Chapter

4

Dynamics

Contents

I. Newton

Dynamics is, in general, the science of motion. If we consider the motion of a particle, the subject is called dynamics of a particle.

The study of dynamics is essential for those interested in defence the motion of projectiles and rockets can be discussed with its help. Dynamics is indispensable for those interested in space flights for calculating the paths of satellites and of all other space vehicles which may be sent to the moon or to outer space. The study of dynamics has been useful to mathematics in another way – the discovery of calculus was, to a large extent, facilitated by the attempts of Fermat, Newton and others to understand continuous motion.

Dynamics

4.1 Introduction

Dynamics is that branch of mechanics which deals with the study of laws governing motions of material system under the action of given forces.

(1) **Displacement :** The displacement of a moving point is its change of position. To know the displacement of a moving point, we must know both the length and the direction of the line joining the two positions of the moving point. Hence the displacement of a point involves both magnitude and direction *i.e.,* it is a vector quantity.

(2) **Speed :** The speed of a moving point is the rate at which it describes its path. The speed of a moving particle or a body does not give us any idea of its direction of motion; so it is a scalar quantity. In M.K.S. or S.I. system, the unit of speed is *metre per second* (m/s).

Average speed : The average speed of moving particle in a time-interval is defined as the distance travelled by the particle divided by the time interval.

If a particle travels a distance s in time-interval t, then Average speed = $\frac{s}{t}$

Note : ❑ The average speed of a particle gives the overall "rapidity" with which the particle moves in the given interval of time.

Instantaneous speed : The instantaneous speed or simply speed of a moving particle is defined as the rate of change of distance along its path, straight or curved.

If s is the distance travelled by a particle along its path, straight or curved, in time t, then

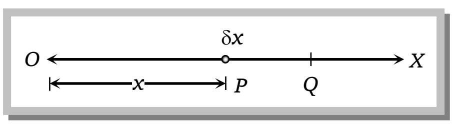
Instantaneous speed = $\frac{ds}{dt}$ or speed at time $t = \frac{ds}{dt}$

Note : ❑ The average speed is defined for a time interval and the instantaneous speed is defined at a particular instant.

4.2 Velocity and Acceleration

(1) **Velocity :** The velocity of a moving point is the rate of its displacement. It is a vector quantity.

Let a particle starting from the fixed point O and moving along the straight line OX describes the distance $OP = x$ in time t. If the particle describes a further distance $PQ = \delta x$ in time δt, then $\frac{\delta x}{\delta t}$ is called the average velocity of the particle in time δt.



And $\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = v$ is called the instantaneous velocity (or simply velocity) of the particle at time t.

Note : ❑ Average velocity in time t is the mean of the initial and final velocity.

(2) **Acceleration :** The rate of change of velocity of a moving particle is called its acceleration. It is a vector quantity.

The acceleration of a moving point at time t at distance x is given by, $f = \frac{v_2 - v_1}{t} \Rightarrow f = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v.\frac{dv}{dx}$

In M.K.S. or S.I. system, the unit of acceleration is m/sec^2.

It should be noted that $a > 0$ if v increases with time and $a < 0$ if v decreases with time. A negative acceleration is called retardation.

Clearly, retardation means decrease in the velocity.

Example: 1 The average speed of a bicycle over a journey of 20 *km*; if it travels the first 10 *km*. at 15 *km/hr* and the second 10 *km*. at 10 *km/hr*, is

(a) 12 *km/hr* (b) 10 *km/hr* (c) 15 *km/hr* (d) None of these

Solution: (a) Time taken by the bicycle to travel the first 10 *km* = $\frac{10}{15}$ *hour* = $\frac{2}{3}$ *hour*

Time taken by the bicycle to travel the second 10 *km* = $\frac{10}{10}$ *hour* = 1 *hour*

Total distance travelled = 10 + 10 = 20 *km*

Total time taken = $\frac{2}{3} + 1 = \frac{5}{3}$ *hour*.

$\therefore$ Average speed of bicycle = $\frac{\text{Distance}}{\text{Time}} = \frac{20}{5/3} = \frac{60}{5} = 12$ *km/hr*.

Example: 2 If a particle moves in a straight line according to the formula, $x = t^3 - 6t^2 - 15t$, then the time interval during which the velocity is negative and acceleration is positive, is

(a) [0, 5] (b) $(2, \infty)$ (c) (2, 5) (d) None of these

Solution: (c) We have $x = t^3 - 6t^2 - 15t$

$\therefore\ v = \frac{dx}{dt} = 3t^2 - 12t - 15$ and $a = \frac{d^2x}{dt^2} = 6t - 12$

Now $v < 0$ and $a > 0 \Rightarrow 3(t^2 - 4t - 5) < 0$ and $6(t-2) > 0 \Rightarrow -1 < t < 5$ and $t > 2 \Rightarrow t \in (2, 5)$.

Example: 3 A particle moves in a fixed straight path so that $S = \sqrt{1+t}$. If v is the velocity at any time t, then its acceleration is

(a) $-2v^3$ (b) $-v^3$ (c) $-v^2$ (d) $2v^3$

Solution: (a) We have $s = \sqrt{1+t}$

$\Rightarrow \frac{ds}{dt} = \frac{1}{2\sqrt{1+t}} \Rightarrow \frac{d^2s}{dt^2} = \frac{-1}{4(1+t)^{3/2}} = \frac{-1}{4}\left(\frac{1}{\sqrt{1+t}}\right)^3 \Rightarrow \frac{d^2s}{dt^2} = \frac{-1}{4}\left(2\frac{ds}{dt}\right)^3 \Rightarrow \frac{d^2s}{dt^2} = -2v^3$.

Example: 4 Two particles are moving with uniform velocities u and v respectively along X and Y axes, each directed towards the origin. If the particles are at distances a and b from the origin, the time at which they will be nearest to each other will be equal to

(a) $\frac{au}{u^2+v^2}$ (b) $\frac{bu}{u^2+v^2}$ (c) $\frac{au+bv}{u^2+v^2}$ (d) $\frac{au}{bv}$

Solution: (c) Let the two particles be at A and B respectively at $t = 0$ and at time t their positions be P and Q respectively.

Then, $AP = ut$ and $BQ = vt$

$\therefore\ OP = a - ut$ and $OQ = b - vt$

Now, $PQ^2 = OP^2 + OQ^2 \Rightarrow \quad PQ^2 = (a - ut)^2 + (b - vt)^2$

$\Rightarrow \frac{d}{dt}(PQ^2) = -2u(a - ut) - 2v(b - vt)$ and $\frac{d^2}{dt^2}(PQ^2) = 2(u^2 + v^2)$

For maximum or minimum value of PQ^2, we must have

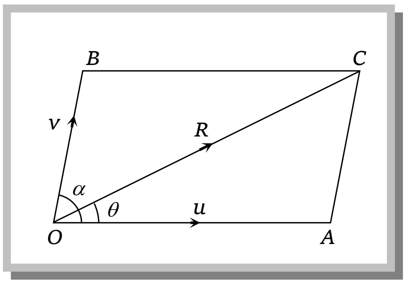
$\frac{d}{dt}(PQ^2) = 0 \Rightarrow -2u(a - ut) - 2v(b - vt) = 0 \Rightarrow t = \frac{au + bv}{u^2 + v^2}$

Clearly, $\frac{d^2}{dt^2}(PQ^2) > 0$ for all t. Hence, PQ is least at $t = \frac{au + bv}{u^2 + v^2}$.

4.3 Resultant and Components of Velocities

(1) **Resultant of velocities :** The resultant of two velocities possessed by a particle is given by the law of parallelogram of velocities.

Parallelogram law of velocities : If a moving particle has two simultaneous velocities represented in magnitude and direction by the two sides of a parallelogram drawn from an angular point, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.



Let a particle at *O* has simultaneously two velocities *u* and *v* represented in magnitude and direction by *OA* and *OB* respectively of the parallelogram *OACB*. The diagonal *OC* represents the resultant velocity.

Let *R* be the resultant velocity.

If $\angle AOB = \alpha$ and $\angle AOC = \theta$, then the magnitude of their resultant velocity is $\boxed{R = \sqrt{u^2 + v^2 + 2uv\cos\alpha}}$ and, the direction of this resultant velocity makes an angle θ with the direction of *u* such that $\boxed{\tan\theta = \frac{v\sin\alpha}{u + v\cos\alpha}}$

Case I : When $\alpha = 0$, then $R_{max} = u + v$

Case II : When $\alpha = \pi$, then $R_{min} = |u - v|$

Case III : When $\alpha = \pi/2$, *i.e.,* *u* and *v* are at right angle to each other, then

$R = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1}\left(\frac{v}{u}\right)$.

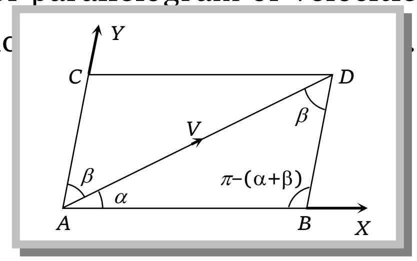
Case IV : When $u = v$, then $R = 2u\cos\frac{\alpha}{2}$ and $\theta = \frac{\alpha}{2}$.

Note : ❑ The angle made by the direction of the resultant velocity with the direction of *v* is given by $\tan^{-1}\left(\frac{u\sin\alpha}{v + u\cos\alpha}\right)$.

❑ If the direction of resultant velocity R makes an angle θ with the direction of $\vec{u}$, then

$$\sin\theta = \frac{v\sin\alpha}{R} \text{ and } \cos\theta = \frac{u + v\cos\alpha}{R}$$

(2) **Components of a velocity in two given directions :** If components of a velocity V in two given directions making angles α and β with it, then by the law of parallelogram of velocities the required components of V are represented by AB and $AC = BD$ alc



$$\frac{AB}{\sin ADB} = \frac{BD}{\sin BAD} = \frac{AD}{\sin ABD} \quad i.e., \quad \frac{AB}{\sin\beta} = \frac{BD}{\sin\alpha} = \frac{AD}{\sin(\alpha+\beta)}$$

$$\therefore \quad AB = AD.\frac{\sin\beta}{\sin(\alpha+\beta)} \text{ and } BD = AD.\frac{\sin\alpha}{\sin(\alpha+\beta)}$$

Hence, the components of velocity V in the directions making angle α and β are $\frac{V\sin\beta}{\sin(\alpha+\beta)}$ and $\frac{V\sin\alpha}{\sin(\alpha+\beta)}$ respectively.

