01

Composite Functions

1.01 Introduction and Previous Knowledge

We have studied the notion of relations and functions, domain, co-domain and range have been introduced in previous class along with different types of specific real valued functions and their graphs. As the concept of function, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Function : A function from a non-empty set A to a non-empty set B is defined as a rule in which every element of a set A is *uniquely* associated with the element of set B.

Domain, Co-domain and Range of a Function : If *f* is a function from set A to set B then set A is called as domain of *f* and set B is called as co-domain of *f*. All those elements of set B which are the images of elements of set A are called as range of *f*. It is written as *f*(A).

Constant Function : In this type of function, every element of domain is associated with only one element of co-domain.

Identify Function : A function defined on set A in such a way that every element of A is associated to itself is known as identity function of A. It is written as I_A

Equal Functions : Two functions *f* and *g* are called equal if.

(i) Domain of *f* = Domain of *g* (ii) Co-domain of *f* = Co-domain of *g* (iii) $f(x) = g(x), \ \forall \ x \in$ domain

Type of Functions on the basis of association of elements

(i) **One-One function :** Let $f : A \to B$ is a function, then *f* is one-one if every element of set A has distinct image in set B

(ii) **Many-One function :** Let $f : A \to B$ is a function, then *f* is called many-one if two or more elements of set A has the same image in set B.

(iii) **Onto function :** A function $f : A \to B$ is said to be onto if every element of set B is the image of some element of set A under the function *f* i.e. for every element of B, there exist some pre-image in A i.e. is onto function if *f*(A)=B.

(iv) **Into function :** A function $f : A \to B$ is said to be into if there exist atleast one element in set B which is not the image of element of set A under the function *f* i.e. *f* is into function if $f(A) \neq B$.

(v) **One-One onto function :** A function $f : A \to B$ is said to be One One-Onto if *f* is one-one and onto. This is also called bijective function.

1.02 Composition of Function

Let A, B, C be three non-empty sets and let $f : A \to B$ and $g : B \to C$ are two functions.

Since *f* is a function from A to B therefore every element x of A threre exists a unique $f(x)$ in set B.

Again since g is a function from B to C therefore every $f(x)$ in set B there exists $g\left[f(x)\right]$ in set C.

Thus we see that for two functions *f* and *g* we get a new function defined from A to C. This function is said to be a composition of functions and represented by (gof) . It is defined as follows.

Definition : If $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of *f* and *g*, denoted by *gof*, is defined as the function $(gof) : A \to C$,

$$(gof)\,x = g\left[f(x)\right], \quad \forall \;\; x \in A$$

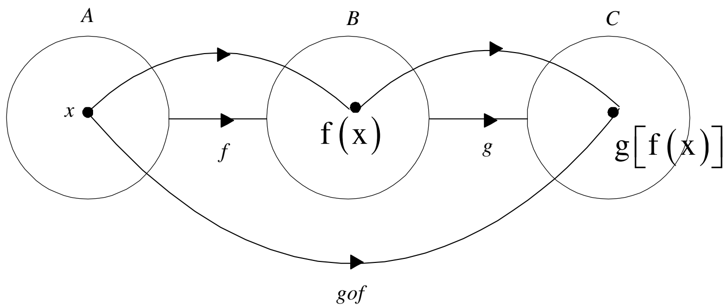

Fig. 1.01

Note : By the definition of $g \circ f$, when every element x of set A have $f(x)$, element of domain of g. so that be find image of g. Hence $g \circ f$ is defind if the range of f is the subset of domain of g is necessary.

Illustrative Examples

Example 1. If $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{7, 8, 9\}$ and $f : A \to B$ and $g : B \to C$ be the functions defined as $f(1) = 4$, $f(2) = 4$, $f(3) = 5$; $g(4) = 8$, $g(5) = 9$ then find $g \circ f$.

Solution : We have $(gof) : A \to C$

$$(gof)(1) = g\left[f(1)\right] = g(4) = 8$$
$$(gof)(2) = g\left[f(2)\right] = g(4) = 8$$
$$(gof)(3) = g\left[f(3)\right] = g(5) = 9$$
$$\therefore \qquad (gof) = \{(1,8),(2,8),(3,9)\}$$

Example 2. If $f : R \to R$, $f(x) = \sin x$ and $g : R \to R$, $g(x) = x^2$ then find $g \circ f$ and $f \circ g$.

Solution : Here the range of f is the subset of domain of g and range of g is the subset of domain of f. Therefore (gof) and (fog) both are defined.

$$(gof)(x) = g\left[f(x)\right] = g(\sin x) = (\sin x)^2 = \sin^2 x$$
$$(fog)(x) = f\left[g(x)\right] = f(x^2) = \sin x^2$$

Here $\qquad (gof) \neq (fog)$

Example 3. If $f : N \to Z$, $f(x) = 2x$

and $\quad g : Z \to Q$, $g(x) = (x+1)/2$ then find $f \circ g$ and $g \circ f$.

Solution : $\quad (gof)(x) = g\left[f(x)\right] = g(2x) = (2x+1)/2, \quad \forall\, x \in N$

here (fog) does not exist.

1.03 Properties of Composite of Functions

(i) The composite of functions is not necessarily commutative

Let $f : A \to B$ and $g : B \to C$ be the two functions, then composite function $(gof): A \to C$ exists and defined because range of *f* is a subset of domain of *g*. But here (fog) does not exist as range of *g* is not a subset of domain of A of *f*, thus if $C \not\subset A, (fog)$ will not exist.

If $C = A$ then $f : A \to B$ and $g : B \to A$

In this case $(gof): A \to A$ and $(fog): B \to B$ both will exist and $(gof) \neq (fog)$ as the domain and co-domain are different.

If $A = B = C$ then $(gof): A \to A$ and $(fog): A \to A$, but it is not necessary that both will be equal.

Example : If $f : R \to R, f(x) = 2x$ and $g : R \to R, g(x) = x^2$ then $(gof): R \to R, (fog): R \to R$ but

$$(gof)(x) = g\left[f(x)\right] = g(2x) = (2x)^2 = 4x^2$$

$$(fog)(x) = f\left[g(x)\right] = f\left(x^2\right) = 2x^2$$

$$\therefore \qquad (fog) \neq (fog)$$

Note : (gof) and (fog) are equal only in specific conditions.

Example : If $f : R \to R, f(x) = x^2$

$g : R \to R, g(x) = x^3$ then $(gof): R \to R, (fog): R \to R$

and $(gof)(x) = g\left[f(x)\right] = g\left(x^2\right) = \left(x^2\right)^3 = x^6$

$$(fog)(x) = f\left[g(x)\right] = f\left(x^3\right) = \left(x^3\right)^2 = x^6$$

$$\therefore \qquad (fog) = (gof)$$

This condition does not occur every time.

