08

Height and Distance

8.01. Introduction

In the preceding chapter, we have studied trigonometric identities and trigonometric ratios for complementary angles. In this chapter we will study problems based on height and distace by using trigonometric ratios, we will see how trigonometry is used for finding the heights and distances of various objects without actually measuring them. Before this, we will study some definitations.

8.02. Important Definitions

Line of sight : The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.

In figure 8.01, if eye is at point *O* and object is at point *P* then *OP* is the line of sight.

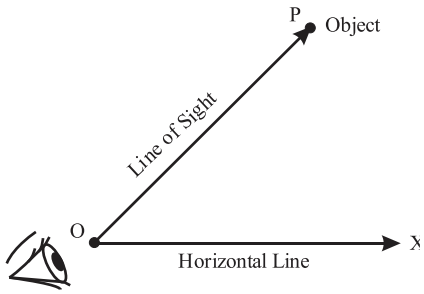

Fig. 8.01

Angle of Elevation

If an oject is above the eye then we raise our head to look at the object. The angle formed by the line of sight with the horizontal line is called angle of elevation.

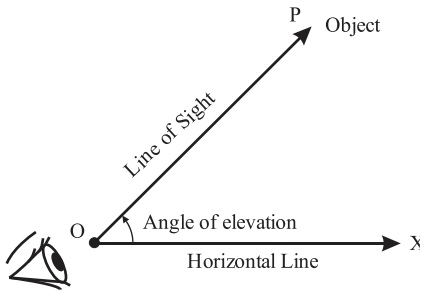

Fig. 8.02

In fig. 8.02, eye is at point O and object is at point P then line of sight OP makes an angle $\angle XOP$ with horizontal line OX then

Angle of elevation $= \angle XOP$

Note : Angle of elevation is also called as angular height of the object.

Angle of depression

The angle of depression of a point on the object being viewed is the angle formed by the line when the point is below the horizontal line.

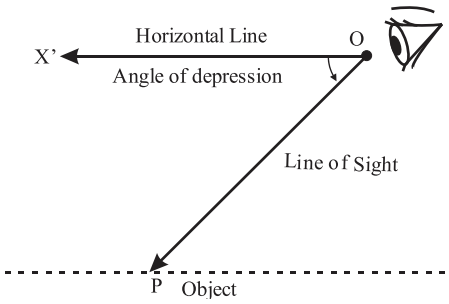

Fig. 8.03

In figure 8.03, eye is at point O and object is at point P then OP is line of sight which makes an angle $X'OP$ with horizontal line OX' then angle of depession $= \angle X'OP$

In solving the problems related to height and distance, following points are to be kept in mind.

(i) First read the question carefully then draw figure and prepare right angled triangle.

(ii) In right angled triangle, express trignometric ratios (sin, cos, tan etc.) of given angle in the terms of given sides.

Note : Complemetary angles : If sum of two angles is 90^o then they are called complelmentary angles.

Following are the Examples with figure of angle of depression subtended at the eye of the observer by the objects.

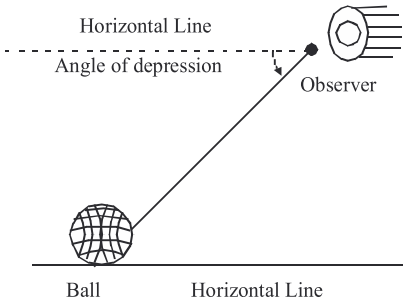

Fig. 8.04

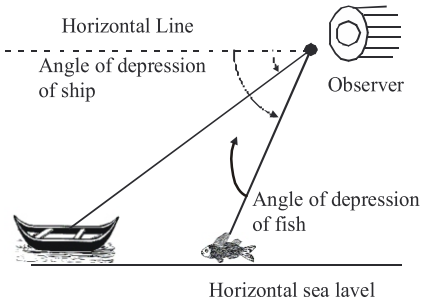

Fig. 8.05

Illustrative Examples

Example 1. The angle of elevation of the top of a tower form a point on the ground which is $10\sqrt{3}$ m away from the foot of the tower is 60°. Find the height of the tower.

Solution : Let AB is a tower. The angle of elevation of the top of a tower from point C on the ground, which is $10\sqrt{3}$ m away from the foot of the tower is 60°. Let h is the height of the tower AB.

In right angled ΔABC

$$\tan 60^{\circ} = \frac{AB}{BC}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{10\sqrt{3}}$$

or $\quad h = 10\sqrt{3} \times \sqrt{3}$

or $\quad h = 10 \times 3 = 30$

Thus, height of the tower AB is 30 m.