Important Tips

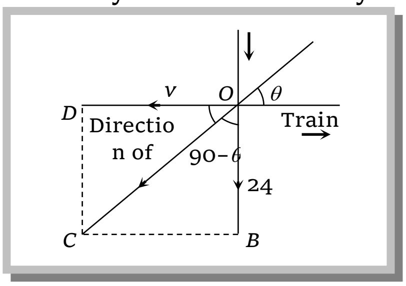
☞ *If components of velocity V are mutually perpendicular, then* $\beta = 90^o - \alpha$. *Thus components of V are* $V\cos\alpha$ *along AX and* $V\sin\alpha$ *along AY.*

Example: 5 Drops of water falling from the roof of the tunnel seems from the window of the train to be falling from an angle making $\tan^{-1}\frac{1}{2}$ from the horizontal. It is known that their velocities are 24 *decimeter/sec*. Assuming the resistance of air negligible, what will be the velocity of the train

(a) 42 *decimeter/sec.* (b) 48 *decimeter/sec.* (c) 45 *decimeter/sec.* (d) 44 *decimeter/sec.*

Solution: (b) Let the velocity of train is *v decimeter/sec.*

Let the real velocity of drops in magnitude and direction may be represented by *OB* and the velocity of train in opposite direction is represented by *OD*, then $OB = 24$, $OD = BC = v$



Completing parallelogram *OBCD*, *OC* will represent the relative velocity of drops.

If the drops are seen in a direction making an angle θ with the horizontal, *i.e.*, $\angle COD = \theta$, then $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ or $\tan\theta = \frac{1}{2}$ or $\cot\theta = 2$

In ΔOBC, from sine formula, $\frac{OB}{\sin\theta} = \frac{BC}{\sin(90-\theta)}$ or $\frac{24}{\sin\theta} = \frac{v}{\cos\theta}$ or

$v = 24\cot\theta = 24 \times 2 = 48$ *decimeter/second.*

Example: 6 If u and v be the components of the resultant velocity w of a particle such that $u = v = w$, then the angle between the velocities is

(a) 60^o (b) 150^o (c) 120^o (d) 30^o

Solution: (a) Let α be the angle between the components of the resultant velocity w.

Then $w^2 = u^2 + v^2 + 2uv\cos\alpha \Rightarrow u^2 = 2u^2(1+\cos\alpha) \Rightarrow 4\cos^2\frac{\alpha}{2} = 1 \Rightarrow \cos\frac{\alpha}{2} = \frac{1}{2} \Rightarrow \alpha = 60^o$.

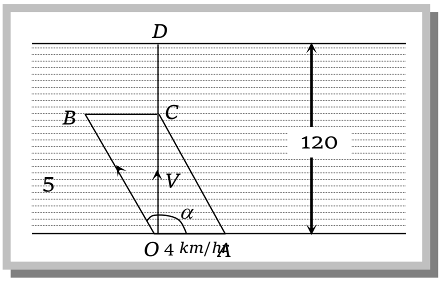
Example: 7 A man swims at a speed of 5 *km/hr*. He wants to cross a canal 120 *metres* wide, in a direction perpendicular to the direction of flow. If the canal flows at 4 *km/hr*, the direction and the time taken by the man to cross the canal are

(a) $\tan^{-1}\left(\frac{3}{4}\right), 2.4$ min. (b) $\pi - \tan^{-1}\left(\frac{3}{4}\right), 144$ sec. (c) $\tan^{-1}\left(\frac{1}{2}\right), 100$ sec. (d) None of these

Solution: (b) Suppose the man swims in a direction making an angle α with the direction of current.

Since the man wants to cross the canal in a direction perpendicular to the direction of flow.

Therefore, $\tan\frac{\pi}{2} = \frac{5\sin\alpha}{4+5\cos\alpha} \Rightarrow 4 + 5\cos\alpha = 0 \Rightarrow \cos\alpha = \frac{-4}{5}$



$\Rightarrow \alpha = \cos^{-1}\left(\frac{-4}{5}\right) \Rightarrow \alpha = \pi - \cos^{-1}\frac{4}{5} \Rightarrow \alpha = \pi - \tan^{-1}\frac{3}{4}$

Let V be the resultant velocity, then

$$V^2 = 4^2 + 5^2 + 2\times 4\times 5\cos\alpha$$

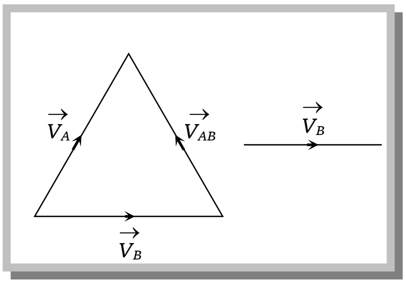
$$V^2 = 16 + 25 + 40\times\frac{-4}{5} = 9 \Rightarrow V = 3km/hr.$$

Time taken by the man to cross the canal = $\frac{120}{3\times 1000}hr = \frac{1}{25}hr = \frac{60}{25}$ min. = 2.4 min. = 144 sec.

4.4 Relative Velocity

Let A and B are two bodies in motion. The velocity of A relative to B is the velocity with which A appears to move as viewed by B *i.e.,* velocity of A relative to B = Resultant of velocity of A and reversed velocity of B.

Apparent velocity of A as seen from B means velocity of A relative to B.



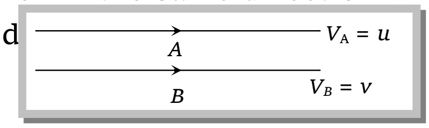
The relative velocity of A with respect to B is also called the velocity of A relative to B and is denoted by $\vec{V}_{AB}$ and $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$.

If two bodies A and B are moving with velocities of magnitudes V_A and V_B respectively inclined at an angle α with each other, then

$$\vec{V}_{AB} = \sqrt{V_A^2 + V_B^2 - 2V_A V_B\cos\alpha}$$

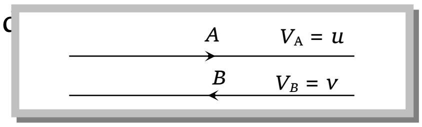
If the direction of the relative velocity of A with respect to B makes an angle θ with the direction of velocity of B, then $\tan\theta = \frac{V_A\sin\alpha}{V_B - V_A\cos\alpha}$

Case I : When the particles A and B move parallel to each other in the same direction with velocities u and v respectively. Here $\alpha = 0$, then $V_{AB} = (u-v)$ in the d



$V_{BA} = (v-u)$ in the direction of B

Case II : When the particles A and B move parallel to each other in opposite directions with velocities u and v respectively. Here $\alpha = \pi$, then $V_{AB} = (u+v)$ in the c



$V_{BA} = (v+u)$ in the direction of B.

Note : ❑ True velocity of A = Resultant of the relative velocity of A with respect to B and the true velocity of B

i.e., $\vec{V}_A = \vec{V}_{AB} + \vec{V}_B$ [Since $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$]

Example: 8 A train of length 200 *m* travelling at 30 *m/sec*. overtakes another of length 300 *m* travelling at 20 *m*/sec. The time taken by the first train to pass the second is

(a) 30 *sec*. (b) 50 *sec*. (c) 10 *sec*. (d) 40 *sec*.

Solution: (b) Distance covered by the first train to pass the second = (200 + 300)*m* = 500 *metres*.

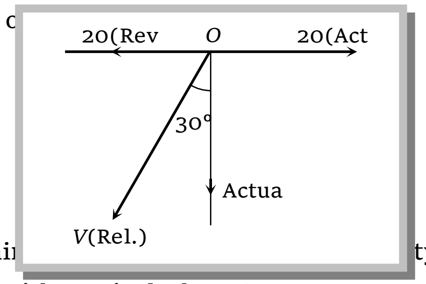
Since both the trains are travelling in the same direction. ∴ Velocity of first train relative to second = (30 – 20) = 10 *m/sec*.

∴ Time taken by the first train to pass the second = $\frac{500}{10} = 50$ *second*.

Example: 9 To a man running at a speed of 20 *km/hr*, the rain drops appear to be falling at an angle of 30^o from the vertical. If the rain drops are actually falling vertically downwards, their velocity in *km/hr*. is

(a) $10\sqrt{3}$ (b) 10 (c) $20\sqrt{3}$ (d) 40

Solution: (c) Velocity of rain relative to man = Actual velocity of rain – Velocity c



Resolving horizontally and vertically,

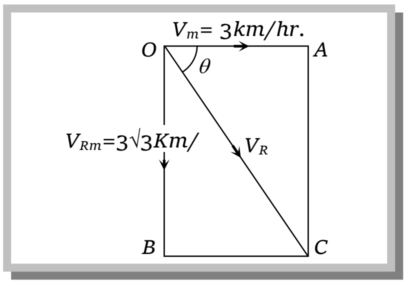
$V\cos 30^o = u$ and $V\sin 30^o$ = 20 *i.e.*, $\frac{V}{2} = 20$, ∴ $V = 40$

$\therefore 40\frac{\sqrt{3}}{2} = u \Rightarrow u = 20\sqrt{3}\ km/hr$.

Example: 10 A man is walking at the rate of 3 *km/hr* and the rain appears to hi ty of $3\sqrt{3}\ km/hr$. If the actual direction of the rain makes an angle θ with vertical, then θ =

(a) 15^o (b) 30^o (c) 45^o (d) 60^o

Solution: (d) Let $\vec{V}_M$ denotes the velocity of man. Then, $V_M = |\vec{V}_M| = 3km/hr$. It appears to the man that rain is falling vertically at the rate of $3\sqrt{3}km/hr$. This means that the relative velocity of rain with respect to man is $3\sqrt{3}km/hr$, in a direction perpendicular to the velocity of the man. Let $\vec{V}_{RM}$ denote the relative velocity of rain with respect to man and $\vec{V}_R$ be the velocity of rain.



Let $\overrightarrow{OA} = \vec{V}_R$, $\overrightarrow{OB} = \vec{V}_{RM}$. Complete the parallelogram *OACB*.

