ldots (2)$$

From equation (1).

$$y = \ln\left(1-x^2\right), 0 \le x \le \frac{1}{2}$$

Find the value of $\frac{dy}{dx}$ as follows:

Differentiate on both sides with respect to x.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\ln\left(1-x^2\right)\right) \\
&= \frac{1}{1-x^2}\frac{d}{dx}\left(1-x^2\right) \\
&= \frac{1}{1-x^2}(-2x) \\
&= \frac{-2x}{1-x^2}
\end{aligned}$$

Substitute the value of $\frac{dy}{dx}$ in equation (2).

$$\begin{aligned}
L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{\frac{1}{2}} \sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\, dx \\
&= \int_0^{\frac{1}{2}} \sqrt{1+\frac{4x^2}{1-2x^2+x^4}}\, dx \\
&= \int_0^{\frac{1}{2}} \sqrt{\frac{1-2x^2+x^4+4x^2}{1-2x^2+x^4}}\, dx \\
&= \int_0^{\frac{1}{2}} \sqrt{\frac{1+2x^2+x^4}{1-2x^2+x^4}}\, dx \\
&= \int_0^{\frac{1}{2}} \sqrt{\frac{\left(x^2+1\right)^2}{\left(1-x^2\right)^2}}\, dx \\
&= \int_0^{\frac{1}{2}} \sqrt{\left(\frac{x^2+1}{1-x^2}\right)^2}\, dx \\
&= \int_0^{\frac{1}{2}} \frac{x^2+1}{1-x^2}\, dx
\end{aligned}$$

The partial fractions of decomposition $\frac{x^2+1}{1-x^2}$ is,

$$\frac{1+x^2}{(1+x)(1-x)}=\frac{1}{x+1}-\frac{1}{x-1}-1$$

That implies,

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{x^2+1}{1-x^2}dx &= \int_0^{\frac{1}{2}}\left(-1+\frac{1}{(1+x)}-\frac{1}{(x-1)}\right)dx \\ &= -\int_0^{\frac{1}{2}}(1)dx+\int_0^{\frac{1}{2}}\left(\frac{1}{(1+x)}\right)dx-\int_0^{\frac{1}{2}}\left(\frac{1}{(x-1)}\right)dx \\ &= -(x)\Big|_0^{\frac{1}{2}}+\ln|(1+x)|\Big|_0^{\frac{1}{2}}-\ln|(x-1)|\Big|_0^{\frac{1}{2}}\end{aligned}$$

$$=-\frac{1}{2}+\ln\left(\frac{3}{2}\right)-\ln\left(\frac{1}{2}\right)+0$$

$$=-\frac{1}{2}+\ln\left(\frac{3/2}{1/2}\right)$$

$$=\boxed{-\frac{1}{2}+\ln(3)}$$

Therefore, the arc length is,

$$L=\boxed{-\frac{1}{2}+\ln(3)}.$$

Answer 18E.

Consider the function:

$$y=f(x),\qquad a\le x\le b$$

The derivative of the above function f' that continuous over the closed interval $[a,b]$, so the formula used to determine the arc length of the curve is:

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Consider the equation:

$$y=1-e^{-x},\qquad 0\le x\le 2$$

Differentiate the above equation with respect to *x*:

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(1-e^{-x}\right)\\ &=\frac{d}{dx}(1)+\frac{d}{dx}\left(e^{-x}\right)\\ &=0-\left(-e^{-x}\right)\\ &=e^{-x}\end{aligned}$$

Determine the equation for the length of the curve:

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^2 \sqrt{1+\left(e^{-x}\right)^2}\,dx$$
$$= \int_0^2 \sqrt{1+e^{-2x}}\,dx$$

Evaluate the above integral to obtain the exact length of the curve:

$$L = \int_0^2 \sqrt{1+e^{-2x}}\,dx$$

Substitute $u = 2x$ in the above integral:

$$u = 2x$$
$$du = 2dx$$

So, the integral changes to:

$$L = \frac{1}{2}\int \sqrt{1+e^{-u}}\,du$$

Substitute $v = 1+e^{-u}$ in the above integral:

$$v = 1+e^{-u}$$
$$dv = -e^{-u}du$$
$$= (1-v)\,du$$

So, the integral changes to:

$$L = \left(\frac{1}{2}\right) \int \frac{\sqrt{v}}{1-v}dv$$
$$= \left(\frac{-1}{2}\right) \int \frac{\sqrt{v}}{v-1}dv$$

Substitute $w^2 = v$ in the above integral:

$$w^2 = v$$
$$2wdw = dv$$

So, the integral changes to:

$$L = \left(\frac{-1}{2}\right)\int\left(\frac{w}{w^2-1}\right)\times 2wdw$$
$$= -\int\left(\frac{w^2}{w^2-1}\right)dw$$

Evaluate the above integral:

$$L = -\int\left(\frac{w^2}{w^2-1}\right)dw$$
$$= -\int\left(\frac{w^2-1+1}{w^2-1}\right)dw$$
$$= -\int\left(\frac{1}{w^2-1}\right)dw - \int dw$$

Factor the first denominator and apply partial fraction decomposition to obtain:

$$L = -\int\frac{1}{(w-1)(w+1)}dw - \int dw$$
$$= \int\frac{1}{2(w+1)}dw - \int\frac{1}{2(w-1)}dw - \int dw$$
$$= \left(\frac{1}{2}\right)\ln(w+1) - \left(\frac{1}{2}\right)\ln(w-1) - w$$

Plug in the values substituted to write the function in terms of x:

$$L = \left(\frac{1}{2}\right)\ln\left(\sqrt{v}+1\right) - \left(\frac{1}{2}\right)\ln\left(\sqrt{v}-1\right) - \sqrt{v}$$
$$= \left(\frac{1}{2}\right)\ln\left(\sqrt{1+e^{-u}}+1\right) - \left(\frac{1}{2}\right)\ln\left(\sqrt{1+e^{-u}}-1\right) - \sqrt{1+e^{-u}}$$
$$= \left(\frac{1}{2}\right)\ln\left(\sqrt{1+e^{-2x}}+1\right) - \left(\frac{1}{2}\right)\ln\left(\sqrt{1+e^{-2x}}-1\right) - \sqrt{1+e^{-2x}}$$

Substitute the limits in the above integral to obtain the value:

$$L = \left[\left(\frac{1}{2}\right)\ln\left(\sqrt{1+e^{-2x}}+1\right) - \left(\frac{1}{2}\right)\ln\left(\sqrt{1+e^{-2x}}-1\right) - \sqrt{1+e^{-2x}}\right]_0^2$$
$$= 2.22$$

Hence, the length of the curve is $\boxed{2.22}$.

Answer 19E.

Consider the following equation of the curve:

$y=\frac{1}{2}x^2$. ...... (1)

Find the arc length of the curve from point $P\left(-1,\frac{1}{2}\right)$ to $Q\left(1,\frac{1}{2}\right)$.

The arc of the curve $y=\frac{1}{2}x^2$ from the point $P\left(-1,\frac{1}{2}\right)$ to $Q\left(1,\frac{1}{2}\right)$, is shown in red color in the following figure:

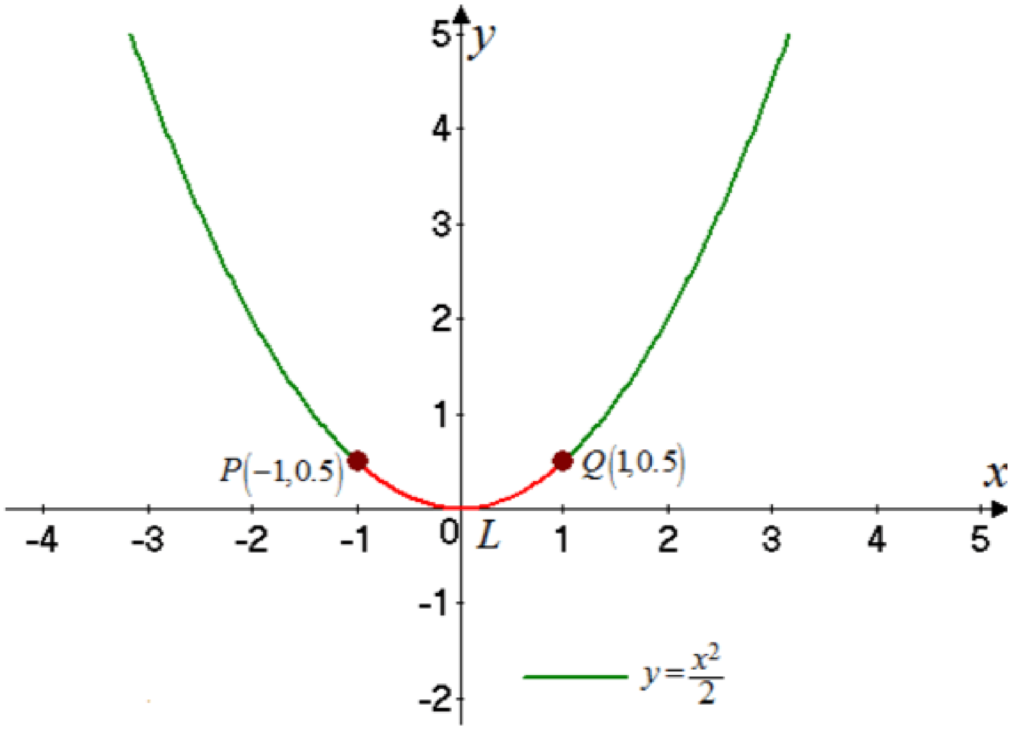


Figure 1

Differentiate (1) with respect to x.

$$\frac{dy}{dx}=\frac{1}{2}(2x)$$
$$=x$$

Find the arc length of the curve from $x=-1$ to $x=1$.