(ii) Composite of Functions is Associative

Thorem 1.1 If three functions f, g, h are such that the function $f \circ (g \circ h)$ and $(f \circ g) \circ h$ are defined then

$$f \circ (g \circ h) = (f \circ g) \circ h$$

Proof : Let the three functions *f*, *g*, *h* are such that:

$$h : A \to B, \ g : B \to C, \ f : C \to D$$

Now both the functions $fo(goh)$ and $(fog)\,oh$ are defiend from A to D.

i.e. $fo(goh): A \to D$ and $(fog)\,oh: A \to D$

Clearly the domain A and co-domain D of both the functions are same, hence to compare them we have to prove that

$$\left[fo\left(goh\right)\right]\left(x\right)=\left[\left(fog\right)oh\right]\left(x\right),\ \forall\, x\in A$$

Let $x\in A,\ y\in B,\ z\in C$ such that

$$h\left(x\right)=y \quad \text{and} \quad g\left(y\right)=z$$

then
$$\left[fo\left(goh\right)\right]\left(x\right)=f\left[\left(goh\right)\left(x\right)\right]$$
$$f\left[g\left\{h\left(x\right)\right\}\right]=f\left[g\left(y\right)\right]=f\left(z\right)$$

$\therefore$
$$\left[fo\left(goh\right)\right]\left(x\right)=f\left(z\right) \tag{1}$$

again
$$\left[\left(fog\right)oh\right]\left(x\right)=\left(fog\right)\left[h\left(x\right)\right]=\left(fog\right)\left(y\right)$$
$$=f\left[g\left(y\right)\right]=f\left(z\right) \tag{2}$$

from (1) and (2)

$$\left[fo\left(goh\right)\right]\left(x\right)=\left[\left(fog\right)oh\right]\left(x\right),\ \forall\, x\in A$$

$\therefore$
$$fo\left(goh\right)=\left(fog\right)oh$$

This can be shown through the following figure:

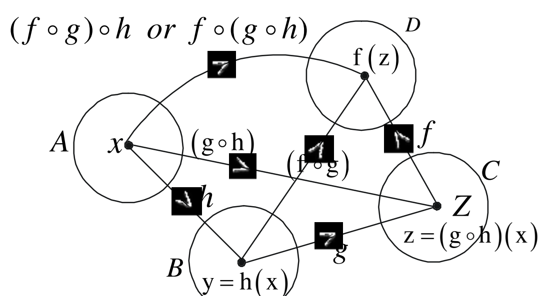

Fig 1.02

(ii) The composite of two bijections is a bijection

Theorem 1.2 If f and g are bijective functions such that $\left(gof\right)$ is defined then $\left(gof\right)$ is also a bijective function.

Proof : Let $f:A\rightarrow B$ and $g:B\rightarrow C$ are the two one-one onto functions then composite function $\left(gof\right)$ is defined from set A to set C such that,

$$\left(gof\right):A\rightarrow C$$

To prove that $\left(gof\right)$ is one-one onto function:

One-one : Let $a_1,a_2\in A$ be such that

$$\left(gof\right)\left(a_1\right)=\left(gof\right)\left(a_2\right)$$

$\Rightarrow$
$$g\left[f\left(a_1\right)\right]=g\left[f\left(a_2\right)\right]$$

$\Rightarrow \qquad f(a_1) = f(a_2) \qquad [\because g \text{ is one-one}]$

$\Rightarrow \qquad a_1 = a_2 \qquad [\because f \text{ is one-one}]$

$\therefore \qquad (gof)$ is one-one

Onto : If $c \in C$ then

$c \in C \Rightarrow \exists\, b \in B$ is such that $g(b) = c \qquad [\because g \text{ is onto}]$

again $\qquad b \in B \Rightarrow \exists\, a \in A$ is such that $f(a) = b \qquad [\because f \text{ is onto}]$

similarly $\qquad c \in C \Rightarrow \exists\, a \in A$ is such that

$$(gof)(a) = g[f(a)] = g(b) = c$$

i.e. every element of C is the image of some element of A, in other words A has the pre-image of every element of C. Therefore (gof) is onto.

$\therefore \qquad (gof)$ is One-one onto function.

Theorem 1.3 If $f : A \to B$ then $foI_A = I_B of = f$, where I_A and I_B are identity functions defined in set A and B.

i.e. composition of any function with the identity function is function itself.

Proof : $\quad \because \quad I_A : A \to A$ and $f : A \to B \qquad \therefore \quad (foI_A) : A \to B$

Let $x \in A$ then

$$(foI_A)(x) = f[I_A(x)] = f(x) \qquad [\because I_A(x) = x, \forall\, x \in A]$$

$\therefore \qquad foI_A = f \qquad (1)$

again $\qquad f : A \to B$ and $I_B : B \to B \qquad \therefore \quad (I_B of) : A \to B$

Let $x \in A$ and $f(x) = y$, where $y \in B$

$\therefore \qquad (I_B of)(x) = I_B[f(x)] = I_B(y) = y \qquad [\because I_B(y) = y, \forall y \in B]$

$= f(x) \qquad (2)$

from (1) and (2) $\qquad (I_B of) = f = (f \circ I_A)$.

Illustrative Examples

Example 4. If $f : R \to R,\ f(x) = x^3$ and $g : R \to R,\ g(x) = 3x - 1$ then find $(gof)(x)$ and $(fog)(x)$. Also prove that $fog \neq gof$.

Solution : Clearly $(gof) : R \to R$ and $(fog) : R \to R$

$$(gof)(x) = g[f(x)] = g(x^3) = 3x^3 - 1$$

again $\qquad (fog)(x) = f[g(x)] = f(3x-1) = (3x-1)^3$

$\because \qquad (3x^3 - 1) \neq (3x-1)^3$

$\therefore \qquad (gof) \neq (fog)$

Example 5. If $f : R \to R, f(x) = x^2 + 2$ and $g : R \to R, g(x) = \dfrac{x}{x-1}$ then find (gof) and (fog).

Solution : Clearly $(gof) : R \to R$ and $(fog) : R \to R$ both exist

Let $x \in R$

then $$(g \circ f)(x) = g[f(x)] = g[x^2 + 2] = \frac{x^2+2}{x^2+2-1} = \frac{x^2+2}{x^2+1}$$

and $$(fog)(x) = f[g(x)] = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2 = \frac{x^2 + 2(x-1)^2}{(x-1)^2}$$

Example 6. Verify the associativity of the following functions:

$f : N \to Z_0, f(x) = 2x; g : \ Z_0 \to Q, g(x) = \dfrac{1}{x}$ and $h : Q \to R, h(x) = e^x$.

Solution : $\because \quad f : N \to Z_0, g : Z_0 \to Q, \ h : Q \to R$

$\because \quad (go\, f) : N \to Q$ and $(hog) : Z_0 \to R$

$\therefore \quad (hog)of : N \to R$

and $\quad h : Q \to R, (go\, f) : N \to Q \quad \therefore ho(go\, f) : N \to R$

Thus both the functions $(hog)of$ and $ho(go\, f)$ are defined on the set from N to R. Now we have to show that

$$[(ho\, g)of](x) = [ho(go\, f)](x), \quad \forall \ x \in N$$

Now $$[(hog)of](x) = (hog[f(x)]) = (hog)(2x) = h[g(2x)] = h\left(\frac{1}{2x}\right) = e^{1/2x} \qquad (1)$$

and $$[ho(gof)](x) = h[(go\, f)(x)] = h[g(f(x))]$$

$$= h[g(2x)] = h\left(\frac{1}{2x}\right) = e^{1/2x} \qquad (2)$$

from (1) and (2)

$$[(hog)o\ f](x) = [ho(go\, f)](x).$$

Thus the associativity of the function f, g, h is verified.

