2. $z + \bar{z} = a + ib + a - ib = 2a = 2Re(z)$ as $Re(z) = a$

$$\therefore \quad \frac{z+\bar{z}}{2} = Re(z)$$

3. $z - \bar{z} = a + ib - a + ib = 2ib = 2i\, Im(z)$ as $Im(z) = b$

$$\therefore \quad \frac{z-\bar{z}}{2i} = Im(z)$$

4. $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow b = -b \Leftrightarrow 2b = 0 \Leftrightarrow b = 0.$

Thus, $z = \bar{z}$ if and only if z is real.

Modulus of a complex number :

Modulus of a complex number $z = a + ib$ is defined as $\sqrt{a^2+b^2}$ and is denoted by $|z|$.

Thus, $|z| = \sqrt{a^2+b^2}$

Note that $|z|$ is a real number and $|z| \geq 0, \quad \forall z \in C$.

As an example, if $z = 3 + 4i$, then $|z| = \sqrt{9+16} = \sqrt{25} = 5$

Notice that if z is a real number (i.e. $z = a + 0i$) then, $|z| = \sqrt{a^2} = |a|$, where $|z|$ is the modulus of the complex number and $|a|$ is the absolute value of the real number (recall that for any real number a we have $\sqrt{a^2} = |a|$).

Properties of modulus :

1. $|z| = 0$ if and only if $z = 0$

2. $|z| \geq |Re(z)|, |z| \geq |Im(z)|$

3. $z\bar{z} = |z|^2$

4. $|z| = |\bar{z}|$

5. $|z| = |-z|$

6. $\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z_2}}{|z_2|^2}$, where $z_2 \neq 0$

7. $|z_1z_2| = |z_1||z_2|$

8. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ where $z_2 \neq 0$

9. $|z_1 + z_2| \leq |z_1| + |z_2|$ **(Triangular inequality)** (Why triangular ?)

10. $|z_1 - z_2| \geq ||z_1| - |z_2||$

Let us verify some of the above properties :

1. $|z| = 0 \Leftrightarrow \sqrt{a^2+b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0, b = 0 \Leftrightarrow z = 0$

2. $|z|^2 = a^2 + b^2 = (Re(z))^2 + (Im(z))^2 \geq (Re(z))^2$

$\therefore \quad |z| \geq |Re(z)|$ Similarly, $|z| \geq |Im(z)|$

3. $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$

4. $|z| = |a + ib| = \sqrt{a^2+b^2}$ and $|\bar{z}| = |a - bi| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

so, $|z| = |\bar{z}|$

7. $|z_1z_2|^2 = (z_1z_2)\,(\overline{z_1z_2})$

$= (z_1z_2)\,(\overline{z_1}\,\overline{z_2})$

$= (z_1\overline{z_1})\,(z_2\overline{z_2})$

$= |z_1|^2\,|z_2|^2$

$\therefore\ |z_1z_2| = |z_1|\,|z_2|$

9. $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1}$

$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}}$ **($\overline{z_1\overline{z_2}} = \overline{z_1}\,\overline{\overline{z_2}} = \overline{z_1}z_2$)**

$= |z_1|^2 + |z_2|^2 + 2Re\,(z_1\overline{z_2})$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1|\,|\overline{z_2}|$

$= |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$

$= (|z_1| + |z_2|)^2$

$\therefore\ |z_1 + z_2| \leq |z_1| + |z_2|$

10. $z_1 - z_2 + z_2 = z_1$

$\therefore\ |z_1 - z_2 + z_2| = |z_1| \leq |z_1 - z_2| + |z_2|$

$\therefore\ |z_1| - |z_2| \leq |z_1 - z_2|$

Similarly, $|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$

But, $|z_1| - |z_2|$ or $|z_2| - |z_1| = \big||z_1| - |z_2|\big|$ **(If $a \in R$ then $|a| = a$ or $-a$)**

$\therefore\ \big||z_1| - |z_2|\big| \leq |z_1 - z_2|$ or $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$.