Then, the diagonal $\overrightarrow{OC}$ represents true velocity V_R of the rain.

$\therefore\ V_R^2 = V_M^2 + V_{RM}^2 + 2V_M.V_{RM}\cos 90^o$ [Using $w^2 = v^2 + u^2 + 2uv\cos\theta$]

$\Rightarrow V_R = \sqrt{3^2 + (3\sqrt{3})^2 + 2\times 3\times 3\sqrt{3}\times 0} = 6km/hr$.

Let θ be the angle between the direction of velocity of man and velocity of rain. Then,

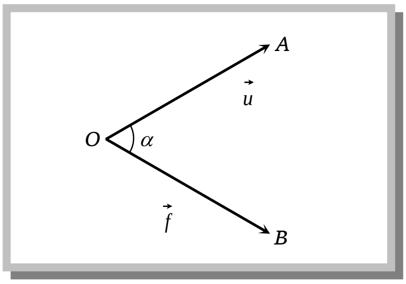
$\tan\theta = \frac{V_{RM}\sin 90^o}{V_M + V_{RM}\cos 90^o} \Rightarrow \tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}$

Thus, the actual velocity of the rain is 6 *km/hr* in a direction making an angle of 60^o with the direction of motion of the man.

Example: 11 The particles start simultaneously from the same point and move along two straight lines, one with uniform velocity $\vec{u}$ and the other from rest with uniform acceleration $\vec{f}$. Let α be the angle between their directions of motion. The relative velocity of the second particle *w.r.t.* the first is least after a time **[AIEEE 2003]**

(a) $\frac{u \sin \alpha}{f}$ (b) $\frac{f \cos \alpha}{u}$ (c) $u \sin \alpha$ (d) $\frac{u \cos \alpha}{f}$

Solution: (d) After *t*, velocity = $f \times t$



$V_{BA} = \vec{f}t + (-\vec{u})$

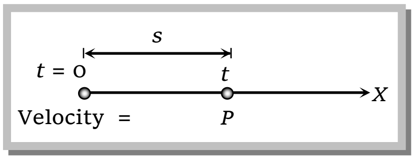
$V_{BA} = \sqrt{f^2t^2 + u^2 - 2fut \cos \alpha}$

For max. and min., $\frac{d}{dt}(V_{BA}^2) = 2f^2t - 2fu \cos \alpha = 0$

$t = \frac{u \cos \alpha}{f}$

4.5 Rectilinear motion with uniform Acceleration

A point or a particle moves in a straight line, starting with initial velocity *u*, and moving with constant acceleration *f* in its direction of motion. If *v* be its final velocity at the end of time *t* and *s* be its distance at the instant, from its starting point, then the equations describing the motion of the particle are,



(i) $v = u + ft$ (ii) $s = ut + \frac{1}{2}ft^2$ (iii) $v^2 = u^2 + 2fs$

(iv) If s_n is the distance travelled by the particle in n^{th} second, then $s_n = u + \frac{1}{2}f(2n - 1)$

Important Tips

☞ *If a particle moves in a straight line with initial velocity u m/sec. and constant acceleration f m/sec², then distance travelled in t seconds is given by,* $s = ut + \frac{1}{2}ft^2 = \frac{1}{2}[2ut + ft^2] = \frac{1}{2}[u + (u + ft)]t = \left(\frac{u+v}{2}\right).t$, *where* $v = u + ft$

= (Average velocity) × t *{Average velocity =* $\frac{u+v}{2}$*}*

Example: 12 A body is moving in a straight line with uniform acceleration. It covers distances of 10 *m* and 12 *m* in third and fourth seconds respectively. Then the initial velocity in *m/sec.* is

(a) 2 (b) 3 (c) 4 (d) 5

Solution: (d) Let initial velocity is *u m/sec.* and acceleration is $f\, m/\sec^2$.

So, $u + \frac{1}{2}f(2 \times 3 - 1) = 10$ or $u + \frac{5}{2}f = 10$ .....(i) and $u + \frac{1}{2}f(2 \times 4 - 1) = 12$ or $u + \frac{7}{2}f = 12$ ......(ii)

Subtracting (i) from (ii), we get $0 + \frac{2}{2}f = 2$ or $f = 2m/\sec^2$

Substituting value of *f* in equation (i) , $u + \frac{5}{2} \times 2 = 10$ or $u + 5 = 10$ or $u = 5m/\sec$.

Example: 13 Two trains A and B, 100 *kms* apart, are travelling to each other with starting speed of 50 *km/hr* for both. The train A is accelerating at 18 km/hr^2 and B is decelerating at 18 m/h^2. The distance where the engines cross each other from the initial position of A is

(a) 50 *kms* (b) 68 *kms* (c) 32 *kms* (d) 59 *kms*

Solution: (d) Let engine of train A travel x *km* and cross engine of train B in t hours.

$u = 50 km/hr$,

Acceleration $f = 18 km/hr^2$, so $s = 50t + \frac{1}{2} \times 18t^2$, $s = 50t + 9t^2$ .....(i)

So distance travelled by engine of train B will be $(100 - s)$ in t hour.

$u = 50 km/hr.$, Acceleration = –18 *km/hr*

$\therefore 100 - s = 50t - \frac{1}{2} \times 18t^2 \Rightarrow 100 - s = 50t - 9t^2$ .....(ii)

Adding (i) and (ii), we get $100 = 100t, \therefore t = 1$ *hour*. From (i), $s = 50 \times 1 + 9 \times 1^2 = 50 + 9 = 59 kms$.

Example: 14 A particle is moving with a uniform acceleration. If during its motion, it moves x, y and z distance in p^{th}, q^{th} and r^{th} seconds respectively, then

(a) $(q-r)x + (r-p)y + (p-q)z = 1$ (b) $(q-r)x + (r-p)y + (p-q)z = -1$

(c) $(q-r)x + (r-p)y + (p-q)z = 0$ (d) $(q+r)x + (r+p)y + (p+q)z = 0$

Solution: (c) Let u be the initial velocity and f be the acceleration of the particle.

Then $s_{pth} = u + \frac{1}{2} f(2p-1) = x$, $s_{qth} = u + \frac{1}{2} f(2q-1) = y$, $s_{rth} = u + \frac{1}{2} f(2r-1) = z$

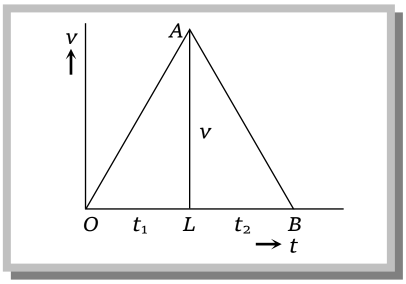
Now, multiplying (i) by $(q-r)$, (ii) by $(r-p)$, (iii) by $(p-q)$ and adding, we get $x(q-r) + y(r-p) + z(p-q) = 0$.

Example: 15 A body travels a distance s in t seconds. It starts from rest and ends at rest. In the first part of the journey, it moves with constant acceleration f and in the second part with constant retardation r. The value of t is given by **[AIEEE 2003]**

(a) $2s\left(\frac{1}{f} + \frac{1}{r}\right)$ (b) $\frac{2s}{\frac{1}{f} + \frac{1}{r}}$ (c) $\sqrt{2s(f+r)}$ (d) $\sqrt{2s\left(\frac{1}{f} + \frac{1}{r}\right)}$

Solution: (d) Portion OA, OB corresponds to motion with acceleration 'f' and retardation 'r' respectively.

Area of $\Delta OAB = s$ and $OB = t$. Let $OL = t_1, LB = t_2$ and $AL = v$



$s = \frac{1}{2} OB.AL = \frac{1}{2} t.v$ and $v = \frac{2s}{t}$

Also, $f = \frac{v}{t_1}, t_1 = \frac{v}{f} = \frac{2s}{tf}$ and $r = \frac{v}{t_2}, t_2 = \frac{v}{r} = \frac{2s}{tr}$

$t = t_1 + t_2 = \frac{2s}{tf} + \frac{2s}{tr}$; $t = \left(\frac{1}{f} + \frac{1}{r}\right)\frac{2s}{t} \Rightarrow t = \sqrt{2s\left(\frac{1}{f} + \frac{1}{r}\right)}$.

4.6 Motion Under Gravity

When a body is let fall in vacuum towards the earth, it will move vertically downward with an acceleration which is always the same at the same place on the earth but which varies slightly from place to place. This acceleration is called acceleration due to gravity. Its value in M.K.S. system is $9.8 m/sec^2$, in C.G.S. system $981 cm/sec^2$ and in F.P.S. system $32 ft/sec^2$. It is always denoted by g.

(1) **Downward motion:** If a body is projected vertically downward from a point at a height h above the earth's surface with velocity u, and after t second its velocity becomes v, the equation of its motion are $v = u + gt$,

$h = ut + \frac{1}{2}gt^2$, $v^2 = u^2 + 2gh$, $s_t = u + \frac{1}{2}g(2t - 1)$. In particular, if the body starts from rest or is simply let fall or dropped, then, $v = gt$, $h = \frac{1}{2}gt^2$, $v^2 = 2gh$. $(\because u = 0)$

(2) **Upward motion :** When a body be projected vertically upward from a point on the earth's surface with an initial velocity u and if the direction of the upward motion is regarded as *+ve,* the direction of acceleration is *–ve* and it is, therefore, denoted by – *g*. The body thus moves with a retardation and its velocity gradually becomes lesser and lesser till it is zero. Thus, for upward motion, the equations of motion are,

$$v = u - gt,\quad h = ut - \frac{1}{2}gt^2,\quad v^2 = u^2 - 2gh,\quad s_t = u - \frac{1}{2}g(2t - 1).$$

(3) **Important deductions**

(i) **Greatest height attained :** Let H be the greatest height. From the result, $v^2 = u^2 - 2gh$.

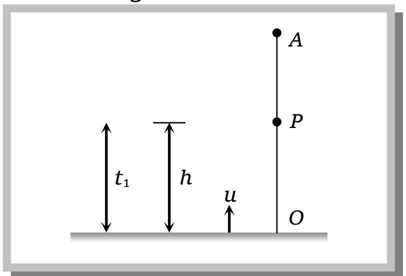
We have, $0 = u^2 - 2gH$ $\therefore$ $H = \frac{u^2}{2g}$. Hence greatest height $(H) = \frac{u^2}{2g}$.

