13

Vector

13.01 Introduction

As we knows that many useful physical quantities in nature are of two types, scalars and vectors. Scalars are those quantities which are completely determined by a single real number when the units of measurement of that quantity are given. Scalars are not related or assigned to any particular direction in space. For example, mass, volume, temperature, density etc are scalars. Scalars depend only on the points in space but not on any particular choice of the coordinate system. Vectors are those quantities which are completely determined if their lengths (also called magnitude) and their directions in space are given. For example displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc. are vectors.

In this chapter, we will study basic concepts about vectors, various operations on vectors and their algebraic and geometric properties.

13.02 Basic Concepts

Let L be any straight line in plane or three dimensional space. This line can be given two directions by means of arrow heads. A line with one of these directions prescribed is called a *directed line*. Now observe that if we restrict the line L to the line segment AB, then a magnitude is prescribed on the line L with one of the two directions, so that we obtain a *directed line segment* (Fig). Thus, a directed line segment has magnitude as well as direction.

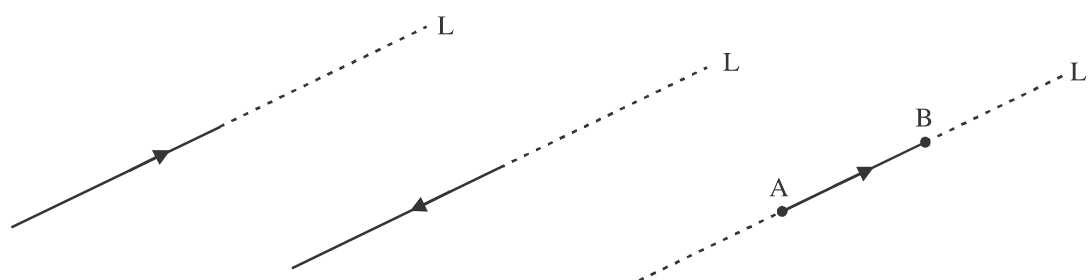


Fig. 13.01

Each directed line segment has following properties:

(i) Length: The length of directed line segment $\vec{AB}$ is the length of line segment represented by AB or $|\vec{AB}|$

(ii) Support: The base of a directed line segment $\vec{AB}$ is a line L whose segment is AB

(iii) Sense: the point A from where the vector $\vec{AB}$ starts is called its *initial point,* and the point B where it ends is called its *terminal point*. A directed line segment $\vec{BA}$ is from A to B where as for it is from B to A

Note: Although $\vec{AB}$ and $\vec{BA}$ have same length and base yet they are different vectors as $\vec{AB}$ and $\vec{BA}$ are opposite senses.

Vector Quantity : A quantity that has magnitude as well as directionis called a vector notice that directed line segment is a vector, denoted as $\vec{AB}$ or simply as $\vec{a}$, and read as vector $\vec{AB}$ or vector $\vec{a}$

Magnitude of the Vector: The distance between initial and terminal points of a vector is called the *magnitude* (or length) of the vector, denoted as $|\vec{a}|$ or $|\overline{AB}|$ where $\vec{a}$ thus the magnitude of vector $=|\vec{a}|=a$

Note : $|\vec{a}|\geq 0$

13.03 Various Types of Vectors

(1) Unit vector : A vector whose magnitude is unity (i.e. 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\hat{a}$. We denote the unit vector in the direction of vector *a*, *b*, *c* as $\hat{a}, \hat{b}, \hat{c}$ and it is given by

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}, \hat{b}=\frac{\vec{b}}{|\vec{b}|}, \hat{c}=\frac{\vec{c}}{|\vec{c}|}$$

$\hat{a}$ is read as *a* cap.

(2) Zero or Null Vector: A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{O}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

also $\qquad |\vec{a}|=0$

i.e. if $\qquad |\overrightarrow{AB}|=0$

then A and B coincides.

(3) Like Vectors: If two vectors have same direction or senses then they are called Like Vectors.

(4) Equal Vectors: Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a}, \vec{b}$.

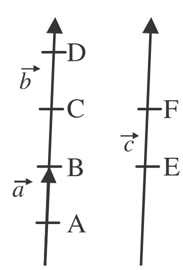


Fig. 13.02

In the fig : (13.02) the initial and terminal points of vectors $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF}$ represented by $\vec{a}, \vec{b}$ and $\vec{c}$ are different but their length is same therefore they are equal vectors.

i.e. $\quad \overrightarrow{AB}=\overrightarrow{CD}=\overrightarrow{EF}$

If $\vec{a}$ and $\vec{b}$ are equal vectors then we write them as $\vec{a}=\vec{b}$.

(5) Unlike Vectors: If the direction of the vectors are opposite then they are called unlike vectors.

(6) Negative Vector: A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$

$\therefore$ If $\vec{a} = \overrightarrow{AB}$ then $\overrightarrow{BA} = -\vec{a}$

Position Vector

From a rectangular coordinate system consider a point P, having coordinates (x, y) with respect to the origin O(0, 0). Then the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the *position vector* of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$\left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2}$$

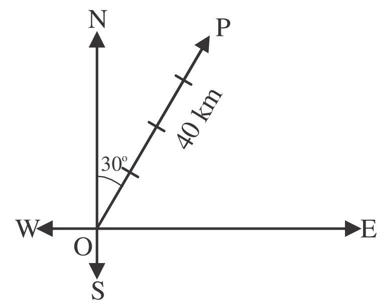


Fig. 13.03

For Example : Represent graphically a displacement of 40 km, 30° east of North.

Solution : The vector $\overrightarrow{OP}$, represents the required displacement (Fig: 13.03)

13.04 Addition of Vectors

(A): Addition of Two Vectors

If there are two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in a plane whcih are denoted by $\vec{a}$ and $\vec{b}$ then we add the two vectors by two methods.

I. Triangle law of Vector Addition: A vector $\overrightarrow{OE}$ simply means the displacement from a point E to the point F. Now consider a situation that a girl moves from O to E and then from E to F (Fig. 13.04). The net displacement made by the girl from point o to the point F is given by the vector $\overrightarrow{OF}$ and expressed as

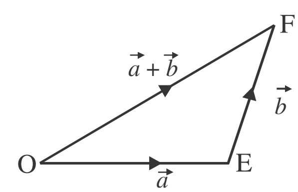


Fig. 13.04

$$\overrightarrow{OE} + \overrightarrow{EF} = \overrightarrow{OF}$$

$$\vec{a} + \vec{b} = \overrightarrow{OF} \quad \text{where} \quad \overrightarrow{OE} = \vec{a} \quad \text{and} \quad \overrightarrow{EF} = \vec{b}$$

This is known as the *triangle law of vector addition*. In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig. 13.04), then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other. According to this law, "If two vectors in same order represents the two sides of a triangle then their sum is represented by the third side of triangle in opposite order".

II. Parallelogram law of Vector Addition: We have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig. 13.05), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point.

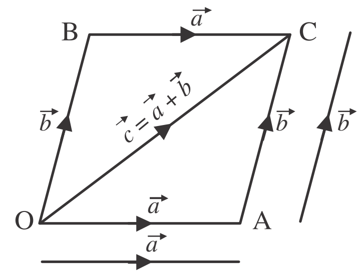


Fig. 13.05

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Now OACB is a Parallelogram and OC is the digonal of OACB Here! $\overrightarrow{OA} = \overrightarrow{BC} = \vec{a}$ and $\overrightarrow{OB} = \overrightarrow{AC} = \vec{b}$.

In triangle OAC using triangle law of addition $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b}$

So, if two vectors are represented in magnitude and direction by two adjacent sides of a parallelogram, then their sum is represented by diagonal of parallelogram which is cointial with the given vectors. This is known as 'parallelogram law of vector addition'.

(B) Addition of more than two Vectors:

For addition of more or more than two vectors the triangle law of addition can be used. This addition of vectors is known as Polygon law of vector addition.

Example : Suppose we have to add vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Let us take point O in a plane. Draw $\overrightarrow{OA} = \vec{a}$ s, also draw $\overrightarrow{AB} = \vec{b}$ similarly draw $\overrightarrow{BC} = \vec{c}$. Now by triangle law of vector addition we have

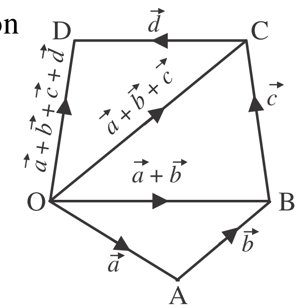


Fig. 13.06

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \vec{a} + \vec{b} = \overrightarrow{OB}$$

$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \overrightarrow{OC}$$

and $$\overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OD} \Rightarrow \vec{a} + \vec{b} + \vec{c} + \vec{d} = \overrightarrow{OD}$$

sNow vector $\overrightarrow{OD}$ denotes the sum of vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Polygon OABCD is called as Polygon of vectors.

Note : If the initial point of first vector and terminal point of last vector coincides then the sum of the vectors is always zero.

13.05 Properties of Vector Addition:

Vector addition has the following properties:

(i) Commutativity: Addition of vectors follows the commutative law i.e. for any two vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Proof : Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

By Triangle law of addition we have

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b} \quad \ldots \text{(i)}$$

Complete the parallelogram OABC, such that

$$\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$$

and $$\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$$

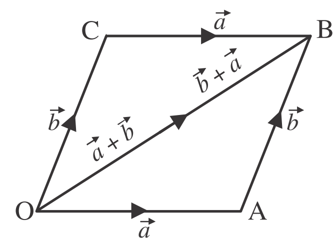


Fig. 13.07

In triangle OCB,

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \vec{b} + \vec{a} \quad \ldots \text{(ii)}$$

From equation (i) and (ii),

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$

Thus addition of vectors is commutative.

(ii) Associativity: Addition of vectors obeys the associative law i.e. let $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors then

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

Proof : Let vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are denoted by $\overrightarrow{OA}, \overrightarrow{AB}$ and $\overrightarrow{BC}$, thus $\overrightarrow{OA}=\vec{a}, \overrightarrow{AB}=\vec{b}$ and $\overrightarrow{BC}=\vec{c}$. Using triangle law of vector addition in triangle OAB and OBC

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$$

and $$\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=(\vec{a}+\vec{b})+\vec{c} \qquad (1)$$

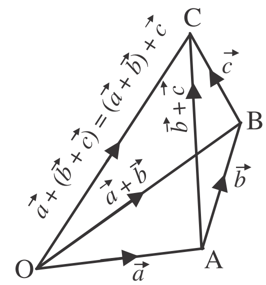


Fig. 13.08

Similarly triangle law of vector addition in triangles ABC and OAC

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\vec{b}+\vec{c}$$

and $$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\vec{a}+(\vec{b}+\vec{c}) \qquad (2)$$

from equation (1) and (2)

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

Thus the addition of vectors is associative.

Note: It is clear from the above rule that addition of vectors $\vec{a},\vec{b},\vec{c}$ does not depend in the order in which they are added. Thus the above addition can be expressed as $\vec{a}+\vec{b}+\vec{c}$.