Here, $a=-1$ and $b=1$.

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\int_{-1}^{1}\sqrt{1+x^2}\,dx$$

Evaluate the integral by using trigonometric substitution.

Substitute, $x = \tan\theta$, then $dx = \sec^2\theta d\theta.$

When $x = -1, \tan\theta = -1,$ so $\theta = -\frac{\pi}{4}.$

When $x = 1, \tan\theta = 1,$ so $\theta = \frac{\pi}{4}.$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= \int_{-1}^{1} \sqrt{1 + x^2} dx$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \tan^2\theta} \left(\sec^2\theta\right) d\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sec^2\theta} \left(\sec^2\theta\right) d\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sec\theta\right)\left(\sec^2\theta\right) d\theta$$

Simply further as shown below:

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta d\theta$$
$$= \frac{1}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\begin{array}{l} \sec\frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| \\ -\sec\left(-\frac{\pi}{4}\right)\tan\left(-\frac{\pi}{4}\right) - \ln\left|\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right| \end{array}\right]$$
$$L = \frac{1}{2}\left[\left(\sqrt{2}\right)(1) + \ln\left|\sqrt{2} + 1\right| - \left(\sqrt{2}\right)(-1) - \ln\left|\left(\sqrt{2}\right) - 1\right|\right]$$
$$= \frac{1}{2}\left[2\sqrt{2} + \ln\left|\sqrt{2} + 1\right| - \ln\left|\sqrt{2} - 1\right|\right]$$
$$\approx 2.29558$$
$$\approx 2.3$$

Therefore, the arc length of the curve $y = \frac{1}{2}x^2$ from the point P to Q is $\boxed{2.3}$.

Answer 20E.

To find the length of the arc of the curve from point $P(1,5)$ to $Q(8,8)$

If a curve has the equation $x=g(y)$, $c\le y\le d$, and $g'(y)$ is continuous, then by the following formula for its length:

$$L=\int_{y=c}^{d}\sqrt{1+(g'(y))^2}\,dy$$

$$=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$ Since $x=(y-4)^{\frac{3}{2}}$, we have $\frac{dx}{dy}=\frac{3}{2}(y-4)^{\frac{1}{2}}$

$$=\int_{y=5}^{8}\sqrt{1+\left(\frac{3}{2}(y-4)^{\frac{1}{2}}\right)^2}\,dy$$ Since the limits $c=5$ and $d=8$

$$=\int_{y=5}^{8}\sqrt{1+\frac{9}{4}(y-4)}\,dy$$

$$=\int_{y=5}^{8}\left(\frac{9y}{4}-8\right)^{\frac{1}{2}}dy$$

$$=\frac{1}{9/4}\left(\frac{\left(\frac{9y}{4}-8\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)_5^8$$ $\left(\text{since }\int(by+c)^n\,dy=\frac{1}{b}\frac{(by+c)^{n+1}}{n+1}+C\right)$

Continuation of the above:

$$=\frac{4}{9}\left(\frac{\left(\frac{9y}{4}-8\right)^{\frac{3}{2}}}{\frac{3}{2}}\right)_5^8$$ Simplify

$$=\frac{2}{3}\left(\frac{4}{9}\right)\left(\left(\frac{9y}{4}-8\right)^{\frac{3}{2}}\right)_5^8$$

$$=\frac{8}{27}\left[\left(\frac{9y}{4}-8\right)\sqrt{\left(\frac{9y}{4}-8\right)}\right]_5^8$$ Since $x^{\frac{3}{2}}=x\sqrt{x}$

$$=\frac{8}{27}\left[\left\{\left(\frac{9(8)}{4}-8\right)\sqrt{\left(\frac{9(8)}{4}-8\right)}\right\}-\left\{\left(\frac{9(5)}{4}-8\right)\sqrt{\left(\frac{9(5)}{4}-8\right)}\right\}\right]$$

$$=\frac{8}{27}\left[\left\{\left(\frac{72-32}{4}\right)\sqrt{\left(\frac{72-32}{4}\right)}\right\}-\left\{\left(\frac{45-32}{4}\right)\sqrt{\left(\frac{45-32}{4}\right)}\right\}\right]$$

$$=\frac{8}{27}\left[\left(10\sqrt{10}\right)-\left(\frac{13\sqrt{13}}{8}\right)\right]$$

$$=\frac{8}{27}\left(31.623-\frac{46.8728}{8}\right)$$

$$=\frac{8}{27}(31.623-5.8591)$$

$$=\frac{8}{27}(25.7639)$$

$$=\frac{206.1112}{27}$$

$$=7.6347$$

Therefore, the length of the arc of the curve from point $P(1,5)$ to $Q(8,8)$ is 7.6347.

Answer 21E.

Given $y=x^2+x^3, \qquad 1\le x\le 2$

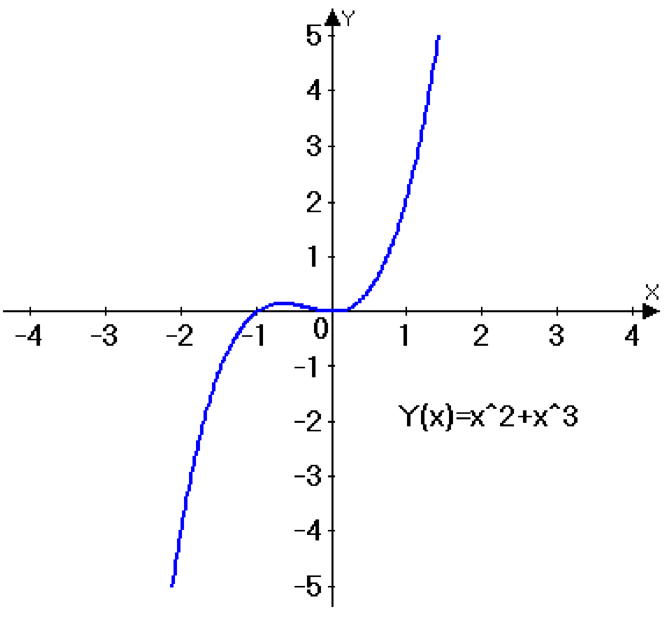
$$\Rightarrow \frac{dy}{dx}=2x+3x^2 \qquad \left[\because \frac{d}{dx}\left(x^n\right)=nx^{n-1}\right]$$

Length of the curve

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_1^2\sqrt{1+\left(2x+3x^2\right)^2}\,dx$$

$$=\boxed{10.0556}$$

Graph is



Answer 22E.

Given $y=x+\cos x, \qquad 0\le x\le \frac{\pi}{2}$

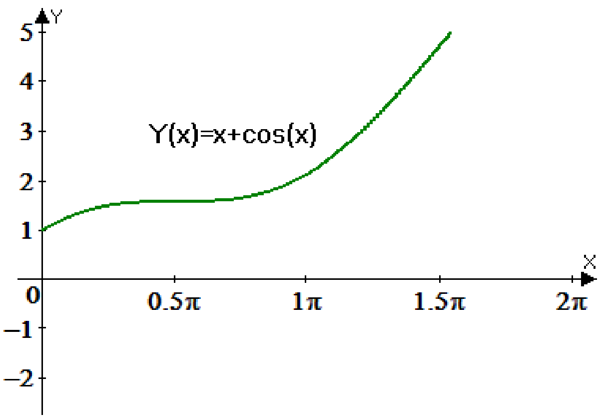
$$\Rightarrow \frac{dy}{dx}=1-\sin x$$

Length of the curve

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\pi/2}\sqrt{1+(1-\sin x)^2}\,dx$$

$$=\boxed{1.728634}$$

Graph of the given function is



Answer 23E.