(ii) **Time to reach the greatest height :** Let T be the time taken by the particle to reach the greatest height.

From the result, $v = u - gt$

We have, $0 = u - gT$ *i.e.,* $T = \frac{u}{g}$. Therefore time to reach the greatest height $(T) = \frac{u}{g}$

(iii) **Time for a given height :** Let t be the time taken by the body to reach at a given height h. Then, $h = ut - \frac{1}{2}gt^2 \Rightarrow gt^2 - 2ut + 2h = 0$. Hence, $t_1 = \frac{u - \sqrt{u^2 - 2gh}}{g}$ and $t_2 = \frac{u + \sqrt{u^2 - 2gh}}{g}$



Clearly t_1 and t_2 are real; if $u^2 \geq 2gh$. If $u^2 = 2gh$, then $t_1 = t_2 = \frac{u}{g}$, which is the time taken to reach the highest point.

So, let $u^2 > 2gh$.

The lesser value of t_1 gives the time when the body is going up *i.e.,* it is time from *O* to *P* and the larger value t_2 gives the total time taken by the body to reach the highest point and then coming back to the given point, *i.e.,* it is the time from *O* to *A* and then *A* to *P*.

(iv) **Time of flight :** It is the total time taken by the particle to reach the greatest height and then return to the starting point again. When the particle returns to the starting point, $h = 0$. Therefore, from the result, $h = ut - \frac{1}{2}gt^2$.

We have, $0 = ut - \frac{1}{2}gt^2$ or $t = 0$ or $\frac{2u}{g}$

$t = 0$ corresponds to the instant with the body starts, and $t = 2\,u/g$ corresponds to the time when the particle after attaining the greatest height reaches the starting point. $\therefore$ Time of flight $= \frac{2u}{g}$.

Example: 16 A stone is dropped from a certain height which can reach the ground in 5 *sec*. If the stone is stopped after 3 *seconds* of its fall and then time taken by the stone to reach the ground for the remaining distance is **[MNR 1985; UPSEAT 2000]**

(a) 2 *seconds* (b) 3 *seconds* (c) 4 *seconds* (d) None of these

Solution: (c) Let distance travelled in 5 *sec*. is h_1 *metre*. $\therefore$ $h_1 = 0 + \frac{1}{2}g \times 5 \times 5$ or $h_1 = \frac{25}{2}g$

Let distance travelled in 3 *seconds* is h_2 *metre*. So, $h_2 = 0 + \frac{1}{2}g \times 3 \times 3$ or $h_2 = \frac{9}{2}g$

So remaining distance $h = h_1 - h_2$ or $h = \frac{25}{2}g - \frac{9}{2}g$ or $h = \frac{16}{2}g = 8g$.

Let time taken by the stone to reach the ground for the remaining distance $8g$ is t second.

$\therefore \quad 8g = 0 + \frac{1}{2}gt^2$ or $t^2 = 16, \therefore t = 4$ *sec*.

Example: 17 A man in a balloon, rising vertically with an acceleration of 4.9 m/sec^2 releases a ball 2 *seconds* after the balloon is let go from the ground. The greatest height above the ground reached by the ball is

(a) 14.7 *m* (b) 19.6 *m* (c) 9.8 *m* (d) 24.5 *m*

Solution: (a) Velocity after 2 *seconds* = (4.9)2 = 9.8 *m/sec*.

Distance covered in 2 *seconds* = $\frac{1}{2}ft^2 = \frac{1}{2}(4.9)(2)^2 = 9.8m$

Again after the release of ball, velocity of the ball = 9.8 *m/sec*.

$v = 0$; Using $v^2 = u^2 - 2gh$, we get $0 = (9.8)^2 - 2(9.8)h \Rightarrow h = \frac{9.8}{2} = 4.9$ *metre*.

Hence greatest height attained = 9.8 + 4.9 = 14.7 *m*.

Example: 18 A particle is dropped under gravity from rest from a height $h(g = 9.8m/\sec^2)$ and then it travels a distance $\frac{9h}{25}$ in the last second. The height h is

(a) 100 *metre* (b) 122.5 *metre* (c) 145 *metre* (d) 167.5 *metre*

Solution: (b) $S_n = u + \frac{1}{2}f(2n-1) = \frac{1}{2}g(2n-1)$ $\quad (\because u = 0, f = g)$

$\frac{9h}{25} = \frac{1}{2}g(2n-1)$ and $h = \frac{1}{2}gn^2$

$\therefore \quad \frac{9}{25}.\frac{1}{2}gn^2 = \frac{1}{2}g(2n-1) \Rightarrow \frac{9}{25}n^2 = (2n-1) \quad \Rightarrow \quad 9n^2 = 50n - 25$

$\Rightarrow \quad 9n^2 - 50n + 25 = 0 \quad \Rightarrow \quad n = \frac{50 \pm \sqrt{2500 - 900}}{18} = \frac{50 \pm 40}{18}$; $\quad n = 5$ or $\frac{5}{9}$

Since $\frac{5}{9} < 1$, $\therefore$ rejecting this value. We have $n = 5$; $\therefore$ $h = \frac{1}{2}(9.8) \times 25 = 122.5\,metre$.

4.7 Laws of Motion

(1) Newton's laws of motion

(i) **First law :** Every body continues in its state of rest or of uniform motion in a straight line except when it is compelled by external impressed forces to change that state.

(ii) **Second law :** The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

$$\frac{d}{dt}(mv) \propto F \Rightarrow m\frac{dv}{dt} \propto F \quad \Rightarrow \quad m\frac{dv}{dt} = KF \Rightarrow F = mf \qquad \text{(for } K = 1)$$

Force = mass × acceleration. The S.I. unit of force is Newton and C.G.S. unit of force is dyne.

1 Newton = 1 $kg\text{-}m/sec^2$, 1 Dyne = 1 $gm\text{-}cm/sec^2$

(iii) **Third law :** To every action there is an equal and opposite reaction, or the actions and reaction are always equal and opposite.

Note : ❑ 1 Newton = 10^5 Dynes.

❑ The action and reaction do not act together on the same body or the same part of the body.

(2) **Weight :** The weight of a body is the force with which it is attracted by the earth towards its centre.

For a body of mass *m*, the weight *W* is given by $W = mg$ (By Newton's second law of motion)

Note : ❑ Let W_1 and W_2 be the weights of two bodies of masses m_1 and m_2 respectively at a place on the earth then, $W_1 = m_1 g$ and $W_2 = m_2 g \Rightarrow \frac{W_1}{W_2} = \frac{m_1}{m_2}$

❑ 1 *gm.wt.* = *g* dynes = 981 dynes

1 *kg. wt.* = *g* Newtons = 9.81 *N*.

(3) **Momentum of a body :** It is the quantity of motion in a body and is equal to the product of its mass (*m*) and velocity (*v*) with which it moves. Thus, momentum of the body is *mv*. The units of momentum are *gm-cm/sec* or *kg-m/sec*.

Momentum = Mass × Velocity = *m.v*.

(4) **Impulse of a body :** The impulse of a force in a given time is equal to the product of the force and the time during which it acts. The impulse of a force *F* acting for a time *t* is therefore *F.t*.

Example: 19 A train weighing *W* tons is moving with an acceleration *f* ft/sec^2. When a carriage of weight *w* tons is suddenly detached from it, then the change in the acceleration of the train is

(a) $\frac{Wf}{W-w}\ ft/\sec^2$ (b) $\frac{W}{W-w}\ ft/\sec^2$ (c) $\frac{wf}{W-w}\ ft/\sec^2$ (d) $\frac{w}{W-w}\ ft/\sec^2$

Solution: (c) Mass of train = *W* × 2240 *lbs*.

∴ Pull of the engine = *W* × 2240 × *f Poundals*

When a carriage of mass *W* tons is detached, mass of the train = (*W* – *w*) tons = (*W* – *w*) . 2240 *lbs*

Pull is the same as before *i.e.*, *W* × 2240 × *f Poundals*.

∴ New acceleration = $f_1 = \frac{W \times 2240 \times f}{(W-w)2240}$

$$f_1 = \frac{Wf}{W-w}\ ft/\sec^2$$

Change in acceleration = $f_1 - f = \frac{Wf}{W-w} - f = \left(\frac{W-W+w}{W-w}\right)f = \frac{wf}{W-w}$.

Example: 20 A hockey stick pushes a ball at rest for 0.01 second with an average force of 50 *N*. If the ball weighs 0.2 *kg*, then the velocity of the ball just after being pushed is

(a) 3.5 *m/sec*. (b) 2.5 *m/sec*. (c) 1.5 *m/sec*. (d) 4.5 *m/sec*.

Solution: (b) If *v m/sec*. is the velocity of the ball just after being pushed, then 50 × 0.01 = 0.2 × *v*[∵ Impulse = change of mo

v = 2.5 *m/sec*.

Example: 21 A mass m is acted upon by a constant force P *lb.wt.*, under which in t seconds it moves a distance of x feet and acquires a velocity v *ft/sec*. Then x is equal to

(a) $\frac{gP}{2mt^2}$ (b) $\frac{mg}{2v^2P}$ (c) $\frac{gt^2}{2Pm}$ (d) $\frac{mv^2}{2gP}$

Solution: (d) Force = P *lb.wt.*= Pg poundals and mass = m *lbs*

$\therefore\ f = \frac{F}{m} = \frac{Pg}{m}\ ft/\sec^2$

Now initial velocity = 0, final velocity = v *ft/sec*.

Distance = x and time = t

Hence $x = \frac{v^2 - u^2}{2f} = \frac{v^2 - 0}{2\frac{Pg}{m}} = \frac{mv^2}{2Pg}$.

Example: 22 An engine and train weight 420 *tons* and the engine exerts a force of 7 *tons*. If the resistance to motion be 14 *lbs. wt. per ton*, then the time, the train will take to acquire a velocity of 30 *m/hr*. from rest is **[BIT Ranchi 1994]**

(a) 2.2 m (b) 2.6 m (c) 2.8 m (d) 3 m

Solution: (a) Effective force = Pull of the engine – Resistance

= 7 tons – 420 × 14 *lbs* = 7 × 2240 *lbs* – 5880 *lbs* = 9800 *lbs* = 9800 × 32 Poundals

Also acceleration = $\frac{P}{m} = \frac{9800 \times 32}{420 \times 2240} = \frac{1}{3}\ ft/\sec^2$

Again initial velocity = 0 and final velocity = 30 *m.p.h.* = 44 *ft/sec*.