Example 5 : Find the conjugate and modulus of (1) $(2 - 3i)^2$ (2) $\frac{-3+7i}{1+i}$

Solution : (1) $(2 - 3i)^2 = 4 - 12i - 9 = -5 - 12i$

$\therefore$ Complex conjugate of $(2 - 3i)^2$ is $-5 + 12i$ and

$|(2 - 3i)^2| = |2 - 3i|^2 = 4 + 9 = 13$

(2) Let $z = \frac{-3+7i}{1+i}$

$= \frac{-3+7i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-3+3i+7i-7i^2}{1-i^2}$

$= \frac{4+10i}{2} = 2 + 5i$

$\therefore\ \overline{z} = 2 - 5i$ and $|z| = \sqrt{2^2+5^2} = \sqrt{29}$

or $|z| = \frac{|-3+7i|}{|1+i|} = \frac{\sqrt{49+9}}{\sqrt{2}} = \sqrt{29}$

Example 6 : If $z = x + yi$ and $| 3z | = | z - 4 |$, then prove that $x^2 + y^2 + x = 2$.

Solution : We have $| 3z | = | z - 4 |$

$\therefore \quad | 3x + 3yi | = | (x - 4) + yi |$

$\therefore \quad 3\sqrt{x^2 + y^2} = \sqrt{(x-4)^2 + y^2}$

$\therefore \quad 9(x^2 + y^2) = (x - 4)^2 + y^2$

$\therefore \quad 9x^2 + 9y^2 = x^2 - 8x + 16 + y^2$

$\therefore \quad 8x^2 + 8x + 8y^2 = 16$

$\therefore \quad x^2 + y^2 + x = 2$

Example 7 : If $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$, verify the following :

(1) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (2) $| z_1 + z_2 | < | z_1 | + | z_2 |$ (3) $| z_1 z_2 | = | z_1 || z_2 |$

Solution : We have $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$

(1) $z_1 z_2 = (3 + 4i)(12 - 5i) = 36 - 15i + 48i - 20i^2$

$= 36 - 15i + 48i + 20$

$= 56 + 33i$

$\therefore \quad \overline{z_1 z_2} = 56 - 33i$

Now, $\overline{z_1}\,\overline{z_2} = (3 - 4i)(12 + 5i) = 36 - 48i + 15i - 20i^2 = 56 - 33i$

Hence, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ is verified.

(2) $z_1 + z_2 = 3 + 4i + 12 - 5i = 15 - i$

$\therefore \quad | z_1 + z_2 | = \sqrt{225 + 1} = \sqrt{226}$

Also, $| z_1 | = \sqrt{9 + 16} = 5$, $| z_2 | = \sqrt{144 + 25} = 13$

Also, $| z_1 | + | z_2 | = 5 + 13 = 18 = \sqrt{324}$

Clearly, $\sqrt{226} < \sqrt{324}$

Hence, $| z_1 + z_2 | < | z_1 | + | z_2 |$ is verified.

(3) $| z_1 z_2 | = \sqrt{56^2 + 33^2} = \sqrt{3136 + 1089} = \sqrt{4225} = 65$ **(by (1))**

Also, $| z_1 || z_2 | = 5 \cdot 13 = 65$

Hence, $| z_1 z_2 | = | z_1 || z_2 |$ is verified.

Example 8 : (1) If $z \in$ C and $| z + 3 | \le 8$, find the maximum and minimum values of $| z - 2 |$.

(2) If $z \in$ C and $| z - 4 | \le 4$, find the maximum and minimum values of $| z + 1 |$.

Solution : (1) We have $| z + 3 | \le 8$

$| z - 2 | = | (z + 3) - 5 | \le | z + 3 | + | -5 |$ **(Triangular Inequality)**

$\le 8 + 5 = 13$

$\therefore \quad | z - 2 | \le 13$

If we take $z = -11$ then $| z + 3 | = | -11 + 3 | = 8$ and $| z - 2 | = 13$

So the maximum value of $| z - 2 |$ subject to $| z + 3 | \leq 8$ is 13.