Given $y = x\sin x, \qquad 0 \le x \le 2\pi$

$$\Rightarrow \frac{dy}{dx} = x\cos x + \sin x$$

Length of the curve

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{2\pi} \sqrt{1+(x\cos x+\sin x)^2}\,dx$$

$$= \boxed{15.37456}$$

Here

$a = 0, b = 2\pi, n = 10, \Delta x = 0.2\pi$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{2\pi} \sqrt{1+(x\cos x+\sin x)^2}\,dx$$

Formula for the Simpson's Rule

$$\int_a^b f(x)dx = \frac{\Delta x}{x}\left[\begin{array}{l}(f(x_0)+f(x_n))+4(f(x_1)+f(x_3)+\ldots)\\ +2(f(x_2)+f(x_4)+\ldots)\end{array}\right]$$

$$= \frac{0.2\pi}{3}\left[\begin{array}{l} f(0)+f(2\pi)+4\left(\begin{array}{l}f(0.2\pi)+f(0.6\pi)+f(\pi)\\ +f(1.4\pi)+f(1.8\pi)\end{array}\right)\\ \qquad +2\left(\begin{array}{l}f(0.4\pi)+f(0.8\pi)+\\ f(1.2\pi)+f(1.6\pi)\end{array}\right)\end{array}\right]$$

$$= \boxed{15.3745}$$

Answer 24E.

Given $y = \sqrt[3]{x}, 1 \le x \le 6 \Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-2/3}$

Length of the curve

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^6 \sqrt{1+\left(\frac{1}{3}x^{-2/3}\right)^2}\,dx$$

$$= \boxed{5.0740944}$$

Here

$a = 1, b = 6, n = 10, \Delta x = 0.5$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^6 \sqrt{1+\left(\frac{1}{3}x^{-2/3}\right)^2}\,dx$$

Formula for the Simpson's Rule:

$$\int_a^b f(x)dx = \frac{\Delta x}{x}\left[(f(x_0)+f(x_n))+4(f(x_1)+f(x_3)+\ldots)+2(f(x_2)+f(x_4)+\ldots)\right]$$

$$= \frac{0.5}{3}\left[\begin{array}{l} f(1)+f(6)+4(f(1.5)+f(2.5)+f(3.5)+f(4.5)+f(5.5))+\\ 2(f(2)+f(3)+f(4)+f(5))\end{array}\right]$$

$$= \boxed{5.0740}$$

Answer 25E.

Given that the function is $y=\ln(1+x^3)$, $\quad 0\le x\le 5$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{1+x^3}3x^2$$

Length of the curve

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^5\sqrt{1+\left(\frac{3x^2}{1+x^3}\right)^2}\,dx$$

$$=\boxed{7.118819}$$

Here

$a=0, b=5, n=10, \Delta x=0.5$

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^5\sqrt{1+\left(\frac{3x^2}{1+x^3}\right)^2}\,dx$$

Formula for the Simpson's Rule:

$$\int_a^b f(x)dx=\frac{\Delta x}{x}\left[\begin{array}{r}(f(x_0)+f(x_n))+4(f(x_1)+f(x_3)+\ldots)\\ +2(f(x_2)+f(x_4)+\ldots)\end{array}\right]$$

$$=\frac{0.5}{3}\left[\begin{array}{l}f(0)+f(5)+4(f(0.5)+f(1.5)+f(2.5)+f(3.5)+f(4.5))+\\ 2(f(1)+f(2)+f(3)+f(4))\end{array}\right]$$

$$=\boxed{7.1188}$$

Answer 26E.

Given that the function is $y=e^{-x^2}$, $\quad 0\le x\le 2$

$$\Rightarrow \frac{dy}{dx}=e^{-x^2}(-2x)$$

Length of the curve

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^2\sqrt{1+\left(e^{-x^2}(-2x)\right)^2}\,dx$$

$$=\boxed{2.2805258}$$

Here

$a=0, b=2, n=10, \Delta x=0.2$

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^2\sqrt{1+\left(e^{-x^2}(-2x)\right)^2}\,dx$$

Formula for the Simpson's Rule:

$$\int_a^b f(x)dx=\frac{\Delta x}{x}\left[\begin{array}{r}(f(x_0)+f(x_n))+4(f(x_1)+f(x_3)+\ldots)\\ +2(f(x_2)+f(x_4)+\ldots)\end{array}\right]$$

$$=\frac{0.2}{3}\left[\begin{array}{l}f(0)+f(2)+4(f(0.2)+f(0.6)+f(1)+f(1.4)+f(1.8))+\\ 2(f(0.4)+f(0.8)+f(1.2)+f(1.6))\end{array}\right]$$

$$=\boxed{2.28}$$

Answer 27E.

(A) We graph the function $y=x(4-x)^{1/3}$

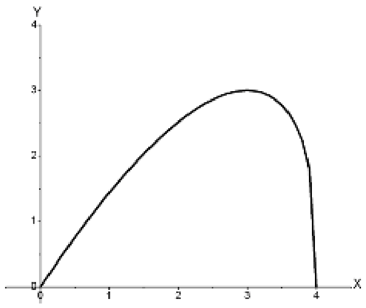


Fig.1

(B)

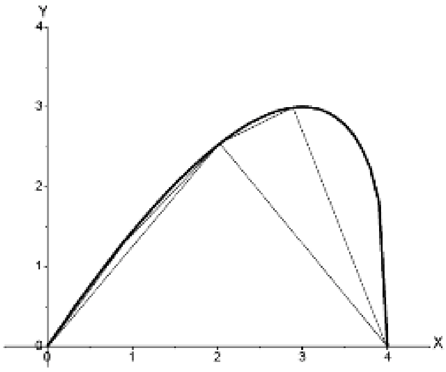


Fig.2

For n = 1, interval is [0, 4]
So the length of the inscribed polygon with (n = 1) = length of the straight line joining the points (0, 0) and (4, 0)

$$\boxed{L_1 = 4}$$

[By the distance formula $L_1 = \sqrt{(4-0)^2+(0+0)^2} = 4$]

For n = 2, intervals are [0, 2] and [2, 4]
So the length of the inscribed polygon with n = 2
L_2 = length of the straight line joining the points (0, 0) and $\left(2, 2\sqrt[3]{2}\right)$ + length of the straight line joining the points $\left(2, 2\sqrt[3]{2}\right)$ and (4, 0) = $D_1 + D_2$ (let)

By the distance formula

$$D_1 = \sqrt{(2-0)^2+\left(2\sqrt[3]{2}-0\right)^2}$$
$$= \sqrt{4+4\left(\sqrt[3]{2}\right)^2} = 3.217080$$

And $$D_2 = \sqrt{(4-2)^2+\left(0-\sqrt[3]{2}\right)^2}$$
$$= \sqrt{4+4\left(\sqrt[3]{2}\right)^2} = 3.217080$$

Then $L_2 = D_1 + D_2 \approx 6.43416$
$\Rightarrow \boxed{L_2 \approx 6.43}$

For $n = 4$, intervals are [0, 1], [1, 2], [2, 3], [3, 4]
We have the points (0, 0), $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\sqrt[3]{2}\right)$, $(3, 3)$, $(4, 0)$
Length of the line joining the points $(0, 0)$ and $\left(1, \sqrt[3]{3}\right)$ is

$$D_1 = \sqrt{(1-0)^2+\left(\sqrt[3]{3}-0\right)^2} \approx 1.75502$$

Length of the line joining the points $\left(1, \sqrt[3]{2}\right)$ and $\left(2, 2\sqrt[3]{2}\right)$ is

$$D_2 = \sqrt{(2-1)^2+\left(2\sqrt[3]{2}-\sqrt[3]{3}\right)^2} \approx 1.4701039$$

Length of the line joining the points $\left(2, 2\sqrt[3]{2}\right)$ and $(3, 3)$ is

$$D_3 = \sqrt{(3-2)^2+\left(3-2\sqrt[3]{2}\right)} \approx 1.109302$$

Length of the line joining the points $(3, 3)$ and $(4, 0)$ is

$$D_4 = \sqrt{(4-3)^2+(0-3)^2} = \sqrt{1+9} = 3.162278$$

Then the length of the inscribed polygon with n = 4 is

$$L_4 = D_1 + D_2 + D_3 + D_4 = 1.75502 + 1.470104 + 1.109302 + 3.162278$$
$$\approx 7.4967 \Rightarrow \boxed{L_4 \approx 7.50}$$

(C) We have $y = x(4-x)^{1/3}$

Then $\frac{dy}{dx} = (4-x)^{1/3} + x\left(\frac{1}{3}\right).(4-x)^{-2/3}(-1)$

$$= (4-x)^{1/3} - \left(\frac{x}{3}\right)(4-x)^{-2/3}$$

Then $\frac{dy}{dx} = \frac{3(4-x)-x}{3(4-x)^{2/3}}$

$$\Rightarrow \frac{dy}{dx} = \frac{4(3-x)}{3(4-x)^{2/3}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left[4(3-x)/\left\{3(4-x)^{2/3}\right\}\right]^2$$

Then arc length $\boxed{L = \int_0^4 \sqrt{1+\left[4(3-x)/\left\{3(4-x)^{2/3}\right\}\right]^2}\,dx}$

(D) With the help of the computer we calculate the integral which gives the value of the integral $\boxed{\approx 7.7988}$

We see that, in part (B) as we increase the number of sides of the polygon we get the length of the arc closer to the actual value

Answer 28E.