(iii) Identity:

For every vector $\vec{a}$, $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$, where $\vec{0}$ is a zero vector is known as identity vector for addition

Proof : From definition of addition of vectors

$$\overrightarrow{OA}=\overrightarrow{OA}+\overrightarrow{AA}=\vec{a}+\vec{0}$$

$\therefore \quad \vec{a}=\vec{a}+\vec{0}$

similarly $\vec{a}=\vec{0}+\vec{a}$

(iv) Additive inverse : For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$

Proof : Let vector $\vec{a}=\overrightarrow{OP}$ then by definition of Negative Vector, $(-\vec{a})$ will be denoted by $\overrightarrow{PO}$

Now $\quad \vec{a}+(-\vec{a})=\overrightarrow{OP}+\overrightarrow{PO}=\overrightarrow{OO}=\vec{O}$

similarly $\quad (-\vec{a})+\vec{a}=\overrightarrow{PO}+\overrightarrow{OP}=\overrightarrow{PP}=\vec{O}$

Fig. 13.10

thus from (1) and (2) $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{O}$

13.06 Subtraction of Vectors

Let $\vec{a}$ and $\vec{b}$ are two vector quantities and let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$. Now if we have to find $\vec{a} - \vec{b}$ then at point B draw a line BD opposite in direction and equal in length to BC which represents the directed line segment as $\overrightarrow{BD} = -\vec{b}$

Join A and D. Now using triangle law of addition in triangle ABD

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$$

Similarly if we have to subtract $\vec{a}$ from $\vec{b}$ i.e. we have to find $(\vec{a} - \vec{b})$ then add the negative of vector $\vec{a}$ i.e. $(-\vec{a})$ to vector $\vec{b}$

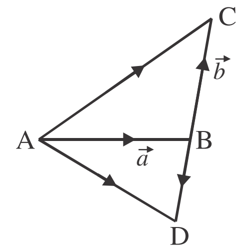


Fig. 13.10

13.07 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ is denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ Note that, $\lambda\vec{a}$ is also a vector, collinear tot he vector $\vec{a}$. The vector, $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magntiude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualization of multiplication of a vector by a scalar is given in Fig. 13.10,

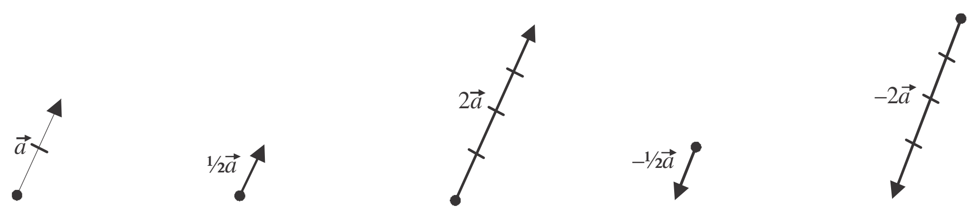


Fig. 13.11

What $\lambda = -1$, then $\lambda\vec{a} = -\vec{a}$ which is a vector having magnitude equal to the magnitude of $\vec{a}$. The vector $-\vec{a}$ called the *negative* (or *additive inverse*) *of vector* $\vec{a}$ and we always have $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{O}$.

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$, i.e. $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

So, $\lambda\vec{a}$ represents the unit vector in the direction of $\vec{a}$

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

13.08 Components of a Vector

Let us take the points A (1, 0 , 0), B(0, 1, 0) and C (0, 0, 1) on the x-axis, y-axis and z-axis respectively. Then, clearly

$$|\vec{OA}|=1,\ |\vec{OB}|=1 \text{ and } |\vec{OC}|=1$$

The vectors $\vec{OA}, \vec{OB}$ and $\vec{OC}$, each having magnitude 1, are called *unit vectors along the axes* OX, OY and OZ respectively and denoted by $\hat{i}, \hat{j}$ and $\hat{k}$ respectively

Let P (x, y,. z) is a point whose position vector is $\vec{OP}$. Therefore

$$\vec{OL} = x\hat{i}$$

$$\vec{OM} = \vec{LQ} = y\hat{j}$$

$$\therefore \quad \vec{OQ} = \vec{OL} + \vec{LQ}$$

$$= x\hat{i} + y\hat{j}$$

again $\vec{OP} = \vec{OQ} + \vec{QP}$

$$= \left(x\hat{i} + y\hat{j}\right) + z\hat{k}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

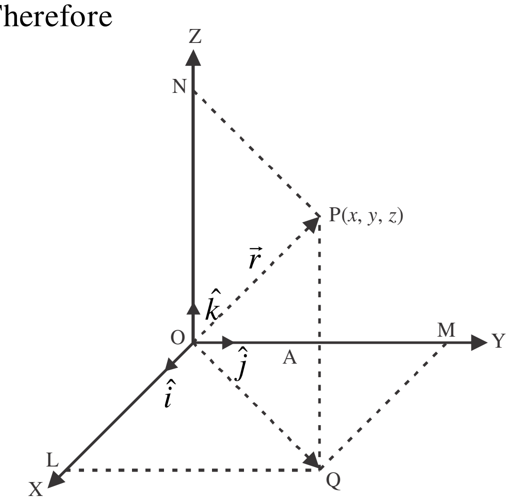


Fig. 13.12

Thus with respect to O we get the position vector of P i.e. $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

This is known as the component form of the vector where x, y and z *are the scalar components of* $\overrightarrow{OP}$ and $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are the vector components of $\overrightarrow{OP}$ Some times x, y and z are also termed as *rectangular components*.

If $\quad \overline{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$\overline{OP} = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

13.09 Vector joining two points

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$ (Fig. 13.12). Joining the points P_1 and P_2 with the origin O, and applying triangle law, from the triangle OP_1P_2, $\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$ we have

Using the properties of vector addition, the above equation becomes

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

i.e. $$\overrightarrow{P_1P_2} = \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_2\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

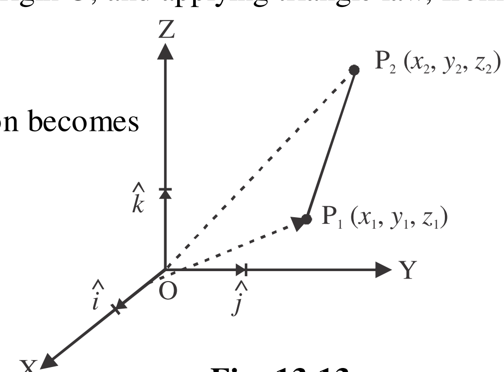


Fig. 13.13

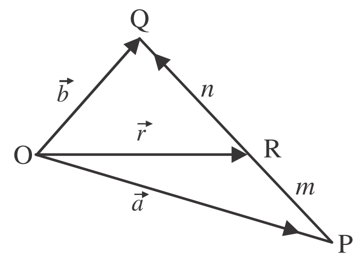
The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

13.10 Section Formula

Let P and O be two points represented by the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ with respect to the origin O. Then the line segment joining the points P and Q may be divided by a third point, say R, in two ways-internally and externally. (Fig. 13.10 (a) and Fig. 13.10 (b)). Here, we intend to find the position vector $\overrightarrow{OR}$ for the point R with respect tot he origin O. We take the two cases one by one.

Case& I: When R divides PQ internally



Fi.g 13.14 (a)

Let R, divides $\overline{PQ}$ internally in the ratio $m : n$ (Fig. 13.13(a))

$$\frac{PR}{RQ} = \frac{m}{n}$$

$\Rightarrow$ $$nPR = mRQ$$

$\Rightarrow$ $$n\overrightarrow{PR} = m\overrightarrow{RQ}$$

$\Rightarrow$ n (*position vector of R –position vector of* P) = m (*position vector of Q –position vector of* R)

$\Rightarrow$ $$n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$$

$\Rightarrow$ $$(m+n)\vec{r} = m\vec{b} + n\vec{a}$$

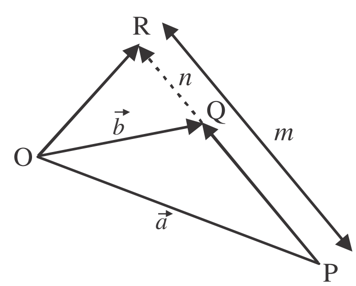
$\Rightarrow$ $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Here, the position vector of the point R which divides P and Q internally in the ratio of $m : n$ is given by

$$\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Case& II: When R, divides PQ externally:

Let the position vector of the point R which divides the line segment PQ externally in the ratio $m : n$ (Fig. 13.14(b) then



Fi.g 13.14 (b)

$$\frac{PR}{QR} = \frac{m}{n}$$

$\Rightarrow$ $$nPR = mQR$$

$\Rightarrow$ $$n\overrightarrow{PR} = m\overrightarrow{QR}$$

$\Rightarrow$ n (*Position vector of R –Position vector of* P) = m (*Position vector of R – Position vector of* Q)

$\Rightarrow$ $$n(\vec{r} - \vec{a}) = m(\vec{r} - \vec{b})$$

$\Rightarrow$ $$m\vec{b} - n\vec{a} = m\vec{r} - n\vec{r}$$

$\Rightarrow$ $$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Note: if R, is the midpoint of PQ, then $m : n$. And therefore, from Case I, the midpoint R of $\overrightarrow{PQ}$, will have its position vector as $\overrightarrow{OR} = \dfrac{\vec{a}+\vec{b}}{2}$.

Illustrative Examples

Example 1. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Solution : The sum of the vectors $= \vec{a} + \vec{b} + \vec{c}$

$$= \left(\hat{i} - 2\hat{j} + \hat{k}\right) + \left(-2\hat{i} + 4\hat{j} + 5\hat{k}\right) + \left(\hat{i} - 6\hat{j} - 7\hat{k}\right)$$

$$= \left(\hat{i} - 2\hat{j} + \hat{k}\right) + \left(-2\hat{i} + 4\hat{j} - 6\hat{k}\right) + \left(\hat{i} + 5\hat{j} - 7\hat{k}\right)$$

$$= 0 - \hat{i} - 4\hat{j} - \hat{k} = -4\hat{j} - \hat{k}$$

Example 2. If vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal then find the value of x, y and z.

Solution : Two vectors are equal if their scalar components are equal.

Thus if $\vec{a}$ and $\vec{b}$ are equal if $x = 2,\ y = 2,\ z = 1$

Example 3. Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$ then is $|\vec{a}| = |\vec{b}|$? Are vector $\vec{a}$ and $\vec{b}$ equal?

Solution : Here $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

Therefore $|\vec{a}| = |\vec{b}|$ But the given vectors are not equal becase their corresponding components are not equal.

Example 4. Find the unit vector in the direction of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$.

Solution : The unit vector along vector $\vec{a}$ is $\hat{a} = \dfrac{1}{|\vec{a}|}\vec{a}$.

now $|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

therefore $\hat{a} = \dfrac{1}{\sqrt{14}}\left(2\hat{i} + 3\hat{j} + \hat{k}\right) = \dfrac{2}{\sqrt{14}}\hat{i} + \dfrac{3}{\sqrt{14}}\hat{j} + \dfrac{1}{\sqrt{14}}\hat{k}$

Example 5. Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ which has magnitude 7 units.