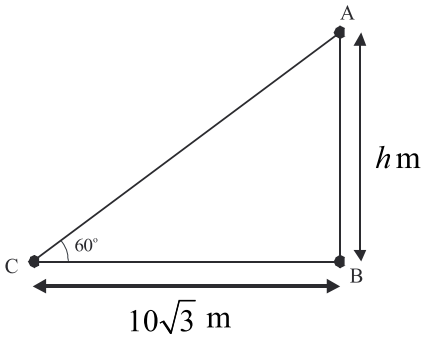

Fig. 8.06

Example 2. The angle of drepression of any boat from a 50 m high bridge is 30°. Find the horizontal distance between boat and bridge.

Solution : Let horizontal distance between boat and bridge is x m.

Given, Angle of depression is 30°

Here $\quad PQ = 50$ m

$$\angle XPO = \angle POQ = 30^{\circ} \text{ (Alternate angles)}$$

In right angled ΔPQO

$\because \quad \tan 30^{\circ} = \frac{PQ}{OQ}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{50}{x}$

or $\quad x = 50\sqrt{3} = 50 \times 1.732 \quad \left(\because \sqrt{3} = 1.732\right)$

or $\quad x = 86.60$

Thus, horizontal distance between boat and bridge is 86.60 m.

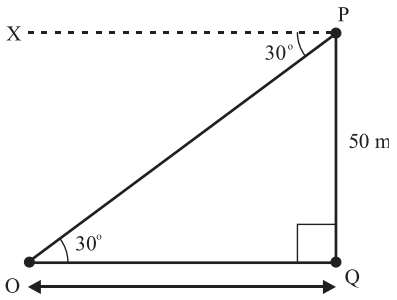

Fig. 8.07

Example 3. The shadow of a 1.5 m tall student standing on a plane ground in found to be 1 m and at the same time shadow of a tower on ground is 5 m, then find height of the tower.

Solution : Given, Length of student AC = 1.5 m

Shadow of student BC = 1 m

In right angled ΔACB

$$\tan\theta = \frac{AC}{BC} \quad \Rightarrow \quad \tan\theta = \frac{1.5}{1}$$

or $\quad \tan\theta = 1.5 \qquad \ldots (1)$

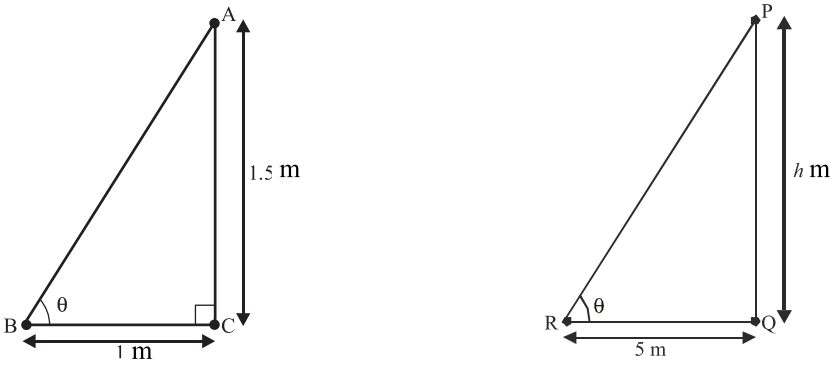

Fig 8.08

Now it is given that

Length of shadow of the tower $QR = 5$ m

Let height of the tower $= PQ = h$ m

In right angled ΔPQR

or $$\tan\theta = \frac{PQ}{QR}$$

or $$\frac{h}{5} = 1.5 \qquad [\because \tan\theta = 1.5 \text{ from equation (i)}]$$

or $$h = 5 \times 1.5$$

or $$h = 7.5$$

Thus, height of tower is 7.5 m.

Example 4. There is a small island in 100 m broad river and there is a tall tree on this island. P and Q lie on the opposite banks of the river such that P, Q and tree are in the same line. If angle of elevation form P and Q at top of the tree are 30° and 45° respectivaly then find height of the tree.

Solution : Let OA is tree whose height is h m.