(A)

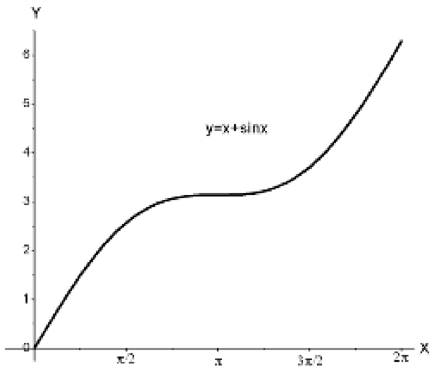


Fig.1

(B) For n = 1, interval is $[0, 2\pi]$

So the length of the inscribed polygon with n = 1

L_1 = distance between the points $(0,0)$ and $(2\pi, 2\pi)$

$$= \sqrt{(2\pi-0)^2 + (2\pi-0)^2}$$

$$= \sqrt{4\pi^2 + 4\pi^2}$$

$$= \sqrt{8\pi^2}$$

$$= 2\pi\sqrt{2}$$

$\boxed{L_1 \approx 8.89}$

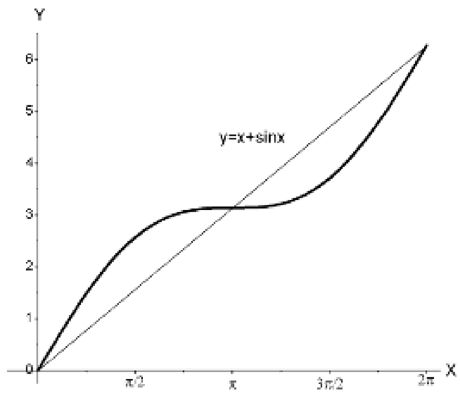


Fig.2

For n = 2

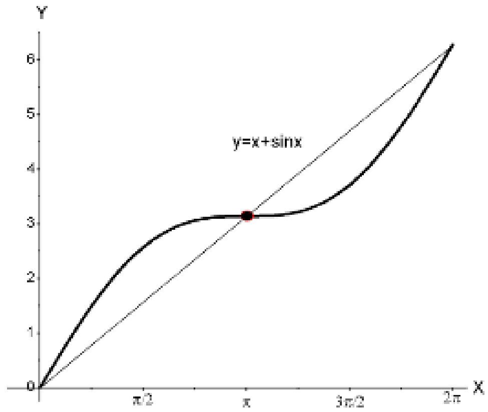


Fig.3

Interval is $[0, 2\pi]$ for n = 2

Subintervals are $[0, \pi]$, $[\pi, 2\pi]$

The length of the inscribed polygon with n = 2 is

L_2 = Distance between $(0,0)$ and (π,π) + distance between (π,π) and $(2\pi, 2\pi)$

= L_1 (from figure and summitry)

$\boxed{L_2 = 8.89}$

For n = 4

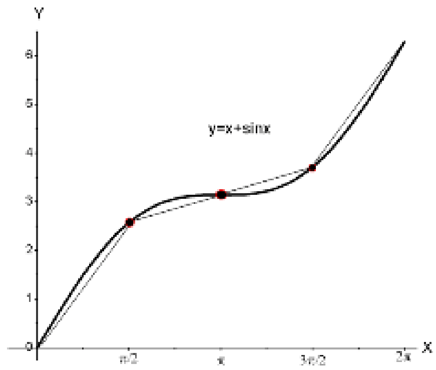


Fig.4

Sub intervals are $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi]$

The length of the inscribed polygon with n = 4 is

L_4 = distance between $(0,0)$ and $\left(\frac{\pi}{2}, 1+\frac{\pi}{2}\right)$ + distance between $\left(\frac{\pi}{2}, 1+\frac{\pi}{2}\right)$ and (π,π)

+ distance between (π,π) and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}-1\right)$

+ distance between $\left(\frac{3\pi}{2}, \frac{3\pi}{2}-1\right)$ and $(2\pi, 2\pi)$

$$= \sqrt{\left(\frac{\pi}{2}\right)^2+\left(1+\frac{\pi}{2}\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(1+\frac{\pi}{2}\right)^2}$$

$$= 2\sqrt{\left(\frac{\pi}{2}\right)^2+\left(1+\frac{\pi}{2}\right)^2}+2\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}$$

$\approx 6.02541 + 3.34258$

$\boxed{L_4 \approx 9.37}$

(C) We have y = x + sin x

Then $\frac{dy}{dx} = 1 + \cos x$

$$\left(\frac{dy}{dx}\right)^2 = (1+\cos x)^2$$

Then the arc length

$$L = \int_0^{2\pi} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\boxed{L = \int_0^{2\pi} \sqrt{1+(1+\cos x)^2}\,dx}$$

(D) With the help of computer we get the value of arc length about $\boxed{9.50759}$
As we increase the number of sides the value of curve length gets closer to the actual value.

Answer 29E.

Consider the equation of the curve is

$$y = \ln x$$

That lies between the points $(1,0)$ and $(2,\ln 2)$.

Need to find the exact arc length of the given curve.

Differentiating y with respect to x, to obtain

$$\frac{dy}{dx} = \frac{1}{x}$$

Write the arc length formula as

$$L = \int_c^d \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Now the arc length is given by

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\left(\frac{1}{x}\right)^2}\,dx$$

$$= \int_1^2 \frac{\sqrt{1+x^2}}{x}\,dx$$

$$= \int_1^2 \frac{\sqrt{1+x^2}}{x^2}\,x dx \quad \ldots\ldots (1)$$

Let

$$1+x^2=t^2$$
$$x^2=t^2-1$$
$$2xdx=2tdt$$
$$xdx=tdt$$

Limits: if $x=1$ then $t=\sqrt{2}$

if $x=2$ then $t=\sqrt{5}$

Substitute the above expressions, to get

$$\int_1^2 \frac{\sqrt{1+x^2}}{x^2}xdx = \int_{\sqrt{2}}^{\sqrt{5}} \frac{\sqrt{t^2}}{t^2-1}tdt$$
$$= \int_{\sqrt{2}}^{\sqrt{5}} \frac{t^2}{t^2-1}dt$$
$$= \int_1^{\sqrt{5}} \left(\frac{1}{t^2-1}+1\right)dt \qquad \text{Write } \frac{t^2}{t^2-1}=\frac{1}{t^2-1}+1$$

Continuous to the above

$$= \int_{\sqrt{2}}^{\sqrt{5}} \frac{1}{t^2-1}dt + \int_{\sqrt{2}}^{\sqrt{5}} 1dt$$
$$= \frac{1}{2}\left[\ln\left|\frac{t-1}{t+1}\right|\right]_{\sqrt{2}}^{\sqrt{5}} + [t]_{\sqrt{2}}^{\sqrt{5}}$$
$$= \frac{1}{2}\left[\ln\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right] + \left[\sqrt{5}-\sqrt{2}\right]$$
$$= \frac{1}{2}\left[\ln\left|\frac{(\sqrt{5}-1)(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)}\right| - \ln\left|\frac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right|\right] + \left[\sqrt{5}-\sqrt{2}\right]$$

Continuous to the above

$$= \left[\frac{1}{2}\ln\left|\frac{(\sqrt{5}-1)^2}{(5-1)}\right| - \frac{1}{2}\ln\left|\frac{(\sqrt{2}-1)^2}{(2-1)}\right|\right] + \left[\sqrt{5}-\sqrt{2}\right]$$
$$= \ln\left|\frac{\sqrt{(\sqrt{5}-1)^2}}{\sqrt{(5-1)}}\right| - \ln\left|\frac{\sqrt{(\sqrt{2}-1)^2}}{\sqrt{(2-1)}}\right| + \left[\sqrt{5}-\sqrt{2}\right]$$
$$= \ln\frac{1}{2}\left|\sqrt{5}-1\right| - \ln\left|\sqrt{2}-1\right| + \sqrt{5}-\sqrt{2}$$

Therefore the exact length of the arc is

$$L = \boxed{\ln\frac{1}{2}\left|\sqrt{5}-1\right| - \ln\left|\sqrt{2}-1\right| + \sqrt{5}-\sqrt{2}}$$

Answer 30E.

