Chapter – 6

Differentiation

Ex 6.1

Question 1.
A firm produces x tonnes of output at a total cost of $C(x) = 1/10\ x^3 - 4x^2 - 20x + 7$ find the
(i) average cost
(ii) average variable cost
(iii) average fixed cost
(iv) marginal cost and
(v) marginal average cost.

Solution:
$c(x) = f(x) + x$

$c(x) = \frac{1}{10} x^3 - 4x^2 - 20x + 7$

Then $f(x) = \frac{1}{10} x^3 - 4x^2 - 20x$ and $k = 7$

(i) Average Cost (AC) = $\frac{\text{Total cost}}{\text{Output}} = \frac{C(x)}{x} = \frac{f(x)+k}{x}$

$$= \frac{\frac{1}{10}x^3 - 4x^2 - 20x + 7}{x}$$

$$= \frac{1}{10}\frac{x^3}{x} - \frac{4x^2}{x} - \frac{20x}{x} + \frac{7}{x}$$

$$= \frac{1}{10}x^2 - 4x - 20 + \frac{7}{x}$$

(ii) Average Variable Cost (AVC) = $\frac{\text{Variable cost}}{\text{Output}} = \frac{f(x)}{x}$

$$= \frac{\frac{1}{10}x^3 - 4x^2 - 20x}{x}$$

$$= \frac{1}{10}\frac{x^3}{x} - \frac{4x^2}{x} - \frac{20x}{x}$$

$$= \frac{1}{10}x^2 - 4x - 20$$

(iii) Average Fixed Cost (AFC) = $\frac{\text{Fixed cost}}{\text{Output}} = \frac{k}{x} = \frac{7}{x}$

(iv) Marginal Cost (MC) = $\frac{dC}{dx}$

$$= \frac{d}{dx}\left(\frac{1}{10}x^3 - 4x^2 - 20x + 7\right)$$

$$= \frac{d}{dx}\left(\frac{1}{10}x^3\right) - \frac{d}{dx}(4x^2) - \frac{d}{dx}(20x) + \frac{d}{dx}(7)$$

$$= \frac{1}{10}\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) - 20\frac{d}{dx}(x) + 0$$

$$= \frac{1}{10}(3)x^{3-1} - 4\,(2)\,x^{2-1} - 20\,(1)$$

$$= \frac{3}{10}x^2 - 8x - 20$$

(v) Marginal Average Cost (MAC) = $\frac{d}{dx}$ (AC)

$$= \frac{d}{dx}\left(\frac{1}{10}x^2 - 4x - 20\right) + \frac{7}{x}$$

$$= \frac{1}{10}\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) - \frac{d}{dx}(20) + \frac{d}{dx}\left(\frac{7}{x}\right)$$

$$= \frac{1}{10}(2x^{2-1}) - 4(1) - 0 - \frac{7}{x^2}$$

$$= \frac{1}{5}x - 4 - \frac{7}{x^2}$$

Question 2.

The total cost of x units of output of a firm is given by C = 2/3 x + 35/2. Find the

(i) cost when output is 4 units

(ii) average cost when output is 10 units

(ii) marginal cost when output is 3 units

Solution:

$C = \frac{2}{3}x + \frac{35}{2}$

i.e., $C(x) = \frac{2}{3}x + \frac{35}{2}$

(i) Cost when output is 4 units, i.e., to find when x = 4, C = ?

$$C(4) = \frac{2}{3}(4) + \frac{35}{2}$$

$$C = \frac{8}{3} + \frac{35}{2}$$

$$C = \frac{8\times 2 + 35\times 3}{3\times 2} = \frac{16+105}{6} = ₹\frac{121}{6}$$

(ii) Average cost when output is 10 units, i.e., to find when x = 10, AC = ?

$$C = \frac{2}{3}x + \frac{35}{2}$$

$$\text{Average Cost (AC)} = \frac{\text{Total cost}}{\text{Output}} = \frac{C(x)}{x} = \frac{f(x)+k}{x}$$

$$= \frac{\frac{2}{3}x + \frac{35}{2}}{x} = \frac{2}{3}\frac{x}{x} + \frac{35}{2}\frac{1}{x}$$

$$AC = \frac{2}{3} + \frac{35}{2}\times\frac{1}{x}$$

$$\text{When } x = 10,\ AC = \frac{2}{3} + \frac{35}{2}\times\frac{1}{10}$$

$$= \frac{2}{3} + \frac{7}{2}\times\frac{1}{2} = \frac{2}{3} + \frac{7}{4} = \frac{2\times 4 + 7\times 3}{3\times 4}$$

$$= \frac{8+21}{12} = \frac{29}{12}$$

Average cost when output is 10 units is ₹ 29/12

(iii) Marginal cost when output is 3 units

$C = \frac{2}{3}x + \frac{35}{2}$

Marginal Cost (MC) = $\frac{d}{dx}$ (C)

$$= \frac{d}{dx}\left(\frac{2}{3}x + \frac{35}{2}\right) = \frac{2}{3}\frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{35}{2}\right)$$

$$= \frac{2}{3}(1) + 0 = \frac{2}{3}$$

Marginal cost when output is 3 units will be ₹ $\frac{2}{3}$

Question 3.

Revenue function 'R' and cost function 'C' are $R = 14x - x^2$ and $C = x(x^2 - 2)$.

Find the

(i) average cost

(ii) marginal cost

(iii) average revenue and

(iv) marginal revenue.

Solution:

$R = 14x - x^2$ and $C = x(x^2 - 2)$

$C = x^3 - 2x$

(i) Average Cost (AC) = $\frac{\text{Total cost}}{\text{Output}} = \frac{C(x)}{x}$

$= \frac{x^3 - 2x}{x}$

$= \frac{x^3}{x} - \frac{2x}{x}$

$= x^2 - 2$

(ii) Marginal Cost (MC) = $\frac{dC}{dx}$

$= \frac{d}{dx}(x^3 - 2x)$

$= \frac{d}{dx}(x^3) - 2\frac{d}{dx}(x)$

$= 3x^2 - 2$

(iii) Average Revenue R = $14x - x^2$

Average Revenue (AR) = $\frac{\text{Total Revenue}}{\text{Output}} = \frac{R(x)}{x}$

$= \frac{14x - x^2}{x}$

$= \frac{14x}{x} - \frac{x^2}{x}$

= 14 – x

(iv) Marginal Revenue (MR) = $\frac{dR}{dx}$

$= \frac{d}{dx}(14x - x^2)$

$= 14\frac{d}{dx}(x) - \frac{d}{dx}(x^2)$

= 14(1) – 2x

= 14 – 2x

Question 4.

If the demand law is given by $p = 10e^{-\frac{x}{2}}$ then find the elasticity of demand.