Solution : The unit vector along vector $\vec{a}$ is $\hat{a} = \dfrac{1}{|\vec{a}|}\vec{a} = \dfrac{1}{\sqrt{5}}\left(\hat{i} - 2\hat{j}\right) = \dfrac{1}{\sqrt{5}}\hat{i} - \dfrac{2}{\sqrt{5}}\hat{j}$

therefore the vector along $\vec{a}$ having magnitude 7 unit $7\hat{a} = 7\left(\dfrac{1}{\sqrt{5}}\hat{i} - \dfrac{2}{\sqrt{5}}\hat{j}\right) = \dfrac{7}{\sqrt{5}}\hat{i} - \dfrac{14}{\sqrt{5}}\hat{j}$

Example 6. Find the unit vector in the direction of the vector $\vec{a}=2\hat{i}+2\hat{j}-5\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}+3\hat{k}$.

Solution : The sum of the given vectors

$$\vec{a}+\vec{b}=\vec{c} \text{ (let) } \therefore \ \vec{c}=4\hat{i}+3\hat{j}-2\hat{k}$$

and $$|\vec{c}|=\sqrt{4^2+3^2+(-2)^2}=\sqrt{29}$$

Required unit vector

$$\hat{c}=\frac{1}{|\vec{c}|}\vec{c}=\frac{1}{\sqrt{29}}\left(4\hat{i}+3\hat{j}-2\hat{k}\right)=\frac{4}{\sqrt{29}}\hat{i}+\frac{3}{\sqrt{29}}\hat{j}-\frac{2}{\sqrt{29}}\hat{k}$$

Example 7. Find the vector directed from point P to Q joining the points P(2, 3, 0) and Q(–1, –2, –4).

Solution: As P is the initial point and Q is the terminal point, therefore

$$\overrightarrow{PQ}=\text{Position vector of Q–Position vector of P}$$

$$\overrightarrow{PQ}=-i-2j-4k-(2i+3j)$$

$$\overrightarrow{PQ}=(-1-2)\hat{i}+(-2-3)\hat{j}+(-4-0)\hat{k}$$

$$\Rightarrow \quad \overrightarrow{PQ}=-3\hat{i}-5\hat{j}-4\hat{k}$$

Example 8. Find the position vector of a point R which divides the line joining two points P and Q in ratio 2 : 1 whose position vectors are $\overrightarrow{OP}=3\vec{a}-2\vec{b}$ and $\overrightarrow{OQ}=\vec{a}+\vec{b}$.

Solution : (i) the position vector of a point R which divides the line joining two points P and Q in the ratio 2 : 1 internally is

$$\overrightarrow{OR}=\frac{2\left(\vec{a}+\vec{b}\right)+\left(3\vec{a}-2\vec{b}\right)}{3}=\frac{5\vec{a}}{3}$$

(ii) the position vector of a point R whcih divides the line joining two points P and Q in the ratio 2 : 1 externally is

$$\overrightarrow{OR}=\frac{2\left(\vec{a}+\vec{b}\right)-\left(3\vec{a}-2\vec{b}\right)}{2-1}=4\vec{b}-\vec{a}$$

Example 9. Show that the points $A\left(2\hat{i}-\hat{j}+\hat{k}\right)$, $B\left(\hat{i}-3\hat{j}-5\hat{k}\right)$, $C\left(3\hat{i}-4\hat{j}-4\hat{k}\right)$ are the vertices of a right angled triangle.

Solution : We have $\overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$

$$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$$

and $$\overrightarrow{CA}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$$

Further, note that $|\overrightarrow{AB}|^2=41=6+35=|\overrightarrow{BC}|^2+|\overrightarrow{CA}|^2$

Here, the triangle is a right angled triangle.

Exercise 13.1

1. Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k};\ \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k};\ \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

2. Write two different vectors having same magnitude.
3. Write two different vectors having same direction.
4. Find the vlaues of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.
5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7)
6. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k};\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.
7. Find the unit vector in the direction of the vector $\vec{c} = \hat{i} + \hat{j} + 2\hat{k}$.
8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$ where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.
9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.
10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.
12. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $P\left(\hat{i} + 2\hat{j} - \hat{k}\right)$ and $Q\left(-\hat{i} + \hat{j} + \hat{k}\right)$ respectively, in the ratio 2 : 1

    (i) internally (ii) externally.
13. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).
14. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$ respectively form the vertices of a right angled triangle.

13.11 Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector.

(I) Scalar product: In this the product of two vectors is a Scalar.

(II) Vector product: In this the product of two vectors is a vector.

13.12 Scalar or dot Product

Definition : If product of two vectors is a scalar quantity then it is called 'scalar or dot-product of vector'.

The scalar product of two non zero vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\cdot\vec{b}$ (read as $\vec{a}$ dot $\vec{b}$) is defined as:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$$

($|\vec{a}| = a$ and $|\vec{b}| = b$ are the magnitudes of $\vec{a}$ and $\vec{b}$)

Note: When both the vectors are Unit vectors, i.e. $|\vec{a}|=1$, $|\vec{b}|=1$

$$\hat{a}.\hat{b} = (1)(1)\cos\theta = \cos\theta$$

13.13 Geometrical interpretation of Scalar Product

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ are two vectors, inclined at an angle θ, the scalar product is given by

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

$$= |\vec{a}||\vec{b}|\cos\theta \qquad (1)$$

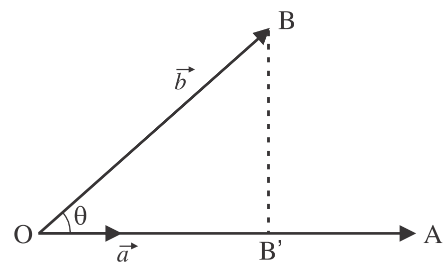


Fig. 13.15

Now from point A and B drop perpendicualr AM and BN on OB and OA then from ΔOMA and ΔONB

$OM = OA\cos\theta$ i.e. projection of $\overline{OM}$ in the direction of $\overline{OB}$

$ON = OB\cos\theta$ i.e projection of $\overline{OB}$ in the direction of $\overline{OA}$

From equation (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}|(|\vec{b}|\cos\theta) = |\vec{a}|(ON)$$

$$= (\text{magnitude of } \vec{a})(\text{projection of } \vec{b} \text{ on } \vec{a}) \qquad (2)$$

Similarly from equation (1)

$$\vec{a}\cdot\vec{b} = |\vec{b}|(|\vec{a}|\cos\theta) = |\vec{b}|(OM)$$

$$= (\text{magnitude of } \vec{b})(\text{projection of } \vec{a} \text{ on } \vec{b}) \qquad (3)$$

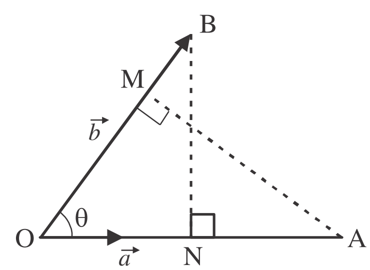


Fig. 13.16

Thus the scalar product of two vectors is the product of modulus of either vector and the project of the other in its direction.

Note: from (2) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$

and from (3) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$

13.14 Some Important Deductions from Scalar Product of Vectros

We know that

$$\vec{a}\cdot\vec{b} = ab\cos\theta \qquad (1)$$

Observeations:

(i) When vectors $\vec{a}$ and $\vec{b}$ are parallel: In this condition the value of $\theta = 0^\circ$, thus from (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 0^\circ = |\vec{a}||\vec{b}| = ab$$

(ii) When vectors $\vec{a}$ and $\vec{b}$ coincides: In this condition the angle between the two vectors is zero i.e. $\theta = 0^\circ$, thus from (1)

$$\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0^\circ = |\vec{a}||\vec{a}| = aa = a^2$$

(iii) When vectors $\vec{a}$ and $\vec{b}$ are linear: In this condition the angle between the two vectors is 180° i.e. $\theta = 180^\circ$ thus from (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 180^\circ = ab(-1) = -ab$$

(iv) When vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicualr: In this condition the angle between the two vectors is 90° i.e. $\theta = \pi/2$ thus from (1)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = |\vec{a}||\vec{b}|0 = 0$$

thus if two vectors are perpendicualr then

$$\vec{a}\cdot\vec{b} = 0$$

Converse: If the scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is zero then the vectors are perpendicular let

let $\qquad \vec{a}\cdot\vec{b} = 0$

$\Rightarrow \qquad |\vec{a}||\vec{b}|\cos\theta = 0$

$\Rightarrow \qquad \cos\theta = 0 \qquad \left[\because |\vec{a}| \neq 0, |\vec{b}| \neq 0\right]$

$\Rightarrow \qquad \theta = \pi/2 \qquad \Rightarrow \qquad \vec{a} \perp \vec{b}$

So $\qquad \vec{a}\cdot\vec{b} = 0 \qquad \Leftrightarrow \qquad \vec{a} \perp \vec{b}$

Note: In view of the observations, for mutually perpendicualr unit vectors $\hat{i}, \hat{j}, \hat{k}$ we have

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

and $\qquad \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

The above result can be expressed in the form of a table also

$\cdot$	i	j	k
i	1	0	0
j	0	1	0
k	0	0	1

13.15. Properties of Scalar Product

(i) **Commutativity:** Thus scalar product of two vector sis commutative.

Proof : We know that

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$
$$= b\,a\cos\theta \qquad (\because ab = ba,)$$
$$= \vec{b}\cdot\vec{a}$$

(ii) **Associativity:** If $\vec{a}$ and $\vec{b}$ are two vectors then let m be any scalar

$$(m\vec{a})\cdot\vec{b} = \vec{a}\cdot(m\vec{b}) = m(\vec{a}\cdot\vec{b})$$

(iii) **Distributivity:** If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors then

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

similarly $\quad (\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$

13.16 Scalar Product of Two Vectors in terms of the Components

Let $\quad \vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, are two vectors

$$\vec{a}\cdot\vec{b} = \left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$$
$$= a_1b_1(\hat{i}\cdot\hat{i})+a_1b_2(\hat{i}\cdot\hat{j})+a_1b_3(\hat{i}\cdot\hat{k})+a_2b_1(\hat{j}\cdot\hat{i})+a_2b_2(\hat{j}\cdot\hat{j})$$
$$+a_2b_3(\hat{j}\cdot\hat{k})+a_3b_1(\hat{k}\cdot\hat{i})+a_3b_2(\hat{k}\cdot\hat{j})+a_3b_3(\hat{k}\cdot\hat{k}) \quad \text{(from property (ii) and (iii))}$$
$$= a_1b_1+a_2b_2+a_3b_3 \qquad \text{(Article 13.15)}$$

$\therefore \quad \vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

Note: $\quad \vec{a}\cdot\vec{a} = \left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)$

$$= a_1a_1+a_2a_2+a_3a_3 = a_1^{\,2}+a_2^{\,2}+a_3^{\,2} = a^2$$

$\therefore \quad (\vec{a})^2 = a^2$

13.17 Angle Between two Vectors:

We know by the definition of scalar product

$$\vec{a}\cdot\vec{b} = a\,b\cos\theta$$

or $\quad \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab} = \left(\dfrac{\vec{a}}{a}\right)\cdot\left(\dfrac{\vec{b}}{b}\right) = \hat{a}\cdot\hat{b}$, where $\hat{a},\hat{b}$ are the unit vectors in the direction of $\vec{a}$ and $\vec{b}$

again if $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$$\vec{a}\cdot\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$
$$= a_1b_1+a_2b_2+a_3b_3 \qquad \text{(Article 13.16)}$$

$\therefore \quad \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

Note: if vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicualr then $a_1b_1 + a_2b_2 + a_3b_3 = 0$

13.18 Components of any Vector $\vec{b}$ along and perpendicular to a Vector $\vec{a}$

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $BM \perp OA$.