In figure, $PQ = 100$ m

$\angle APO = 30^\circ$ and $\angle AQO = 45^\circ$

Now, in right angled ΔPOA and ΔQOA

$$\tan 30^\circ = \frac{OA}{OP} \text{ and } \tan 45^\circ = \frac{OA}{OQ}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{OP} \text{ and } 1 = \frac{h}{OQ}$$

$$OP = h\sqrt{3} \text{ and } OQ = h$$

$\therefore$ From figure $PQ = OP + OQ$

$$100 = h\sqrt{3} + h$$

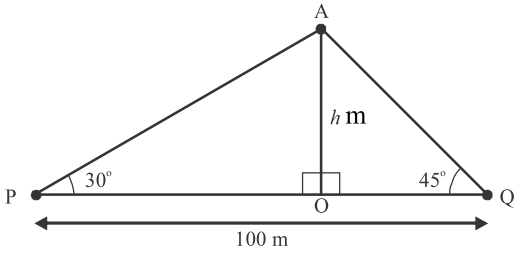

Fig 8.09

$$100 = h\left(\sqrt{3}+1\right)$$

$$\therefore \quad h = \frac{100}{\sqrt{3}+1} = \frac{100}{\left(\sqrt{3}+1\right)} \times \left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$

$$h = \frac{100\left(\sqrt{3}-1\right)}{2}$$

$$h = 50\left(\sqrt{3}-1\right) = 36.6 \text{ m} \qquad \left(\because \sqrt{3} = 1.732\right)$$

Thus, height of the tree is 36.6 m.

Example 5. A car is moving on a straight road which goes towards a tower. At a distance of 500 m from tower, driver of car observe that angle of elevation of top of tower is 30° after driving the car for 10 sec. towards tower then he observe that the angle of elevation of top of tower became 60°. Find the speed of the car.

Solution : Let height of tower $AB = h$ m and distance covered by car in 10 sec. (DC) = x m.

$$BD = 500 \text{ m}$$

$$\therefore \quad BC = (500 - x) \text{ m}$$

$$\angle ADB = 30^\circ, \angle ACB = 60^\circ$$

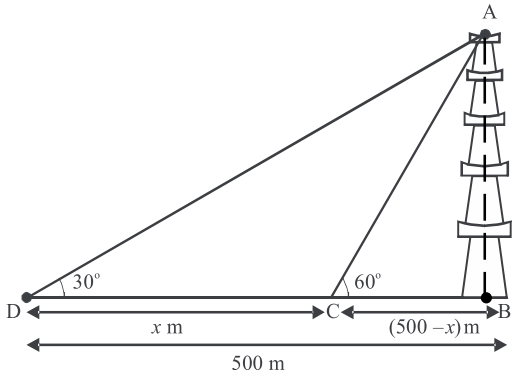

Fig 8.10

Now, in right angled $\triangle ABD$

$$\frac{AB}{BD} = \tan 30$$

$$\frac{h}{500} = \frac{1}{\sqrt{3}} \Rightarrow h = \frac{500}{\sqrt{3}} \ldots (1)$$

Now, in right angled ΔABC

$$\frac{AB}{BC} = \tan 60$$

or $$\frac{h}{500-x} = \sqrt{3} \Rightarrow h = \left(500 - x\right)\sqrt{3} \ldots (2)$$

From equation (1) and (2)

$$\frac{500}{\sqrt{3}} = (500 - x)\sqrt{3} \Rightarrow 500 = (500 - x)\sqrt{3}.\sqrt{3}$$

or $$500 = (500 - x).3$$

or $$500 = 1500 - 3x$$

or $$3x = 1500 - 500 = 1000$$

or $$x = \frac{1000}{3}$$

Distance covered by car in 10 sec $= \frac{1000}{3}$ m

? Distance covered by car in 1 min. $= \frac{1000 \times 60}{3 \times 10} = 2000$ m $= 2$ km

Thus speed of car = 2 km/min.

Example 6. The angle of elevations of the top of a tower form two pointrs C and D from base of tower and in the same straight line at a distance a and b respectively , are complement to each other. Prove that height of tower is $\sqrt{ab}$.

Solution : Let the height of tower $AB = h$ meter and points C and D are in such a way that $BC = a$, $BD = b$.