Exericse 1.1

1. If $f : R \to R$ and $g : R \to R$ are the two functions defined below then find $(fog)(x)$ and $(gof)(x)$

   (i) $f(x) = 2x + 3, \ g(x) = x^2 + 5$

   (ii) $f(x) = x^2 + 8, \ g(x) = 3x^3 + 1$

   (iii) $f(x) = x, \ g(x) = |x|$

   (iv) $f(x) = x^2 + 2x + 3, g(x) = 3x - 4.$

2. If $A=\{a,b,c\},\ B=\{u,v,w\}$

and $f:A\to B$ and $g:B\to A$ are defined as

$$f=\{(a,v),(b,u),(c,w)\}\ ;\ \ g=\{(u,b),(v,a),(w,c)\}$$

then find (fog) and (gof) .

3. If $f:R^+\to R^+$ and $g:R^+\to R^+$ are defiend as

$$f(x)=x^2 \text{ and } g(x)=\sqrt{x}$$

then find gof and fog . Are they equal ?

4. If $f:R\to R$ and $g:R\to R$ are two functions such that $f(x)=3x+4$ and $g(x)=\frac{1}{3}(x-4)$ then find $(fog)(x)$ and $(gof)(x)$ also find $(gog)(1)$.

5. If three functions f,g,h defined from R to R in such a way that $f(x)=x^2$, $g(x)=\cos x$ and $h(x)=2x+3$ then find the value of $\{ho\,(gof)\}\sqrt{2\pi}$.

6. If f and g are defiend as given below then find $(gof)(x)$.

(i) $f:R\to R,\ f(x)=2x+x^{-2}$, $g:R\to R,\ g(x)=x^4+2x+4.$

7. If $A=\{1,2,3,4\},\ f:R\to R,\ f(x)=x^2+3x+1$

$g:R\to R,\ g(x)=2x-3$ then find

(i) $(fog)(x)$ (ii) $(gof)(x)$ (iii) $(fof)(x)$ (iv) $(gog)(x)$.

1.04 Inverse function

(a) Inverse of an element

Let A and B be two sets and f is a function from A to B. i.e. $f:A\to B$ If an element 'a' of set A is associated to an element 'b' of set B under f then b is the f image of a under the function f is expressed as $b=f(a)$ and element 'a' is called as pre-image or inverse of 'b' under f and is denoted by $a=f^{-1}(b)$.

Inverse of an element may be unique, more than one or no one under a function. In fact, this all depend upon the function is one-one, many one, onto or into.

The function f is defined as shown in the figure.

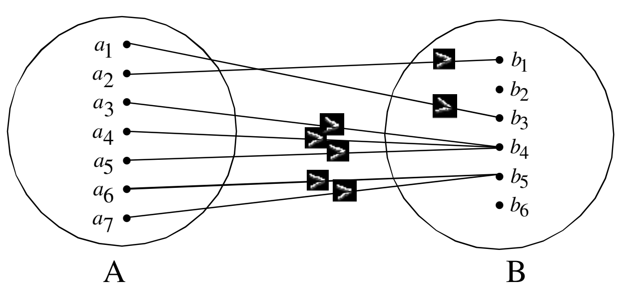

Fig. 1.03

We see that $f^{-1}(b_1) = a_2,$

$f^{-1}(b_2) = \phi,\ f^{-1}(b_3) = a_1,$

$f^{-1}(b_4) = \{a_3, a_4, a_5\},\ f^{-1}(b_5) = \{a_6, a_7\},$

$f^{-1}(b_6) = \phi.$

Example : If $A = \{-1, 1, -2, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and $f : A \to B,\ f(x) = x^2$ are defined as

$f^{-1}(1) = \{-1, 1\},\ f^{-1}(4) = \{-2, 2\},\ f^{-1}(6) = \phi$ and $f^{-1}(9) = \{3\}.$

Example : If $f : C \to C,\ f(x) = x^2 - 1$ then find $f^{-1}(-5)$ and f^{-1}a8f .

Solution : Let $f^{-1}(-5) = x$ then $f(x) = -5$

$\Rightarrow \quad x^2 - 1 = -5 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4}$

$\Rightarrow \quad x = \pm 2i$. both are in C.

again let $f^{-1}(8) = x$ then $f(x) = 8.$

$\Rightarrow \quad x^2 - 1 = 8 \Rightarrow x^2 = 9,\ x = \pm 3$ both are in C

$\therefore \quad f^{-1}(8) = \{-3, 3\}$

i.e. $f^{-1}(-5) = \{2i, -2i\}$ and $f^{-1}(8) = \{-3, 3\}.$

(b) Inverse function

Let A and B be two sets and $f : A \to B$. If we correlate the element of B to their pre-image in A under any rule then we find that there is some element in B which is not associated with any element in A. It happen when it is not onto. therefore, it is necessary that f is onto if all element of B would associate any element of A. Just like that if f is many-one then some element of B is associated with one or more element of A Therefore, an element of B is associated only one element of A only if f is one-one.

Thus we see that if $f : A \to B$ is One-One Onto function then we can define a new function from B to A in which every element y of B is related to its pre-image $f^{-1}(y)$ in A. This function is called as Inverse of f and is denoted by f^{-1} .

Definition : If $f : A \to B$ is one-one onto function and inverse of f is f^{-1}, then B is a function defined in A in which $b \in B$, is related to $a \in A$ where $f(a) = b.$

$$\therefore \quad f^{-1} : B \to A,\ f^{-1}(b) = a \Leftrightarrow f(a) = b$$

It is represented as $f^{-1} : \{(b, a) | (a, b) \in f\}$ in terms of ordered pair.

Note: The function f^{-1} is said to be the inverse of f, only when it is one-one onto.

1.05 Domain and Range of inverse function

It is clear from the definition that

$$\text{Domain of } f^{-1} = \text{Range of } f$$

and $$\text{Range of } f^{-1} = \text{Domain of } f$$

For Example : If $A = \{1,2,3,4\}$, $B = \{2,5,10,17\}$ and $f(x) = x^2 + 1$ then

$$f(1) = 2,\ f(2) = 5,\ f(3) = 10,\ f(4) = 17$$

$\therefore$ $$f = \{(1,2),(2,5),(3,10),(4,17)\}$$

Clearly *f* is one-one onto therefore its inverse exists i.e. $f^{-1}: B \to A$ and

$$f^{-1} = \{(2,1),(5,2),(10,3),(17,4)\}.$$

For Example : Let $f : R \to R,\ f(x) = 3x + 4,$ then we can easily prove that *f* is one-one onto function.

Therefore $f^{-1}: R \to R$ exists

Let $x \in R$ (Domain of *f*) and $y \in R$ (Co-domain of *f*)

Let $f(x) = y,$ $\therefore\ x = f^{-1}(y)$

Now $f(x) = y \Rightarrow 3x + 4 = y \Rightarrow x = \dfrac{y-4}{3}$

$$\Rightarrow\ f^{-1}(y) = \frac{y-4}{3}$$

$\therefore$ $f^{-1} : R \to R,\ f^{-1}(x) = \dfrac{x-4}{3}$ is defined.

1.06 Properties of Inverse Functions

Theorem 1. The inverse of a bijection is unique.

Proof : Let $f : A \to B$ is one-one, onto function then to prove that inverse of *f* is unique.

If possible let $g : B \to A$ and $h : B \to A,\ f$ are two inverse function of *f*. Let *y* be any element of B.