$\therefore$ by triangle law of addition in ΔOBM $\vec{b} = \overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$, where $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are the perpendicualr vectors of vector $\vec{b}$ along vector $\vec{a}$

Now $\overrightarrow{OM} = (OM)\hat{a} = (b\cos\theta)\hat{a}$

$= b\left(\dfrac{\vec{a}\cdot\vec{b}}{ab}\right)\hat{a}$ (Article 13.17)

$= \left(\dfrac{\vec{a}\cdot\vec{b}}{a}\right)\hat{a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$ $\left[\because \hat{a} = \dfrac{\vec{a}}{a}\right]$

and $\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM}$

$= \vec{b} - \left(\dfrac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$

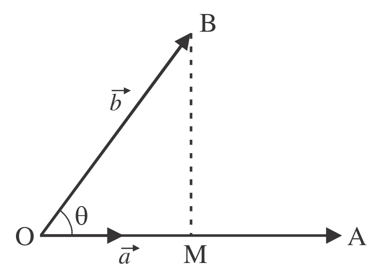


Fig. 13.17

Thus components of vector $\vec{b}$ in the direction of vector $\vec{a}$ and perpendicular along $\vec{a}$ are $\left(\dfrac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$ and $\vec{b} - \left(\dfrac{\vec{a}\cdot\vec{b}}{a^2}\right)\vec{a}$

Illustrative Examples

Example 10. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$ then find the value of $\vec{a}\cdot\vec{b}$.

Solution: $\vec{a}\cdot\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k})\cdot(3\hat{i} + 2\hat{j} + \hat{k})$

$= (1)(3) + (2)(2) + (3)(1) = 3 + 4 + 3 = 10$

Thus the value of $\vec{a}\cdot\vec{b}$ is 10.

Example 11. For what value of λ are the vectors $2\hat{i} + \lambda\hat{j} + 5\hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ mutually perpendicular?

Solution: the vectors are perpendicular if their product is zero

$(2\hat{i} + \lambda\hat{j} + 5\hat{k})\cdot(-\hat{i} + \hat{j} + \hat{k}) = 0$

or $(2)(-1) + (\lambda)(1) + (5)(1) = 0$

or $2 + \lambda + 5 = 0$

or $\lambda = -3$

Thus at $\lambda = -3$ the vectors are perpendicular to each other.

Example 12. Find the angle between the vectors $3\hat{i}+\hat{j}+3\hat{k}$ and $2\hat{i}+2\hat{j}-\hat{k}$.

Solution: Let $\vec{a}=3\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+2\hat{j}-\hat{k}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a}\cdot\vec{b}=ab\cos\theta$$

$$\Rightarrow \quad \cos\theta=\frac{\vec{a}\cdot\vec{b}}{ab}=\frac{(3\hat{i}+\hat{j}+3\hat{k})\cdot(2\hat{i}+2\hat{j}-\hat{k})}{\sqrt{9+1+9}\sqrt{4+4+1}}$$

$$=\frac{(3)(2)+(1)(2)+(3)(-1)}{\sqrt{19}\sqrt{9}}=\frac{5}{3\sqrt{19}}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{5}{3\sqrt{19}}\right)$$

Example 13. Show that-

(i) $(\vec{a}+\vec{b})=a^2+2\vec{a}\cdot\vec{b}+b^2$

and (ii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=a^2-b^2$

Solution: (i) $(\vec{a}+\vec{b})^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$$=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=a^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}+b^2 \qquad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$$

$$=a^2+2\vec{a}\cdot\vec{b}+b^2$$

(ii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}$

$$=a^2-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}-b^2$$

$$=a^2-b^2 \qquad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$$

Example 14. If θ is the angle between the two vectors $\hat{a}$ and $\hat{b}$ then prove that

$$\sin(\theta/2)=\frac{1}{2}|\hat{a}-\hat{b}|$$

Solution : $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$$=\hat{a}.\hat{a}-\hat{a}.\hat{b}-\hat{b}.\hat{a}+\hat{b}\hat{b}$$

$$=|\hat{a}|^2-2\hat{a}.\hat{b}+|\hat{b}|^2 \qquad \left[\because \hat{a}.\hat{b}=\hat{b}.\hat{a}\right]$$

$$=1-2\hat{a}\cdot\hat{b}+1 \qquad \left[\because |\hat{a}|=1=|\hat{b}|\right]$$

$$=2-2(1)(1)\cos\theta=2(1-\cos\theta)$$

$$=2\cdot\left(2\sin^2\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \left|\hat{a}-\hat{b}\right| = 2\sin\frac{\theta}{2} \quad \text{or} \quad \sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}-\hat{b}\right|$$

Example 15. (i) If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicualr vectors with equal magnitudes, then prove that vector $\vec{a}+\vec{b}+\vec{c}$ makes equal angle with vectors $\vec{a},\vec{b}$ and $\vec{c}$.

(ii) $\vec{a},\vec{b},\vec{c}$ are the vectors of magnitude 3, 4, 5 resp. If every vector is perpendicular on the sum of the other two then find the magnitude of vector $\vec{a}+\vec{b}+\vec{c}$.

Solution: (i) $\vec{a},\vec{b},\vec{c}$ are mutually perpendicualr therefore $\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$

again the magnitude of vectors $\vec{a},\vec{b},\vec{c}$ are equal $a=b=c$

and $\quad \left(\vec{a}+\vec{b}+\vec{c}\right)^2 = \left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}$$

$$= a^2+b^2+c^2 = 3a^2 \qquad \left[\because a=b=c \text{ तथा } \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0 \text{ इत्यादि}\right]$$

$$\Rightarrow \quad \left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{3}a$$

$$\because \quad (\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a} = a^2$$

Let θ_1 be the angle between $\vec{a}+\vec{b}+\vec{c}$ and $\vec{a}$

$$\therefore \quad (\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} = |\vec{a}+\vec{b}+\vec{c}||\vec{a}|\cos\theta_1$$

$$\Rightarrow \quad a^2 = (\sqrt{3}a)(a)\cos\theta_1$$

$$\Rightarrow \quad \cos\theta_1 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly if vector $\vec{a}+\vec{b}+\vec{c}$ makes angle θ_2 and θ_3 with $\vec{b}$ and $\vec{c}$ then it can be proved that $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

i.e. vector $\vec{a}+\vec{b}+\vec{c}$ makes equal angle with the vectors $\vec{a},\vec{b}$ and $\vec{c}$

(ii) $\quad \vec{a}\cdot\left(\vec{b}+\vec{c}\right)=0,\ \vec{b}\cdot\left(\vec{a}+\vec{c}\right)=0$ and $\vec{c}\cdot\left(\vec{a}+\vec{b}\right)=0$

adding all the three $2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=0$

and $\quad \vec{a}\cdot\vec{a}=a^2=9,\ b^2=16,\ c^2=25$

$$\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)$$

$$\Rightarrow \qquad |\vec{a}+\vec{b}+\vec{c}|^2=9+16+25+0=50$$

$$\Rightarrow \qquad \left|\vec{a}+\vec{b}+\vec{c}\right|=\sqrt{50}=5\sqrt{2} \text{ units}$$

Exercise 13.2

1. If the magnitude of two vectors is 4 and 5 units then find their scalar product if the angle between the two vectors is

   (i) 60^o (ii) 90^o (iii) 30^o

2. Find the value of $\vec{a}.\vec{b}$ if $\vec{a}$ and $\vec{b}$ respectively are

   (i) $2\hat{i}+5\hat{j};3\hat{i}-2\hat{j}$ (ii) $4\hat{i}+3\hat{k};\hat{i}-\hat{j}+\hat{k}$ (iii) $5\hat{i}+\hat{j}-2\hat{k};\,2\hat{i}-3\hat{j}$

3. Prove that $(\vec{a}\cdot\vec{b})^2\leq|\vec{a}|^2\,|\vec{b}|^2$

4. If the coordinates of P and Q are (3, 4) and (12, 9) respectively. Find the value of $\angle POQ$ where O is the origin.

5. For what value of λ are the vectors $\vec{a}$ and $\vec{b}$ mutually perpendicular.

   (i) $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k};\quad \vec{b}=4\hat{i}-2\hat{j}-2\hat{k}$ (ii) $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k};\quad \vec{b}=3\hat{i}+2\hat{j}-\lambda\hat{k}$

6. Find the projection of vector $4\hat{i}-2\hat{j}+\hat{k}$ on the vector $3\hat{i}+6\hat{j}-2\hat{k}$.

7. If $\vec{a}=2\hat{i}-16\hat{j}+5\hat{k}$ and $\vec{b}=3\hat{i}+\hat{j}+2\hat{k}$ then find a vector $\vec{c}$ where $\vec{a},\vec{b},\vec{c}$ denote the sides of right angle triangle.

8. If $\left|\vec{a}+\vec{b}\right|=\left|\vec{a}-\vec{b}\right|$, then prove that $\vec{a}$ and $\vec{b}$ are mutually perpendicular to each other.

9. If the coordinates of the points A, B, C and D are $(3, 2, 4)$, $(4, 5, -1)$, $(6, 3, 2)$ and $(2, 1, 0)$ respectively Then prove that lines AB and CD are mutually perpendicular.

10. For any vector $\vec{a}$ prove that $\vec{a}=(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}$

11. Using the vector method Prove that sum of the diagonals of the parallelogram is equal to the sum of square of its sides.

13.19 Vector or Cross Product of two Vectors

Definition : The vector product of two non zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and defined as

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}, \qquad (1)$$

If the angle between $\vec{a}$ and $\vec{b}$ is $\theta\left(0\leq\theta\leq\pi\right)$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b}$ and $\hat{n}$ form a right handed screw system i.e., the right handed screw system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.

In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta \quad \Rightarrow \quad \sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \qquad (2)$$

from (1) $$\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}||\vec{b}|\sin\theta}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

thus the unit vector perpendicualr to vector $\vec{a}$ and $\vec{b}$ is $=\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ (3)

13.20 Geometrical Interpretation of Vector Product

Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}$ are two non parallel and non-zero vectors, the angle between them is θ and $\hat{n}$ is the unit vector perpendicular to vectors $\vec{a}$ and $\vec{b}$ then,

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$$

$$=(OA)(OB)\sin\theta \qquad (1)$$

Area of $OACB$

Cosnidering OA and OB as the sides of the parallelogram $OACB$,
Area of $OACB$ = 2 (Area of ΔOAB)

$$=2\left(\frac{1}{2}OA\cdot OB\sin\theta\right)=OA\cdot OB\sin\theta \qquad (2)$$

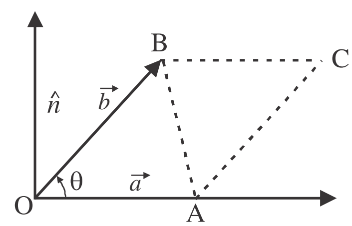


Fig. 13.18

from (1) and (2) the magnitude of $\vec{a}\times\vec{b}=\left|\vec{a}\times\vec{b}\right|$

13.21 Some Important Deductions from Vector Product

(i) The product of two parallel vectors is always zero:

Proof : If $\vec{a}$ and $\vec{b}$ are two parallel vectors and let θ be the angle between them then $\theta=0^{\circ}$ or $\theta=\pi^{\circ}$ thus in both the situations the value of the $\sin\theta$ will be zero.