If $\angle ACB = \theta$ then $\angle ADB = 90^\circ - \theta$

Again, in right angled ΔABC

$$\tan\theta = \frac{AB}{BC} = \frac{h}{a} \qquad \ldots (1)$$

Again, in right angled ΔABD

$$\tan\left(90^\circ - \theta\right) = \frac{AB}{BD}$$

or $$\cot\theta = \frac{h}{b} \qquad \ldots (2)$$

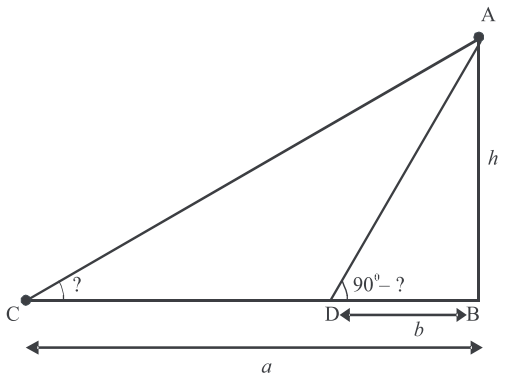

Fig. 8.11

On multiplying equation (1) and (2), we get

$$\tan\theta \times \cot\theta = \frac{h}{a} \times \frac{h}{b}$$

or $$1 = \frac{h^2}{ab} \Rightarrow h^2 = ab$$

or $$h = \sqrt{ab}$$

Example 7. Two poles of equal height are standing opposite each other on either side of the road which is 80 m wide. From a point between them on the road, the angle of elevation of the top of the poles are 60° and 30° respectively. Find the height of the poles and the distances of the point from the poles.

Solution: Let BC and DE are two poles of same height (in meter). From a point between the poles,on the road BD,the angles of elevations of the top of the poles are 60° and 30°

Thus, $\angle CAB = 60^\circ$ and $\angle EAD = 30$ $BC = DE = h$ m. $BD = 80$ m.

Let $AD = x$ m

$\therefore$ $AB = BD - AD = (80 - x)$ m

In right angle ΔADE

$$\tan 30^\circ = \frac{DE}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\therefore \qquad h = \frac{x}{\sqrt{3}} \qquad \ldots (1)$$

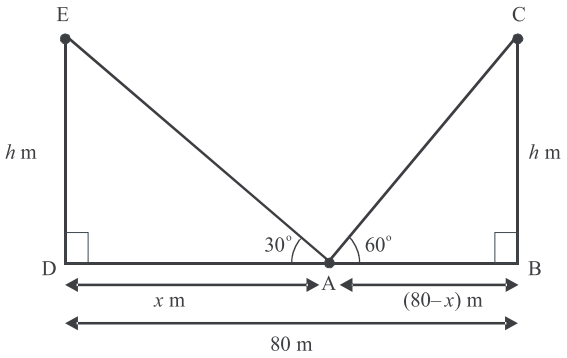

Fig. 8.12

Again, in right angled ΔABC

$$\tan 60^\circ = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{h}{(80-x)}$$

$$h = (80-x)\sqrt{3} \text{ m} \qquad \ldots (2)$$

From equation (1) and (2)

$$\frac{x}{\sqrt{3}} = \sqrt{3}(80-x)$$

$$x = \sqrt{3}.\sqrt{3}(80-x)$$

$$x = 3(80-x)$$

$$x\ ?\ 240?\ 3x$$

$$\Rightarrow \quad x+3x = 240$$

$$4x = 240$$

$$x = \frac{240}{4} = 60$$

From equation (1)

$$h = \frac{60}{\sqrt{3}} = \frac{60}{\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}} = \frac{60\sqrt{3}}{3}$$

$$h = 20\sqrt{3}$$

Thus height of poles $(h) = 20\sqrt{3}$ m and distance of the point form the poles is 20 m and 60 m.

Example 8. The angle of elevation of a cloud form a point $'h'$ m above a lake is and the angle of depression of its reflection in the lake is β. Prove that height of the cloud from surface of water is $\frac{h(\tan\beta + \tan\propto)}{\tan\beta - \tan\propto}$.

Solution : Let AB is surface of lake and P is point of observation.

Given $AP = h$ m. Let position of cloud is c and c' is its shadow in lake

$\therefore CB = C'B$. Let PM is perpendicular from P to CB, it is given that

$\angle CPM = \alpha$ and $\angle MPC' = \beta$ Let $CM = x$

It is clear that $CB = CM + MB = CM + PA = x + h$

In ΔCMP $\tan\alpha = \dfrac{CM}{PM}$

or $\tan\alpha = \dfrac{x}{AB}$ $(\because PM = AB)$

$\therefore$ $AB = x\cot\alpha$ . . . (1)

In $\Delta PMC'$, $\tan\beta = \dfrac{C'M}{PM} = \dfrac{x+2h}{AB}$

$\therefore$ $AB = (x+2h)\cot\beta$ . . . (2)