Use $v = u + ft$, we get $44 = 0 + \frac{1}{3}t$

$\therefore\ t = 132$ *second* = $2\frac{12}{60}$ *minute*; $t = 2\frac{1}{5}$ *minute* = 2.2 *minute*.

4.8 Motion of a Body released from a Balloon or a Lift

(1) When a lift is ascending with uniform acceleration of f m/sec^2 and after t *second* a body is dropped from it, then at the time when the body is dropped:

(i) Initial velocity of the body is same as that of the lift and is in the same direction. So, the velocity of the body is ft *m/sec*.

(ii) Initial velocity of the body relative to the lift = Velocity of the body – Velocity of the lift = $ft - ft = 0$

(iii) Acceleration of the body = g m/sec^2 in downward direction.

(iv) Acceleration of the lift = f m/sec^2 in upward direction.

(v) Acceleration of the body relative to the lift

= Acceleration of the body – Acceleration of the lift = $g - (-f) = f + g$ in downward direction.

(2) When a lift is ascending with uniform acceleration of $f\ m/\sec^2$ and after t *second* a body is thrown vertically upward with velocity $v\ m/\sec$, then at that time, we have the following :

(i) Initial velocity of the body = v + velocity of lift = $v + ft$, in upward direction

(ii) Initial velocity of the body relative to the lift = Velocity of the body – Velocity of lift = $(v + ft) - ft = v$ *m/sec*.

(iii) Acceleration of the body relative to the lift in vertically downward direction is $(f + g)$ m/sec^2.

(3) When a lift is descending with uniform acceleration $f\, m/\sec^2$ and after time t a body is dropped from it. Then at that time, we have the following

(i) Velocity of the body = Velocity of the lift = ft m/sec in downward direction

(ii) Acceleration of the body relative to the lift in downward direction

= Acceleration of the body – Acceleration of lift = $g - f\ m/\sec^2$

4.9 Apparent weight of a Body resting on a moving Horizontal plane or a Lift

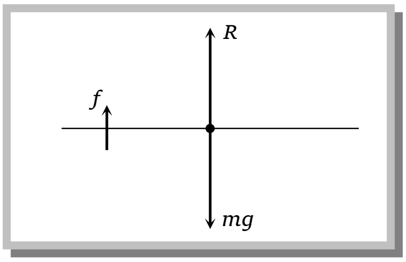
Let a body of mass m be placed in a lift moving with an acceleration f and R is the normal reaction, then

(1) **When the lift is rising vertically upwards :** Effective force in upward direction = Sum of the external forces in the same direction.

$\Rightarrow\ mf = R - Mg \Rightarrow R = m(f + g)$

Clearly, the pressure R exerted by the body on the plane is greater than the actual weight mg of the body. This pressure is also known as apparent weight.

If a man of mass m is standing on a lift which is moving vertically upwards with an acceleration, then $R = m(g + f)\ \Rightarrow\ R = mg\left(1 + \frac{f}{g}\right)$



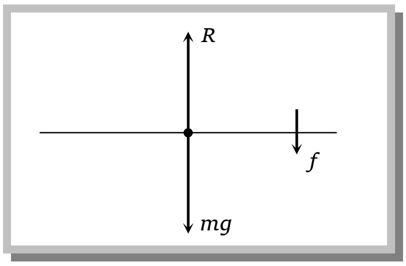
$\Rightarrow R$ = (weight of the man) $\left(1 + \frac{f}{g}\right)$

$\Rightarrow R$ = Weight of the man + $\frac{f}{g}$ (weight of the man)

Thus, the apparent weight of the man is $\frac{f}{g}$ times more than the actual weight.

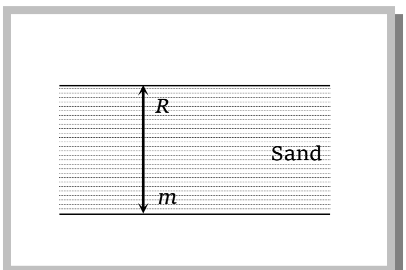
(2) **When the lift is descending vertically downwards :** Effective force in downward direction = Sum of the forces in the same direction.

$\Rightarrow\quad mf = mg - R \quad \Rightarrow \quad R = m(g - f)$ ......(ii)



Clearly, the pressure exerted by the body on the plane is less than its actual weight when the plane moves vertically downwards.

If a man of mass m in is standing on a lift, which is moving vertically downward with an acceleration f, then the pressure $R = m(g - f)$ = $mg\left(1 - \frac{f}{g}\right)$ $\Rightarrow R$ = weight of the man – $\frac{f}{g}$ (weight of the man)



Thus, the apparent weight of the man is $\frac{f}{g}$ times less than his actual weight.

Note : ❑ Effective force stopping a falling body = $F - mg$

❑ If R be the resistance of sand on a body of mass m falling in sand, then effective force = $R - mg$.

Important Tips

- *If the plane moves vertically upward with retardation equal to g i.e.,* $f = -g$*, then from* $R = m(f+g)$*, we get* $R = 0$. *Thus there is no pressure of the body on the plane when the plane rises vertically with retardation equal to g.*
- *If the plane moves down freely under gravity i.e., with acceleration equal to g, then from* $R = m(g-f)$*, we get* $R = 0$. *Thus there is no pressure of the body on the plane, when it moves vertically downwards with an acceleration equal to g.*

Example: 23 A paracute weighing 112 *lbs. wt*. falling with a uniform acceleration from rest, describes 16 *ft*. in the first 4 *sec*. Then the resultant pressure of air on the parachute is

(a) 85 *lbs. wt.* (b) 95 *lbs. wt.* (c) 105 *lbs. wt.* (d) 115 *lbs. wt.*

Solution: (c) Let *f* be the uniform acceleration. Here $u = 0$, $s = 16$, $t = 4\,\text{sec}$.

Since $s = ut + \frac{1}{2}ft^2$, $\therefore\ 16 = 0 + \frac{1}{2}f(4)^2$. $\therefore\ f = 2\,ft/\sec^2$

Equation of motion is $mf = mg - R$ (Downward)

$\therefore\ R = m(g-f)$ = 112 (32 – 2) = 112 × 30 Poundals $= \frac{112 \times 30}{32}$ *lbs. wt.* = 105 *lbs. wt.*

Example: 24 A man falls vertically under gravity with a box of mass '*m*' on his head. Then the reaction force between his head and the box is

(a) *mg* (b) 2 *mg* (c) 0 (d) 1.5 *mg*

Solution: (c) Let reaction be *R*. Since the man falls vertically under gravity $\therefore\ f = g$

From $mg - R = mf$, we have $mg - R = mg$. $\therefore R = 0$.

Example: 25 A man of mass 80 *kg*. is travelling in a lift. The reaction between the floor of the lift and the man when the lift is ascending upwards at 4 m/sec^2 is

(a) 1464.8 *N* (b) 1784.8 *N* (c) 1959.8 *N* (d) 1104.8 *N*

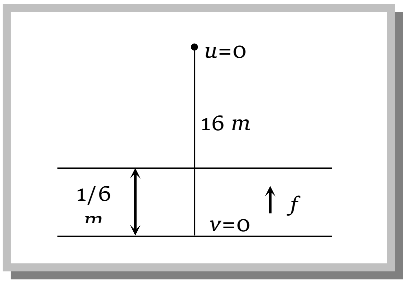
Solution: (d) When the lift is ascending, we have $R = mg + ma = m(g + a)$ = $80(9.81 + 4) = 80(13.81)$ = 1104.8 *N*.

Example: 26 A particle of mass *m* falls from rest at a height of 16 *m* and is brought to rest by penetrating $\frac{1}{6}m$ into the ground. If the average vertical thrust exerted by the ground be 388 *kg. wt*, then the mass of the particle is

(a) 2 *kg* (b) 3*kg* (c) 4*kg* (d) 8*kg*

Solution: (c) Let $f\ m/\sec^2$ be the retardation produced by the thrust exerted by the ground and let $v\,m/\sec$. be the velocity with which the mass *m* penetrates into the ground. Then, $v^2 = 0^2 + 2g \times 16$ $[\because v^2 = u^2 + 2gh]$

$\Rightarrow v = \sqrt{2 \times 9.8 \times 16} = 4\sqrt{19.6} = 17.70\ m/\sec$.



Since the mass *m* penetrates through $\frac{1}{6}m$ in the ground.

Therefore, its final velocity is zero.

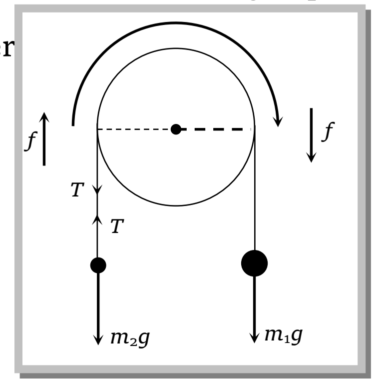
So, $0^2 = v^2 - 2fs \Rightarrow 0 = (4\sqrt{19.6})^2 - 2f \times \frac{1}{6}$

$\Rightarrow f = \frac{16 \times 19.6 \times 6}{2} = 940.8\ m/\sec^2 \Rightarrow (388 - m) = \frac{940.8\ m}{9.8}$ $[\because\ g = 9.8\ m/\sec^2]$

$\Rightarrow 388 - m = 96m \Rightarrow m = \frac{388}{97} = 4$. Hence, the mass of the particle is 4 *kg*.