Solution:

$p = 10e^{-\frac{x}{2}}$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$p = 10e^{-\frac{x}{2}}$

$$\frac{dp}{dx} = 10(e^{-\frac{x}{2}})\left(-\frac{1}{2}\right)$$

$$= -\frac{10}{2}e^{-\frac{x}{2}} = -5e^{-\frac{x}{2}}$$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{1}{\left(\frac{dp}{dx}\right)}$

$$= \left(\frac{-10e^{-\frac{x}{2}}}{x}\right)\left(\frac{1}{-5e^{-\frac{x}{2}}}\right)$$

$$= \frac{-10}{-5x} \times \frac{e^{-\frac{x}{2}}}{e^{-\frac{x}{2}}}$$

$$\eta_d = \frac{2}{x}$$

Question 5.
Find the elasticity of demand in terms of x for the following demand laws and also find the value of x where elasticity is equals to unity.
(i) $p = (a - bx)^2$
(ii) $p = a - bx^2$

Solution:

(i) $p = (a - bx)^2$

$= \frac{dp}{dx} = 2(a - bx)^{2-1} \frac{d}{dx}(a - bx)$

$= 2(a - bx)(0 - b(1))$

$= -2b(a - bx)$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$$= \frac{-(a-bx)^2}{x} \times \frac{1}{\left(\frac{dp}{dx}\right)} = \frac{-(a-bx)^2}{x} \times \frac{1}{-2b(a-bx)}$$

$$\eta_d = \frac{(a-bx)}{2bx}$$

When the elasticity of demand is equals to unity,

$\frac{a-bx}{2bx} = 1$

$a - bx = 2bx$

$2bx = a - bx$

$2bx + bx = a$

$3bx = a$

$x = \frac{a}{3b}$

$\therefore$ The value of x when elasticity is equal to unity is $\frac{a}{3b}$

(ii) $p = a - bx^2$

$\frac{dp}{dx} = 0 - b\frac{d}{dx}(x^2)$

$= -b(2x)$

$= -2bx$

Elasticity of demand: $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$= \frac{-p}{x} \times \frac{1}{\left(\frac{dp}{dx}\right)} = \frac{-(a-bx^2)}{x} \times \frac{1}{-2bx}$

$\eta_d = \frac{a-bx^2}{2bx^2}$

When elasticity is equals to unit,

$\frac{a-bx^2}{2bx^2} = 1$

$a - bx^2 = 2bx^2$

$2bx^2 = a - bx^2$

$2bx^2 + bx^2 = a$

$3bx^2 = a$

$x^2 = \frac{a}{3b}$

$x = \sqrt{\frac{a}{3b}}$

$\therefore$ The value of x when elasticity is equal to unity is $\sqrt{\frac{a}{3b}}$

Question 6.
Find the elasticity of supply for the supply function $x = 2p^2 + 5$ when $p = 3$.

Solution:
$x = 2p^2 + 5$

$\frac{dx}{dp} = 2 \times 2p + 0 = 4p$

Elasticity of supply: $\eta s = \frac{p}{x} \cdot \frac{dx}{dp}$

$= \frac{p}{2p^2+5} \times 4p$

$= \frac{4p^2}{2p^2+5}$

When p = 3, elasticity of supply, ηs = $\frac{4\times3^2}{2(3)^2+5}$

$= \frac{4\times9}{18+5}$

$= \frac{36}{23}$

Question 7.

The demand curve of a commodity is given by p = $\frac{50-x}{5}$, find the marginal revenue for any output x and also find marginal revenue at x = 0 and x = 25?

Solution:

Given that p = $\frac{50-x}{5}$

Revenue, R = px

$= \left(\frac{50-x}{5}\right) x$

$= \frac{50x-x^2}{5}$

$= \frac{1}{5}(50x - x^2)$

Marginal Revenue (MR) = $\frac{d}{dx}$ (R)

$= \frac{d}{dx} \frac{1}{5} (50x - x^2)$

$= \frac{1}{5} \frac{d}{dx} (50x - x^2)$

$= \frac{1}{5} (50 - 2x)$

Marginal revenue when x = 0 is, MR = $\frac{1}{5}$ (50 – 2 × 0)

$= \frac{1}{5} \times 50$

= 10

When x = 25, marginal revenue is MR = $\frac{1}{5}$ (50 – 2 × 25)

$= \frac{1}{5}$ (50 – 50)

= 0

Question 8.

The supply function of certain goods is given by x = $a\sqrt{p-b}$ where p is unit price, a and b are constants with p > b. Find elasticity of supply at p = 2b.

Solution:

Given that x = $a\sqrt{p-b}$

Elasticity of supply: $\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$

$$x = a\sqrt{p-b}$$

$$\frac{dx}{dp} = a\left(\frac{1}{2\sqrt{p-b}}\right)$$

$$\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{p}{a\sqrt{p-b}} \times a \times \frac{1}{2\sqrt{p-b}}$$

$$= \frac{p}{2(p-b)}$$

Hint for differentiation

Use $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

(or) $x = a\sqrt{p-b}$

$$x = a(p-b)^{\frac{1}{2}}$$

$$\frac{dx}{dp} = a \cdot \frac{1}{2}(p-b)^{\frac{1}{2}-1}$$

$$= \frac{a}{2}(p-b)^{-\frac{1}{2}}$$

$$= \frac{a}{2}\frac{1}{\sqrt{p-b}}$$

When p = 2b, Elasticity of supply: $\eta_s = \frac{2b}{2(2b-b)} = \frac{2b}{2b} = 1$

Question 9.

Show that MR = p $\left[1 - \frac{1}{\eta_d}\right]$ for the demand function p = 400 – 2x – 3x^2 where p is unit price and x is quantity demand.

Solution:

Given p = 400 – 2x – 3x^2

Revenue, R = px = (400 – 2x – 3x^2)x = 400x – 2x^2 – 3x^3

Marginal Revenue (MR) = $\frac{d}{dx}$ (R)

= $\frac{d}{dx}$ (400x – 2x^2 – 3x^3)

= 400 – 4x – 9x^2

Now $\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$

$$p = 400 - 2x - 3x^2$$

$$\frac{dp}{dx} = 0 - 2 - 6x$$

$$\therefore \eta_d = -\frac{p}{x}\cdot\frac{1}{\left(\frac{dp}{dx}\right)}$$

$$= -\left[\frac{400-2x-3x^2}{x}\right]\times\frac{1}{(-2-6x)}$$

$$= \frac{400-2x-3x^2}{-x}\times\frac{1}{(-2-6x)} = \frac{400-2x-3x^2}{2x+6x^2}$$

$$\therefore 1-\frac{1}{\eta_d}=1-\frac{1}{\left(\frac{400-2x-3x^2}{2x+6x^2}\right)}=1-\frac{2x+6x^2}{400-2x-3x^2}$$

$$=\frac{400-2x-3x^2-(2x+6x^2)}{400-2x-3x^2}=\frac{400-4x-9x^2}{400-2x-3x^2}$$

$$\therefore p\left[1-\frac{1}{\eta_d}\right]=(400-2x-3x^2)\left(\frac{400-4x-9x^2}{400-2x-3x^2}\right)$$

$$=400-4x-9x^2=\text{MR (using (1))}$$

Thus for the function $p = 400 - 2x - 3x^2$

$$MR = p\left[1-\frac{1}{\eta_d}\right]$$

Question 10.
For the demand function $p = 550 - 3x - 6x^2$ where x is quantity demand and p is unit price. Show that MR = $p\left[1-\frac{1}{\eta_d}\right]$

Solution:
Given $p = 550 - 3x - 6x^2$
Revenue, $R = px = (550 - 3x - 6x^2)x = 550x - 3x^2 - 6x^3$

Marginal Revenue (MR) = $\frac{d}{dx}$ (R)

$= \frac{d}{dx}(550x - 3x^2 - 6x^3)$

$= 550 - 6x - 18x^2$

Now $\eta d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$p = 550 - 3x - 6x^2$

$\frac{dp}{dx} = 0 - 3 - 12x$

$\therefore \eta_d = -\frac{p}{x} \cdot \frac{1}{\left(\frac{dp}{dx}\right)}$

$$= -\left[\frac{550-3x-6x^2}{x}\right] \times \frac{1}{(-3-12x)}$$

$$= \frac{550-3x-6x^2}{-x} \times \frac{1}{(-3-12x)} = \frac{550-3x-6x^2}{3x+12x^2}$$

$$\therefore 1 - \frac{1}{\eta_d} = 1 - \frac{1}{\left(\frac{550-3x-6x^2}{3x+12x^2}\right)} = 1 - \frac{3x+12x^2}{550-3x-6x^2}$$

$$= \frac{550-3x-6x^2-3x-12x^2}{550-3x-6x^2} = \frac{550-6x-18x^2}{550-3x-6x^2}$$

$$\therefore p\left[1-\frac{1}{\eta_d}\right] = \left(\frac{(550-3x-6x^2)(550-6x-18x^2)}{550-3x-6x^2}\right)$$

$$= 550-6x-18x^2 = \text{MR}$$

Question 11.