From equations (1) and (2), we get

$$x\cot\alpha = (x+2h)\cot\beta$$

$$x(\cot\alpha - \cot\beta) = 2h\cot\beta$$

or $$x\left(\frac{1}{\tan\alpha} - \frac{1}{\tan\beta}\right) = \frac{2h}{\tan\beta}$$

or $$x\left[\frac{\tan\beta - \tan\alpha}{\tan\alpha\tan\beta}\right] = \frac{2h}{\tan\beta}$$

or $$x = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha}$$

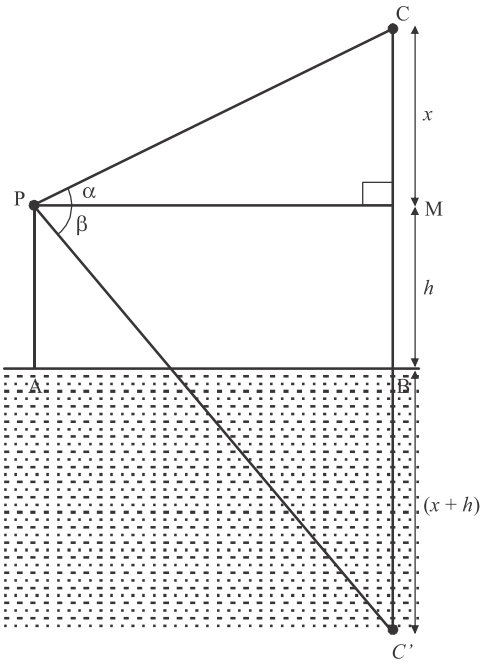

Fig. 8.13

Thus height of cloud form surface of water

$$CB = x + h = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha} + h = \frac{h(\tan\alpha + \tan\beta)}{\tan\beta - \tan\alpha}$$

Exercise 8

1. The shadow of a verticle pillar is same the height of pillar, then angle of elevation of Sun will be :

   (a) 45° (b) 30° (c) 60° (d) 50°

2. From a point on the ground which is 100 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°, then height of tower is :

   (a) $100\sqrt{3}$ m (b) $\dfrac{100}{\sqrt{3}}$ m (c) $50\sqrt{3}$ m (d) $\dfrac{200}{\sqrt{3}}$ m

3. A 15 m long ladder touches the top of a vertical wall. If this ladder makes an angle of 60° with the wall then height of the wall is :

(a) $15\sqrt{3}$ m (b) $\frac{15\sqrt{3}}{2}$ m (c) $\frac{15}{2}$ m (d) 15 m

4. From the top of 10 m height tower, angle of depression at a point on earth is 30°. Distance of point from base of tower is

(a) $10\sqrt{3}$ m (b) $\frac{10}{\sqrt{3}}$ m (c) 10 m (d) $5\sqrt{3}$ m

5. A bridge above the river makes an angle of 45° with the bank of river. If length of bridge above the river is 150 m then breadth of river will be

(a) 75 m (b) $50\sqrt{2}$ m (c) 150 m (d) $75\sqrt{2}$ m

6. Top of two towers of height 20 m and 14 m are joined by a wire. If wire makes an angle of 30° with horizontal line then length of wire is :

(a) 12 m (b) 10 m (c) 8 m (d) 6 m

7. The angle of elevation of the top of the tower form two points distance a and b from the base of tower $(a > b)$ are 30° and 60° then height of tower is :

(a) $\sqrt{a+b}$ (b) $\sqrt{a-b}$ (c) $\sqrt{ab}$ (d) $\sqrt{\frac{a}{b}}$

8. From the top of a 25 m high pillar the angle of elevation of top of the tower is same as the angle of depression of foot of tower then height of tower is :

(a) 25 m (b) 100 m (c) 75 m (d) 50 m

9. If ratio of length of a verticle rod and length of its shadow is $1 : \sqrt{3}$ then angle of elevation of sun is :

(a) 30° (b) 45° (c) 60° (d) 90°

10. The slope of a hill makes an angle of 60° with horizontal . If to reach at top, 500 m distance have to covered then height of the hill is :

(a) $500\sqrt{3}$ m (b) $\frac{500}{\sqrt{3}}$ m (c) $250\sqrt{3}$ m (d) $\frac{250}{\sqrt{3}}$ m

11. A tower is vertically placed on a horizontal plane. If angle of elevation of sun is 30° and length of shadow of tower is 45 m the find height of the tower.

12. The upper part of a tree is broken by windstrom and it makes an angle of 60° with the ground. The distance form the bottom of the tree to the point where the top touches the ground is 10 m.