We have the equation of the curve $y = x^{4/3}$

Then $\frac{dx}{dy} = \frac{4}{3}x^{1/3}$

Thus $\left(\frac{dx}{dy}\right)^2 = \frac{16}{9}x^{2/3}$

So $1+\left(\frac{dx}{dy}\right)^2 = 1+\frac{16}{9}x^{2/3}$

$$= \frac{1}{9}\left(9+16x^{2/3}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \frac{1}{3}\sqrt{\left(9+16x^{2/3}\right)}$$

Then arc length

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{1}{3}\int_0^1 \sqrt{9+16x^{2/3}}\,dx$$

Let $4x^{1/3} = t \quad \Rightarrow \quad \frac{4}{3}x^{-2/3}dx = dt$

$$\Rightarrow dx = \frac{3}{4}x^{2/3}dt = \frac{3}{4}\left(\frac{t}{4}\right)^2 dt$$

$$\Rightarrow dx = \frac{3}{64}t^2 dt$$

And when x = 0, t = 0 and when x = 1, t = 4

$$\Rightarrow L = \frac{1}{3}\int_0^4 \sqrt{3^2+t^2}\left(\frac{3}{64}t^2\right)dt$$

$$= \frac{1}{64}\int_0^4 t^2\sqrt{3^2+t^2}\,dt$$

Using the formula

$$\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}\left(a^2+2u^2\right)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right)+c$$

$$L = \frac{1}{64}\left[\frac{t}{8}\left(9+2t^2\right)\sqrt{9+t^2} - \frac{81}{8}\ln\left(t+\sqrt{9+t^2}\right)\right]_0^4$$

$$= \frac{1}{64}\left[\frac{1}{2}(41).5 - \frac{81}{8}\ln(4+5) - 0 + \frac{81}{8}\ln\left(\sqrt{9}\right)\right]$$

$$= \frac{1}{64}\left[\frac{205}{2} - \frac{81}{8}(\ln 9 - \ln 3)\right]$$

$$\boxed{L = \frac{205}{128} - \frac{81}{512}\ln 3}$$

Answer 31E.

Equation of the curve is $x^{2/3} + y^{2/3} = 1$

Domain of the curve is $[-1,1]$ and range is $[-1,1]$

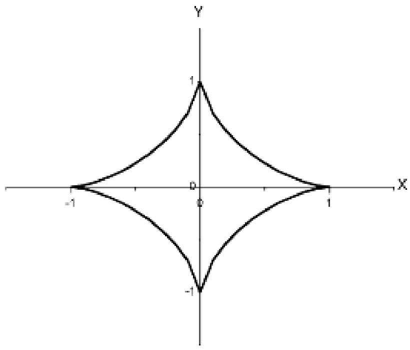


Fig.1

We see that the curve is symmetric in all four quadrants so total length of the curve = 4 × length of arc from $x = 0$, to $x = 1$

Rewrite the equation of curve $x^{2/3}+y^{2/3}=1$,

Then $$y=\left(1-x^{2/3}\right)^{3/2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$$

$$\Rightarrow \frac{dy}{dx}=-x^{-1/3}\left(1-x^{2/3}\right)^{1/2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2=x^{-2/3}\left(1-x^{2/3}\right) \qquad \left[x^{2/3}+y^{2/3}=1\right]$$

$$=x^{-2/3}-1$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=x^{-2/3}$$

Then the length of the arc from $x=0$ to $x=1$ is

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow L=\int_0^1\sqrt{x^{-2/3}}\,dx$$

$$\Rightarrow L=\int_0^1 x^{-1/3}dx$$

$$\Rightarrow L=\lim_{t\to 0^+}\int_t^1 x^{-1/3}dx$$

$$\Rightarrow L=\lim_{t\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_t^1$$

$$\Rightarrow L=\lim_{t\to 0^+}\left[3x^{2/3}/2\right]_t^1$$

$$\Rightarrow L=\frac{3}{2}$$

Then total length of the curve is $4\times L=4\times\frac{3}{2}$

$$\boxed{=6}$$

Answer 32E.

(A)

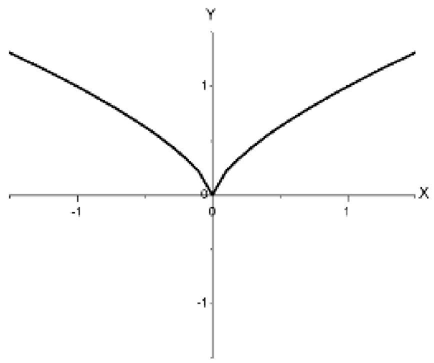


Fig.1

(B) We have $y^3=x^2$

So $$y=x^{2/3}$$

Then $$\frac{dy}{dx}=\frac{2}{3}x^{-1/3}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{4}{9}x^{-2/3}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{4}{9}x^{-2/3}$$

$$=\frac{9x^{2/3}+4}{9x^{2/3}}$$

Then $$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}=\frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}$$

Then by the arc length formula $L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Or $$L=\int_0^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx$$

Since $\frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}$ is not defined for x = 0, so this integral is an improper integral

Thus $\Rightarrow L=\lim_{t\to 0^+}\int_t^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx$

Let $3x^{1/3}=u$, then $x=u^3/27 \Rightarrow dx=\frac{u^2}{9}du$

And when x = t, then $u=3t^{1/3}=k(let)$ so $k\to 0^+$, as $t\to 0^+$

And when x = 1, and y = 3

Then $$L=\lim_{k\to 0^+}\int_k^3 \frac{\sqrt{u^2+2^2}}{u}.\frac{u^2}{9}du$$
$$=\frac{1}{9}\lim_{k\to 0^+}\int_k^3 u\sqrt{u^2+2^2}du$$

Let $u=2\tan\theta \Rightarrow du=2\sec^2\theta\, d\theta$

When u = k, $\theta=\tan^{-1}\left(\frac{k}{2}\right)=v$ (let) so $v\to 0^+$ as $u\to 0^+$

And when $u=3$, $\theta=\tan^{-1}\left(\frac{3}{2}\right)$

So $$L=\frac{1}{9}\lim_{v\to 0^+}\int_v^{\tan^{-1}(3/2)} 2\tan\theta\sqrt{4\tan^2\theta+4}.\left(2\sec^2\theta\right)d\theta$$
$$=\frac{1}{9}\lim_{v\to 0^+}\int_v^{\tan^{-1}(3/2)} 8\sec^2\theta.\sec\theta\tan\theta d\theta \qquad \left\{1+\tan^2\theta=\sec^2\theta\right\}$$
$$\Rightarrow L=\frac{8}{9}\lim_{v\to 0^+}\int_v^{\tan^{-1}(3/2)}\sec^2\theta\sec\theta\tan\theta\ d\theta$$
$$\Rightarrow L=\frac{8}{9}\lim_{v\to 0^+}\left[\frac{\sec^3\theta}{3}\right]_v^{\tan^{-1}\frac{3}{2}} \qquad \text{Since, } \frac{d}{d\theta}\sec^3\theta=3\sec^2\theta\sec\theta\tan\theta$$

Therefore $$L=\frac{8}{27}\lim_{v\to 0^+}\left[\sec^3\left(\tan^{-1}\frac{3}{2}\right)-\sec^3 v\right]$$
$$=\frac{8}{27}\left[\sec^3\left(\tan^{-1}\frac{3}{2}\right)-1\right]$$
$$=\frac{8}{27}\left[\left\{1+\tan^2\left(\tan^{-1}\frac{3}{2}\right)\right\}\sqrt{1+\tan^2\left(\tan^{-1}\frac{3}{2}\right)}-1\right]$$
$$=\frac{8}{27}\left[\left(1+\left(\frac{3}{2}\right)^2\right)\sqrt{1+\frac{9}{4}}-1\right]$$
$$=\frac{8}{27}\left[\frac{13}{4}\frac{\sqrt{13}}{2}-1\right]$$
$$=\boxed{\frac{1}{27}\left[13\sqrt{13}-8\right]\approx 1.44}$$

Again we have $y^3=x^2 \Rightarrow x=y^{3/2}$

Then $$\frac{dx}{dy}=\frac{3}{2}\sqrt{y}$$
$$\Rightarrow\left(\frac{dy}{dx}\right)^2=\frac{9y}{4}$$
$$\Rightarrow 1+\left(\frac{dx}{dy}\right)^2=1+\frac{9}{4}y$$
$$=\frac{1}{4}(4+9y)$$