$\therefore \quad \vec{a}\times\vec{b}=ab\sin\theta\hat{n}=(0)\hat{n}=\vec{O}$ [zero vector]

Converse : If the product of two vectors is zero then the vectors are parallel as

$\vec{a}\times\vec{b}=\vec{O}, \quad \Rightarrow \quad ab\sin\theta\hat{n}=\vec{O} \quad \Rightarrow \quad \sin\theta=0$ $[\because a\neq 0, b\neq 0]$

$\Rightarrow \quad \theta=0$ या $\theta=\pi$

i.e. $\vec{a}$ and $\vec{b}$ are parallel vectors

Note: (i) $\vec{a}\times\vec{a}=\vec{O}$, (ii) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{O}$

(ii) The magnitude of product of two vectors is equal to the product of the two magnitude of the two vectors.

Proof : If $\vec{a}$ and $\vec{b}$ are two perpendicualr vectors then $\theta = 90^\circ$.

$$\therefore \quad \vec{a}\times\vec{b} = \left(ab\sin 90^\circ\right)\hat{n}$$

$$= \left(ab\right)\hat{n}$$

$$\Rightarrow \quad \left|\vec{a}\times\vec{b}\right| = ab$$

Magnitude of vectors $\vec{a}\times\vec{b}$ = (magnitude of $\vec{a}$) (magnitude of $\vec{b}$ u)] Here $\hat{n}$, is a unit vector along $\vec{a}$ and $\vec{b}$ and obeys the left hand rule.

Special Condition :

$$\hat{i}\times\hat{j} = (1)(1)\sin 90^\circ \hat{k} = \hat{k}$$

similarly $\hat{j}\times\hat{k} = \hat{i}$ and $\hat{k}\times\hat{i} = \hat{j}$

again $\hat{j}\times\hat{i} = -\hat{k}$ (opposite to $\hat{i}\times\hat{j}$)

similarly $\hat{k}\times\hat{j} = -\hat{i}$ and $\hat{i}\times\hat{k} = -\hat{j}$

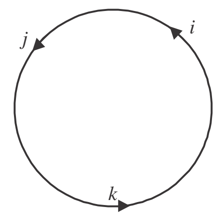


Fig. 13.19

This can be understood by the fig. 13.19.

13.22 Algebraic Properties of Vector Product

(i) Commutativity: Vector product is not commutative i.e.

$$\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$$

(ii) Associativity: Vector product is associative with respect to any scalar m i.e.

$$m(\vec{a}\times\vec{b}) = (m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b})$$

(iii) Distributivity: Vector product obyes the distributive law:

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

13.23 Vector Product of two Vectors in Terms of Components

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors then

$$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1(\hat{i}\times\hat{i}) + a_1b_2(\hat{i}\times\hat{j}) + a_1b_3(\hat{i}\times\hat{k}) + a_2b_1(\hat{j}\times\hat{i})$$

$$+a_2b_2(\hat{j}\times\hat{j}) + a_2b_3(\hat{j}\times\hat{k}) + a_3b_1(\hat{k}\times\hat{i}) + a_3b_2(\hat{k}\times\hat{j}) + a_3b_3(\hat{k}\times\hat{k})$$

$$= a_1b_1(\vec{0}) + a_1b_2(\vec{k}) + a_1b_3(-\hat{j}) + a_2b_1(-\hat{k}) + a_2b_2(\vec{0}) + a_2b_3(\hat{i}) + a_3b_1(\hat{j}) + a_3b_2(-\hat{i}) + a_3b_3(\vec{0})$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

which is a determinant form of $\vec{a}\times\vec{b}$.

13.24 Angle between two Vectors

If θ is the angle between $\vec{a}$ and $\vec{b}$

$$\vec{a}\times\vec{b} = ab\sin\theta\hat{n}$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = |ab\sin\theta||\hat{n}| = ab|\sin\theta||\hat{n}|$$

$$\Rightarrow \quad \sin^2\theta = \frac{|\vec{a}\times\vec{b}|^2}{(a^2)(b^2)}$$

$$= \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1{}^2 + a_2{}^2 + a_3{}^2)(b_1{}^2 + b_2{}^2 + b_3{}^2)}$$

13.25 Vector area of a Triangle

(i) If $\vec{a}$ and $\vec{b}$ are the sides of the triangle

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ then $\vec{a}\times\vec{b} = ab\sin\theta\hat{n}$

Now area of $(\Delta OAB) = \frac{1}{2}ab\sin\theta\hat{n} = \frac{1}{2}(\vec{a}\times\vec{b})$,

here $\hat{n}$ is the unit vector

Note: Now area of $\Delta OBA = \frac{1}{2}\left(\vec{b}\times\vec{a}\right) = -\frac{1}{2}\left(\vec{a}\times\vec{b}\right)$

(ii) If the position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ of triangle ABC are given

The sides of ΔABC, AB and AC

$$\overrightarrow{AB} = \vec{b} - \vec{a} \quad \text{and} \quad \overrightarrow{AC} = \vec{c} - \vec{a}$$

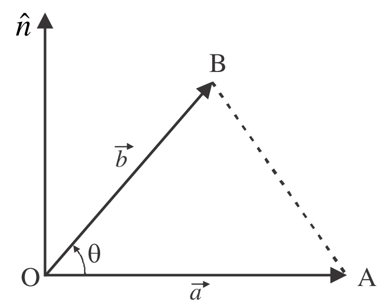


Fig. 13.20

$\therefore$ Area of triangle ΔABC $= \frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$

$$= \frac{1}{2}\left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right]$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{a}\times\vec{a}\right]$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} + \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right] \qquad \left[\because \vec{a}\times\vec{a} = \vec{O}\right]$$

$$= \frac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$$

13.26 Condition of Collinearity of Three points

If points A, B and C are collinear then the Area of triangle will be zero.

Let the position vectors of ΔABC are $\vec{a}, \vec{b}$ and $\vec{c}$, therefore area fo ΔABC $= 0$

$$\Rightarrow \quad \frac{1}{2}\left(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)=0$$

$$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$$

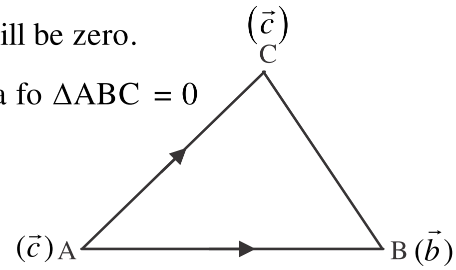


Fig. 13.21

Illustrative Examples

Example 16. Find the value of $(2\hat{i}-3\hat{j}+4\hat{k})\times(3\hat{i}+4\hat{j}-4\hat{k})$.

Solution : $(2\hat{i}-3\hat{j}+4\hat{k})\times(3\hat{i}+4\hat{j}-4\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 4\\ 3 & 4 & -4\end{vmatrix}$

$$=(12-16)\hat{i}+(12+8)\hat{j}+(8+9)\hat{k} = -4\hat{i}+20\hat{j}+17\hat{k}$$

thus required value $-4\hat{i}+20\hat{j}+17\hat{k}$

Example 17. If $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+2\hat{k}$ then find the unit vector $\hat{n}$ perpendicular to vectors $\vec{a}$ and $\vec{b}$.

Solution : By the definition of vector product

$$\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$=\frac{(3\hat{i}+\hat{j}+2\hat{k})\times(2\hat{i}-2\hat{j}+2\hat{k})}{\left|(3\hat{i}+\hat{j}+2\hat{k})\times(2\hat{i}-2\hat{j}+2\hat{k})\right|}$$

again $\quad (3\hat{i}+\hat{j}+2\hat{k})\times(2\hat{i}-2\hat{j}+2\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 2\\ 2 & -2 & 2\end{vmatrix}$

$$=(2+4)\hat{i}+(4-6)\hat{j}+(-6-2)\hat{k}$$

$$=6\hat{i}-2\hat{j}-8\hat{k}$$

$$\hat{n}=\frac{6\hat{i}-2\hat{j}-8\hat{k}}{\left|6\hat{i}-2\hat{j}-8\hat{k}\right|}$$

$$=\frac{6\hat{i}-2\hat{j}-8\hat{k}}{\sqrt{36+4+64}}=\frac{6\hat{i}-2\hat{j}-8\hat{k}}{\sqrt{104}}$$

$$= \frac{3\hat{i} - \hat{j} - 4\hat{k}}{\sqrt{26}}$$, which is the required solution

Thus the required perpendicular unit vector is $\frac{1}{\sqrt{26}}(3\hat{i} - \hat{j} - 4\hat{k})$.

Example 18. If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, then Prove that $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ are parallel.

Solution :
$$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = (\vec{a}\times\vec{b} - \vec{a}\times\vec{c}) - (\vec{d}\times\vec{b} - \vec{d}\times\vec{c})$$
$$= \vec{a}\times\vec{b} - \vec{a}\times\vec{c} + \vec{b}\times\vec{d} + (-\vec{c})\times\vec{d}$$
$$= (\vec{a}\times\vec{b} - \vec{c}\times\vec{d}) + (\vec{b}\times\vec{d} - \vec{a}\times\vec{c})$$
$$= \vec{O} + \vec{O} = \vec{O}$$

$\therefore$ $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ are parallel vectors

Example 19. If $\vec{a}\times\vec{b} = \vec{c}\times\vec{b}$ then Prove that $\vec{a}-\vec{c} = \lambda\vec{b}$, where λ is a scalar

Solution: $\vec{a}\times\vec{b} = \vec{c}\times\vec{b}$

$\Rightarrow$ $\vec{a}\times\vec{b} - \vec{c}\times\vec{b} = 0$

$\Rightarrow$ $(\vec{a}-\vec{c})\times\vec{b} = 0$

$\therefore$ $\vec{a}-\vec{c}$ and $\vec{b}$ are parallel therefore $\vec{a}-\vec{c} = \lambda\vec{b}$, where λ is a scalar

Note: (i) If $\vec{a}-\vec{c}$ and $\vec{b}$ are in the same direction then λ is positive

(ii) If $\vec{a}-\vec{c}$ and $\vec{b}$ are opposite then λ is negative

Example 20. If $A(1, 2, 2)$, $B(2, -1, 1)$ and $C(-1, -2, 3)$ are any three points in a plane then find a vector perpendicualr to the plane ABC whose magnitude is 5 units.