For the demand function $x = \frac{25}{p^4}$, $1 \le p \le 5$, determine the elasticity of demand.

Solution:

The demand function, $x = \frac{25}{p^4}$, $1 \le p \le 5$

The elasticity demand, $\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$

$$x = \frac{25}{p^4}$$

$$x = 25 \times p^{-4}$$

$$\frac{dx}{dp} = (25)(-4)p^{-4-1}$$

$$= 25 \times -4 \times p^{-5}$$

$$= 25 \times (-4) \times \frac{1}{p^5}$$

Hint for differentiation

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$$

$$x = \frac{25}{p^4}$$

$$\frac{dx}{dp} = 25\left(\frac{-4}{p^5}\right)$$

$$\therefore \eta_d = -\frac{p}{x}\cdot\frac{dx}{dp}$$

$$= \frac{-p}{\left(\frac{25}{p^4}\right)} \times 25 \times (-4) \times \frac{1}{p^5}$$

$$= \frac{-p^5}{25} \times 25 \times (-4) \times \frac{1}{p^5} = 4$$

Question 12.
The demand function of a commodity is p = 200 – x/100 and its cost is C = 40x + 120 where p is a unit price in rupees and x is the number of units produced and sold. Determine
(i) profit function
(ii) average profit at an output of 10 units
(iii) marginal profit at an output of 10 units and
(iv) marginal average profit at an output of 10 units.

Solution:

The demand function, $p = 200 - \frac{x}{100}$

Cost is $C = 40x + 120$

Revenue function, $R(x) = px$

$= \left(200 - \frac{x}{100}\right) x$

$= 200x - \frac{x^2}{100}$

(i) Profit function = R(x) – C(x)

$= 200x - \frac{x^2}{100} - (40x + 120)$

$= 200x - \frac{x^2}{100} - 40x - 120$

$= 160x - \frac{x^2}{100} - 120$

(ii) Average profit (AP) = $\frac{\text{Total Profit}}{\text{Output}}$

$= \frac{1}{x}\left(160x - \frac{x^2}{100} - 120\right)$

$= 160 - \frac{x}{100} - \frac{120}{x}$

Average profit at an output of 10 units

When x = 10, average profit = $160 - \frac{10}{100} - \frac{120}{10}$

$= 160 - \frac{1}{10} - 12$

$= 148 - \frac{1}{10}$

$= 148 - 0.1$

= ₹ 147.9

(iii) Marginal profit [MP] = $\frac{dP}{x}$

$= \frac{d}{dx}\left(160x - \frac{x^2}{100} - 120\right)$

$= 160 - \frac{2x}{100}$

$= 160 - \frac{x}{50}$

Marginal profit when x = 10, is = $160 - \frac{10}{50}$

$= 160 - \frac{1}{5}$

$= 160 - 0.2$

= ₹ 159.8

(iv) Average profit AP = $160 - \frac{x}{100} - \frac{120}{x}$

Marginal average profit (MAP) = $\frac{d}{dx}$ (AP)

$= \frac{d}{dx}\left(160 - \frac{x}{100} - \frac{120}{x}\right)$

$= 0 - \frac{1}{100} - 120\left(-\frac{1}{x^2}\right)$ [$\because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$]

$= \frac{-1}{100} + \frac{120}{x^2}$

When x = 10, marginal average profit is $= -\frac{1}{100} + \frac{120}{10^2}$

$= \frac{-1}{100} + \frac{120}{100}$

$= \frac{-1+120}{100}$

$= \frac{119}{100}$

= ₹ 1.19

Question 13.
Find the values of x, when the marginal function of $y = x^3 + 10x^2 - 48x + 8$ is twice the x.

Solution:
$y = x^3 + 10x^2 - 48x + 8$

Marginal function, $\frac{dy}{dx} = 3x^2 + 10(2x) - 48$

$= 3x^2 + 20x - 48$

Given that, the marginal function is twice the x.

Therefore, $3x^2 + 20x - 48 = 2x$

$3x^2 + 18x - 48 = 0$

Divide throughout by 3, $x^2 + 6x - 16 = 0$

$(x + 8)(x - 2) = 0$

x = -8 (or) x = 2

The values of x are -8, 2.

Question 14.

The total cost function y for x units is given by $y = 3x\left(\frac{x+7}{x+5}\right) + 5$. Show that the marginal cost decreases continuously as the output increases.

Solution:

The total cost function, $y = 3x\left(\frac{x+7}{x+5}\right) + 5$

To prove the marginal cost decreases continuously as the output increase we should prove dy/dx is positive.

$$y = 3x\left(\frac{x+7}{x+5}\right) + 5$$

$$= 3x\left(\frac{(x+5)+2}{x+5}\right) + 5 = 3x\left(\frac{x+5}{x+5} + \frac{2}{x+5}\right) + 5$$

$$y = 3x\left(1 + \frac{2}{x+5}\right) + 5$$

$$y = 3\left(x + \frac{2x}{x+5}\right) + 5$$

$$\frac{dy}{dx} = 3\frac{d}{dx}\left[x + \frac{2x}{x+5}\right] + \frac{d}{dx}(5)$$

$$= 3\left[1 + 2\frac{d}{dx}\left(\frac{x}{x+5}\right)\right] + 0 = 3\left[1 + 2\left(\frac{(x+5)1 - x(1)}{(x+5)^2}\right)\right]$$

$$= 3\left[1 + 2\left(\frac{x+5-x}{(x+5)^2}\right)\right] = 3\left[1 + 2\left(\frac{5}{(x+5)^2}\right)\right]$$

$$= 3\left[1 + \frac{10}{(x+5)^2}\right], \text{ which is positive.}$$

∴ The marginal cost decreases continuously of the output increases.

Question 15.
Find the price elasticity of demand for the demand function x = 10 – p where x is the demand p is the price. Examine whether the demand is elastic, inelastic, or unit elastic at p = 6.

Solution:
The demand function is x = 10 – p
Price elasticity of demand,

$$\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$x = 10 - p$$

$$\frac{dx}{dp} = 0 - 1 = -1$$

$$\eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= -\frac{p}{10 - p} \times (-1) = \frac{p}{10 - p}$$

Price elasticity of demand when p – 6 is $\eta_d = \frac{6}{10-6} = \frac{6}{4} = 1.5$

$\therefore |\eta_d| = 1.5 > 1$, the demand is elastic.

Question 16.

Find the equilibrium price and equilibrium quantity for the following functions.

Demand: x = 100 – 2p and supply: x = 3p – 50.