Then $\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\frac{1}{2}\sqrt{4+9y}$

And arc length $L=\frac{1}{2}\int_0^1\sqrt{4+9y}dy$ we used $L=\int_a^b\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$

Let $4+9y=t$, then $9dy=dt$
So when y = 0, t = 4 and when y = 1, t = 13
Therefore $L=\frac{1}{2}\int_4^{13}\sqrt{t}\frac{dt}{9}$

$$\Rightarrow L=\frac{1}{18}\int_4^{13}t^{1/2}dt$$
$$=\frac{1}{18}\left[\frac{2t^{3/2}}{3}\right]_4^{13}$$
$$=\frac{1}{18}\cdot\frac{2}{3}\left[\sqrt{t^3}\right]_4^{13}$$
$$=\frac{1}{27}\left[\sqrt{13^3}-\sqrt{4^3}\right]=\boxed{\frac{1}{27}\left[13\sqrt{13}-8\right]\approx 1.44}$$

(C) Arc length $L=2\int_0^1\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy+\int_1^4\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$

$$=2\int_0^1\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy+\int_1^4\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$
$$=\int_0^1\sqrt{4+9y}dy+\frac{1}{2}\int_1^4\sqrt{4+9y}dy \qquad \text{(From part B)}$$

Let 4 + 9y = t then 9dy = dt
When y = 0 then t = 4, when y = 4 then t = 40
And when y =1, t=13

So $L=\frac{1}{2}\left[2\int_4^{13}\sqrt{t}\frac{dt}{9}+\int_{13}^{40}\sqrt{t}\frac{dt}{9}\right]$

$$\Rightarrow L=\frac{1}{18}\left[2\int_4^{13}t^{1/2}dt+\int_{13}^{40}t^{1/2}dt\right]$$
$$\Rightarrow L=\frac{1}{18}\left\{2\left[\frac{2}{3}t^{3/2}\right]_4^{13}+\left[\frac{2}{3}t^{3/2}\right]_{13}^{40}\right\}$$
$$\Rightarrow L=\frac{1}{27}\left[2\sqrt{13^3}-2\sqrt{4^3}+\sqrt{40^3}-\sqrt{13^3}\right]$$
$$\Rightarrow L=\frac{1}{27}\left[13\sqrt{13}+40\sqrt{40}-16\right]$$
$$\boxed{\Rightarrow L=\frac{1}{27}\left[13\sqrt{13}+80\sqrt{10}-16\right]}$$

Answer 33E.

Let $f(x)=2x^{3/2}$

Then $f'(x)=3x^{1/2}$

$$\Rightarrow \left[f'(x)\right]^2=9x$$
$$\Rightarrow 1+\left[f'(x)\right]^2=1+9x$$
$$\Rightarrow \sqrt{1+\left[f'(x)\right]^2}=\sqrt{1+9x}$$

Then arc length function is $L(x)=\int_1^x\sqrt{1+\left[f'(t)\right]^2}dt$

$$\Rightarrow L(x)=\int_1^x\sqrt{1+9t}dt$$

Let $1+9t=y$

$$\Rightarrow 9dt=dy$$
$$\Rightarrow dt=\frac{dy}{9}$$

And when $t=1$ then $y=10$ and when $t=x$ then y = (1 + 9x)

Then $L(x)=\frac{1}{9}\int_{10}^{(1+9x)}\sqrt{y}dy$

$$=\frac{1}{9}\left[\frac{2y^{3/2}}{3}\right]_{10}^{(1+9x)}$$

$$=\frac{2}{27}\left[(1+9x)^{3/2}-10\sqrt{10}\right]$$

$$\boxed{L(x)=\frac{2}{27}\left[(1+9x)^{3/2}-10\sqrt{10}\right]}$$

Answer 34E.

Consider,

$y=\ln(\sin(x)),0<x<\pi$

We have to find the arc length function with starting point $\left(\frac{\pi}{2},0\right)$

We know that if f' is continuous on $[a,b]$, then the length of the curve $y=f(x),a\le x\le b$ is

$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$ ...... (1)

The objective is to find the length of arc of function with starting point $\left(\frac{\pi}{2},0\right)$.

Here, $y=\ln(\sin(x))$

$\Rightarrow\frac{dy}{dx}=\frac{\cos x}{\sin x}$

$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\cot^2 x$

$\Rightarrow\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\csc x$

It means at $x=\frac{\pi}{2}$, $y=0$

So, the limits of arc length are $\frac{\pi}{2}$ and x

From (1), the length of arc of function with starting point $\left(\frac{\pi}{2},0\right)$ is,

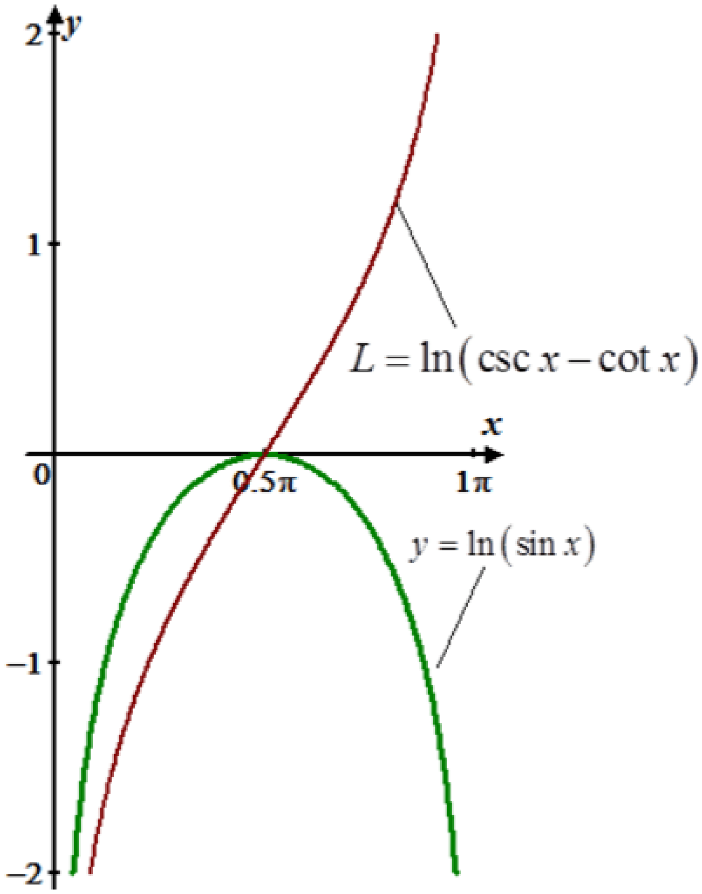
$L=\int_{\frac{\pi}{2}}^{x}\csc x dx$

$=\ln|\csc x-\cot x|$

Hence, the arc length function is $\boxed{\ln|\csc x-\cot x|}$.

(b)

Now sketch the graph of the function $y = \ln(\sin x)$ and the arc length function $L = \ln(\csc x - \cot x)$ is as shown below:



Answer 35E.

Consider the function.

$y = \sin^{-1} x + \sqrt{1-x^2}$

Find the arc length of the curve $y = \sin^{-1} x + \sqrt{1-x^2}$ with starting point $(0,1)$.

Recall the arc length formula.

If f' is continuous on $[a,b]$, then the length of the curve, $y = f(x)$, $a \le x \le b,$ is as follows:

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx \quad \ldots\ldots \ (1)$$

Differentiate the function, $y = \sin^{-1} x + \sqrt{1-x^2}$ with respect to x, to get the following:

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\sin^{-1} x + \sqrt{1-x^2}\right)$$

$$f'(x) = \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{1-x}{\sqrt{1-x^2}}$$

The length of the curve $y=\sin^{-1}x+\sqrt{1-x^2}$, $0\le x$ is as follows:

$$L=\int_0^x\sqrt{1+\left[\frac{1-t}{\sqrt{1-t^2}}\right]^2}dt \qquad \text{From}(1)\text{ and }f'(t)=\frac{1-t}{\sqrt{1-t^2}}$$

$$L=\int_0^x\sqrt{\frac{1-t^2+(1-t)^2}{1-t^2}}dt$$

$$L=\int_0^x\sqrt{\frac{1-t^2+1-2t+t^2}{1-t^2}}dt \quad \text{Since } (a-b)^2=a^2-2ab+b^2$$

$$L=\int_0^x\sqrt{\frac{2-2t}{1-t^2}}dt$$

$$L=\int_0^x\sqrt{\frac{2}{1+t}}dt \qquad \text{Since } a^2-b^2=(a+b)(a-b)$$

$$L=\sqrt{2}\int_0^x\frac{1}{\sqrt{1+t}}dt$$

$$L=\sqrt{2}\left(2\sqrt{1+t}\right)\Big|_0^x$$

$$L=2\sqrt{2}\sqrt{1+x}-2\sqrt{2}$$

$$L=2\sqrt{2}\left(\sqrt{1+x}-1\right)$$

Therefore the arc length of the curve $y=\sin^{-1}x+\sqrt{1-x^2}$ with starting point $(0,1)$ as

$$\boxed{L=2\sqrt{2}\left(\sqrt{1+x}-1\right)}.$$

Answer 36E.