Let $g(y) = x_1$ and $h(y) = x_2$

Now $g(y) = x_1 \Rightarrow f(x_1) = y$ [$\because$ *g* is the inverse of *f*]

and $h(y) = x_2 \Rightarrow f(x_2) = y$ [$\because$ *h*, is the inverse of *f*]

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ [$\because$ *f* is one-one]

i.e. $g(y) = h(y),\ \forall\ y \in B$

$\therefore$ $g = h$

Thus inverse of *f* is unique.

Theorem 1.5 If $f : A \to B$ is bijective function and $f^{-1}: B \to A,\ f$ is the inverse of *f* then $fof^{-1} = I_B$ and $f^{-1}of = I_A$, where I_A and I_B respectively are the identity function on *A* and *B*.

Proof : $f: A \to B$ and $f^{-1}: B \to A$

$\therefore$ $fof^{-1}: B \to B$ and $f^{-1}of: A \to A$

For every $a \in A$ there exist a unique $b \in B$.

Where $f(a) = b$ or $f^{-1}(b) = a$

$$\therefore \quad \left(fof^{-1}\right)(b) = f\left[f^{-1}(b)\right] = f(a) = b$$

$$\therefore \quad \left(fof^{-1}\right)(b) = b, \qquad \forall\, b \in B$$

$$\therefore \quad fof^{-1} = I_B$$

similarly $\left(fof^{-1}\right)(a) = f^{-1}\left[f(a)\right] = f^{-1}(b) = a$

$$\therefore \quad \left(fof^{-1}\right)(a) = a, \qquad \forall\, a \in A$$

$$\therefore \quad f^{-1}of = I_A \,.$$

Theorem 1.6 The inverse of a bijection is also a bijection.

Proof : Let $f: A \to B$ is one-one, onto function and $g: B \to A$ is the inverse function of *f*. To prove that *g* is also one-one onto function.

Let $a_1, a_2 \in A;\ b_1, b_2 \in B$ are elements such that

$g(b_1) = a_1$ i.e. $f(a_1) = b_1$ [$\because$ g is the inverse of f]

and $g(b_2) = a_2$ i.e. $f(a_2) = b_2$ [$\because$ g is the inverse of f]

Now $g(b_1) = g(b_2) \Rightarrow a_1 = a_2$

$\Rightarrow$ $f(a_1) = f(a_2) \Rightarrow b_1 = b_2$

$\therefore$ g is one-one

again $a \in A \Rightarrow \exists\, b \in B$ for which $f(a) = b$

Now $f(a) = b \Rightarrow g(b) = a$

$a \in A \Rightarrow \exists\, b \in B$ such that $g(b) = a$

$\therefore$ g is Onto

Thus inverse function *g* is also one-one onto.

Theorem 1.7 If *f* and *g* are two one-one, onto functions such that the composite function gof is defined then there exist an inverse of gof i.e.

$$(gof)^{-1} = f^{-1}og^{-1}$$

Proof : Let $f: A \to B$ and $g: B \to C$ are two one-one, onto functions. Given that $(gof): A \to C$ is defined therefore from theorem 1.2, gof exist and given by

$$(gof)^{-1}: C \to A$$

To prove that $(gof)^{-1} = f^{-1}og^{-1}$

Now $f : A \to B$ is one-one onto function $\Rightarrow$ $f^{-1} : B \to A$ exists

Again $g : B \to C$ is one-one onto function $\Rightarrow$ $g^{-1} : C \to B$ exists

$\therefore$ $\left(f^{-1} o g^{-1}\right) : C \to A$ exists

similarly domain and co-domain of $(gof)^{-1}$ and $\left(f^{-1} o g^{-1}\right)$ are same.

Let $a \in A, b \in B, c \in C$ are elements such that

$$f(a) = b \quad \text{and} \quad g(b) = c$$

$\therefore$ $(gof)(a) = g\left[f(a)\right] = g(b) = c$

$\Rightarrow$ $(gof)^{-1}(c) = a$ (1)

again $f(a) = b \Rightarrow f^{-1}(b) = a$ (2)

$g(b) = c \Rightarrow g^{-1}(c) = b$ (3)

$\therefore$ $\left(f^{-1} o g^{-1}\right)(c) = f^{-1}\left[g^{-1}(c)\right] = f^{-1}(b)$ [from (3)]

$= a$ [from (2)] (4)

Therefore from (1) and (4), for any element x of C.

$$(gof)^{-1}(x) = \left(f^{-1} o g^{-1}\right)(x)$$

This proves that $(gof)^{-1} = f^{-1} \circ g^{-1}$.

Illustrative Examples

Example 7. If $f : R \to R, f(x) = x^2 + 5x + 9$ then find the value of $f^{-1}(8)$ and $f^{-1}(9)$.

Solution : Let $f^{-1}(8) = x \Rightarrow f(x) = 8$

$\Rightarrow$ $x^2 + 5x + 9 = 8 \Rightarrow x^2 + 5x + 1 = 0$

$\Rightarrow$ $x = \dfrac{-5 \pm \sqrt{25 - 4}}{2} = \dfrac{-5 \pm \sqrt{21}}{2}$

$\therefore$ $f^{-1}(8) = \left\{\frac{1}{2}\left(-5 + \sqrt{21}\right), \frac{1}{2}\left(-5 - \sqrt{21}\right)\right\}$

again Let $f^{-1}(9) = x \Rightarrow f(x) = 9$

$\Rightarrow$ $x^2 + 5x + 9 = 9 \Rightarrow x = 0,\ x = -5$

$\therefore$ $f^{-1}(9) = \{0, -5\}$.

Example 8. If $f : R \to R, f(x) = x^2 + 1$ then find the value of $f^{-1}(-5)$ and $f^{-1}(26)$.

Solution : Let $f^{-1}(-5) = x$ then $f(x) = -5$

$\Rightarrow$ $x^2 + 1 = -5 \Rightarrow x^2 = -6 \Rightarrow x = \pm\sqrt{-6}$

then $\sqrt{-6}$ is not a real number

$\therefore \quad \pm\sqrt{-6} \notin R \qquad \therefore \quad f^{-1}(-5) = \phi$

again let $\quad f^{-1}(26) = x \quad$ then $f(x) = 26$

$\Rightarrow \quad x^2 + 1 = 26 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$

$$\therefore \quad f^{-1}(26) = \{-5, 5\}$$

Example 9. If $f : R \to R, f(x) = x^3 + 2$ then prove that *f* is one-one onto function. Also find the inverse of *f*.

Solution : Let $x_1, x_2 \in R$ then $f(x_1) = f(x_2)$

$$\Rightarrow x_1^3 + 2 = x_2^3 + 2 \qquad \Rightarrow x_1^3 = x_2^3 \qquad \Rightarrow x_1 = x_2$$

$\therefore\ f$ is one-one

again let $y \in R$ then $\exists\ (y-2)^{1/3} \in R$ is such that

$$f\left[(y-2)^{1/3}\right] = (y-2) + 2 = y$$

Thus function is onto

$\therefore \quad f$ is one-one onto function

since *f* is bijective then $f^{-1} : R \to R$ is defined as

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

but $\quad f(x) = x^3 + 2 \quad \Rightarrow \quad x^3 + 2 = y$

$$\Rightarrow \quad x = (y-2)^{1/3}$$

$\Rightarrow \quad f^{-1}(y) = (y-2)^{1/3} \quad \Rightarrow \quad f^{-1}(x) = (x-2)^{1/3}$

$\therefore \quad f^{-1} : R \to R, f^{-1}(x) = (x-2)^{1/3}$.