Solution:

Demand x = 100 – 2p

Supply x = 3p – 50

At equilibrium, demand = supply

100 – 2p = 3p – 50

-2p – 3p = -100 – 50

-5p = -150

$p = \frac{-150}{-5} = 30$

$\therefore$ Equilibrium price $p_E = 30$

Supply, x = 3p – 50

Put p = 30, we get

x = 3(30) – 50 = 90 – 50 = 40

$\therefore$ Equilibrium quantity $x_E = 40$

Question 17.
The demand and cost functions of a firm are x = 6000 – 30p and C = 72000 + 60x respectively. Find the level of output and price at which the profit is maximum.

Solution:
We know that profit is maximum when marginal Revenue (MR) = Marginal Cost (MC)
The demand function, x = 6000 – 30p
30p = 6000 – x

$p = \frac{1}{30}(6000 - x)$

$p = \frac{6000}{30} - \frac{x}{30}$

$p = 200 - \frac{x}{30}$ .... (1)

Revenue, R = px

$= \left(200 - \frac{x}{30}\right) x$

$= 200x - \frac{x^2}{30}$

Marginal Revenue (MR) = $\frac{dR}{dx}$

$= \frac{d}{dx}\left(200x - \frac{x^2}{30}\right)$

$= \frac{200d}{dx}(x) - \frac{1}{30}\frac{d}{dx}(x^2)$

$= 200(1) - \frac{1}{30}(2x)$

$= 200 - \frac{x}{15}$

Cost function, C = 72000 + 60x

Marginal cost, $\frac{dC}{dx} = \frac{d}{dx}(72000 + 60x)$

$= 0 + 60(1)$

= 60

But marginal revenue = marginal cost

$200 - \frac{x}{15} = 60$

$-\frac{x}{15} = 60 - 200$

$-\frac{x}{15} = -140$

$-x = -140 \times 15$

$x = 140 \times 15 = 2100$

The output is 2100 units.

By (1) we have $p = 200 - \frac{x}{30}$

When x = 2100,

Profit, $p = 200 - \frac{2100}{30} = 200 - 70 = 130$

p = ₹ 130

Question 18.
The cost function of a firm is $C = x^3 - 12x^2 + 48x$. Find the level of output ($x > 0$) at which average cost is minimum.

Solution:
The cost function is $C = x^3 - 12x^2 + 48x$
Average cost is minimum,
When Average Cost (AC) = Marginal Cost (MC)
Cost function, $C = x^3 - 12x^2 + 48x$

Average Cost, $AC = \frac{x^3 - 12x^2 + 48x}{x} = x^2 - 12x + 48$

Marginal Cost (MC) = $\frac{dC}{dx}$

$= \frac{d}{dx}(x^3 - 12x^2 + 48x)$

$= 3x^2 - 24x + 48$

But AC = MC

$x^2 - 12x + 48 = 3x^2 - 24x + 48$

$x^2 - 3x^2 - 12x + 24x = 0$

$-2x^2 + 12x = 0$

Divide by -2 we get, $x^2 - 6x = 0$

$x (x - 6) = 0$

$x = 0$ (or) $x - 6 = 0$

$x = 0$ (or) $x = 6$

But $x > 0$

$\therefore x = 6$

Output = 6 units

Ex 6.2

Question 1.
The average cost function associated with producing and marketing x units of an item is given by AC = 2x – 11 + 50/x. Find the range of values of the output x, for which AC is increasing.

Solution:

AC increases when $\frac{d}{dx}$ (AC) > 0

$C = 2x - 11 + \frac{50}{x}$

$\frac{dC}{dx} = 2 - 0 + 50(\frac{-1}{x^2}) = 2 - \frac{50}{x^2}$

$\frac{d}{dx}(AC) > 0$

$2 - \frac{50}{x^2} > 0$

$2 > \frac{50}{x^2}$

$2x^2 > 50$

$x^2 > 25$

$x > 5$

Question 2.
A television manufacturer finds that the total cost for the production and marketing of x number of television sets is $C(x) = 300x^2 + 4200x + 13500$. If each product is sold for ₹ 8,400. show that the profit of the company is increasing.

Solution:
$C(x) = 300x^2 + 4200x + 13,500$
Selling price of one product = ₹ 8,400
Selling price of x numbers of products = 8400x
Profit, P = Selling price – Cost price
$= 8400x - (300x^2 + 4200x + 13500)$
$= 8400x - 300x^2 - 4200x - 13500$
$P = -300x^2 + 4200x - 13500$

Differentiating with respect to x we get
$P'(x) = dP/dx = -600x + 4200$
$dP/dx = 0$ gives $-600x + 4200 = 0$
$-600x = -4200$
$x = 7$

The point $x = 7$ divide the real numbers into the intervals $(0, 7)$, $(7, \infty)$. Here x cannot be negative.

Now $P'(x) = -600x + 4200$
Take $x = 2$ in $(0, 7)$
$P'(2) = -600 \times 2 + 4200$
$= -1200 + 4200$
$= 3000$, positive
$\therefore$ P'(x) is increasing in (0, 7) the profit of the company increasing when each product is sold for ₹ 8,400.

Question 3.
A monopolist has a demand curve $x = 106 - 2p$ and average cost curve $AC = 5 + x/50$, where p is the price per unit output and x is the number of units of output. If the total revenue is $R = px$, determine the most profitable output and the maximum profit.

Solution:

x = 106 – 2p

(or) 2p = 106 – x

p = $\frac{1}{2}$ (106 – x)

Revenue, R = px

= $\frac{1}{2}$(106 – x) x

= 53x – $\frac{x^2}{2}$

Average Cost, AC = 5 + $\frac{x}{50}$

Cost C = (AC)x

= $\left(5+\frac{x}{50}\right) x$

= 5x + $\frac{x^2}{50}$

Profit (P) = Revenue – Cost

$\frac{dP}{dx}=48-\frac{13(2x)}{25}$

$\frac{d\mathbf{P}}{dx}$ = 0 gives

48 – $\frac{13(2x)}{25}$ = 0

48 = $\frac{13\times 2x}{25}$

x = $\frac{48\times 25}{13\times 2}$ = 46.1538 = 46 (approximately)

Also $\frac{d^2P}{dx^2}=0-\frac{(13)^2}{25}$, negative since $\frac{d^2P}{dx^2}$ is negative, profit is maximum at x = 46 units.

Profit = 48x – $\frac{13}{25}$ x^2

When x = 46,

Profit = 48 × 46 – $\frac{13}{25}$ × 46 × 46

= 2208 – $\frac{27508}{25}$

= 2208 – 1100.32

= ₹ 1107.68

Question 4.

A tour operator charges ₹ 136 per passenger with a discount of 40 paise for each passenger in excess of 100. The operator requires at least 100 passengers to operate the tour. Determine the number of passengers that will maximize the amount of money the tour operator receives.

Solution:

Let x be the required number of passengers

Tour operator charges

$$= 136 - \frac{40}{100}(x - 100), \text{ for } x \geq 100$$

$$= 136 - \frac{4x}{10} + \frac{4}{10} \times 100$$

$$= 136 - \frac{4x}{10} + 40$$

$$= 176 - \frac{2x}{5}$$

Amount of money, A = (Number of passengers) × (Tour operator charges)

$$A = x\left(176 - \frac{2x}{5}\right)$$

$$A = 176x - \frac{2x^2}{5}$$

$$\frac{dA}{dx} = 176 - \frac{4x}{5}$$

When $\frac{dA}{dx} = 0$ we get,

$$176 - \frac{4x}{5} = 0$$

$$176 = \frac{4x}{5}$$

$$4x = 176 \times 5$$

$$x = \frac{176 \times 5}{4} = 220$$

$$\frac{d^2A}{dx^2} = -\frac{4}{5}, \text{ negative}$$

∴ The amount of money is maximum when the number of passengers is 220.