4.10 Motion of two particles connected by a String

(1) **Two particles hanging vertically :** If two particles of masses m_1 and $m_2\ (m_1 > m_2)$ are suspended freely by a light inextensible string which passes over a smooth fixed light pulley, then the particle of mass $m_1 (> m_2)$ will move downwards, with Acceler

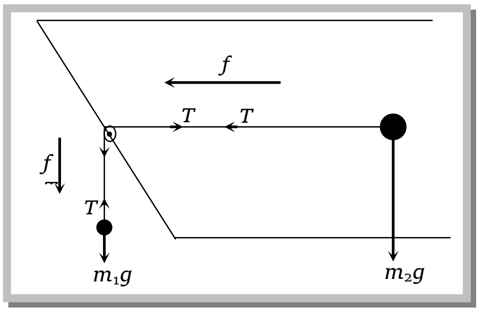


Tension in the string $T = \frac{2m_1m_2g}{m_1 + m_2}$

Pressure on the pulley $= 2T = \frac{4m_1m_2g}{m_1 + m_2}$

(2) **One particle on smooth horizontal table :** A particle of mass m_1 attached at one end of light inextensible string is hanging vertically, the string passes over a pulley at the end of a smooth

horizontal table, and a particle of mass m_2 placed on the table is attached at the other end of the string. Then for the system,

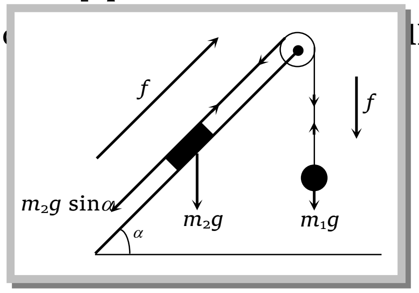


Acceleration $f = \dfrac{m_1 g}{m_1 + m_2}$

Tension in the string $T = \dfrac{m_1 m_2 g}{m_1 + m_2}$

Pressure on the pulley $= T\sqrt{2} = \dfrac{\sqrt{2} m_1 m_2 g}{m_1 + m_2}$.

(3) **One particle on an inclined plane :** A particle of mass m_1 attached at one end of a light inextensible string is hanging vertically, the string passes over a pulley at the end of a smooth plane inclined at an angle α to the horizontal, and a particle of mass m_2 placed on this inclined plane is attached at the other end of the string. If the particle lly downwards, then for the system,



Acceleration $f = \dfrac{(m_1 - m_2 \sin\alpha)}{m_1 + m_2} g$

Tension in the string $T = \dfrac{m_1 m_2 (1 + \sin\alpha)}{m_1 + m_2} g$

Pressure on the pulley $= \sqrt{\{2(1 + \sin\alpha)\}} T = \dfrac{\sqrt{2} m_1 m_2 g (1 + \sin\alpha)^{3/2}}{m_1 + m_2}$

Example: 27 The sum of two weights of an Atwood's machine is 16 *lbs*. The heavier weight descends 64 *ft* in 4 *seconds*. The value of weights in *lbs* are

(a) 10, 6 (b) 9, 7 (c) 8, 8 (d) 12, 4

Solution: (a) Let the two weights be m_1 and m_2 (*lbs*.)

$(m_1 > m_2)$, $\therefore$ $m_1 + m_2 = 16$ ......(i)

Now, the distance moved by the heavier weight in 4 *seconds* from rest is 64 *ft*.

If acceleration = *f*, then $64 = 0 + \frac{1}{2} f(4)^2 \Rightarrow f = 8 ft/\sec^2$

But $f = \dfrac{m_1 - m_2}{m_1 + m_2} g \Rightarrow \dfrac{m_1 - m_2}{16} \times 32 = 8 \Rightarrow m_1 - m_2 = 4$ ......(ii)

From (i) and (ii), $m_1 = 10$ *lbs*, $m_2 = 6 \, lbs$.

Example: 28 Two weights W and W' are connected by a light string passing over a light pulley. If the pulley moves with an upward acceleration equal to that of gravity, the tension of the string is

(a) $\dfrac{WW'}{W + W'}$ (b) $\dfrac{2WW'}{W + W'}$ (c) $\dfrac{3WW'}{W + W'}$ (d) $\dfrac{4WW'}{W + W'}$

Solution: (d) Suppose W descends and W' ascends. If f is the acceleration of the weights relative to the pulley, then since the pulley ascends with an acceleration f, the actual acceleration of W and W' are $(f - g)$ (downwards) and $(f + g)$ (upwards) respectively. Let T be tension in the string.

$\therefore\ Wg - T = W(f - g)$ and $T - W'g = W'(f + g)$ or $g - \frac{T}{W} = f - g$ and $\frac{T}{W'} - g = f + g$

Subtracting, We get $\frac{T}{W} + \frac{T}{W'} - 2g = 2g$ or $T = \frac{4WW'}{W + W'}$ $kg.\ wt.$

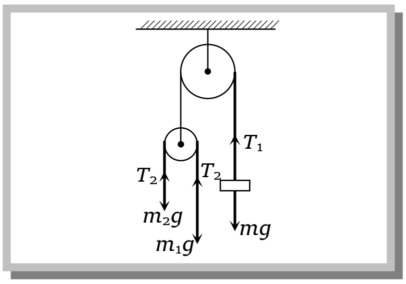
Example: 29 To one end of a light string passing over a smooth fixed pulley is attached a particle of mass M, and the other end carries a light pulley over which passes a light string to whose ends are attached particles of masses m_1 and m_2. The mass M will remain at rest or will move with a uniform velocity, if **[MNR 1993]**

(a) $\frac{1}{M} = \frac{1}{m_1} + \frac{1}{m_2}$ (b) $\frac{2}{M} = \frac{1}{m_1} + \frac{1}{m_2}$ (c) $\frac{4}{M} = \frac{1}{m_1} + \frac{1}{m_2}$ (d) $\frac{8}{M} = \frac{1}{m_1} + \frac{1}{m_2}$

Solution: (c) If the mass M has no acceleration, then $Mg = T_1$

But $T_1 = 2T_2 = 2\left(\frac{2m_1m_2}{m_1 + m_2} g\right) = \frac{4m_1m_2}{m_1 + m_2} g$

$\therefore\ Mg = \frac{4m_1m_2}{m_1 + m_2} g$ or $\frac{4}{M} = \frac{m_1 + m_2}{m_1m_2} = \frac{1}{m_1} + \frac{1}{m_2}$



4.11 Impact of Elastic Bodies

(1) **Elasticity :** It is that property of bodies by virtue of which they can be compressed and after compression they recover or tend to recover their original shape. Bodies possessing this property are called elastic bodies.

(2) **Law of conservation of momentum :** It states "the total momentum of two bodies remains constant after their collision of any other mutual action." Mathematically $m_1.u_1 + m_2.u_2 = m_1.v_1 + m_2.v_2$

where m_1 = mass of the first body, u_1 = initial velocity of the first body, v_1 = final velocity of the first body.

m_2, u_2, v_2 = Corresponding values for the second body.

(3) **Coefficient of restitution :** This constant ratio is denoted by e and is called the coefficient of restitution or coefficient of elasticity. The values of e varies between the limits 0 and 1. The value of e depends upon the substances of the bodies and is independent of the masses of the bodies. If the bodies are perfectly elastic, then $e = 1$ and for inelastic bodies $e = 0$.

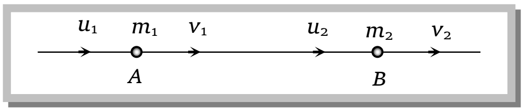
(4) **Impact :** When two bodies strike against each other, they are said to have an impact. It is of two kinds : Direct and Oblique.

(5) **Newton's experimental law of impact :** It states that when two elastic bodies collide, their relative velocity along the common normal after impact bears a constant ratio of their relative velocity before impact and is in opposite direction.

If u_1 and u_2 be the velocities of the two bodies before impact along the common normal at their point of contact and v_1 and v_2 be their velocities after impact in the same direction, $v_1 - v_2 = -e(u_1 - u_2)$

Case I : If the two bodies move in direction shown in diagram given below, then

(a) $v_1 - v_2 = -e(u_1 - u_2)$ and (b) $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$



Case II : If the direction of motion of two bodies before and after the impact are as shown below, then by the laws of direct impact, we have

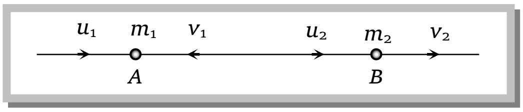
(a) $v_1 - v_2 = -e[u_1 - (-u_2)]$ or $v_1 - v_2 = -e(u_1 + u_2)$

and (b) $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2(-u_2)$ or $m_1 v_1 + m_2 v_2 = m_1 u_1 - m_2 u_2$

Case III : If the two bodies move in directions as shown below, then by the laws of direct impact, we have

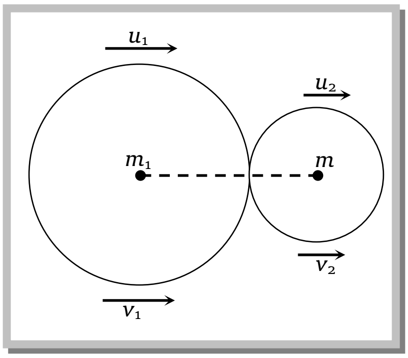
(a) $v_1 - (-v_2) = -e[-u_1 - (-u_2)]$ or $v_1 + v_2 = -e(u_2 - u_1)$

and (b) $m_1 v_1 + m_2(-v_2) = m_1(-u_1) + m_2(-u_2)$ or $m_1 v_1 - m_2 v_2 = -(m_1 u_1 + m_2 u_2)$



(6) **Direct impact of two smooth spheres :** Two smooth spheres of masses m_1 and m_2 moving along their line of centres with velocities u_1 and u_2 (measured in the same sense) impinge directly. To find their velocities immediately after impact, *e* being the coefficient of restitution between them.