Example 10. If $f : Q \to Q,\ f(x) = 2x$ and $g : Q \to Q,\ g(x) = x + 2$ then verify the following

$$(gof)^{-1} = f^{-1}og^{-1}$$

Solution : Since *f* and *g* are two linear functions therefore *f* and *g* are one-one onto functions thus their inverse f^{-1} and g^{-1} exist

$$f^{-1} = Q \to Q,\quad f^{-1}(x) = \frac{x}{2}, \qquad \forall\ x \in Q \tag{1}$$

$$g^{-1} = Q \to Q,\quad g^{-1}(x) = x - 2 \qquad \forall\ x \in Q \tag{2}$$

We know that composition of two bijective functions is also bijective, therefore $(gof) : Q \to Q$ is also bijective and its inverse exists

$\therefore \quad (gof)^{-1} : Q \to Q \qquad \because \quad (gof)(x) = g[f(x)] = g(2x) = 2x + 2$

$$\therefore \quad (gof)^{-1}(x) = (x-2)/2 \tag{3}$$

again $f^{-1}og^{-1}:Q\to Q$

and $\left(f^{-1}og^{-1}\right)(x)=f^{-1}\left[g^{-1}(x)\right]=f^{-1}(x-2)$ [from (2)]

$=(x-2)/2$ [from (1), (4)]

from (3) and (4) $(gof)^{-1}(x)=\left(f^{-1}og^{-1}\right)(x),\ \forall\, x\in Q$

$\therefore$ $(gof)^{-1}=f^{-1}og^{-1}.$

Exericse 1.2

1. If $A=\{1,2,3,4\},B=\{a,b,c\}$, then define four one-one onto functions from A to B and also find their inverse function.
2. If $f:R\to R, f(x)=x^3-3$ then prove that f^{-1} exists and find formula of f^{-1} and the values of $f^{-1}(24)$, $f^{-1}(5)$.
3. If $f:R\to R$ is defined as follows

   (i) $f(x)=2x-3$ (ii) $f(x)=x^3+5.$

   then prove that *f* is bijective and also find f^{-1} .
4. If $A=\{1,2,3,4\},B=\{3,5,7,9\},C=\{7,23,47,79\}$ and $f:A\to B,\ f(x)=2x+1, g:B\to C, g(x)=x^2-2$ then write $(gof)^{-1}$ and $f^{-1}og^{-1}$ in the form of ordered pair.
5. If $f:R\to R, f(x)=ax+b, a\neq 0$ is defined then prove that *f* is bijective also find the formula of f^{-1} .
6. If $f:R\to R, f(x)=\cos(x+2)$ then does f^{-1} exist?
7. Find f^{-1} (if exist) when $f:A\to B$, where

   (i) $A=\{0,-1,-3,2\},B=\{-9,-3,0,6\},f(x)=3x.$

   (ii) $A=\{1,3,5,7,9\},B=\{0,1,9,25,49,81\},f(x)=x^2.$

   (iii) $A=B=R,\ f(x)=x^3.$

1.07 Binary operation

Let S be a non-empty set. A function defined from $S\times S$ to S where S is a binary operation i.e. set S is defined in such a way that for every ordered pair (a, b) of set S there exist a unique element in S Generally the binary operation is denoted by symbols $*, o$ or $\oplus$. We denote $*$ by $a*b$ for all $(a,b)\in S\times S$.

Definition : A binary operation $*$ on set S is a function $*\ :\ S\times S\to S$ we denote $*(a,b)$ by a * b i.e.

$$a\in S, b\in S \quad\Rightarrow\quad a*b\in S,\quad \forall a,b\in S$$

For Example :

1. Addition $(+)$, substraction $(-)$ and multiplication $(\times)$ of integers are the binary operation on a set of integers Z which relates the elements a,b of Z with $(a+b),(a-b)$ and ab

2. For a power set of any set S, the union of sets $(\bigcup)$ and intersection $(\bigcap)$ are binary operations in P(S) because

$$A \in P(S),\ B \in P(S) \Rightarrow A \bigcup B \in P(S) \text{ and } A \bigcap B \in P(S)$$

3. In a set of rational number Q, $*$, is defined as

$$a * b = \frac{ab}{2}, \qquad \forall\, a, b \in Q$$

Q is a binary operation as for all $a \in Q, b \in Q \Rightarrow ab/2 \in Q$

4. In a set of real numbers R, $*$, where $*$ is defined as

$$a*b = a + b - ab, \qquad \forall\, a, b \in R$$

R is a binary operation as

$$a \in R,\ b \in R \Rightarrow (a + b - ab) \in R$$

5. In a set of natural numbers N addition and multiplication are binary operations

$$a \in N,\ b \in N \Rightarrow (a+b) \in N,\ \forall\, a, b \in N$$
$$a \in N,\ b \in N \Rightarrow (a \cdot b) \in N,\ \forall\, a, b \in N$$

But difference and divsion are not bianry operations on N.

6. Division is not a binary operation in any of the sets Z, Q, R, C, N but in Q_0, R_0 and C_0 it is a bianry operation.

7. Let S is a set of all defined function in a set A, then composite funcion S is a binary operation as

$$f, g \in S \Rightarrow f : A \to A,\ g : A \to A$$

$$\Rightarrow \qquad (gof) : A \to A$$

1.08 Types of binary operations

(i) Commutativity

Let S be a non-empty set in which a binary operation $*$ is defined $a, b \in S$ then we know that $(a,b) \neq (b,a)$ untill we have $a = b$. Thus it is not necessary that (a,b) and (b,a) defined under $*$ have same image. In other words it is not ncessary that

$$a*b = b*a, \qquad \forall\, a, b, \in S$$

If $a*b = b*a,\ \forall\, a, b, \in S$ then $*$ is commutative operation is S.

Defintion : A bianry operation in set S is said to be commutiatve if $a*b = b*a,\ \forall\, a, b, \in S$.

For Example 1. In a set of Real numbers R addition and multiplication are commutative operations but difference is not.

2. In a power set P(S) of Set S Union of sets $(\bigcup)$ and intersection $(\bigcap)$ are commutative operations but difference of sets is not commutative.

(ii) Associativity

Let S be a non-empty set in which a binary operation $*$ is defined. Let $a, b, c, \in S$. If three elements a, b, c are there but binary is defined for two numbers but here are three elements of S.

Therefore we have to foucs on $a * (b * c)$ or $(a * b) * c$ its not always true that

$a*(b*c)=(a*b)*c,\ \forall\ a, b,c \in S$. If $a*(b*c)=(a*b)*c, \forall\ a,b,c\ \in S$ then operation $*$ is associative.