Question 5.
Find the local minimum and local maximum of $y = 2x^3 - 3x^2 - 36x + 10$.

Solution:
$y = 2x^3 - 3x^2 - 36x + 10$
$\frac{dy}{dx} = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$
$\frac{dy}{dx} = 0$ gives $6(x^2 - x - 6) = 0$
$6(x - 3)(x + 2) = 0$
$x = 3$ (or) $x = -2$
$\frac{d^2y}{dx^2} = 6(2x - 1)$

Case (i): when $x = 3$,
$\left(\frac{d^2y}{dx^2}\right)_{x=3} = 6(2 \times 3 - 1)$
$= 6 \times 5$
$= 30$, positive
Since $\frac{d^2y}{dx^2}$ is positive y is minimum when $x = 3$.
The local minimum value is obtained by substituting $x = 3$ in y.
Local minimum value $= 2(3^3) - 3(3^2) - 36(3) + 10$
$= 2(27) - (27) - 108 + 10$
$= 27 - 98$
$= -71$

Case (ii): when $x = -2$,
$\left(\frac{d^2y}{dx^2}\right)_{x=-2} = 6(-2 \times 2 - 1)$
$= 6 \times -5$
$= -30$, negative
Since $\frac{d^2y}{dx^2}$ is negative, y is maximum when $x = -2$.

Local maximum value = $2(-2)^3 - 3(-2)^2 - 36(-2) + 10$

$= 2(-8) - 3(4) + 72 + 10$

$= -16 - 12 + 82$

$= -28 + 82$

$= 54$

Question 6.

The total revenue function for a commodity is $R = 15x + \frac{x^2}{3} - \frac{1}{36}x^4$. Show that at the highest point average revenue is equal to the marginal revenue.

Solution:

$R = 15x + \frac{x^2}{3} - \frac{1}{36}x^4$

Average Revenue = AR = $\frac{R}{x}$

$= \frac{15x + \frac{x^2}{3} - \frac{1}{36}x^4}{x}$

$= 15 + \frac{x}{3} - \frac{1}{36}x^3$

To test maxima or minima for AR = $\frac{d(AR)}{dx}$

$= 0 + \frac{1}{3} - \frac{3x^2}{36}$

$= \frac{1}{3} - \frac{x^2}{12}$

$\frac{d(AR)}{dx} = 0$

$\frac{1}{3} - \frac{x^2}{12} = 0$

$\frac{1}{3} = \frac{x^2}{12}$

$x^2 = \frac{12}{3}$

$x^2 = 4$

$x = 2$

$\frac{d^2(\mathrm{AR})}{dx^2} = 0 - \frac{2x}{12} = -\frac{x}{6}$

When x = 2, $\frac{d^2(\mathrm{AR})}{dx^2} = -\frac{2}{6} = -\frac{1}{3}$, negative

∴ AR is maximum when x = 2

Now, AR = 15 + $\frac{x}{3} - \frac{1}{36}x^3$

When x = 2, AR = $15 + \frac{2}{3} - \frac{2^3}{36}$

= $15 + \frac{2}{3} - \frac{8}{36}$

= $15 + \frac{24-8}{36}$

= 15 + $\frac{16}{36}$

= 15 + $\frac{4}{9}$ .... (1)

R = $15x + \frac{x^2}{3} - \frac{1}{36}x^4$

Marginal Revenue (MR) = $\frac{d\mathrm{R}}{dx}$

= $15 + \frac{2x}{3} - \frac{4x^3}{36}$

= $15 + \frac{2}{3}x - \frac{x^3}{9}$

When x = 2, MR = 15 + $\frac{2}{3} \times 2 - \frac{2^3}{9}$

= 15 + $\frac{4}{3} - \frac{8}{9}$

= 15 + $\frac{12-8}{9}$

= 15 + $\frac{4}{9}$ .... (2)

From (1) and (2) at the highest point average revenue is equal to the marginal revenue.

Ex 6.3

Question 1.
The following table gives the annual demand and unit price of 3 items

Items	Annual Demand (units)	Unit Price ₹
A	800	0.02
B	400	1.00
C	13,800	0.20

Ordering cost is ₹ 5 per order and holding cost is 10% of unit price. Determine the following:
(i) EOQ in units
(ii) Minimum average cost
(iii) EOQ in rupees
(iv) EOQ in years of supply
(v) Number of orders per year

Solution:
Item A:
Demand rate, R = 800
Ordering cost, C_3 = ₹ 5
Carrying cost C_1 = 10% of unit price
= 10/100 × 0.02

(i) EOQ in units

$$EOQ = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2\times800\times5\times100}{10\times0.02}} = \sqrt{\frac{800\times100\times100}{0.02\times100}}$$

$$= \sqrt{\frac{800\times100\times100}{2}} = \sqrt{400\times100\times100}$$

$$= 20 \times 10 \times 10$$

$$= 2000 \text{ units}$$

(ii) Minimum Average Cost = C_0 = $\sqrt{2RC_3C_1}$

= $\sqrt{2 \times 800 \times 5 \times \frac{10}{100} \times 0.02}$

= $\sqrt{800 \times 0.02}$

= $\sqrt{16.00}$

= ₹ 4

(iii) EOQ in rupees = EOQ × Unit price

= 2000 × 0.02

= 2000 × $\frac{2}{100}$

= ₹ 40

(iv) $\frac{\text{EOQ}}{\text{Demand}} = \frac{2000}{800}$ = 2.5

(v) $\frac{\text{Demand}}{\text{EOQ}} = \frac{800}{2000}$ = 0.4

Item B:
Demand rate, R = 400
Ordering cost, C_3 = ₹ 5
Carrying cost C_1 = 10% of 1.00

(i) EOQ in units

$$\text{EOQ} = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 400 \times 5}{\frac{10}{100} \times 1}} = \sqrt{\frac{2 \times 400 \times 5 \times 100}{10}}$$

= 20 × 10

= 200 units

(ii) Minimum Average Cost = C_0 = $\sqrt{2RC_3C_1}$

= $\sqrt{2 \times 400 \times 5 \times \frac{10}{100} \times 1}$

= $\sqrt{400}$

= ₹ 20

(iii) EOQ in rupees = EOQ × unit price

= 200 × 1

= ₹ 200

(iv) $\frac{\text{EOQ}}{\text{Demand}} = \frac{200}{400}$ = 0.5

(v) $\frac{\text{Demand}}{\text{EOQ}} = \frac{400}{200}$ = 2

Item C:
Annual Demand, R = 800
Ordering cost, C_3 = ₹ 5
Carrying cost, C_1 = 10% of unit price
= 10/100 × 0.20
= 2/100

(i) EOQ in units

$$\text{EOQ} = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 13800 \times 5}{\frac{2}{100}}} = \sqrt{\frac{2 \times 13800 \times 5 \times 100}{2}}$$

$$= \sqrt{138 \times 100 \times 5 \times 100}$$

$$= 10 \times 10\sqrt{138 \times 5} = 100\sqrt{690}$$

= 100 × 26.2678
= 100 × 26.27
= 2627

(ii) Minimum Average Cost = C_0 = $\sqrt{2RC_3C_1}$

= $\sqrt{2 \times 13800 \times 5 \times \frac{10}{100} \times 0.2}$

= $\sqrt{2 \times 138 \times 5 \times 2}$

= $\sqrt{2760}$

= 52.535

= ₹ 52.54

(iii) EOQ in rupees = 2627 × 0.20 = ₹ 25.40 [∵ Unit price = 0.20]

(iv) $\frac{\text{EOQ}}{\text{Demand}} = \frac{2627}{13800} = 0.19036 = 0.19$

(v) $\frac{\text{Demand}}{\text{EOQ}} = \frac{13800}{2627} = 5.2531 = 5.25$

Question 2.
A dealer has to supply his customer with 400 units of a product per week. The dealer gets the product from the manufacturer at a cost of ₹ 50 per unit. The cost of ordering from the manufacturers in ₹ 75 per order. The cost of holding inventory is 7.5 % per year of the product cost. Find
(i) EOQ
(ii) Total optimum cost.