Let v_1 and v_2 be the velocities of the two spheres immediately after impact, measured along their line of centres in the same direction in which u_1 and u_2 are measured. As the spheres are smooth, the impulsive action and reaction between them will be along the common normal at the point of contact. From the principle of conservation of momentum,



$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \quad .....(i)$$

Also from Newton's experimental law of impact of two bodies,

$$v_2 - v_1 = e[u_1 - u_2] \quad .....(ii)$$

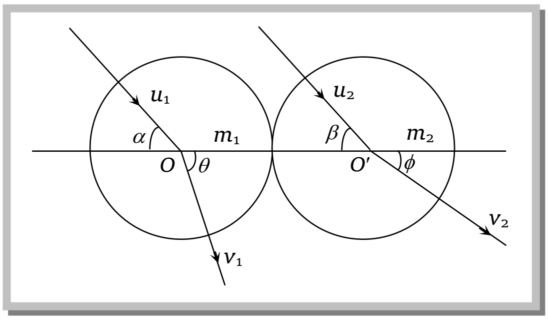
Multiplying (ii) by m_2 and subtracting from (i), we get

$$\Rightarrow (m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2 u_2(1 + e)$$

$$\therefore v_1 = \frac{(m_1 - em_2)u_1 + m_2 u_2(1 + e)}{m_1 + m_2} \quad .....(iii)$$

(7) **Oblique impact of two spheres :** A smooth spheres of mass m_1, impinges with a velocity u_1 obliquely on a smooth sphere of mass m_2 moving with a velocity u_2. If the direction of motion before impact make angles α and β respectively with the line joining the centres of the spheres, and if the coefficient of restitution be *e*, to find the velocity and directions of motion after impact,

Let the velocities of the sphere after impact be v_1 and v_2 in directions inclined at angles θ and ϕ respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line joining the centres of the two balls and therefore, velocities in that direction are unaltered.



$v_1 \sin\theta = u_1 \sin\alpha$ .....(i)

$v_2 \sin\phi = u_2 \sin\beta$ .....(ii)

And by Newton's law, along the line of centres,

$v_2 \cos\phi - v_1 \cos\theta = -e(u_2 \cos\beta - u_1 \cos\alpha)$ .....(iii)

Again, the only force acting on the spheres during the impact is along the line of centres. Hence the total momentum in that direction is unaltered.

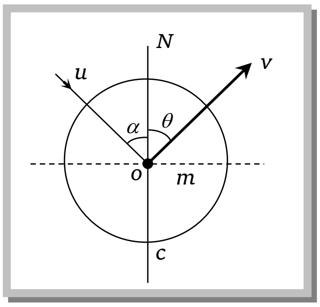
$\therefore m_1 v_1 \cos\theta + m_2 v_2 \cos\phi = m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta$ .....(iv)

The equations (i), (ii), (iii) and (iv) determine the unknown quantities.

(8) **Impact of a smooth sphere on a fixed smooth plane :** Let a smooth sphere of mass *m* moving with velocity *u* in a direction making an angle α with the vertical strike a fixed smooth horizontal plane and let *v* be the velocity of the sphere at an angle θ to the vertical after impact.

Since, both the sphere and the plane are smooth, so there is no change in velocity parallel to the horizontal plane.

$\therefore v \sin\theta = u \sin\alpha$ .....(i)



And by Newton's law, along the normal *CN*, velocity of separation

= *e*.(velocity of approach)

$\therefore v\cos\theta - 0 = eu\cos\alpha$

$v\cos\theta = eu\cos\alpha$ ....(ii)

Dividing (i) by (ii), we get $\cot\theta = e\cot\alpha$

***Particular case* :** If $\alpha = 0$ then from (i), $v\sin\theta = 0 \Rightarrow \sin\theta = 0$

$\therefore \theta = 0$; $\because v \neq 0$ and from (ii) $v = eu$

Thus if a smooth sphere strikes a smooth horizontal plane normally, then it will rebound along the normal with velocity, *e* times the velocity of impact *i.e.* velocity of rebound = *e*.(velocity before impact).

Rebounds of a particle on a smooth plane : If a smooth ball falls from a height *h* upon a fixed smooth horizontal plane, and if *e* is the coefficient of restitution, then whole time before the rebounding ends $= \sqrt{\left(\frac{2h}{g}\right)}.\frac{1+e}{1-e}$

And the total distance described before finishing rebounding $= \frac{1+e^2}{1-e^2}h$.

Example: 30 A ball is dropped from a height *h* on a horizontal plane and the coefficient of restitution for the impact is *e*, the velocity with which the ball rebounds from the floor is

(a) *eh* (b) *egh* (c) $e\sqrt{gh}$ (d) $e\sqrt{2gh}$

Solution: (d) Let u be the velocity with which the ball strikes the ground. Then, $u^2 = 2gh \Rightarrow u = \sqrt{2gh}$ .....(i)

Suppose the ball rebounds with velocity v, then $V = -e(-u-0) \Rightarrow v = ue \Rightarrow v = e\sqrt{2gh}$, [from (i)].

Example: 31 A sphere impinges directly on a similar sphere at rest. If the coefficient of restitution is $\frac{1}{2}$, the velocities after impact are in the ratio

(a) 1 : 2 (b) 2 : 3 (c) 1 : 3 (d) 3 : 4

Solution: (c) From principle of conservation of momentum ,

$mv_1 + m'v_1' = mu_1 + m'u_1'$ ......(i)

From Newton's rule for relative velocities before and after impact.

$v_1 - v_1' = -e(u_1 - u_1')$ ......(ii)

Here $m = m'$, $e = \frac{1}{2}$ and let $u_1' = 0$

Then from (i) $mv_1 + mv_1' = mu_1$ or $v_1 + v_1' = u_1$ ......(iii)

From (ii), $v_1 - v_1' = \frac{-1}{2}u_1$ ....(iv)

Adding (iii) and (iv) , $2v_1 = \frac{1}{2}u_1$ or $v_1 = \frac{1}{4}u_1$ ......(v)

From (iii) and (v), $v_1' = u_1 - \frac{1}{4}u_1 = \frac{3}{4}u_1$. Hence $v_1 : v_1' = 1:3$.

Example: 32 A ball impinges directly upon another ball at rest and is itself reduced to rest by the impact. If half of the K.E. is destroyed in the collision, the coefficient of restitution, is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

Solution: (d) Let masses of the two balls be m_1 and m_2 .

By the given question, we have

m_1	m_2
u_1	$u_2 = 0$
$v_1 = 0$	v_2

$\therefore$ $m_1u_1 + m_2.0 = m_1.0 + m_2v_2$ or $m_1u_1 = m_2v_2$ .....(i) and $0 - v_2 = e(0 - u_1)$ *i.e.,* $v_2 = eu_1$ .....(ii)

Again for the given condition, K.E. (before impact) = 2 × K.E. after impact

$\therefore$ $\frac{1}{2}m_1u_1^2 + 0 = 2\left(0 + \frac{1}{2}m_2v_2^2\right)$ .....(iii) or $\frac{1}{2}m_1u_1^2 = 2.\frac{1}{2}m_2v_2^2$

But from (i), $m_1u_1 = m_2v_2$, $\therefore$ $\frac{1}{2}u_1 = v_2$ *i.e.,* $u_1 = 2v_2$. Putting in (ii), we get $v_2 = e.2v_2$, $\therefore$ $e = \frac{1}{2}$.

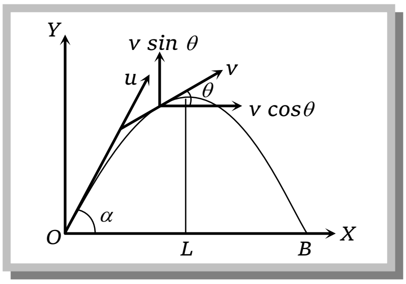
4.12 Projectile Motion

When a body is thrown in the air, not vertically upwards but making an acute angle α with the horizontal, then it describes a curved path and this path is a Parabola.

The body so projected is called a projectile. The curved path described by the body is called its trajectory.

The path of a projectile is a parabola whose equation is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$



Its vertex is $A\left(\frac{u^2 \sin\alpha \cos\alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g}\right)$.

Focus is $S\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2 \cos 2\alpha}{2g}\right)$

(1) **Some important deductions about projectile motion:** Let a particle is projected with velocity u in a direction making angle α with OX. Let the particle be at a point $P(x,y)$ after time t and v be its velocity making an angle θ with OX.

(i) Time of flight $= \frac{2u\sin\alpha}{g}$

(ii) Range of flight *i.e.*, Horizontal Range $R = \frac{u^2 \sin 2\alpha}{g}$

Maximum horizontal Range $= \frac{u^2}{g}$ and this happens, when $\alpha = \frac{\pi}{4}$

(iii) Greatest height $= \frac{u^2 \sin^2 \alpha}{2g}$

Time taken to reach the greatest height $= \frac{u\sin\alpha}{g}$

(iv) Let $P(x,y)$ be the position of the particle at time t. Then $x = (u\cos\alpha)t$, $y = (u\sin\alpha)t - \frac{1}{2}gt^2$

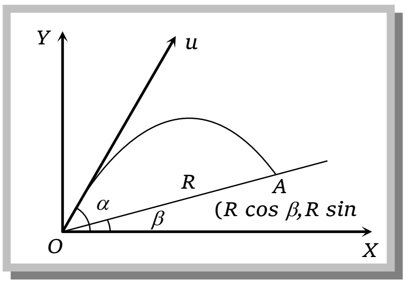
(v) Horizontal velocity at time $t = \frac{dx}{dt} = u\cos\alpha$

Vertical velocity at time $t = \frac{dy}{dt} = u\sin\alpha - gt$

(vi) Resultant velocity at time $t = \sqrt{u^2 - 2ugt\sin\alpha + g^2t^2}$. And its direction is $\theta = \tan^{-1}\left(\frac{u\sin\alpha - gt}{u\cos\alpha}\right)$

(vii) Velocity of the projectile at the height h $= \sqrt{u^2 - 2gh}$ and its direction is $\theta = \tan^{-1}\left[\frac{\sqrt{u^2\sin^2\alpha - 2gh}}{u\cos\alpha}\right]$

(2) **Range and time of flight on an inclined plane**



Let *OX* and *OY* be the coordinate axes and *OA* be the inclined plane at an angle β to the horizon *OX*. Let a particle be projected from *O* with initial velocity *u* inclined at an angle α with the horizontal *OX*. The equation of the trajectory is $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$.

(i) Range and time of flight on an inclined plane with angle of inclination β are given by

$R = \frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$ and maximum range up the plane $= \frac{u^2}{g(1+\sin\beta)}$, where $2\alpha - \beta = \frac{\pi}{2}$

(ii) Time of flight $T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$

(iii) Range down the plane $= \frac{2u^2\cos\alpha\sin(\alpha+\beta)}{g\cos^2\beta}$

(iv) Maximum range down the plane $= \frac{u^2}{g(1-\sin\beta)}$, where $2\alpha + \beta = \frac{\pi}{2}$

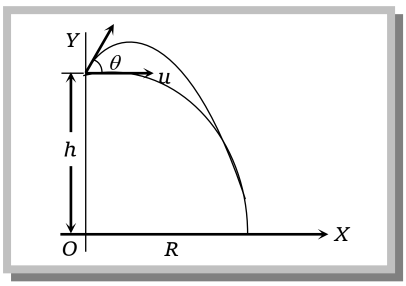
(v) Time of flight $= \frac{2u\sin(\alpha+\beta)}{g\cos\beta}$, down the plane

(vi) Condition that the particle may strike the plane at right angles is $\cot\beta = 2\tan(\alpha-\beta)$.