Solution :
$\overrightarrow{AB}$ = (position vector of B) – (position vector of A)
$$= (2\hat{i} - \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 2\hat{k})$$
$$= \hat{i} - 3\hat{j} - \hat{k}$$

and $\overrightarrow{AC}$ = (position vector of C) – (position vector of A)
$$= (-\hat{i} - 2\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 2\hat{k})$$
$$= -2\hat{i} - 4\hat{j} + \hat{k}$$

$\because$ $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both are in plane ABC thus vectors $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicualr to the plane

therefore $\overrightarrow{AB}\times\overrightarrow{AC} = (\hat{i} - 3\hat{j} - \hat{k})\times(-2\hat{i} - 4\hat{j} + \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -1 \\ -2 & -4 & 1 \end{vmatrix}$$

$$= -7\hat{i} + \hat{j} - 10\hat{k}$$

Unit vector perpendicualr to the plane ABC

$$\hat{n} = \frac{-7\hat{i} + \hat{j} - 10\hat{k}}{\sqrt{49+1+100}} = \frac{-1}{\sqrt{150}}\left(7\hat{i} - \hat{j} + 10\hat{k}\right)$$

magnitude of the vector with 5 in the direction perpendicualr to it is

$$= 5\left[\frac{-1}{\sqrt{150}}\left(7\hat{i} - \hat{j} + 10\hat{k}\right)\right] = \frac{-1}{\sqrt{6}}\left(7\hat{i} - \hat{j} + 10\hat{k}\right)$$

Example 21. Prove that the Area of rectangle $ABCD$ is $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$ where AC and BD are the diagonals.

Solution: Area of rectangle $ABCD$ = Area of ΔACD + Area of ΔABC

$$= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \frac{1}{2}\left[\overrightarrow{AC} \times \overrightarrow{AD} - \overrightarrow{AC} \times \overrightarrow{AB}\right]$$

$$= \frac{1}{2}\left[\overrightarrow{AC} \times \left(\overrightarrow{AD} - \overrightarrow{AB}\right)\right] = \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$$

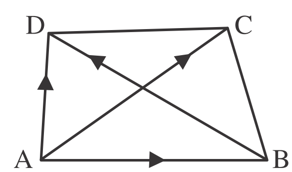


Fig. 13.22

Thus Area of Rectangle $= \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$

Exercise 13.3

1. Find the vector product of $3\hat{i} + \hat{j} - \hat{k}$ and $2\hat{i} + 3\hat{j} + \hat{k}$.

2. Find the unit vector perpendicualr to the vectors $\hat{i} - 2\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$.

3. For vectors $\vec{a}$ and $\vec{b}$ Prove that $\left(\vec{a} \times \vec{b}\right)^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

4. Prove that $\vec{a} \times \left(\vec{b} + \vec{c}\right) + \vec{b} \times \left(\vec{c} + \vec{a}\right) + \vec{c} \times \left(\vec{a} + \vec{b}\right) = 0$.

5. If $\hat{a}, \hat{b}, \hat{c}$ are the unit vectors such that $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = 0$ and the angle between $\hat{b}$ and $\hat{c}$ is $\pi/6$ then prove that $\hat{a} = \pm 2\left(\vec{b} \times \vec{c}\right)$.

6. Find the value of $|\vec{a}\times\vec{b}|$ if $|\vec{a}|=10, |\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=12$.

7. Find the vector with magnitude 9 units whcih is perpendicular to the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.

8. Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$. Also explain geometrically.

9. For any vector $\vec{a}$ prove that $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$.

10. If the two sides of the triangle are given by $\hat{i}+2\hat{j}+2\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$ then find the area of the triangle.

13.27 Product of Three Vectors

The product of three vectors can have the following six conditions:

(i) $\vec{a}(\vec{b}\cdot\vec{c})$ (ii) $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ (iii) $\vec{a}\times(\vec{b}\cdot\vec{c})$

(iv) $\vec{a}(\vec{b}\times\vec{c})$ (v) $\vec{a}\cdot(\vec{b}\times\vec{c})$ (vi) $\vec{a}\times(\vec{b}\times\vec{c})$

By observation the following facts are to be considered

(i) $\vec{a}(\vec{b}\cdot\vec{c})$ is meaningless, because $\vec{b}\cdot\vec{c}$ is a scalar quantity, thus here $\vec{a}$ is a vector whose magnitude is a product of $(\vec{b}\cdot\vec{c})$, but this condition does not specify the product of three vectors.

(ii) $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ is meaningless, because $\vec{b}\cdot\vec{c}$ is a scalar whereas to find the scalar product with $\vec{a}$ a vector term is required.

(iii) $\vec{a}\times(\vec{b}.\vec{c})$ is meaningless, because $\vec{b}\cdot\vec{c}$ is a scalar and to get the vector product with $\vec{a}$, a vector term is required.

(iv) $\vec{a}(\vec{b}\times\vec{c})$ is meaningless, because $\vec{b}\times\vec{c}$ is a vector term and $\vec{a}$ is also a vector, but there is no sign of $(\cdot)$ or $(\times)$ so nothing can be predicted about the result.

(v) $\vec{a}\cdot(\vec{b}\times\vec{c})$ is meaningful, because $\vec{b}\times\vec{c}$ is a vector and $\vec{a}$ is also a vector and the product of these two vectors is possible and the result is a scalar. This is known as the scalar triple product.

(vi) $\vec{a}\times(\vec{b}\times\vec{c})$ is meaningful, because $\vec{b}\times\vec{c}$ is a vector and $\vec{a}$ is also a vector, the vector product of these terms is possible and the result is also a vector, this is called as vector triple product.

Thus from the above analysis only the product of two types of vectors is possible.

13.28 Scalar Triple Product

Definition: If the vector product of two vector quantities is again multipled with the scalar quantity then this product is known as scalar triple product.

As both vector and scalar product are found in this triple products so it is also known as mixed product.

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors then $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ is known as scalar triple product of vectors $\vec{a}, \vec{b}, \vec{c}$ and is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$, also $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$ and $[\vec{b}\ \vec{a}\ \vec{c}] = \vec{b}\cdot(\vec{a}\times\vec{c})$ ।

Note: It is also known as Box Product, it is to be noted that the terms inside the box should not be seperated by comma.

13.29 Geometrical Interpretation of Scalar Triple Product

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. Draw a rectangular parallelopiped with concurrent edges $\vec{a}, \vec{b}, \vec{c}$

Now the vector area of parallelogram OBDC $= \vec{b}\times\vec{c}$

$\therefore\ \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = |\vec{a}||\vec{b}\times\vec{c}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$

$= |\vec{b}\times\vec{c}|(|a|\cos\theta)$

= (area of parallelogram OBDC) (height of rectangular parallelopiped)

= (area of base × height)

$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c})$ = volume of rectangular parallelopiped whose concurrent edges are $\vec{a}, \vec{b}$ and $\vec{c}$

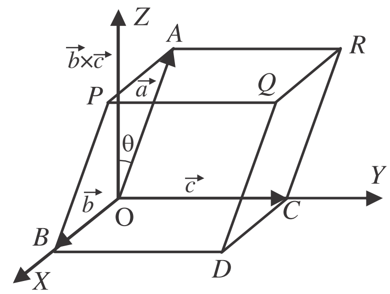


Fig. 13.23

similarly we can show $\vec{b}.\left(\vec{c}\times\vec{a}\right) = \vec{c}.\left(\vec{a}\times\vec{b}\right)$ the concurrent edges of rectangular parallelopipped

$$\vec{a}.\left(\vec{b}\times\vec{c}\right) = \vec{b}.\left(\vec{c}\times\vec{a}\right) = \vec{c}.\left(\vec{a}\times\vec{b}\right)$$

or $$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

is equal to volume of rectangular parallelopiped whose concurrent edges are given.

13.30 Properties of Scalar Triple Product

(i) $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right) \quad (1)$$

again $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{b}\times\vec{c}\right)\cdot\vec{a} \quad (2)$$

similarly $$\vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \left(\vec{c}\times\vec{a}\right)\cdot\vec{b} \quad (3)$$

and $$\vec{c}\cdot\left(\vec{a}\times\vec{b}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c} \quad (4)$$

from equation (1) and (4) $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$$

i.e. $$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$$

If the cyclic order remains unchanged then dot and cross signs can be changed.

(ii) If the cyclic order changes then the sign of scalar triple product changes.

$\because$ $$(\vec{b}\times\vec{c}) = -(\vec{c}\times\vec{b})$$

$\therefore \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(iii) In scalar triple product if two vectors are parallel then the product is zero.

Let $\vec{a},\vec{b},\vec{c}$ are three vectors and $\vec{b}$ and $\vec{c}$ are parallel then $\vec{b} = \lambda\vec{c}$, where λ is a scalar,

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{a}\cdot\left(\lambda\vec{c}\times\vec{c}\right) = \lambda\left(\vec{a}\cdot\vec{0}\right) = 0 \qquad \because \left[\vec{c}\times\vec{c} = \vec{0}\right]$$

Note: If two vectors are same then also the result is zero.

13.31 Volume of a Tetrahedron

Let in tetrahedron OABC, O be the origin and $A\left(\vec{a}\right), B\left(\vec{b}\right)$ and $C\left(\vec{c}\right)$ are other vertices.

Volume of Tetrahedron $\left(V\right) = \frac{1}{3}$ (area of base) × (height)

$$= \frac{1}{3}\left[\frac{1}{2}\left(\vec{a}\times\vec{b}\right)\right]\cdot\vec{c} = \frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

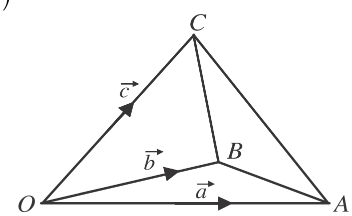


Fig. 13.24

Thus Volume fo Tetrahedron = (1 / 6) (Volume of rectangular parallelopipped whose three concurrent edges are $\vec{a},\vec{b},\vec{c}$)

Note: If the four vertices of a tetrahedron are $A\left(\vec{a}\right), B\left(\vec{b}\right), C\left(\vec{c}\right)$ and $D\left(\vec{d}\right)$ then the volume is

$$= \frac{1}{6}[\vec{a}-\vec{b} \quad \vec{a}-\vec{c} \quad \vec{a}-\vec{d}]$$

13.32 Necessary and sufficient condition for the three non-parallel and non-zero vector $\vec{a},\vec{b},\vec{c}$ to be coplanar is $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

Necessary Condition : Let $\vec{a},\vec{b}$ and $\vec{c}$ are three non-zero non-parallel coplaner vectors then $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane i.e. $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = 0$

($\because \vec{a}$ is in a plane and $\vec{b}\times\vec{c}$ N is perpendicular to the plane and scalar product of two vectors is always zero)

$\Rightarrow \quad \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

Sufficient condition : Let

$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0 \quad \Rightarrow \quad \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = 0$

$\Rightarrow \quad \vec{a}\perp\left(\vec{b}\times\vec{c}\right)$, But $\vec{b}\times\vec{c}$, is perpendicular to vectors $\vec{b}$ and $\vec{c}$ i.e. vector $\vec{a}$ lies in the plane of vector $\vec{b}$ and $\vec{c}$ therefore $\vec{a},\vec{b}$ and $\vec{c}$ are coplaner.