Solution:
Demand = 400 units per week
Annual demand = 400 × 52 per year
Ordering cost per order C_3 = 175
Inventory cost C_1 = 7.5% per year of the cost
= 7.5% of 50 per year

$= \frac{7.5}{100} \times 50$
$= \frac{7.5 \times 50}{100 \times 52}$ (per week)

(i) EOQ in units

$$\text{EOQ} = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 400 \times 52 \times 75}{\frac{7.5}{100} \times 50}} = \sqrt{\frac{2 \times 400 \times 52 \times 75 \times 100}{7.5 \times 50}}$$

$$= \sqrt{\frac{2 \times 400 \times 52 \times 75 \times 100 \times 10}{75 \times 50}} = \sqrt{832000} = 912 \text{ (appr)}$$

EOQ = 912 units

(ii) Total optimum cost = Purchasing cost + Minimum annual cost

$= 400 \times 50 + \sqrt{2RC_3C_1}$

$= 20000 + \sqrt{2 \times 400 \times 75 \times \frac{7.5}{100 \times 52} \times 50}$

$= 20000 + \sqrt{2 \times 4 \times 75 \times \frac{7.5}{52} \times 50}$

$= 20000 + \sqrt{\frac{225000}{52}}$

$= 20000 + \sqrt{4326.92307}$

$= 20000 + \sqrt{4326.9231}$

$= 20000 + 65.7793$

= ₹ 20,065.78 per week

Ex 6.4

Question 1.

If z = (ax + b) (cy + d), then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

Given, z = (ax + b) (cy + d)

Differentiating partially with respect to x we get,

$\frac{\partial z}{\partial x} = (cy + d) \frac{\partial}{\partial x} (ax + b)$ [∵ (cy + d) is constant]

= (cy + d) (a + 0)

= a(cy + d)

Differentiating partially with respect to y we get,

$\frac{\partial z}{\partial y} = (ax + b) \frac{\partial}{\partial y} (cy + d)$

= (ax + b)(c + 0)

= c(ax + b)

Question 2.

If u = e^{xy}, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ = u(x^2 + y^2).

Solution:

Given, u = e^{xy}

Differentiating partially with respect to x, we get,

$\frac{\partial u}{\partial x}$ = yexy (Treating y as constant)

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(ye^{xy})$

= $y\frac{\partial}{\partial x}(e^{xy})$

= y(ye^{xy})

= y^2e^{xy} ……… (1)

We have u = e^{xy}

Differentiating partially with respect to y,

$\frac{\partial u}{\partial y} = xe^{xy}$

Again differentiating partially with respect to x, we get,

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{xy})$

= $x\frac{\partial}{\partial y}(e^{xy})$

= x^2e^{xy} ……… (2)

Adding (1) and (2) we get,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ = e^{xy}(x^2 + y^2)

= u(x + y) [∵ u = e^{xy}]

Question 3.

Let u = x cos y + y cos x. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution:

u = x cos y + y cos x

Differentiating partially with respect to y, we get,

$\frac{\partial u}{\partial y}$ = $\frac{\partial}{\partial y}$ (xcosy) + $\frac{\partial}{\partial y}$ (y cos x)

= x $\frac{\partial}{\partial y}$(cos y) + cos x $\frac{\partial}{\partial y}$(y)

= x(-sin y) + cos x

Again differentiating partially with respect to x, we get

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ = $\frac{\partial}{\partial x}$(-x sin y) + $\frac{\partial}{\partial x}$ (cos x)

= $\frac{\partial}{\partial x}$ (-x sin y) + $\frac{\partial}{\partial x}$ (cos x)

= -sin y $\frac{\partial}{\partial x}(x)$ + (-sin x)

= -sin y (1) + (-sin x)

= -sin y – sin x .... (1)

Now u = x cos y + y cos x

Differentiating partially with respect to x we get

$\frac{\partial u}{\partial x}$ = cos y $\frac{\partial}{\partial x}$ (x) + y $\frac{\partial}{\partial x}$ (cos x)

= cos y (1) + y(-sin x)

= cos y – y sin x

Again differentiating partially with respect to y we get,

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ = $\frac{\partial}{\partial y}$ (cos y – y sin x)

= $\frac{\partial}{\partial y}$ (cos y) – $\frac{\partial}{\partial y}$ (y sin x)

= -sin y – sin x $\frac{\partial}{\partial y}$ (y)

= -sin y – sin x (1)

= -sin y – sin x ....(2)

From (1) and (2), $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Hence verified.

Question 4.
Verify Euler's theorem for the function $u = x^3 + y^3 + 3xy^2$.

Solution:
$u = x^3 + y^3 + 3xy^2$

i.e., $u(x, y) = x^3 + y^3 + 3xy^2$
$u(tx, ty) = (tx)^3 + (ty)^3 + 3(tx)(ty)^2$
$= t^3x^3 + t^3y^3 + 3tx(t^2y^2)$
$= t^3(x^3 + y^3 + 3xy^2)$
$= t^3u$
∴ u is a homogeneous function in x and y of degree 3.

∴ By Euler's theorem, $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3u$

Verification:

$u = x^3 + y^3 + 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 + 0 + 3y^2 \frac{\partial}{\partial x}(x)$

$= 3x^2 + 3y^2(1)$

$= 3x^2 + 3y^2$ .... (1)

$\therefore x \cdot \frac{\partial u}{\partial x} = 3x^3 + 3xy^2$

$\frac{\partial u}{\partial y} = 0 + 3y^2 + 3x(2y) = 3y^2 + 6xy$

$y \cdot \frac{\partial u}{\partial y} = 3y^3 + 6xy^2$ .... (2)

∴ (1) + (2) gives

$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3x^3 + 3y^3 + 9xy^2$

$= 3(x^3 + y^3 + 3xy^2)$

$= 3u$

Hence Euler's theorem is verified.

Question 5.

Let $u = x^2y^3 \cos(\frac{x}{y})$. By using Euler's theorem show that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$

Solution:

Given, $u = x^2y^3 \cos(\frac{x}{y})$

i.e., $u(tx, ty) = (tx)^2 (ty)^3 \cos(\frac{tx}{ty})$

$= t^2x^2t^3y^3 \cos(\frac{x}{y})$

$= t^5x^2y^3 \cos(\frac{x}{y})$

$= t^5$ u

∴ u is a homogeneous function in x and y of degree 5.

∴ By Euler's theorem, $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 5u$

Hence Proved.

Ex 6.5

Question 1.
Find the marginal productivities of capital (K) and labour (L) if $P = 8L - 2K + 3K^2 - 2L^2 + 7KL$ when K = 3 and L = 1.