Definition : A binary operation $*$ defined on set S is said to be associative if $a*(b*c)=(a*b)*c,\ \forall a,b,c \in S.$

For Example

1. Addition and multiplication of set of integers Z are associative but difference is not as

$$a+(b+c)=(a+b)+c,\ \ \forall\ a,b,c \in Z$$

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c, \qquad \forall\ a,b,c\ \in Z$$

but $$a-(b-c)\neq(a-b)-c$$

2. For a power set P(S) of set S the union and intersection of sets are associative as for $A, B, C \in P(S)$ we have

$$(A\bigcup B)\bigcup C = A\bigcup(B\bigcup C)$$

and $$(A\bigcap B)\bigcap C = A\bigcap(B\bigcap C).$$

3. If A is a non-empty set and S is a set of all functions defined on A then operation defined on set S is a composite function and is associative as

$$(fog)\ oh = fo(goh), \forall\ f, g, h \in S.$$

(iii) Identity element for a binary operation

Let $*$, be a binary operation in set *S*. If there exist an element e in *S* such that

$$a*e=e*a=a,\ \forall\ a\ \in S,$$

then *e* is called as identity element in *S* under the operation *

For Example 1. In a set of integers *Z*, 0 and 1 are the identity elements of A under addition and multiplication because

for all $a\ \in Z$ $\qquad 0+a = a+0 = a$

and $\qquad 1\cdot a\ =\ a\cdot 1\ =\ a$

2. In a set of natural numbers *N* there is no identity element in addition operation but for mulitplication operation 1 is the identity element.

3. For power set P(S), *S* and ϕ are the identity elements of Union and Intersection because for all $A \in P(S)$

$$A\bigcap S\ =\ S\bigcap A\ =\ A \qquad \text{and} \qquad A\bigcup\phi = \phi\bigcup A = A.$$

4. For a set of rational numbers *Q*, $*$ is a binary operation defined

$$a*b=\frac{ab}{2},\ \forall\ a,b \in Q$$

Here $2\ \in Q$ is an identity element as for all $a\ \in Q$

$$2*a=\frac{2\cdot a}{2}=a \qquad \text{and} \qquad a*2=\frac{a\cdot 2}{2}=a.$$

Theorem 1.8 If an identity element of a binary operation in a set exist then it is unique.

Proof : Let *e* and *e*' be the identity element in the binary operation $*$ in a set *S*

$$e * e' = e' = e' * e \qquad [\because\ e \text{ is identity in } S \text{ and } e' \in S] \qquad (1)$$

again $$e' * e = e = e * e' \qquad [\because\ e' \text{ is identity in } S \text{ and } e \in S] \qquad (2)$$

from (1) and (2) $\quad e = e'$

Thus if the identity element of any operation exists then it is unique.

(iv) Inverse Element

Let $*$, be the binary operation in set S and let e be its identity element. Let $a \in S$. Let b be an element in set S such that

$$a * b = b * a = e$$

then b is known as the inverse of a and is denoted by a^{-1} .

The inverse element of a exist in S then a, is known as invertible element, therefore

$$\text{a} \in \text{S is invertible} \iff a^{-1} \in S$$

Note: Let $*$ be the binary operation in set S and let e be its identity element then $e * e = e * e = e$.

For Example 1. In a set of integers Z for every element a, $(-a) \in Z$, is an inverse element

$$a + (-a) = (-a) + a = 0 \text{ (identity)}$$

Thus every element of Z has inverse in addition operation.

2. In a set of rational numbers Q every non-zero number has inverse for multiplication operation and

$$a \in Q \ \ \text{a}a \neq 0\text{f} \ \Rightarrow a^{-1} = 1/a \ \text{ because } \ a \cdot (1/a) = (1/a) \cdot a = 1$$

3. For positive set of rational numbers Q^+ a binary operation is defined as

$$a * b = ab/2, \qquad \forall\ a, b \in Q^+$$

We know that identity element of this operation is 2. The inverse of $a \in Q^+$ is $(4/a) \in Q^+$ as

$$\frac{4}{a} * a = \frac{(4/a) \times a}{2} = 2 \text{ (identity) and } \ a * \frac{4}{a} = \frac{a \times (4/a)}{2} = 2 \text{ (identity)}$$

Theorem 1.9 : Inverse of any invertible element with respect to a associative operation is uniuqe.

Proof : Let $*$, be a binary operation in Set S, which have identity element e. Let a is an inverse element of S. Let b and c are inverse element of a under S, is possible.

Now, $$b * (a * c) = b * e = b \qquad \left[\because c = a^{-1}\right]$$

and $$\text{a}b * a\text{f} * c = e * c = c \qquad \left[\because b = a^{-1}\right]$$

But by property of Associativity,

$$b * (a * c) = (b * a) * c$$

thus $\quad b = c$

So, inverse of an invertible element is unique.

1.09 Addition and Multiplication operations in modulo system

If a and b are integers and $(a-b)$ is a positive integers divisible by m then $a \equiv b$ (mod m) is denoted by a symbol and read as a is congruent to modulo m.

therefore $a \equiv b \pmod{m} \quad \Leftrightarrow \quad m \mid (a-b)$

For Example : $18 \equiv 6 \pmod{2} \quad \because 18-6=12, 2$ is divisible by 2

$-14 \equiv 6 \pmod{4} \quad \because -14-6=-20, 4$ is divisible by 4

again if m is a positive integer and a, b are two integers then by division algorithm there exist r, q such that

$$a+b=mq+r, \quad 0 \le r < m$$

then r is called as the remainder of addition modulo m of a and b and symbolically $a+b=r \pmod{m}$ or $a+_m b=r$

therefore $a +_m b = \begin{cases} a+b, & ;\text{if } a+b<m \\ r, & ;\text{if } a+b \ge m \end{cases}$, where r, is the non-negative remainder obtained by dividing $a + b$ by m

For Example $2+_4 3=1 \qquad [\because 2+3=5=1\times 4+1]$

$-10+_4 3=1 \qquad [\because -10+3=-7=-2\times 4+1]$

similarly m is a positive integer then for two numbers a, b if

$$a \cdot b = mq+r, \quad 0 \le r < m$$

then r is called as the remainder of multiplication modulo m of a and b, symbolically it is written as $a \cdot b = r \pmod{m}$ or is denoted by $a \times_m b = r$

$\therefore \quad a \times_m b = \begin{cases} ab, & ;\text{if } ab<m \\ r, & ;\text{if } ab \ge m \end{cases}$, Where r is the remainder when (a, b) is divided by m

Example $5 \times_4 3 = 3 \qquad [\because 15 = 4\times 3+3]$

$5 \times_3 6 = 0 \qquad [\because 5\times 6=30=10\times 3+0]$

1.10 Composition table for a finite set

When a given set A is finite, we can express a binary operation on the finite set A by a table called the *operation table* or *composition table* for the operation. For example:

Example 1. $S=\{(1,\omega,\omega^2);\times\}$ where ω is the cube root of unity

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

2. $S=\{(0,1,2,3);+_4\}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Just like we have following result from composite table:

(i) If table is symmetrical with respect to principle diagonal then defined operation is commutative under the set.

(ii) If row initiated from a_i is superimposed to uppermost row and column initiated from a_j is superimposed to left most column then, identify element of operation is in set S.

(iii) Any element of S is invertible if there is an identity element in corresponding row and column of table.