We have $y=150-\frac{1}{40}(x-50)^2$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{40}(x-50)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{1}{400}(x-50)^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=\frac{400+(x-50)^2}{400}$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{\frac{400+(x-50)^2}{400}}=\frac{1}{20}\sqrt{400+(x-50)^2}$$

Then distance traveled by the kite from $x=0$ to $x=80$

$$L=\int_0^{80}\frac{1}{20}\sqrt{400+(x-50)^2}dx$$

Let $x-50=t$ then $dx=dt$,
When $x=0$, $t=-50$ and when $x=80$, $t=30$

$$\Rightarrow L=\frac{1}{20}\int_{-50}^{30}\sqrt{400+t^2}dt$$

Using the formula $\int \sqrt{a^2+u^2}du = \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+c$

$$\Rightarrow L = \frac{1}{20}\left[\frac{t}{2}\sqrt{400+t^2}+\frac{400}{2}\ln\left(t+\sqrt{400+t^2}\right)\right]_{-50}^{30}$$

$$= \frac{1}{20}\left[15\sqrt{1300}+200\left\{\ln\left(30+\sqrt{1300}\right)\right\}+25\sqrt{2900}-200\left\{\ln\left(-50+\sqrt{2900}\right)\right\}\right]$$

$$= \frac{1}{20}\left[150\sqrt{13}+200\left\{\ln\left(30+10\sqrt{13}\right)\right\}+250\sqrt{29}-200\left\{\ln\left(10\sqrt{29}-50\right)\right\}\right]$$

$$= \frac{1}{20}\left[150\sqrt{13}+250\sqrt{29}+200\left\{\ln\frac{\sqrt{13}+3}{\sqrt{29}-5}\right\}\right]$$

$$= \frac{15}{2}\sqrt{13}+\frac{25}{2}\sqrt{29}+10\left\{\ln\frac{6.60555}{0.38516}\right\}$$

$$= 122.7761$$

$\boxed{L \approx 122.78}$ ft

Answer 37E.

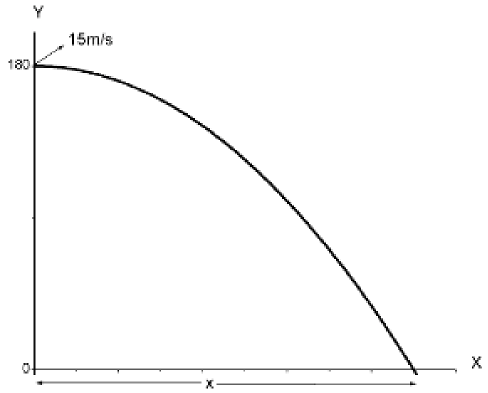


Fig.1

We have $y = 180 - \frac{x^2}{45}$

This is a projectile motion, the horizontal distance $x = u\sqrt{\frac{2h}{g}}$

Where $u = 15$ m/s (horizontal speed) $h = 180$m (height) and g = acceleration due to gravity ≈ 10 m/s^2

Then $\quad x \approx 15\sqrt{\frac{2\times 180}{10}} = 90$m

We have to find the length of the arc from $x = 0$ to $x = 90$

We have $\quad y = 180 - \frac{x^2}{45}$

Then $\quad \frac{dy}{dx} = \frac{-2x}{45}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{45^2}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4x^2}{45^2}$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4x^2}{45^2}} = \sqrt{1+\left(\frac{2x}{45}\right)^2}$$

Then arc length $L = \int_0^{90}\sqrt{1+\left(\frac{2x}{45}\right)^2}\,dx$

Let $\quad \frac{2x}{45} = t \Rightarrow \frac{2}{45}dx = dt \Rightarrow dx = \frac{45}{2}dt$

And when $x = 0$ then $t = 0$ when $x = 90$ the $t = 4$

Then $\quad L = \frac{45}{2}\int_0^4 \sqrt{1+t^2}\,dt$

Using formula $\int \sqrt{a^2+u^2}du = \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+c$

$$\Rightarrow L = \frac{45}{2}\left[\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\right]_0^4$$

$$= \frac{45}{2}\left[\frac{4}{2}\sqrt{17}+\frac{1}{2}\ln\left(4+\sqrt{17}\right)-\left(0+\frac{1}{2}\ln 1\right)\right]$$

$$= \frac{45}{2}\left[\frac{4}{2}\sqrt{17}+\frac{1}{2}\ln\left(4+\sqrt{17}\right)\right]$$

$$\Rightarrow L \approx 209.1\text{m}$$

So distance traveled by the prey is about $\boxed{209.1\text{ m}}$

Answer 38E.

Consider the an equation of the arc is

$$y = 211 - 20.96\cosh 0.03291765x.$$

Need to set up and estimate the length correct to the nearest meter.

Differentiate y with respect to x, to get

$$\frac{dy}{dx} = (-20.96\sinh 0.03291765x)(0.03291765)$$

$$\frac{dy}{dx} = (-0.6899539440\sinh 0.03291765x)$$

And given that $|x| \le 91.20$

That is the x limits are $-91.20 \le x \le 91.20$.

Write the formula for the arc length as

$$L = \int_c^d \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Therefore, the distance travelled by the prey is

$$L = \int_{-91.20}^{91.20} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{-91.20}^{91.20} \sqrt{1+(-0.6899539440\sinh 0.03291765x)^2}\,dx$$

Now, calculate the integration by using Maple software:

Maple input command as:

```
>evalf(int(sqrt(1+(-0.6899539440*sinh(0.03291765*x))^2),x=-91.20..91.20));
```

Maple gives an output:

```
> evalf(int(sqrt(1 + (-0.6899539440·sinh(0.03291765·x))^2), x=-91.20..91.20));
451.1370409
```

Hence, the distance travelled by the prey is $\boxed{451.1370409\text{m}}$

Answer 39E.

A manufacture of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in.thick.

The roof of the metal takes the shape of the sine wave $y = \sin\left(\frac{\pi x}{7}\right)$.

Since, the profile of the roofing makes two peaks over the course of 28 in.

The length w is equals to the arc length of the graph of $y = \sin\left(\frac{\pi x}{7}\right)$ for $0 \le x \le 28$.

The arc length formula, to find the length of the curve $y=f(x)$, $a \le x \le b$, is,

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Find the derivative of *y*.

$$y=\sin\left(\frac{\pi x}{7}\right)$$

$$\frac{dy}{dx}=\frac{\pi}{7}\cos\left(\frac{\pi x}{7}\right)$$

The arc length

$$w=\int_0^{28}\sqrt{1+\left(\frac{\pi}{7}\cos\left(\frac{\pi x}{7}\right)\right)^2}\,dx$$

$$=\int_0^{28}\sqrt{1+\frac{\pi^2}{49}\cos^2\left(\frac{\pi x}{7}\right)}\,dx$$

$$=\frac{1}{7}\int_0^{28}\sqrt{49+\pi^2\cos^2\left(\frac{\pi x}{7}\right)}\,dx$$

Use Maple calculator to evaluate the integral as follows:

$$> \frac{1}{7}\cdot\int_0^{28}\left(\sqrt{49+\pi^2\cdot\left(\cos\left(\frac{\pi\cdot x}{7}\right)\right)^2}\right)dx$$

$$\frac{8\sqrt{\pi^2+49}\ \mathrm{EllipticE}\left(\frac{\pi}{\sqrt{\pi^2+49}}\right)}{\pi}$$

> *evalf*[10](**(1)**)

29.36072662

Answer 40E.