Example: 33 Two stones are projected from the top of a cliff *h metres* high, with the same speed *u* so as to hit the ground at the same spot. If one of the stones is projected horizontally and the other is projected at an angle θ to the horizontal then $\tan\theta$ equals

(a) $\sqrt{\frac{2u}{gh}}$ (b) $2g\sqrt{\frac{u}{h}}$ (c) $2h\sqrt{\frac{u}{g}}$ (d) $u\sqrt{\frac{2}{gh}}$

Solution: (d) $R = u\sqrt{\frac{2h}{g}} = (u\cos\theta)\times t$; $t = \frac{1}{\cos\theta}\sqrt{\frac{2h}{g}}$ .....(i)



Now $h = (-u\sin\theta)t + \frac{1}{2}gt^2$, '*t*' from (i)

$h = -\frac{u\sin\theta}{\cos\theta}\sqrt{\frac{2h}{g}} + \frac{1}{2}g\left[\frac{2h}{g\cos^2\theta}\right]$, $h = -u\sqrt{\frac{2h}{g}}\tan\theta + h\sec^2\theta$

$h = -u\sqrt{\frac{2h}{g}}\tan\theta + h\tan^2\theta + h$, $\tan^2\theta - u\sqrt{\frac{2}{hg}}\tan\theta = 0$; $\therefore \tan\theta = u\sqrt{\frac{2}{hg}}$.

Example: 34 A gun projects a ball at an angle of 45^o with the horizontal. If the horizontal range is 39.2 *m*, then the ball will rise to

[AMU 1999]

(a) 9.8 *m* (b) 4.9 *m* (c) 2.45 *m* (d) 19.6 *m*

Solution: (a) Horizontal range = $\frac{u^2\sin 2\alpha}{g} = 39.2 \Rightarrow \frac{u^2\sin 90^o}{g} = 39.2 \Rightarrow \frac{u^2}{g} = 39.2$

Greatest height attained = $\frac{u^2\sin^2\alpha}{2g}$ = $\frac{39.2}{2}\sin^2 45^o$ = $(19.6)\frac{1}{2} = 9.8m$.

Example: 35 If the range of any projectile is the distance equal to the height from which a particle attains the velocity equal to the velocity of projection, then the angle of projection will be

(a) 60^o (b) 75^o (c) 36^o (d) 30^o

Solution: (b) Range = $\frac{u^2\sin 2\alpha}{g}$, where α is the angle of projection.

Also, $u^2 = 2gh \Rightarrow \frac{u^2}{2g} = h$

By the given condition, $\frac{u^2}{2g} = \frac{u^2 \sin 2\alpha}{g} \Rightarrow \frac{1}{2} = \sin 2\alpha$, $\therefore$ $2\alpha = 30^o$ or 150^o

$\therefore$ $\alpha = 15^o$ or 75^o . Hence, required angle of projection = 75^o.

Example: 36 A particle is thrown with the velocity v such that its range on horizontal plane is twice the maximum height obtained. Its range will be

(a) $\frac{2v^2}{3g}$ (b) $\frac{4v^2}{3g}$ (c) $\frac{4v^2}{5g}$ (d) $\frac{v^2}{7g}$

Solution: (c) Range = 2 × (maximum height)

$\therefore$ $\frac{v^2 \sin 2\alpha}{g} = 2\frac{v^2 \sin^2 \alpha}{2g} \Rightarrow \sin 2\alpha = \sin^2 \alpha$

$\Rightarrow 2\sin\alpha\cos\alpha - \sin^2\alpha = 0 \Rightarrow \sin\alpha(2\cos\alpha - \sin\alpha) = 0 \Rightarrow \sin\alpha = 0$ or $\tan\alpha = 2$

But $\alpha \neq 0$, $\therefore \sin\alpha \neq 0$, $\therefore \tan\alpha = 2$. Now $R = \frac{v^2 \sin 2\alpha}{g} = \frac{v^2}{g}.\frac{2\tan\alpha}{1+\tan^2\alpha} \Rightarrow \frac{v^2}{g}.\frac{2.2}{1+4} = \frac{4v^2}{5g}$.

Example: 37 Let u be the velocity of projection and v_1 be the velocity of striking the plane when projected so that the range up the plane is maximum, and v_2 be the velocity of striking the plane when projected so that the range down the plane is maximum, then u is the

(a) A.M. of v_1, v_2 (b) G.M. of v_1, v_2 (c) H.M. of v_1, v_2 (d) None of these

Solution: (b) Let u be the velocity of projection and β be the inclination of the plane with the horizon. Let R be the maximum range up the plane. Then, $R = \frac{u^2}{g(1+\sin\beta)}$

The coordinates of the point where the projectile strikes the plane are ($R\cos\beta, R\sin\beta$) . It is given that v_1 is the velocity of striking the plane when projected so that the range up the plane is maximum.

$\therefore$ $v_1^2 = u^2 - 2gR\sin\beta$

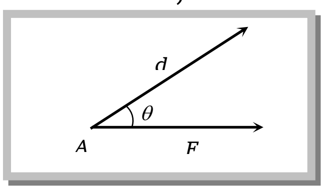
$\Rightarrow$ $v_1^2 = u^2 - 2g\sin\beta.\frac{u^2}{g(1+\sin\beta)} \Rightarrow v_1^2 = u^2\left(\frac{1-\sin\beta}{1+\sin\beta}\right)$ .....(i)

Replacing β by $-\beta$ in (i) , we get $v_2^2 = u^2\left(\frac{1+\sin\beta}{1-\sin\beta}\right)$ .....(ii)

From (i) and (ii), we get $v_1^2.v_2^2 = u^4 \Rightarrow v_1v_2 = u^2$ *i.e.*, u is the G.M. of v_1, v_2.

4.13 Work, Power and Energy

(1) **Work :** Work is said to be done by a force when its point of application undergoes a displacement. In other words, when a body is displaced due to the action of a force, then the force is said to do work. Work is a scalar quantity.



Work done = force × displacement of body in the direction of force

$W = \vec{F}.\vec{d}$

$W = Fd\cos\theta$, where θ is the angle between F and d.

(2) **Power :** The rate of doing work is called power. It is the amount of work that an agent is capable of doing in a unit time. 1 *watt* = 10^7 *ergs per sec* = 1 *joule* per *sec*., 1 H.P. = 746 watt.

(3) **Energy :** Energy of a body is its capacity for doing work and is of two kinds :

(i) **Kinetic energy :** Kinetic energy is the capacity for doing work which a moving body possesses by virtue of its motion and is measured by the work which the body can do against any force applied to stop it, before the velocity is destroyed. $K.E.=\frac{1}{2}mv^2$

(ii) **Potential energy :** The potential energy of a body is the capacity for doing work which it possesses by virtue of its position of configuration. $P.E.= mgh$.

Example: 38 Two bodies of masses m and $4m$ are moving with equal momentum. The ratio of their K.E. is **[MNR 1990; UPS**

(a) 1 : 4 (b) 4 : 1 (c) 1 : 1 (d) 1 : 2

Solution: (b) Momentum of first body = mu, where u is the velocity of the first body.

$\therefore$ Momentum of second body = $4mv$, where v is the velocity of the second body.

By the given condition, $mu = 4mv \Rightarrow u = 4v$

$\therefore$ Ratio of K.E. = $\frac{\frac{1}{2}mu^2}{\frac{1}{2}.4mv^2} = \frac{(4v)^2}{4v^2} = 4$. Hence, the ratio of K.E. is 4 : 1.

Example: 39 A bowler throws a bumper with a speed of $24 m/\sec$. The moment the ball touches the ground, it losses its energy by 1.5 *kgm*. If the weight of the ball is 225 *gm*, then speed of the ball at which it hits the bat is **[UPSEAT 2000]**

(a) 2.22 *m/sec.* (b) 22.2 *m/sec.* (c) 4.44 *m/sec.* (d) 44.4 *m/sec.*

Solution: (b) Weight of the ball = $w = 225\ gm$ = 0.225 *kg*

Velocity of the ball = v = 25 *m/sec.*

Loss of energy = 1.5 *kg.m*

$\therefore$ K.E. of the ball before it touches the ground = $\frac{1}{2}\frac{wv^2}{g} = \frac{1}{2}.\frac{0.225}{9.81}\times(25)^2 = 7.17 kgm$.

K.E. of the ball after bumping = K.E. before touching the ground – loss of K.E. = 7.17 – 1.5 = 5.67 *kg-m.*

Let v_1 be the velocity of the ball at which it hits the bat.

Now K.E. of the ball at the time of hitting the bat $= \frac{1}{2}\frac{w}{g}v_1^2 \Rightarrow$ $5.67 = \frac{1}{2}\times\frac{0.225}{9.81}v_1^2$ $\Rightarrow$

$v_1^2 = 494 \Rightarrow v_1 = 22.2\ m/\sec$.

Example: 40 A particular starts from rest and move with constant acceleration. Then the ratio of the increase in the K.E. in m^{th} and $(m+1)^{th}$ second is

(a) $m : m$ (b) $m+1 : m+1$ (c) $2m-1 : 2m+1$ (d) None of these

Solution: (c) Required ratio = $\frac{\text{Increase in K.E. in } m^{th} \text{second}}{\text{Increase in K.E. in } (m+1)^{th}\text{ second}} = \frac{\text{Work done against the force in } m^{\text{th}} \text{second}}{\text{Work done against the force in } (m+1)^{th} \text{second}}$

$$= \frac{P\times\left[0+\frac{1}{2}f(2m-1)\right]}{P\times\left[0+\frac{1}{2}f(2(m+1)-1)\right]} = \frac{2m-1}{2m+2-1} = \frac{2m-1}{2m+1}.$$