Find the original height of the tree $\left(\sqrt{3} = 1.732\right)$

13. From a point on the ground which is 120 m away form the foot of the unfinished tower, the angle of elevation of the top of the tower is found to be 30°. Find how much height of tower have to increased so that its angle of elevation at same point become 60° ?

14. The angle of elevation of the top of a tower form a point situated at 100 m far form the foot of tower is 30°. Find the height of the tower.

15. The angle of elevation of the top of a pillar form a point on the ground is 15^{o} on walking 100 m towards the tower, the angle of elevation is found to be 30^{o}. Find the height of the tower (where $\tan 15^{0} = 2 - \sqrt{3}$)
16. The shadow of a verticle tower on level ground is increased by 40 m, when the altitude of the sun changes form 60^{o} to 30^{o}. Find the height of the tower.
17. The angle of depression of two ships from the top of light house situated at 60 m heigtht from sea level, are 30^{o} and 45^{o} if two ships are on the same side of the light house then find the distance between two ships.
18. A 1.5 m tall boy is standing at some distance away form a 30 m high building when he moves towards the building then angle of elevation from his eye become 60^{o} to 30^{o}. Find how much distance he covered towards the building ?
19. Angle of elevation of top of a tower form a 7 m high building is 60^{o} and angle of depression of its foot is 45^{o}. Find the height of the tower.
20. From the top of a hill, in east side at two points of angle of depression are 30^{o} and 45^{o}. If distance between two points is 1 km, then find height of the hill.
21. The angle of elevation of a cloud from a point 20 m above a lake (point A) is 30^{o}. If the angle of depression of its reflection form point A is 60^{o} then find the distance of cloud form point A.
22. From a point on a bridge across a river, the angles of depression of the banks on opposite side of the river are 30^{o} and 45^{o} respectively. If the bridge is at height of 4 m form the bank, find the width of the river.
23. A man on the deck of the ship is 10 m above water -level. He observes that the angle of elevation of the top of hill is 60^{o} and the angle of depression of the base is 30^{o} then find the distance of the hill form the ship and height of the hill.
24. A verticle straight tree 12 m high is broken by strong wind in such a way that its top touches the ground and makes an angle of 60^{o} with the ground. Find at what height from the ground did the tree break ? $\left(\sqrt{3} = 1.732\right)$
25. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30° , which is approaching the foot of the tower with a uniform speed. Six seconds later,the angle of depression of the car is found to be 60^{o}. Find the time taken by the car to reach the foot of the tower from this point.
26. The angles of elevation of the top of the tower form two points at a distance of 4 m and 9 m form the base of the tower in the same straight line are complementary. Prove that the height of tower is 6 m.
27. A tower and a building on the opposite side of road are situated. The angles of depression form the top of tower at the roof and base of building are 45^{o} and 60^{o} respectively. If height of building is 12 m then find the height of the tower $\left(\sqrt{3} = 1.732\right)$
28. If angle of elevation of sun changes from 30^{o} to 60^{o}. Then at these angles of elevation find the difference in the length of shadow of 15 m high pillar.

Important Points

1. The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.
2. Angle subtended by an eye with the horizontal to see an object in the upward direction is called angle of elevation.
3. Angle subtended by an eye with the horizontal to see an object in the downward direction is called angle of depression.
4. $\sin 30^\circ = 0.5774 = \cos 60^\circ$

   $\sin 45^\circ = 0.7071 = \cos 45^\circ$

   $\sin 60^\circ = 0.8660 = \cos 30^\circ$

   $\sqrt{2} = 1.4141,\ \sqrt{3} = 1.732$

Answers

Exercise 8

1. (a) 45°
2. (a) $100\sqrt{3}$ m
3. (c) $\frac{15}{2}$ m
4. (a) $10\sqrt{3}$ m
5. (d) $75\sqrt{2}$
6. (a) 12 m
7. (c) $\sqrt{ab}$
8. (d) 50 m
9. (a) 30°
10. (c) $250\sqrt{3}$ m
11. $15\sqrt{3}$ m
12. 37.32 m
13. 138.56 m
14. 57.73 m
15. 50 m
16. 34.64 m
17. 43.92 m
18. $19\sqrt{3}$ m
19. $7\left(\sqrt{3}+1\right)$ m
20. 1.366 km
21. 40 m
22. 10.92 m
23. $10\sqrt{3}$ m, 40 m
24. 5.569 m
25. 3 m
27. 28.392 m
28. 17.32 m