Illustrative Examples

Example 22. Prove that $\left[\hat{i}\ \hat{j}\ \hat{k}\right]+\left[\hat{j}\ \hat{k}\ \hat{i}\right]+\left[\hat{k}\ \hat{i}\ \hat{j}\right]=3.$

Solution : $\left[\hat{i}\ \hat{j}\ \hat{k}\right]=\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)=\hat{i}\cdot\hat{i}=1$

$\because \quad \left[\hat{i}\ \hat{j}\ \hat{k}\right]=\left[\hat{j}\ \hat{k}\ \hat{i}\right]=\left[\hat{k}\ \hat{i}\ \hat{j}\right]$

$\therefore \quad \left[\hat{i}\ \hat{j}\ \hat{k}\right]+\left[\hat{j}\ \hat{k}\ \hat{i}\right]+\left[\hat{k}\ \hat{i}\ \hat{j}\right]=1+1+1=3$

Example 23. If $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+2\hat{j}+\hat{k}$ then find the value of $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ and $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$, also show that $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

Solution :

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}1 & -2 & 1\\ 1 & 1 & 1\\ 1 & 2 & 1\end{vmatrix}=0 \qquad (\because \text{ first and third columns are same})$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}1 & 2 & 1\\ 1 & -2 & 1\\ 1 & 1 & 1\end{vmatrix}=0 \qquad (\because \text{ first and third columns are same})$$

$\therefore \quad \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

Example 24. Prove that $\left[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

Solution : since $\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)=\vec{b}\times\left(\vec{c}+\vec{a}\right)+\vec{c}\times\left(\vec{c}+\vec{a}\right)$ (distributive law)

$$=\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{c}\right)+\left(\vec{c}\times\vec{a}\right) \qquad \text{(distributive law)}$$

$$=\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{a}\right) \qquad (1)$$

$\therefore \quad \left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=\left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right\}$

$$=\left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{a}\right)\right\} \qquad \text{(from (1))}$$

$$=\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}\right)+\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{a}\right)+\left(\vec{a}+\vec{b}\right).\left(\vec{c}\times\vec{a}\right) \qquad \text{(distributive law)}$$

$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]+0+0+0+0+\left[\vec{b}\ \vec{c}\ \vec{a}\right] \qquad (\because \text{ property of triple product})$$

$$=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

Example 25. For what value of λ are the vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c}=3\hat{i}+\lambda\hat{j}+5\hat{k}$ coplaner.

Solution : Condition of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplaner is $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

i.e. $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix}=0$ or $\begin{vmatrix} 3 & \lambda & 5 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}=0$

$\Rightarrow \quad 3(3-2)+\lambda(1+6)+5(4+1)=0 \qquad \Rightarrow 3+7\lambda+25=0$

$\Rightarrow \quad \lambda=-4$

thus for $\lambda=-4$ the three vectolrs $\vec{a}, \vec{b}$ and $\vec{c}$ are coplaner.

Example 26. Prove that the points $A(4,8,12), B(2,4,6), C(3,5,4), D(5,8,5)$ are coplaner.

Solution : If the points $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}$ are coplaner, again by the codnition $\left[\overrightarrow{BA}\ \ \overrightarrow{BC}\ \ \overrightarrow{BD}\right]=0$

now $\overrightarrow{BA}=\left(4\hat{i}+8\hat{j}+12\hat{k}\right)-\left(2\hat{i}+4\hat{j}+6\hat{k}\right)=2\hat{i}+4\hat{j}+6\hat{k}$

$\overrightarrow{BC}=\left(3\hat{i}+5\hat{j}+4\hat{k}\right)-\left(2\hat{i}+4\hat{j}+6\hat{k}\right)=\hat{i}+\hat{j}-2\hat{k}$

$\overrightarrow{BD}=\left(5\hat{i}+8\hat{j}+5\hat{k}\right)-\left(2\hat{i}+4\hat{j}+6\hat{k}\right)=3\hat{i}+4\hat{j}-\hat{k}$

$$\therefore \qquad \left[\overrightarrow{BA}\,\overrightarrow{BC}\,\overrightarrow{BD}\right]=\begin{vmatrix} 2 & 4 & 6 \\ 1 & 1 & -2 \\ 3 & 4 & -1 \end{vmatrix}=2(7)+4(-5)+6(1)=0$$

Thus the four points are coplaner.

Example 27. If four points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ are copaner, then prove that

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left[\vec{b}\ \vec{c}\ \vec{d}\right]+\left[\vec{c}\ \vec{a}\ \vec{d}\right]+\left[\vec{a}\ \vec{b}\ \vec{d}\right]$$

Solution : Four points are coplaner thus vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are are coplaner.

$$\Rightarrow \qquad \left[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}\right]=0$$

$$\Rightarrow \qquad \left[\left(\vec{b}-\vec{a}\right)\ \left(\vec{c}-\vec{a}\right)\ \left(\vec{d}-\vec{a}\right)\right]=0$$

$$\Rightarrow \qquad \left(\vec{b}-\vec{a}\right)\cdot\left\{\left(\vec{c}-\vec{a}\right)\times\left(\vec{d}-\vec{a}\right)\right\}=0$$

$$\Rightarrow \qquad \left(\vec{b}-\vec{a}\right)\cdot\left\{\vec{c}\times\vec{d}-\vec{c}\times\vec{a}-\vec{a}\times\vec{d}+\vec{a}\times\vec{a}\right\}=0$$

$$\Rightarrow \vec{b}\cdot(\vec{c}\times\vec{d})-\vec{b}\cdot(\vec{c}\times\vec{a})-\vec{b}\cdot(\vec{a}\times\vec{d})-\vec{a}\cdot(\vec{c}\times\vec{d})=0$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{d}]+[\vec{c}\ \vec{a}\ \vec{d}]+[\vec{a}\ \vec{b}\ \vec{d}]$$

Example 28. Find the volume of the rectangular parallelopipped whose concurrent edges are $2\hat{i}-3\hat{j}+4\hat{k}, \hat{i}+2\hat{j}-\hat{k}$ P and $2\hat{i}-\hat{j}+2\hat{k}$.

Solution : Let $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+2\hat{k}$, volume of parallelopipped $=[\vec{a}\,\vec{b}\,\vec{c}]$

$$=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ 2 & -1 & 2\end{vmatrix}=2(3)+3(-4)+4(-5)=6-12-20=-26 \text{ unit}$$

Since Volume is positive, hence the result is 26 units.

Example 29. Find the volume of tetrahedron if the vertices are $O(0,0,0)$, $A(1,2,1), B(2,1,3)$ and $C(-1,1,2)$.

Solution : Here $O(0,0,0)$ is the origin and the position vector are $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}+2\hat{k}$.

$$\text{volume of tetrahedron} \quad =\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]=\frac{1}{6}\begin{vmatrix}1 & 2 & 1\\ 2 & 1 & 3\\ -1 & 1 & 2\end{vmatrix}$$

$$=\frac{1}{6}[1(-1)+2(-7)+1(3)]=-2 \text{ unit}$$

Since the volume is positive thus the result is 2 units.

Exercise 13.4

1. Prove that

   (i) $[\hat{i}\ \hat{j}\ \hat{k}]+[\hat{i}\ \hat{k}\ \hat{j}]=0$ (ii) $[2\hat{i}\ \hat{j}\ \hat{k}]+[\hat{i}\ \hat{k}\ \hat{j}]+[\hat{k}\ \hat{j}\ 2\hat{i}]=-1$

2. If $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}, \vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$ then find $[\vec{a}\,\vec{b}\,\vec{c}]$.

3. Prove that the vectors $-2\hat{i}-2\hat{j}+4\hat{k}, -2\hat{i}+4\hat{j}-2\hat{k}$ and $4\hat{i}-2\hat{j}-2\hat{k}$ are coplaner.

4. For what value of λ are the vectors copalner

   (i) $\vec{a}=2\hat{i}-\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c}=3\hat{i}+\lambda\hat{j}+5\hat{k}$

   (ii) $\vec{a}=\hat{i}-\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{i}-\hat{j}+\lambda\hat{k}$

5. Prove that the following four points are coplaner

   (i) $A(-1,4,-3), B(3,2,-5), C(-3,8,-5), D(-3,2,1)$

(ii) $A(0,-1,0), B(2,1,-1), C(1,1,1), D(3,3,0)$

6. Prove that $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ are the vector sides of a right angle triangle.

7. Find the volume of the rectangular parallelopipped whose three concurrent edges are given by the vectors:

(i) $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

(ii) $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - \hat{k}$

13.33 Vector Triple Product

Definition : The product of vector with the vector product of two vectors is known as vector triple product.

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then their vector product will be $\vec{a} \times (\vec{b} \times \vec{c}), (\vec{b} \times \vec{c}) \times \vec{a}, (\vec{a} \times \vec{b}) \times \vec{c}$ etc.

Geometrical Proof:

Here $\vec{a} \times (\vec{b} \times \vec{c})$, is perpendicular to vector $\vec{a}$ and vector $(\vec{b} \times \vec{c})$

$\therefore$ $\vec{a} \times (\vec{b} \times \vec{c}) = \lambda\vec{b} + \mu\vec{c}$ where λ and μ are scalar

Note: It is clear from the vector triple product $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$, it is not associative.

13.34 For vectors $\vec{a}, \vec{b}, \vec{c}$ Prove that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

now $\vec{a} \times (\vec{b} \times \vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \{(b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}\}$$

$$= \sum \{a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)\}\hat{i}$$

$$= \sum \{b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)\}\hat{i}$$

$$= \sum \{(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1\}\hat{i}$$ (adding and subtracting *a, b, c*)

$$= \sum \{(\vec{a} \cdot \vec{c})b_1 - (\vec{a} \cdot \vec{b})c_1\}\hat{i} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$\therefore$ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

similarly $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=-\vec{c}\times\left(\vec{a}\times\vec{b}\right)=-\left\{\left(\vec{c}\cdot\vec{b}\right)\vec{a}-\left(\vec{c}\cdot\vec{a}\right)\vec{b}\right\}=\left(\vec{c}\cdot\vec{a}\right)\vec{b}-\left(\vec{c}\cdot\vec{b}\right)\vec{a}$

Illustrative Examples

Example 30. If $\vec{a}=3\hat{i}+2\hat{j}+\hat{k},\vec{b}=\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{c}=2\hat{i}+\hat{j}-\hat{k}$ then find the value of $\vec{a}\times(\vec{b}\times\vec{c})$

Solution :

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$\vec{a}\cdot\vec{c}=\left(3\hat{i}+2\hat{j}+\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}-\hat{k}\right)$$

$$=(3)(2)+(2)(1)+(1)(-1)=7$$

$$\vec{a}\cdot\vec{b}=\left(3\hat{i}+2\hat{j}+\hat{k}\right)\cdot\left(\hat{i}-2\hat{j}+2\hat{k}\right)$$

$$=(3)(1)+(2)(-2)+(1)(2)=1$$

$\therefore$ $$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$=7\left(\hat{i}-2\hat{j}+2\hat{k}\right)-1\left(2\hat{i}+\hat{j}-\hat{k}\right)=5\hat{i}-15\hat{j}+15\hat{k}$$

Example 31. Prove that $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, if and only if $\left(\vec{c}\times\vec{a}\right)\times\vec{b}=\vec{O}$

Solution : Let $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

$\Rightarrow$ $\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$\Rightarrow$ $-\left(\vec{b}\cdot\vec{c}\right)\vec{a}=-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$\Rightarrow$ $\left(\vec{b}\cdot\vec{c}\right)\vec{a}-\left(\vec{b}\cdot\vec{a}\right)\vec{c}=\vec{O}$

$\therefore$ $\left(\vec{c}\times\vec{a}\right)\times\vec{b}=\vec{O}$

Example 32. Prove that the vectors $\vec{a}\times\left(\vec{b}\times\vec{c}\right),\vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplaner.