Now, $| z - 2 | \geq 0$ is always true.

For $z = 2$, $| z + 3 | \leq 8$ is true and $| z - 2 | = 0$.

So the minimum value of $| z - 2 |$ subject to $| z + 3 | \leq 8$ is 0.

(2) We have $| z - 4 | \leq 4$

$$| z + 1 | = | (z - 4) + 5 | \leq | z - 4 | + | 5 | \qquad \text{(Triangular Inequality)}$$

$$\leq 4 + 5 = 9$$

$$\therefore \quad | z + 1 | \leq 9$$

If we take $z = 8$ then $| z - 4 | = 4$ and $| z + 1 | = 9$.

So the maximum value of $| z + 1 |$ subject to $| z - 4 | \leq 4$ is 9.

$| z + 1 | \geq 0$. If we let $z = -1$, $| z + 1 |$ would be zero.

But, $| z - 4 | = | -1 - 4 | = 5 \nleq 4$.

Thus the condition $| z - 4 | \leq 4$ is violated if $z = -1$.

Now, $| z + 1 | = | (z - 4) + 5 | = | (z - 4) - (-5) | \geq || z - 4 | - | -5 ||$

$$(| z_1 - z_2 | \geq || z_1 | - | z_2 ||)$$

$$\geq 5 - 4 = 1$$

$$\therefore \quad | z + 1 | \geq 1$$

If we take $z = 0$ then $| z - 4 | = 4$ and $| z + 1 | = 1$.

So, the minimum value of $| z + 1 |$ subject to $| z - 4 | \leq 4$ is 1.

Example 9 : If $z (\neq -1)$ is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then show that $| z | = 1$.

Solution : Let $z = x + iy$.

Then $\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$

Since $\frac{z-1}{z+1}$ is purely imaginary, we have $Re\left(\frac{z-1}{z+1}\right) = 0$

$$\therefore \quad \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0$$

$$\therefore \quad x^2 + y^2 = 1$$

$$\therefore \quad | z | = 1 \qquad (| z | = \sqrt{x^2 + y^2})$$

2.7 Argand Plane and Polar representation

Historically, the geometric representation of a complex number as a point in the plane is useful because it relates the whole idea of a complex number as an ordered pair in R^2. We know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa. (Figure 2.1)

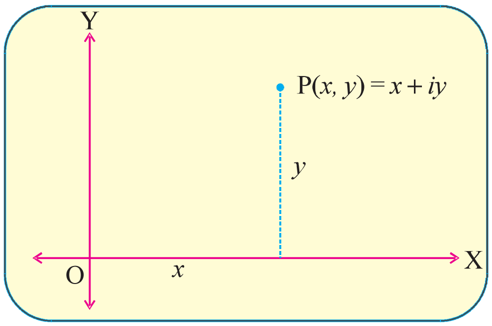

Figure 2.1

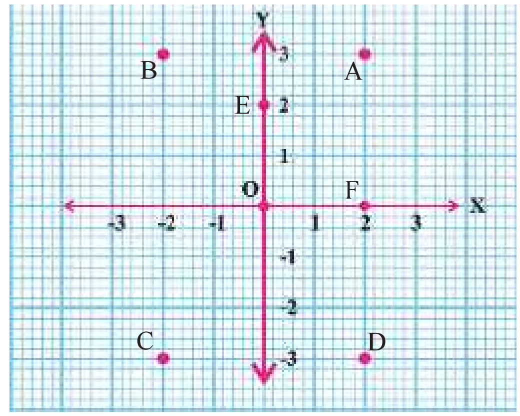

Figure 2.2

Some complex numbers such as $2 + 3i$, $-2 + 3i$, $-2 - 3i$, $2 - 3i$, $0 + 2i$, $2 + i0$ which correspond to the ordered pairs (2, 3), (−2, 3), (−2, −3), (2, −3), (0, 2), (2, 0) are represented geometrically by the points A, B, C, D, E, F respectively in the figure 2.2.