Illustrative Examples

Example 11. In a set of real number R, $*$ operation is defined as

$$a*b = a+b-ab, \quad \forall\ a,b \in R \text{ and } a \neq 1$$

(i) check the commutativity and associativity of $*$

(ii) find the identity element in $*$ if any

(iii) find the inverse element of $*$ with respect to R

Solution : (i) If $a,b \in R$ then by definition

$$a*b = a+b-ab = b+a-b\cdot a \qquad \text{(commutative)}$$
$$= b*a$$

$\therefore$ $*$ is a commutative

again $(a*b)*c = (a+b-ab)*c$

$$= (a+b-ab)+c-(a+b-ab)\cdot c$$
$$= a+b-ab+c-ac-bc+abc$$
$$= a+b+c-bc-ca-ab+abc \qquad (1)$$

and $a*(b*c) = a*(b+c-bc)$

$$= a+(b+c-bc)-a\cdot(b+c-bc)$$
$$= a+b+c-bc-ca-ab+abc \qquad (2)$$

from (1) and (2) it is clear that $(a*b)*c = a*(b*c)$

$\therefore$ $*$ is associative

(ii) Let e be the identity element of $*$ then for $a \in R$

$a*e = a$, by definition of identity

$\Rightarrow \quad a+e-ae = a \quad \Rightarrow \quad e(1-a) = 0$

$\Rightarrow \quad e = 0 \in R \qquad [\because a \neq 1]$

0 is the identity element of $*$

(iii) Let $a \in R$ and let x be the inverse element of a then by definition

$a*x = 0$ (identity)

$\Rightarrow \quad a+x-ax = 0 \quad \Rightarrow \quad x(a-1) = a$

$$\Rightarrow \quad x = \frac{a}{a-1} \in R, \quad \because \quad a \neq 1$$

$\therefore$ $a \in R\ (a \neq 1)$ is invertible

Example 12. If $S=\{(a,b)|a,b\in R,\ a\neq 0\}$ and an operation $*$ is defined in S in such a way that

$(a,b)*(c,d)=(ac,bc+d)$ then

(i) check the commutativity and associativity of $*$

(ii) find the identity element in $*$ if any

(iii) find the inverse element of $*$ with respect to R

Solution : (i) Let $(a,b),(c,d)\in S$

then $(a,b)*(c,d)=(ac,bc+d)$ and $(c,d)*(a,b)=(ca,da+b)$

similarly $(a,b)*(c,d)\neq(c,d)*(a,b)$

$\therefore$ operation $*$ is not commutative

again let $(a,b),(c,d),(e,f)\in S$

now $[(a,b)*(c,d)]*(e,f)=(ac,bc+d)*(e,f)$

$$=(ace,(bc+d)e+f)=(ace,bce+de+f) \quad (1)$$

and $(a,b)*[(c,d)*(e,f)]=(a,b)*(ce,de+f)$
$=(ace,bce+de+f)$

$\therefore$ from (1) and (2) $[(a,b)*(c,d)]*(e,f)=(a,b)*[(c,d)*(e,f)]$ (2)

Thus $*$ is associative operation.

(ii) Let (x,y) be the identity element in S then for all $(a,b)\in S$

$(a,b)*(x,y)=(a,b)$ [by the definition of identity]

$\Rightarrow$ $(ax,bx+y)=(a,b)$

$\Rightarrow$ $ax=a$ and $bx+y=b$

Now $ax=a$ $\Rightarrow$ $x=1$ $[\because a\neq 0]$

and $bx+y=b$ $\Rightarrow$ $b+y=b$ $[\because x=1]$

$\Rightarrow$ $y=0$

$\therefore$ $(x,y)=(1,0)\in S$

$\therefore$ identity element of S is $(1,0)$

because $(a,b)*(1,0)=(a,b)$ and $(1,0)*(a,b)=(a,b)$.

(iii) Let $(a,b)\in S$ and inverse element of (a,b) is (x,y) then by the definition of inverse

$(a,b)*(x,y)=(1,0)$ [identity]

$\Rightarrow$ $(ax,bx+y)=(1,0)$

$\Rightarrow$ $ax=1,\ bx+y=0$

$ax=1,\ \Rightarrow$ $x=(1/a)$ $(a\neq 0)$

and $bx+y=0\ \Rightarrow\ y=(-b/a)$ $(a\neq 0)$

$\therefore$ inverse of (a,b) is $(1/a,-b/a)$

Example 13. If $S=\{A,B,C,D\}$ where $A=\phi, B=\{a,b\}$ $C=\{a,c\}, D=\{a,b,c\}$ prove that the union of set $\cup$ is a binary operation in S but intersection of set $\cap$ is not a binary operation in S.

Solution : We see that

$$A\cup B=\phi\cup\{a,b\}=\{a,b\}=B,\ A\cup C=C, A\cup D=D$$

$$B\cup C=\{a,b\}\cup\{a,c\}=\{a,b,c\}=D$$

$$B\cup D=\{a,b\}\cup\{a,b,c\}=\{a,b,c\}=D,\ C\cup D=D$$

Thus union of set $\cup$ is a binary operation is S but $B\cap C=\{a,b\}\cap\{a,c\}=\{a\}\notin S$ therefore intersection of set $\cap$ is not a binary operation in S

Exercise 1.3

1. Determine whether or not each of the definition of $*$ given below gives a bianry operation. In the event that $*$ is not a binary operation, give justification for this.
   (i) $a*b = a$, on N
   (ii) $a*b = a + b - 3$, on N
   (iii) $a*b = a + 3b$, on N
   (iv) $a*b = a/b$, on Q
   (v) $a*b = a-b$, on R
2. For each binary operation * defined below, determine whether it is commutative or associative?
   (i) * on N where $a*b=2^{ab}$
   (ii) * on N where $a*b=a+b+a^2b$
   (iii) * on Z where $a*b=a-b$
   (iv) * on Q where $a*b=ab+1$
   (v) * on R where $a*b=a+b-7$
3. If in a set of integers Z an operation * is defined as $*, a*b=a+b+1,\ \forall a,b\in Z$ then prove that *, is commutative and associative. Also find its identity element. Find the inverse of any integer.
4. A binary operation defined on a set $R-\{1\}$ is as follows:-

   $$a*b=a+b-ab,\quad \forall\ a,b\in R-\{1\}$$

   Prove that * is commutative and associative also find its identity element and find inverse of any element a.
5. Four functions are defined in set R_0 as follows

   $$f_1(x)=x,\ f_2(x)=-x,\ f_3(x)=1/x,\ f_4(x)=-1/x$$

   Form the composition table for the 'compositive functions f_1, f_2, f_3, f_4 also find the identity element and inverse of every element.

Miscellaneous Exercise – 1

1. If $f:R\to R, f(x)=2x-3;\ g:R\to R, g(x)=x^3+5$ then the value of $(fog)^{-1}(x)$ is

   (a) $\left(\frac{x+7}{2}\right)^{1/3}$ (b) $\left(x-\frac{7}{2}\right)^{1/3}$ (c) $\left(\frac{x-2}{7}\right)^{1/3}$ (d) $\left(\frac{x-7}{2}\right)^{1/3}$.

2. If $f(x)=\frac{x}{1-x}=\frac{1}{y}$, then the vlaue of $f(y)$ is

(a) x (b) $x-1$ (c) $x+1$ (d) $\frac{1-x}{2x-1}$

3. If $f(x)=\frac{x-3}{x+1}$ then the vlaue of $f[f\{f(x)\}]$ is equal to

(a) x (b) $1/x$ (c) $-x$ (d) $-1/x$.