Solution : Let $\vec{P}=\vec{a}\times\left(\vec{b}\times\vec{c}\right),\vec{Q}=\vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{R}=\vec{c}\times\left(\vec{a}\times\vec{b}\right)$, then

$$\vec{P}+\vec{Q}+\vec{R}=\left\{\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}\right\}+\left\{\left(\vec{b}\cdot\vec{a}\right)\vec{c}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}\right\}+\left\{\left(\vec{c}\cdot\vec{b}\right)\vec{a}-\left(\vec{c}\cdot\vec{a}\right)\vec{b}\right\}=\vec{O}$$

$\Rightarrow$ $\vec{P}=(-1)\vec{Q}+(-1)\vec{R}$

$\Rightarrow$ $\vec{P},\vec{Q}$ and $\vec{R}$ are in one plane

$\Rightarrow$ $\vec{P},\vec{Q},\vec{R}$ are coplaner

Example 33. Prove that $\left[\left(\vec{a}\times\vec{b}\right)\left(\vec{b}\times\vec{c}\right)\left(\vec{c}\times\vec{a}\right)\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$

Solution :

$$\left[\left(\vec{a}\times\vec{b}\right)\left(\vec{b}\times\vec{c}\right)\left(\vec{c}\times\vec{a}\right)\right]=\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\vec{d}\times\left(\vec{b}\times\vec{c}\right)\right\}\cdot\left(\vec{c}\times\vec{a}\right),\qquad \left(\text{Let } \vec{d}=\vec{a}\times\vec{b}\right)$$

$$=\left\{\left(\vec{d}\cdot\vec{c}\right)\vec{b}-\left(\vec{d}\cdot\vec{b}\right)\vec{c}\right\}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\left[\vec{a}\;\vec{b}\;\vec{c}\right]\vec{b}-\left[\vec{a}\;\vec{b}\;\vec{b}\right]\vec{c}\right\}\cdot\left(\vec{c}\times\vec{a}\right)$$

$$\left[\because \vec{d}\cdot\vec{c}=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\left[\vec{a}\;\vec{b}\;\vec{c}\right] \text{ and } \vec{d}\cdot\vec{b}=\left(\vec{a}\times\vec{b}\right)\cdot\vec{b}=\left[\vec{a}\;\vec{b}\;\vec{b}\right]=0\right]$$

$$=\left[\vec{a}\;\vec{b}\;\vec{c}\right]\left\{b\cdot\left(\vec{c}\times\vec{a}\right)\right\}\quad\left\{\because\left[\vec{c}\;\vec{c}\;\vec{a}\right]=0\right\}$$

$$=\left[\vec{a}\;\vec{b}\;\vec{c}\right]\left[\vec{b}\;\vec{c}\;\vec{a}\right]=\left[\vec{a}\;\vec{b}\;\vec{c}\right]^2\qquad\left[\because\left[\vec{b}\;\vec{c}\;\vec{a}\right]=\left[\vec{a}\;\vec{b}\;\vec{c}\right]\right]$$

Exercise 13.5

1. Find the value of $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ it

   (i) $\vec{a}=3\hat{i}-\hat{j}+\hat{k},\vec{b}=\hat{i}+3\hat{j}-\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}+3\hat{k}$

   (ii) $\vec{a}=2\hat{i}+\hat{j}-3\hat{k},\vec{b}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}-4\hat{k}$

2. Prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ it

   (i) $\vec{a}=2\hat{i}+5\hat{j}-7\hat{k},\vec{b}=-3\hat{i}+4\hat{j}+\hat{k},\vec{c}=-\hat{i}-2\hat{j}-3\hat{k}$

   (ii) $\vec{a}=2\hat{i}+3\hat{j}-5\hat{k},\vec{b}=-\hat{i}+\hat{j}+\sqrt{2}\hat{k},\vec{c}=4\hat{i}-2\hat{j}+\sqrt{3}\hat{k}$

3. Verify the formula $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$ where

   (i) $\vec{a}=\hat{i}+\hat{j}-2\hat{k},\vec{b}=2\hat{i}-\hat{j}+\hat{k},\vec{c}=\hat{i}+3\hat{j}-\hat{k}$

   (ii) $\vec{a}=\hat{i}-2\hat{j}+\hat{k},\vec{b}=2\hat{i}+\hat{j}-\hat{k},\vec{c}=3\hat{i}+5\hat{j}+2\hat{k}$

4. For any vector $\vec{a}$ prove that

   $$\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)=2\vec{a}$$

5. Prove that

   $$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\vec{0}$$

6. Prove that $\vec{a},\vec{b},\vec{c}$ are coplaner if and only if $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ are coplaner

7. Prove that

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}-\left[\vec{a}\,\vec{c}\,\vec{d}\right]\vec{d}$$

8. If the magnitude of two vectors $\vec{a}$ and $\vec{b}$ are $\sqrt{3}$ and 2 and $\vec{a}\cdot\vec{b}=\sqrt{6}$ then find the angle between vector $\vec{a}$ and $\vec{b}$.

9. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

10. Find the projection of vector $\hat{i}+\hat{j}$ on $\hat{i}-\hat{j}$.

11. Fidn the projection of vector $\hat{i}+3\hat{j}+7\hat{k}$ on $7\hat{i}-\hat{j}+8\hat{k}$.

12. Find the valueof $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

13. Find the magnitude of the two vectors $\vec{a}$ and $\vec{b}$ if there magnitude is same and the angle between them is 60° and their scalar products is $\frac{1}{2}$.

14. For a unit vector $\vec{a}$, if $\left(\vec{x}-\vec{a}\right)\cdot\left(\vec{x}+\vec{a}\right)=12$ then find the valeu of $|\vec{x}|$.

15. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}, \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+3\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to vector $\vec{c}$ then find the valeu of λ.

16. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

17. If the vertices of triangle ABC are $(1,2,3)(-1,0,0)(0,1,2)$ then find $\angle ABC$.

Important Points

1. $\vec{a}\cdot\vec{b}=ab\cos\theta,\ \therefore\ \vec{a}\cdot\vec{b}=0\Leftrightarrow\vec{a}\perp\vec{b}\left(\vec{a}\neq 0\neq\vec{b}\right)$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{ab}$$

.	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

2. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$

3. $\vec{a}\times\vec{b}=\left(ab\sin\theta\right)\hat{n}$

$$\sin\theta=\frac{\left|\vec{a}\times\vec{b}\right|}{ab}\text{ and }\hat{n}=\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$$

$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

X	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

$\hat{j}\times\hat{i}=-\hat{k}, \hat{k}\times\hat{j}=-\hat{i}, \hat{i}\times\hat{k}=-\hat{j}$

$\hat{i}\times\hat{i}=\vec{O}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}$

$\vec{a}\times\vec{b}=\vec{O}\Leftrightarrow\vec{a}\parallel\vec{b} \qquad \left(\vec{a}\neq\vec{O}\neq\vec{b}\right)$

4. $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

5. Area of Parallelogram of two vectors is $=\vec{a}\times\vec{b}$, where $\vec{a}$ and $\vec{b}$ are the adjacent sides of the parallelogram.

6. Area of ΔABC =Qy$=\frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$, where $\vec{a},\vec{b},\vec{c}$, are position vectors of vertices of triangle.

7. The collinearity of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is given by $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{O}$

8. Area of parallelogram whose diagonals are $\vec{a}$ and $\vec{b}$ $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$

9. We represent the scalar or dot product of three vectors $\vec{a},\vec{b},\vec{c}$ is $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ and $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

10. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

    $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

11. Volume of rectangular parallelopipped $=\left[\vec{a}\ \vec{b}\ \vec{c}\right]$, (where $\vec{a},\vec{b},\vec{c}$ denoted its concurrent edges).

12. Volume of Tetrahedron$=\frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ where $\vec{a},\vec{b},\vec{c}$ are its concurrent edges.

13. The triangular product of three vectors $\vec{a},\vec{b},\vec{c}$ is $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$.

14. In vectors, vector product does not follows associative property i.e. $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$

Answers

Exericse 13.1

(1) $|\vec{a}| = \sqrt{3}; |\vec{b}| = \sqrt{62}; |\vec{c}| = 1$ (2) any two vectors (3) any two vectors (4) $x = 2,\ y = 3$

(5) -7, 6 rFkk $-7i$, $6j$ (6) $-4\hat{j} - \hat{k}$ (7) $\dfrac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}$ (8) $\dfrac{\left(\hat{i} + \hat{j} + \hat{k}\right)}{\sqrt{3}}$

(9) $\dfrac{\hat{i} + \hat{k}}{\sqrt{2}}$ (10) $\dfrac{8(5\hat{i} - \hat{j} + 2\hat{k})}{\sqrt{30}}$ (11) $-4\hat{i} + 6\hat{j} - 8\hat{k} = -2(2\hat{i} - 3\hat{j} + 4\hat{k})$

(12) (i) $\dfrac{1}{3}\hat{i},\ \dfrac{4}{3}\hat{j},\ \dfrac{1}{3}\hat{k}$ (ii) $-3\hat{i} + 3\hat{k}$ (13) $3\hat{i} + 2\hat{j} + \hat{k}$ (14) $\dfrac{-\hat{i} + 4\hat{j} + \hat{k}}{3},\ -3\hat{i} + 3\hat{k}$

(15) (3, 2, 1)

Exercise 13.2

(1) (i) 10 ; (ii) 0 ; (iii) $10\sqrt{3}$ (2) (i) -4 ; (ii) 7 ; (iii) 7 (4) $\theta = \cos^{-1}\left(\dfrac{72}{75}\right)$

(5) (i) 3 ; (ii) 3 (6) $\dfrac{2}{7}$ (7) $5\hat{i} - 15\hat{j} + 7\hat{k}$

Exercise 13.3

(1) $4\hat{i} - 5\hat{j} + 7\hat{k}$ (2) $\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ (6) 16 (7) $-3\hat{i} + 6\hat{j} + 6\hat{k}$ (10) $\dfrac{5\sqrt{5}}{2}$

Exercise 13.4

(2) -7 (5) (i) -4 ; (ii) 1 (8) (i) 30 ; (ii) 14

Exercise 13.5

(1) (i) $-2\hat{i} - 2\hat{j} + 4\hat{k}$; (ii) $8\hat{i} - 19\hat{j} - \hat{k}$

(8) $\dfrac{\pi}{4}$ (9) $\cos^{-1}\left(\dfrac{5}{7}\right)$ (10) 0 (11) $\dfrac{60}{\sqrt{114}}$ (12) $6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

(13) $|\vec{a}| = 1, |\vec{b}| = 1$ (14) $\sqrt{13}$ (15) -4 (16) $-\dfrac{3}{2}$ (17) $\cos^{-1}\left(\dfrac{10}{\sqrt{102}}\right)$