The plane having a complex number assigned to each of its point is called the **Complex Plane** or the **Argand Plane**. The points on the x-axis correspond to the complex numbers of the form $a + i0$ (real numbers) and the points on the y-axis correspond to the complex numbers of the form $0 + ib$ (purely imaginary numbers). The X-axis and Y-axis in the Argand plane are called the real axis and the imaginary axis respectively.

(**Jean-Robert Argand** (1768 – 1822) was a gifted amateur mathematician. In 1806, while managing a bookstore in Paris, he published the idea of geometrical interpretation of complex numbers known as the Argand diagram.)

Geometrical representation of modulus of a complex number :

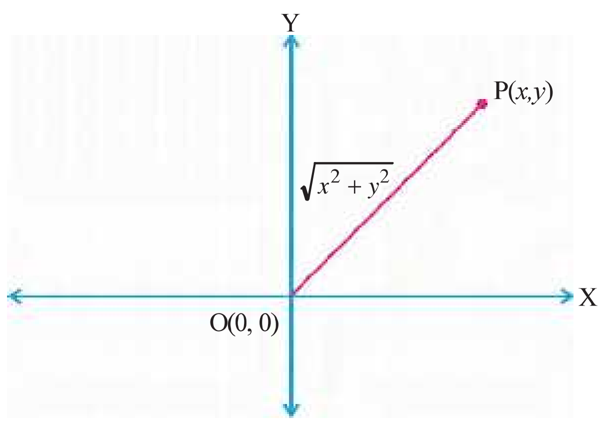

Figure 2.3

In the Argand plane, the modulus of the complex number $x + iy$ is the distance between the point P(x,y) and the origin O(0,0). (Figure 2.3)

Geometrical representation of the conjugate of a complex number :

The representations of a complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ in the Argand plane are the points P(x,y) and Q(x,$-y$) respectively. Geometrically, the point Q(x,$-y$) is called the **mirror image** of the point P(x,y) with respect to the real axis. (Figure 2.4)

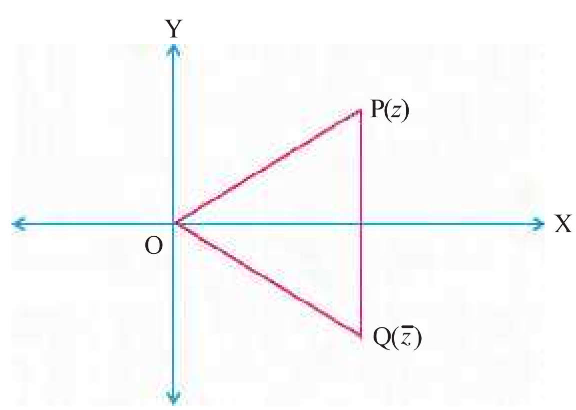

Figure 2.4

Geometrical representation of the sum of two complex numbers :

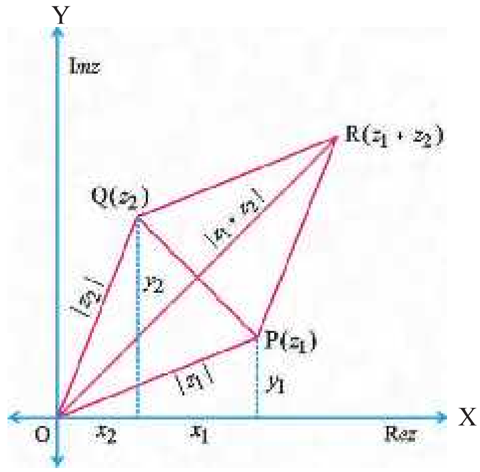

Figure 2.5

From the figure 2.5, in the argand plane P, Q and R represent z_1, z_2 and $z_1 + z_2$ respectively, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Mid-point of $\overline{OR}$ and $\overline{PQ}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

$\therefore$ $\overline{OR}$ and $\overline{PQ}$ bisect each other.