Solution:
$P = 8L - 2K + 3K^2 - 2L^2 + 7KL$

Marginal productivity of labour, $\frac{\partial P}{\partial L}$ = 8 – 0 + 0 – 2(2L) + 7K(1)

= 8 – 4L + 7K

Marginal productivity of labour when K = 3 and L = 1 is

$\left(\frac{\partial P}{\partial L}\right)_{(3,1)}$ = 8 – 4 + 21

= 29 – 4

= 25

Marginal productivity of capital, $\frac{\partial P}{\partial K}$ = 0 – 2(1) + 3(2K) – 0 + 7L(1)

= -2 + 6K + 7L

Marginal productivity of capital when K = 3 and L = 1 is

$\left(\frac{\partial P}{\partial K}\right)_{(3,1)}$

= -2 + 18 + 7

= -2 + 25

= 23

Question 2.
If the production of a firm is given by $P = 4LK - L^2 + K^2$, $L > 0$, $K > 0$, Prove that $L \frac{\partial P}{\partial L} + K \frac{\partial P}{\partial K} = 2P$.

Solution:
$P = 4LK - L^2 + K^2$
$P(K, L) = 4LK - L^2 + K^2$
$P(tK, tL) = 4(tL)(tK) - t^2L^2 + t^2K^2$
$= t^2(4LK - L^2 + K^2)$
$= t^2P$
$\therefore$ P is a homogeneous function in L and K of degree 2.
$\therefore$ By Euler's theorem, $L \frac{\partial P}{\partial L} + K \frac{\partial P}{\partial K} = 2P$.

Question 3.
If the production function is $z = 3x^2 - 4xy + 3y^2$ where x is the labour and y is the capital, find the marginal productivities of x and y when x = 1, y = 2.

Solution:

Marginal productivity of labour, $\frac{\partial z}{\partial x} = 6x - 4y$
Marginal productivity of labour when x = 1, y = 2 is
$\left(\frac{\partial z}{\partial x}\right)_{(1,2)} = 6(1) - 4(1)$
$= 6 - 4$
$= 2$
Marginal productivity of capital, $\frac{\partial z}{\partial y} = 0 - 4x(1) + 3(2y)$
$= -4x + 6y$
Marginal productivity of qapital when x = 1, y = 2 is
$\left(\frac{\partial z}{\partial y}\right)_{(1,2)} = -4(1) + 6(2)$
$= -4 + 12$
$= 8$

Question 4.

For the production function $P = 3(L)^{0.4} (K)^{0.6}$, find the marginal productivities of labour (L) and capital (K) when L = 10 and K = 6. [use: $(0.6)^{0.6} = 0.736$, $(1.67)^{0.4} = 1.2267$]

Solution:

Given that $P = 3(L)^{0.4} (K)^{0.6}$.......(1)

Differentiating partially with respect to L we get,

$$\frac{\partial P}{\partial L} = 3(K)^{0.6}\left(\frac{\partial}{\partial L}(L)^{0.4}\right)$$

$$= 3(K^{0.6})\,(0.4)\,(L^{0.4-1})$$

$$= 1.2\,(K^{0.6})\,(L^{-0.6})$$

$$= 1.2\,(K^{0.6})\,\frac{1}{L^{0.6}}$$

$$= 1.2\left(\frac{K}{L}\right)^{0.6}$$

When L = 10, k = 6, $\frac{\partial P}{\partial L} = 1.2\left(\frac{6}{10}\right)^{0.6}$

$= 1.2(0.6)^{0.6}$

$= 1.2(0.736)$

i.e., the marginal productivity of labour = 0.8832

Again differentiating partially with respect to 'k' we get,

Marginal productivity of labour when L = 10, K = 6 is

$$\frac{\partial P}{\partial k} = 3\,L^{0.4}\left(\frac{\partial}{\partial k}(k)^{0.6}\right)$$

$$= 3L^{0.4}\,(0.6)\,k^{0.6-1}$$

$$= 1.8\,(L^{0.4})\,k^{-0.4}$$

$$= 1.8\,(L^{0.4})\left(\frac{1}{k^{0.4}}\right)$$

$$= 1.8\left(\frac{L}{K}\right)^{0.4}$$

Marginal productivity of capital when k = 10, k = 6

$= 1.8\left(\frac{10}{6}\right)^{0.4}$

$= 1.8(1.66666)^{0.4}$

$= 1.8(1.67)^{0.4}$

$= 1.8 \times 1.2267$

$= 2.2081$

Question 5.

The demand for a quantity A is $q = 13 - 2p_1 - 3p_2^2$. Find the partial elasticities $\frac{Eq}{Ep_1}$ and $\frac{Eq}{Ep_2}$ when $p_1 = p_2 = 2$.

Solution:

$$q = 13 - 2p_1 - 3p_2^2$$

$$\frac{\partial q}{\partial p_1} = 0 - 2(1) - 0 = -2$$

$$\frac{\partial q}{\partial p_2} = 0 - 0 - 3(2)\,p_2 = -6p_2$$

$$\frac{Eq}{Ep_1} = -\frac{p_1}{q}\frac{\partial q}{\partial p_1} = \frac{-p_1}{13-2p_1-3p_2^2}(-2)$$

When $p_1 = p_2 = 2$,

$$\frac{Eq}{Ep_1} = \left(\frac{-2}{13-2\times2-3\times2^2}\right)(-2)$$

$$= \frac{-2\times-2}{13-4-3\times4} = \frac{4}{13-4-12} = \frac{4}{13-16}$$

$$= \frac{4}{-3} = \frac{-4}{3}$$

$$\frac{Eq}{Ep_2} = -\frac{p_2}{q}\frac{\partial q}{\partial p_2} = \left(\frac{-p_2}{13-2p_1-3p_2^2}\right)(-6p_2)$$

When $p_1 = p_2 = 2$,

$$\frac{Eq}{Ep_2} = \left(\frac{-2}{13-2\times 2-3\times 2^2}\right)(-6\times 2)$$

$$= \frac{-2\times -12}{13-4-12} = \frac{24}{-3} = -8$$

Question 6.

The demand for a commodity A is $q = 80 - p_1^2 + 5p_2 - p_1p_2$. Find the partial elasticities $\frac{\mathbf{Eq}}{\mathbf{E}p_1}$ and $\frac{\mathbf{Eq}}{\mathbf{E}p_2}$ when $p_1 = 2$, $p_2 = 1$.

Solution:

$$q = 80 - p_1^2 + 5p_2 - p_1p_2$$

$$\frac{\partial q}{\partial p_1} = 0 - 2p_1 + 0 - (1)p_2$$

$$= -2p_1 - p_2$$

$$\frac{\partial q}{\partial p_2} = 0 - 0 + 5(1) - p_1(1)$$

$$= 5 - p_1$$

$$\therefore \frac{Eq}{\partial p_1} = -\frac{p_1}{q}\frac{\partial q}{\partial p_1} = \frac{-p_1}{80-p_1^2+5p_2-p_1p_2}\times(-2p_1-p_2)$$

When $p_1 = 2$, $p_2 = 1$,

$$\frac{Eq}{\partial p_1} = \frac{-2}{80-2^2+5(1)-2\times 1}(-2\times 2-1)$$

$$= \frac{-2}{80-4+5-2}\times(-4-1) = \frac{10}{79}$$

$$\frac{Eq}{\partial p_2} = -\frac{p_2}{q}\frac{\partial q}{\partial p_2} = \frac{-p_2}{80-p_1^2+5p_2-p_1p_2}(5-p_1)$$