4. If $f(x)=\cos(\log x)$ then the valeu of $f(x)\cdot f(y)-\frac{1}{2}[f(x/y)+f(x\cdot y)]$ is

(a) –1 (b) 0 (c) 1/2 (d) –2.

5. If $f:R\to R, f(x)=2x+1$ and $g:R\to R, g(x)=x^3$, then $(gof)^{-1}(27)$ is equal to

(a) 2 (b) 1 (c) –1 (d) 0.

6. If $f:R\to R$ and $g:R\to R$, where $f(x)=2x+3$ and $g(x)=x^2+1$ then the value of $(gof)(2)$ is

(a) 38 (b) 42 (c) 46 (d) 50.

7. If an operation * defiend on Q_0 as $*, a*b=ab/2, \ \forall a,b\in Q_0$ then the identity element is

(a) 1 (b) 0 (c) 2 (d) 3.

8. A binary operation defind on R as $a*b=1+ab, \ \forall a,b\in R$ then * is

(a) commutative but no associative (b) associative but not commutative

(c) neither commutative nor associative (d) commutative and associative

9. In the set of integers Z the operation subtraction is

(a) commutative and associative (b) associative but not commutative

(c) neither commutative nor associative (d) commutative but not associative

10. If an operation $*$ defined on a set of rational numbers Q as $*$, $a*b=a+b-ab, \ \forall a,b\in Q$. The inverse of $a(\neq 1)$ with respect to this is

(a) $\frac{a}{a-1}$ (b) $\frac{a}{1-a}$ (c) $\frac{a-1}{a}$ (d) $\frac{1}{a}$

11. Which of the following is commutaitve defiend in a set of R

(a) $a*b=a^2b$ (b) $a*b=a^b$ (c) $a*b=a+b+ab$ (d) $a*b=a+b+a^2b$

12. For the given three functions justify the associativity of composite function operation

$$f:N\to Z_0,\ f(x)=2x \ ; \ g:Z_0\to Q,\ g(x)=1/x \ ; \ h:Q\to R,\ h(x)=e^x$$

13. If $f:R^+\to R^+$ and $g:R^+\to R^+$ are defined as below

$f(x)=x^2, g(x)=\sqrt{x}$ then find gof and fog , are these functions equal ?

14. If $f:R\to R, f(x)=\cos(x+2)$ then justify with reason that whether it is invertible or not.

15. If two functions f and g are defined on $A=\{-1,1\}$ and A where $f(x)=x^2$, $g(x)=\sin\left(\frac{\pi x}{2}\right)$, then prove that g^{-1} exist but f^{-1} does not. Also find g^{-1} .

16. If $f : R \to R$ and $g : R \to R$ are functions such that $f(x) = 3x + 4$ and $g(x) = \frac{(x-4)}{3}$, then find $(fog)(x)$ and $(gof)(x)$. Also fidn the vlaue of $(gog)(1)$.

Important Points

1. If f and g are two functions then gof is defined only when range of f is the subset of domain of g.
2. Composite functions need not satisfy commutative law.
3. Composite function obeys associative law i.e. $(fog)oh = fo(goh)$
4. If two functions are bijective then their compsoite functions are also bijective.
5. Inverse of bijective function is unique.
6. The inverse of one-one onto function is also one-one onto.
7. $(gof)^{-1} = f^{-1}og^{-1}$
8. In set A a binary operation is defined from $A \times A$ to A
9. An element $e \in S$ is the identity element for binary operation *

   If $a*e = e*a = a, \quad \forall\, a \in S$
10. If $a*b = b*a = e$, then b is inverse of a under $*$ on S.
11. Inverse of a is denoted by a^{-1}.
12. In a set S defined an operation *

$$a*(b*c) = (a*b)*c, \ \forall\, a, b, c \in S$$

   then * operation is associative

Answers

Exercise 1.1

1. $(i)\ (gof)(x) = 4x^2 + 12x + 14, (fog)(x) = 2x^2 + 13$ $\quad (ii)\ (gof)(x) = 3(x^2+8)^3 + 1, (fog)(x) = 9x^6 + 6x^3 + 9$

$(iii)\ (gof)(x) = |x|, (fog)(x) = |x|$ $\quad (iv)\ (gof)(x) = 3x^2 + 6x - 13, (fog)(x) = 9x^2 - 18x + 5$

2. $fog = \{(u,u), (v,v), (w,w)\};\ gof = \{(a,a), (b,b), (c,c)\}$

3. $(fog)(x) = x, (gof) = x,$ Yes, its an identity function

4. $(fog)(x) = x, (gof) = x, (gog)(1) = -5/3$

5. 5

6. $(i)\ (gof)(x) = (2x + x^{-2})^4 + 2(2x + x^{-2}) + 4$

7. $(i)\ (fog)(x) = 4x^2 - 6x + 1$ $\quad (ii)\ (gof)(x) = 2x^2 + 6x - 1$

$(iii) (fog)(x) = (x)^4 + 6x^3 + 14x^2 + 15x + 5$ $\quad (iv)\ (gog)(x) = 4x - 9$

Exericse 1.2

1. $f_1 = \{(1,a),(2,b),(3,c),(4,d)\};\ f_1^{-1} = \{(a,1),(b,2),(c,3),(d,4)\}$

   $f_2 = \{(1,a),(2,c),(3,b),(4,d)\};\ f_2^{-1} = \{(a,1),(c,2),(b,3),(d,4)\}$

   $f_3 = \{(1,d),(3,b),(2,a)(4,c)\};\ f_3^{-1} = \{(d,1),(b,3),(a,2),(c,4)\}$

   $f_4 = \{(1,a),(3,a),(2,b)(4,c)\};\ f_4^{-1} = \{(a,1),(a,3),(b,2),(c,4)\}$

2. $f^{-1}(x) = (3+x)^{1/3},\ f^{-1}(24) = 3,\ f^{-1}(5) = 2$

3. $f^{-1}(x) = \dfrac{x+3}{2},\ f^{-1}(x) = (x-5)^{1/3}$

4. $(gof)^{-1} = \{(7,1),(23,2),(47,3),(79,4)\} = f^{-1}og^{-1}$

5. $f^{-1}(x) = \dfrac{x-b}{a}$

6. NO

7. $(i)\ f^{-1} = \{(-9,-3),(-3,-1),(0,0),(6,2)\}$ $(ii)\ f^{-1}$ does not exist $(iii)\ f^{-1}(x) = x^{1/3}$

Exercise 1.3

1. (i) yes (ii) no (iii) yes (iv) no (v) yes

2. (i) commutative but not associative (ii) neither commutative nor associative
   (iii) neither commutative nor associative (iv) commutative but not associative
   (v) commutative and associative

3. $e = -1,\ a^{-1} = -(a+2)$

4. $e = 0,\ a^{-1} = \dfrac{a}{a-1}$

Miscellaneous Exericse 1

1. (d) 2. (d) 3. (a) 4. (b) 5. (b) 6. (d) 7. (c)

8. (a) 9. (c) 10. (a) 11. (c) 13. $(fog)(x) = (gof)(x) = x$ 14. No

15. $g^{-1}(x) = \dfrac{2}{\pi}\sin^{-1} x$ 16. $(fog)(x) = (gof)(x) = x;\ (gog)(1) = \dfrac{-5}{3}$