Here, we have assumed that O, P and Q are non-collinear points.

The absolute values of z_1, z_2 and $z_1 + z_2$ are geometrically given by $|z_1| = OP$, $|z_2| = OQ = PR$ and $|z_1 + z_2| = OR$. We know that the sum of any two sides of a triangle is greater than the third side. Hence, in ΔORP, we have $OR < OP + PR$ implying $|z_1 + z_2| < |z_1| + |z_2|$. That is why this inequality for the absolute values of complex numbers is called the triangular inequality. (When does equality occur in $|z_1 + z_2| \le |z_1| + |z_2|$?)

Polar representation of a complex number :

There is an alternate form to represent a complex number $z = x + iy$ which is known as polar representation. Let us understand how we can express any complex number into polar form. Let $z = x + iy$ be a non-zero complex number represented by the point P(x, y). (Figure 2.6) Draw $\overline{PM} \perp \overleftrightarrow{OX}$. Then $OM = x$ and $PM = y$. Draw $\overline{OP}$. Let $OP = r$ and $m\angle MOP = \theta$. Then $x = r\cos\theta$ and $y = r\sin\theta$.

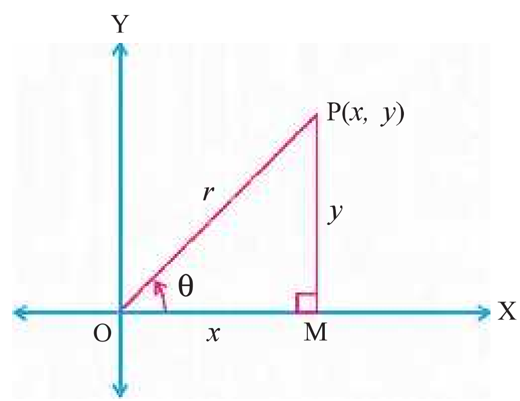

Figure 2.6

Therefore $z = x + iy = r(\cos\theta + i\sin\theta)$

Note : Here P lies in the first-quadrant.

$\therefore$ $x > 0$, $y > 0$. But if P(x, y) lies anywhere in the Argand plane except for origin, then also $x = r\cos\theta$, $y = r\sin\theta$ are true.

$\therefore$ $z = x + iy = r(\cos\theta + i\sin\theta)$

Here, $r^2 = x^2 + y^2$ **(r = OP > 0)**

$\therefore$ $r = \sqrt{x^2 + y^2}$ **($r > 0$)**

$\therefore$ $r = \sqrt{x^2 + y^2} = |z|$ and $\tan\theta = \frac{y}{x}$

The form $z = r(\cos\theta + i\sin\theta)$ is called the **polar form** of the complex number z. Also θ is known as **amplitude** or **argument of z**, written as **arg(z)**. Since *sine* and *cosine* functions are periodic, there are many values of θ satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Each of these θ is an argument of z. The unique value of θ such that $-\pi < \theta \le \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$ is known as the

principal value of arg(z). While reducing a complex number to polar form, we always take the principal value of arg(z). Unless specified the notation arg(z) means principal value of arg(z). To find the value of arg(z), one has to take care of the position of the point in the plane. **Argument of the complex number 0 is not defined (Why ?)**

$$arg(x + i0) = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases} \qquad arg(0 + iy) = \begin{cases} \frac{\pi}{2}, & \text{if } y > 0 \\ -\frac{\pi}{2}, & \text{if } y < 0 \end{cases}$$

$\therefore$ Argument of positive real number is 0 and that of negative real number is π. Similarly argument of purely imaginary number yi is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as $y > 0$ or $y < 0$ respectively.

Also, $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$ and $-\pi < \theta \le \pi$.

(i) If $x > 0$, $y > 0$, then we can get θ, $0 < \theta < \frac{\pi}{2}$, such that $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$.