When $p_1 = 2$, $p_2 = 1$,

$$\frac{Eq}{Ep_2} = \frac{-1}{80 - 2^2 + 5 \times 1 - 2 \times 1}(5 - 2)$$

$$= \frac{-1}{79} \times (3) = \frac{-3}{79}$$

Ex 6.6

Question 1.
Average fixed cost of the cost function $C(x) = 2x^3 + 5x^2 - 14x + 21$ is:
(a) $\frac{2}{3}$
(b) $\frac{5}{x}$
(c) $-\frac{14}{x}$
(d) $\frac{21}{x}$

Answer:
(d) $\frac{21}{x}$
Hint:
Average fixed cost $\frac{k}{x} = \frac{21}{x}$

Question 2.
Marginal revenue of the demand function $p = 20 - 3x$ is:
(a) $20 - 6x$
(b) $20 - 3x$
(c) $20 + 6x$
(d) $20 + 3x$

Answer:
(a) $20 - 6x$
Hint:
$p = 20 - 3x$
Revenue, $R = px = (20 - 3x)x = 20x - 3x^2$
Marginal revenue $= dR/dx = 20 - 6x$

Question 3.
If demand and the cost function of a firm are $p = 2 - x$ and $C = -2x^2 + 2x + 7$ then its profit function is:
(a) $x^2 + 7$
(b) $x^2 - 7$
(c) $-x^2 + 7$
(d) $-x^2 - 7$

Answer:
(b) $x^2 - 7$
Hint:
Profit = Revenue – Cost
$= (2 - x)x - (-2x^2 + 2x + 7)$
$= 2x - x^2 + 2x^2 - 2x - 7$
$= x^2 - 7$

Question 4.
If the demand function is said to be inelastic, then:
(a) $|\eta_d| > 1$
(b) $|\eta_d| = 1$
(c) $|\eta_d| < 1$
(d) $|\eta_d| = 0$

Answer:
(a) $|\eta_d| > 1$

Question 5.
The elasticity of demand for the demand function $x = 1/p$ is:
(a) 0
(b) 1
(c) $-\frac{1}{p}$
(d) ∞

Answer:
(b) 1

Hint:

$$\eta_d = \frac{-p}{x}\frac{dx}{dp}$$

$$= \frac{-p}{x}\left(-\frac{1}{p^2}\right) = \frac{1}{px} = \frac{1}{1} = 1$$

Question 6.
Relationship among MR, AR and ηd is:

(a) $\eta_d = \frac{\text{AR}}{\text{AR}-\text{MR}}$

(b) η_d = AR – MR

(c) MR = AR = η_d

(d) AR = $\frac{\text{MR}}{\eta_d}$

Answer:
(a) $\eta_d = \frac{\text{AR}}{\text{AR}-\text{MR}}$

Question 7.
For the cost function C = $\frac{1}{25}e^{5x}$, the marginal cost is:

(a) $\frac{1}{25}$

(b) $\frac{1}{5}e^{5x}$

(c) $\frac{1}{125}e^{5x}$

(d) $25e^{5x}$

Answer:

(b) $\frac{1}{5}e^{5x}$

Hint:

C = $\frac{1}{25}e^{5x}$

Marginal cost $\frac{dc}{dx} = \frac{1}{25}\left(5e^{5x}\right) = \frac{1}{5}e^{5x}$

Question 8.
Instantaneous rate of change of $y = 2x^2 + 5x$ with respect to x at x = 2 is:
(a) 4
(b) 5
(c) 13
(d) 9

Answer:
(c) 13
Hint:
$y = 2x^2 + 5x$
$\frac{dy}{dx} = 4x + 5$
$\left(\frac{dy}{dx}\right)_{x=2} = 4 \times 2 + 5 = 13$

Question 9.
If the average revenue of a certain firm is ₹ 50 and its elasticity of demand is 2, then their marginal revenue is:
(a) ₹ 50
(b) ₹ 25
(c) ₹ 100
(d) ₹ 75

Answer:
(b) ₹ 25
Hint:
$$MR = AR\left[1 - \frac{1}{\eta_d}\right]$$
$$= 50\left[1 - \frac{1}{2}\right] = 50 \times \frac{1}{2} = ₹\ 25$$

Question 10.
Profit P(x) is maximum when:
(a) MR = MC
(b) MR = 0
(c) MC = AC
(d) TR = AC

Answer:
(a) MR = MC

Question 11.
The maximum value of f(x) = sin x is:
(a) 1
(b) $\frac{\sqrt{3}}{2}$
(c) $\frac{1}{\sqrt{2}}$
(d) $-\frac{1}{\sqrt{2}}$

Answer:
(a) 1
Hint:
When x = π/2, sin x is maximum.
Maximum value is sin π/2 = 1

Question 12.
If f(x, y) is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is equal to:
(a) (n – 1)f
(b) n(n – 1)f
(c) nf
(d) f

Answers:
(c) nf

Question 13.
If $u = 4x^2 + 4xy + y^2 + 4x + 32y + 16$, then $\frac{\partial^2 u}{\partial y \partial x}$ is equal to:
(a) 8x + 4y + 4
(b) 4
(c) 2y + 32
(d) 0

Answer:
(b) 4
Hint:
$u = 4x^2 + 4xy + y^2 + 4x + 32y + 16$

$$\frac{\partial u}{\partial x} = 8x + 4y + 4$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(8x + 4y + 4)$$

$$\frac{\partial^2 u}{\partial y \partial x} = 4$$

Question 14.
If $u = x^3 + 3xy^2 + y^3$ then $\frac{\partial^2 u}{\partial y \partial x}$ is:

(a) 3
(b) 6y
(c) 6x
(d) 2

Answer:
(b) 6y
Hint:
$u = x^3 + 3xy^2 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \Rightarrow \frac{\partial}{\partial y}(3x^2 + 3y^2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = 6y$$

Question 15.
If u = e^{x^2} then $\frac{\partial u}{\partial x}$ is equal to:
(a) $2xe^{x^2}$
(b) e^{x^2}
(c) $2e^{x^2}$
(d) 0

Answer:
(a) $2xe^{x^2}$
Hint:
u = e^{x^2}
$\frac{\partial u}{\partial x} = 2xe^{x^2}$

Question 16.
Average cost is minimum when:
(a) Marginal cost = marginal revenue
(b) Average cost = marginal cost
(c) Average cost = Marginal revenue
(d) Average Revenue = Marginal cost

Answer:
(b) Average cost = marginal cost

Question 17.
A company begins to earn profit at:
(a) Maximum point
(b) Breakeven point
(c) Stationary point
(d) Even point

Answer:
(b) Breakeven point

Question 18.
The demand function is always:

(a) Increasing function
(b) Decreasing function
(c) Non-decreasing function
(d) Undefined function

Answer:
(b) Decreasing function

Question 19.
If $q = 1000 + 8p_1 - p_2$ then, $\frac{\partial q}{\partial p_1}$ is:
(a) -1
(b) 8
(c) 1000
(d) $1000 - p_2$

Answer:
(b) 8
Hint:
$q = 1000 + 8p_1 - p_2$
$\frac{8q}{\partial p_1} = 8$

Question 20.
If R = 5000 units/year, C_1 = 20 paise, C_3 = ₹ 20 then EOQ is:
(a) 5000
(b) 100
(c) 1000
(d) 200

Answer:
(c) 1000
Hint:

$$\text{EOQ} = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 5000 \times 20}{0.20}} = \sqrt{\frac{2 \times 5000 \times 20 \times 10}{2}}$$

$$= \sqrt{10000 \times 100} = \sqrt{1000000} = ₹\ 1000$$