(A) Equation of the shape of a catenary is $y=c+a\cosh\left(\frac{x}{a}\right)$

Then $\frac{dy}{dx}=a\sinh\left(\frac{x}{a}\right).\frac{1}{a}$

$$=\sinh\left(\frac{x}{a}\right)$$

Therefore $1+\left(\frac{dy}{dx}\right)^2=\sinh^2\left(\frac{x}{a}\right)+1$

$$=\cosh^2\left(\frac{x}{a}\right)$$

[Since $\cos h^2 x-\sin h^2 x=1$]

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2}=\cosh\left(\frac{x}{a}\right)$$

And the length of the wire L is given by

$$L=\int_{-b}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_{-b}^{b}\cosh\left(\frac{x}{a}\right)dx$$

$$=2\int_0^b\cosh\left(\frac{x}{a}\right)dx \qquad \left(\int_{-a}^{a}f(x)\,dx=2\int_0^a f(x)\,dx \text{ f f(x) is even}\right)$$

$$=2\left[a\sinh\left(\frac{x}{a}\right)\right]_0^b$$

$$\boxed{L=2a\sinh\left(\frac{b}{a}\right)}$$

(B) We have been given length of wire L = 51 ft

Distance between (-b, 0) to (b, 0) = 50 ft

$$\Rightarrow 2b = 50 \Rightarrow \boxed{b = 25}\ \text{ft}$$

And from (1) (in part A)

$$L = 2a\sinh(b/a)$$

$$\Rightarrow 2a\sinh(25/a) = 51$$

$$\Rightarrow 2a\sin h(25/a) - 51 = 0$$

For solving this equation for a, we sketch the curve of

$$f(a) = 2a\sinh\left(\frac{25}{a}\right) - 51$$

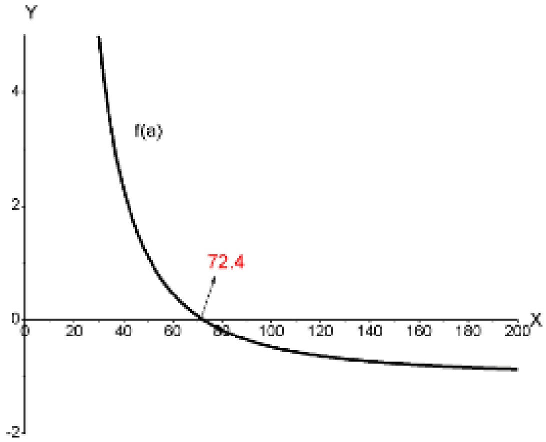


Fig.1

We see that a = 72.4

Another given conditions is $y(0) = 20$

$$\Rightarrow c + a\cos h(0) = 20$$

$$\Rightarrow c + a = 20 \Rightarrow C = (20 - a)$$

$$\Rightarrow c = 20 - 72.4 = -52.4$$

Then $y(25) = -52.4 + (72.4)\cos h\left(25/72.4\right) \approx 24.359\,\text{ft}$

$\boxed{y(25) \approx 24.36\ \text{ft}}$ Above the ground.

Answer 41E.

We have $y = \int_1^x \sqrt{t^3 - 1}\,dt$, $1 \le x \le 4$

Then $\dfrac{dy}{dx} = \dfrac{d}{dx}\int_1^x \sqrt{t^3 - 1}\,dt$

$$\Rightarrow \frac{dy}{dx} = \sqrt{x^3 - 1} \qquad \text{(By fundamental theorem)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = x^3 - 1$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^3 - 1$$

$$= x^3$$

$$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x^{3/2}$$

We know that the arc lengh formula is

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Then length of curve $L = \int_1^4 x^{3/2}dx$

$$= \left[\frac{x^{5/2}}{5/2}\right]_1^4 \qquad \left(\int x^n dx = \frac{x^{n+1}}{n+1}\right)$$

$$= \left[\frac{2x^{5/2}}{5}\right]_1^4$$

$$= \frac{2}{5}\left[4^{5/2}-1\right]$$

$$= \frac{2}{5}[32-1]$$

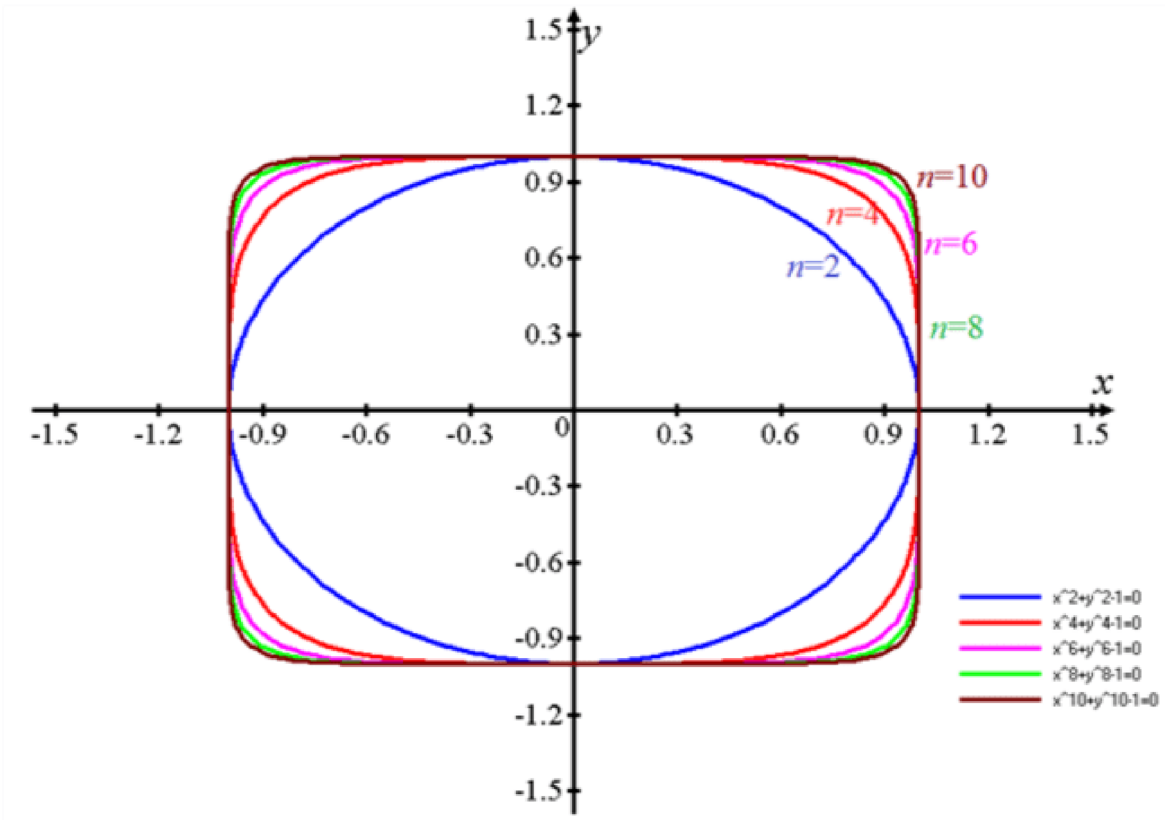
$$= \frac{62}{5}$$

That is length of the curve is $\boxed{L = 12.4}$

Answer 42E.

Consider the curves with equations, $x^n + y^n = 1$, $n = 2, 4, 6, \ldots$. These are called the fat circles.

Sketch the graph of the equations, $x^n + y^n = 1$ for $n = 2, 4, 6, 8,$ and 10.



Set the integral for the length, L_{2k} of the fat circle with $n = 2k$.

Then,

$$x^{2k} + y^{2k} = 1$$

Write the expression for the arc length formula, to find the length of the curve, $y = f(x)$, $a \le x \le b$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{...... (1)}$$

Find the derivative of *y*.

$$x^{2k} + y^{2k} = 1$$
$$y^{2k} = 1 - x^{2k}$$
$$y = \sqrt[2k]{1 - x^{2k}}$$

Differentiate with respect to *x*.

$$\frac{dy}{dx} = \frac{1}{2k}\left(1 - x^{2k}\right)^{\frac{1}{2k}-1} \frac{d}{dx}\left(1 - x^{2k}\right)$$
$$\frac{dy}{dx} = \frac{1}{2k}\left(1 - x^{2k}\right)^{\frac{1}{2k}-1}\left(-2kx^{2k-1}\right)$$
$$\frac{dy}{dx} = -x^{2k-1}\left(1 - x^{2k}\right)^{\frac{1}{2k}-1}$$

Substitute the value of derivative (1), to find the length of the curve.

Find the length from $(-1,0)$ to $(1,0)$, and then, double it.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$L_{2k} = 2\int_{-1}^{1} \sqrt{1 + \left[-x^{2k-1}\left(1 - x^{2k}\right)^{(1/2k)-1}\right]^2}\, dx$$
$$= 2\int_{-1}^{1} \sqrt{1 + x^{4k-2}\left(1 - x^{2k}\right)^{(1/k)-2}}\, dx$$

From the graph, notice that, the fat circles get closer, and, closer to a square of side length 2.

So, the sum of the lengths of the four sides of the square is 8.

Therefore, $\lim_{k \to \infty} L_{2k} = \boxed{8}$.