(ii) If $x < 0$, $y > 0$, then we find α such that $cos\alpha = \frac{|x|}{r}$, $sin\alpha = \frac{|y|}{r}$.

$0 < \alpha < \frac{\pi}{2}$. Let $\theta = \pi - \alpha$. Then $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$.

(iii) If $x < 0$, $y < 0$, then we find α such that $cos\alpha = \frac{|x|}{r}$, $sin\alpha = \frac{|y|}{r}$.

$0 < \alpha < \frac{\pi}{2}$. Let $\theta = -\pi + \alpha$. Then $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$

(iv) If $x > 0$, $y < 0$, then we find α such that $cos\alpha = \frac{|x|}{r}$, $sin\alpha = \frac{|y|}{r}$.

$0 < \alpha < \frac{\pi}{2}$. Let $\theta = -\alpha$. Then $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$.

Example 10 : Write the following complex numbers in polar form. Determine the modulus and the principal value of the argument in each case :

(1) $1 + i$ (2) $-1 + \sqrt{3}i$ (3) $-\sqrt{3} - i$ (4) $1 - i$

(5) -3 (6) $-2i$ (7) 1 (8) $2i$

Solution : (1) Let $z = 1 + i = x + iy$

$\therefore$ $x = 1, y = 1$

$\therefore$ $r = \sqrt{x^2 + y^2} = \sqrt{2}$

$cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$ and $sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$

$\therefore$ P(θ) lies in the first quadrant.

$\therefore$ $\theta = \frac{\pi}{4}$

$\therefore$ The polar form of z is $\sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$.

$|z| = r = \sqrt{2}$, $arg\, z = \theta = \frac{\pi}{4}$.

(2) Let $z = -1 + \sqrt{3}i = x + iy$

$\therefore$ $x = -1, y = \sqrt{3}$

$\therefore$ $r = |z| = \sqrt{1+3} = 2$

$cos\theta = \frac{-1}{r} = \frac{-1}{2}$ and $sin\theta = \frac{\sqrt{3}}{r} = \frac{\sqrt{3}}{2}$

$\therefore \quad cos\alpha = \frac{1}{2}, sin\alpha = \frac{\sqrt{3}}{2}$ **$(|x| = 1, |y| = \sqrt{3})$**

$\therefore \quad \alpha = \frac{\pi}{3}$

Since $x < 0$, $y > 0$, P(θ) lies in the second quadrant.

$\therefore \quad \theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore$ The polar form of z is $2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$.

Also, $|z| = r = 2$, $arg\, z = \theta = \frac{2\pi}{3}$.

(3) Let $z = -\sqrt{3} - i = x + iy$

$\therefore \quad x = -\sqrt{3}, y = -1$

$\therefore \quad r = |z| = \sqrt{3+1} = 2$

$cos\theta = \frac{-\sqrt{3}}{2}$ and $sin\theta = \frac{-1}{2}$

$\therefore \quad cos\alpha = \frac{\sqrt{3}}{2}, sin\alpha = \frac{1}{2}$ **$(|x| = \sqrt{3}, |y| = 1)$**

$\therefore \quad \alpha = \frac{\pi}{6}$

Since $x < 0$, $y < 0$, P(θ) lies in the third quadrant.

$\therefore \quad \theta = -\pi + \alpha = -\pi + \frac{\pi}{6} = \frac{-5\pi}{6}$

$\therefore$ The polar form of z is $2\left(cos\left(-\frac{5\pi}{6}\right) + isin\left(-\frac{5\pi}{6}\right)\right)$.

Also, $|z| = r = 2$, $arg\, z = \theta = \frac{-5\pi}{6}$.

