Continuity and Differentiability

✓ **Continuity :** Suppose f is a real function on a subset of the real numbers and let c be a point in the domain f. Then f is continuous at c if $\boxed{\lim\limits_{x \to c} f(x) = f(c)}$

✓ **Discontinuity :** A function said to be discontinuous at point $x = a$, if it is not continuous at this point. This point $x = a$ where the function is not continuous is called the point of discontinuity.

✓ **Theorem 1** Suppose f and g be two real functions continuous at a real no. then,

(1) $f+g$ is continuous at $x=c$ (3) $f \cdot g$ is continuous at $x=c$

(2) $f-g$ is continuous at $x=c$ (4) $\left(\dfrac{f}{g}\right)$ is continuous at $x=c$, $\{$provided $g(c) \neq 0\}$

✓ **Theorem 2** Suppose f and g are real valued functions such that $(f \circ g)$ is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

✓ **Differentiability :** Suppose f is a real function and c is a point in its domain. The derivative of f at c defined by $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\dfrac{d}{dx}[f(x)]_c$. The function defined by $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ whenever the limit exists is defined to be the derivative of f. The derivative denoted by $f'(x)$ or $\dfrac{d}{dx}[f(x)]$ or if $y = f(x)$ by $\dfrac{dy}{dx}$ or y'.

✓ **Algebra of derivatives :**

(i) $\boxed{(u \pm v)' = u' \pm v'}$

(ii) $\boxed{(uv)' = u'v + uv'}$ (Leibnitz or product rule)

(iii) $\boxed{\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}}$, whenever $v \neq 0$ (Quotient Rule)

✓ **Theorem 3.** If a function f is differentiable at a point c, then it is also continuous at that point.

📍 **Note :** Every differentiable function is continuous.

✓ **Chain Rule :** Let f be a real valued function of which is a composite of two functions u and v i.e. $f = v \circ u$; Suppose $t = u(x)$ and if $\dfrac{dt}{dx}$ and $\dfrac{dv}{dt}$ exist, we have $\dfrac{df}{dx} = \dfrac{dv}{dt} \cdot \dfrac{dt}{dx}$

Suppose f is real valued function which is a composite of three functions u, v and w; i.e. $f = (w \circ u) \circ v$ and if $t = v(x)$ and $s = u(t)$ then

$$\dfrac{df}{dx} = \dfrac{d(w \circ u)}{dt} \cdot \dfrac{dt}{dx} = \dfrac{dw}{ds} \cdot \dfrac{ds}{dt} \cdot \dfrac{dt}{dx}$$

✓ **Some properties of Logarithmic function**

$\boxed{\log_a P = \dfrac{\log_b P}{\log_b a}}$ $\boxed{\log_b pq = \log_b p + \log_b q}$ $\boxed{\log_b P^2 = \log_b P + \log_b P = 2\log p}$

$\boxed{\log_b P^n = n \log p}$ $\boxed{\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y}$ $\boxed{\log_b x = \dfrac{1}{\log_x b}}$

📍 Note :

Exponential form	logarithim form
$2^3 = 8$	$\log_2 8 = 3$
$b^1 = b$	$\log_b b = 1$
$b^0 = 1$	$\log_b 1 = 0$

✔️ Some standard derivative

$\dfrac{d}{dx}(c) = 0$ $c = $ constant

$\dfrac{d}{dx}(x^n) = nx^{n-1}$

$\dfrac{d}{dx}(e^x) = e^x$

$\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

$\dfrac{d}{dx}(a^x) = a^x \log_e a$

$\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x}\log_a e$

$\dfrac{d}{dx}(\sin x) = \cos x$

$\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\dfrac{d}{dx}(\cot x) = -\csc^2 x$

$\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$\dfrac{d}{dx}(\csc x) = -\csc x \cdot \cot x$

$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

$\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

$\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

$\dfrac{d}{dx}(\csc^{-1} x) = \dfrac{-1}{x\sqrt{x^2-1}}$

✔️ logarithmic differentiation

$y = f(x) = [u(x)]^{v(x)}$

taking log both sides,

$\log y = v(x) \log[u(x)]$

using chain rule to differentiate

$\dfrac{1}{y} \cdot \dfrac{dy}{dx} = v(x) \cdot \dfrac{1}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]$

$$\boxed{\dfrac{dy}{dx} = y\left[\dfrac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]\right]}$$

✔️ Derivative of functions in Parametric forms

$x = f(t) = g(t)$ parametric form with t as a parameter.

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\left[\text{whenever } \dfrac{dx}{dt} \neq 0\right]$

$\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$ $\left[\text{as } \dfrac{dy}{dt} = g'(t) \text{ and } \dfrac{dx}{dt} = f'(t)\right]$ $[\text{provided } f'(t) \neq 0]$

✔️ Second order derivative

Let $y = f(x)$

$\dfrac{dy}{dx} = f'(x)$ ———— (i)

differentiate (i) again w.r.t to x,

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}[f'(x)] \Rightarrow \boxed{\dfrac{d^2y}{dx^2} = f''(x)}$ Denoted D^2y or y''

📍 Note : Higher order derivative may be defined similarly

✔️ Rolle's Theorem : If $f : [a\ b] \to R$ is continuous on $[a, b]$ and differentiable on (a,b) such that $f(a) = f(b)$ then there exists some C in (a,b) such that $f'(c) = 0$

✔️ Langrange Theorem or Mean value theorem : If $f : [a,b] \to R$ is continuous on $[a, b]$ and differentiable on (a,b). Then there exists some c in (a, b) such that

$$\boxed{f'(c) = \dfrac{f(b) - f(a)}{b - a}}$$