(4) Let $z = 1 - i = x + iy$

$\therefore \quad x = 1, y = -1$

$\therefore \quad r = |z| = \sqrt{1+1} = \sqrt{2}$

$cos\theta = \frac{1}{r} = \frac{1}{\sqrt{2}}$ and $sin\theta = \frac{-1}{r} = \frac{-1}{\sqrt{2}}$

$\therefore \quad cos\alpha = \frac{1}{\sqrt{2}}, sin\alpha = \frac{1}{\sqrt{2}}$ **$(|x| = 1, |y| = 1)$**

$\therefore \quad \alpha = \frac{\pi}{4}$

Since $x > 0$, $y < 0$, P(θ) lies in the fourth quadrant.

$\therefore \quad \theta = -\alpha = -\frac{\pi}{4}$

$\therefore$ The polar form of z is $\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\right)$.

Also, $|z| = r = \sqrt{2}$, $arg\, z = \theta = -\frac{\pi}{4}$.

(5) Let $z = -3$. Here $z = x + i0$ and $x < 0$.

$\therefore$ Its polar form is $3(cos\pi + i\, sin\pi)$

Also, $|z| = 3$, $arg\, z = \theta = \pi$.

(6) Let $z = -2i$. Here $z = 0 + iy$ and $y < 0$.

$\therefore$ Its polar form is $2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$.

Also, $|z| = 2$, $arg\ z = \theta = -\frac{\pi}{2}$.

(7) Let $z = 1$. Here $z = x + i0$ and $x > 0$. So Its polar form is $1(\cos 0 + i\sin 0)$.

Also, $|z| = 1$, $arg\ z = \theta = 0$.

(8) Let $z = 2i$. Here $z = 0 + iy$ and $y > 0$. So its polar form is $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

Also, $|z| = 2$, $arg\ z = \theta = \frac{\pi}{2}$.

Exercise 2.2

1. Find the absolute value and the principal argument of the following complex numbers :

(1) $\frac{1+7i}{(2-i)^2}$ (2) $\left(\frac{2+i}{3-i}\right)^2$ (3) $\sqrt{3} - i$ (4) $\frac{(1+i)(1+\sqrt{3}i)}{1-i}$ (5) $-3\sqrt{2} + 3\sqrt{2}\,i$

2. If $z = 3 + 2i$, then verify the following :

(1) $|z| = |\bar{z}|$ (2) $-|z| \le Re(z) \le |z|$ (3) $z^{-1} = \frac{\bar{z}}{|z|^2}$

3. If $z_1 = 3 + 2i$ and $z_2 = 2 - i$, then verify the following :

(1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (2) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

4. If z is a non-zero complex number, show that $\overline{(z^{-1})} = (\bar{z})^{-1}$.

5. If $(a + ib)^2 = \frac{1+i}{1-i}$, show that $a^2 + b^2 = 1$.

6. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, then is it necessary that $z_1 = z_2$? Justify your answer.

7. A complex number $z = a + ib$ is such that $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$. Show that $a^2 + b^2 - 2b = 1$.

8. Find the maximum value of $|1 + z + z^2 + z^3|$, if $z \in C$ and $|z| \le 3$.

9. (1) If $z = a + ib$ and $2|z - 1| = |z - 2|$, prove that $3(a^2 + b^2) = 4a$.

(2) If $z \in C$ such that $|2z - 3| = |3z - 2|$, prove that $|z| = 1$.

(3) If $z \in C$ such that $|2z - 1| = |z - 2|$, prove that $|z| = 1$.

10. Show that complex number $-3 + 2i$ is closer to the origin than $1 + 4i$.

11. Represent the points $-2 + 3i$, $-2 - i$ and $4 - i$ in the Argand diagram and prove that they are vertices of a right angled triangle.

12. Find the complex number z whose modulus is 4 and argument is $\frac{5\pi}{6}$.

13. If $(1 - 5i)z_1 - 2z_2 = 3 - 7i$, find z_1 and z_2, where z_1 and z_2 are conjugate complex numbers.

14. If $(a + ib)^2 = x + iy$ prove that $x^2 + y^2 = (a^2 + b^2)^2$.

15. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

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2.8 Square Roots of a Complex Number

If $(a + ib)^2 = z = x + iy$, we say that $a + ib$ is a square root of z.

Let $z = x + iy$ and let a square root of z be the complex number $a + ib$, if it exists.

$\therefore\quad x + iy = (a + ib)^2$

$\therefore\quad x + iy = (a^2 - b^2) + (2ab)i$

$\therefore\quad a^2 - b^2 = x$ and $2ab = y$ **(i)**

Now, $a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = \sqrt{x^2 + y^2} = |\,z\,|$ **(by (i)) (ii)**

From (i) and (ii) we get $2a^2 = |\,z\,| + x$ i.e. $a = \pm\sqrt{\frac{|z| + x}{2}}$ and $b = \pm\sqrt{\frac{|z| - x}{2}}$

If $y > 0$, then a and b both positive or both negative as $y = 2ab$.

Therefore, the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right)$.

If $y < 0$, then out of a and b, one is positive and another is negative.

Therefore, the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right)$.

Now, we have proved that every complex number has two square roots.

Example 11 : Find the square roots of (1) $\sqrt{3} - i$ (2) $7 + 24i$

Solution : (1) Let $z = \sqrt{3} - i$. Here $x = \sqrt{3}$, $y = -1 < 0$

$|\,z\,| = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$

We know that if $y < 0$, then the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|\,z\,| + x}{2}} - i\sqrt{\frac{|\,z\,| - x}{2}}\right)$.

Hence the square roots of $\sqrt{3} - i$ are $\pm\left(\sqrt{\frac{2 + \sqrt{3}}{2}} - i\sqrt{\frac{2 - \sqrt{3}}{2}}\right)$.

Now $2 + \sqrt{3} = \frac{4 + 2\sqrt{3}}{2} = \frac{(\sqrt{3} + 1)^2}{2}$

$\therefore\quad$ The square roots of $z = \sqrt{3} - i$ are $\pm\left(\frac{\sqrt{3} + 1}{2} - i\,\frac{\sqrt{3} - 1}{2}\right)$. **($\sqrt{2}\,\sqrt{2} = 2$)**

(2) Let $z = 7 + 24i$. Here $x = 7$, $y = 24 > 0$

$|\,z\,| = \sqrt{x^2 + y^2} = \sqrt{49 + 576} = 25$

We know that if $y > 0$, then the square roots of $x + iy$ are $\pm\left(\sqrt{\frac{|\,z\,| + x}{2}} + i\sqrt{\frac{|\,z\,| - x}{2}}\right)$.

Hence the square roots of $7 + 24i$ are $\pm\left(\sqrt{\frac{25 + 7}{2}} + i\sqrt{\frac{25 - 7}{2}}\right) = \pm(4 + 3i)$.

Example 12 : Find the square roots of (1) 1 (2) -1 (3) i (4) $-i$

(1) Let $z = 1$

$\therefore\quad |\,z\,| = 1$. Let the square roots of z be $a + ib$.

$\therefore\quad (a + ib)^2 = 1$

$\therefore\quad a^2 - b^2 + 2abi = 1 = 1 + 0i$

$\therefore$ $a^2 - b^2 = 1$, $2ab = 0$. From $2ab = 0$ we have $a = 0$ or $b = 0$.

From $a = 0$, we have $-b^2 = 1$ which is not possible as $b \in$ R. So $a \neq 0$.

$\therefore$ $2ab = 0$ gives $b = 0$

$\therefore$ $a^2 = 1$

$\therefore$ $a = \pm 1$

$\therefore$ $a + ib = \pm 1$

$\therefore$ Square roots of 1 are ± 1.

Note : In R, we know square roots of 1 are ± 1.

(2) Let $z = -1$. Let the square root of z be $a + ib$.

$\therefore$ $(a + ib)^2 = -1$

$\therefore$ $a^2 - b^2 + 2abi = -1$

$\therefore$ $a^2 - b^2 = -1$, $2ab = 0$

$2ab = 0$ gives $a = 0$ or $b = 0$

But $b = 0$ gives $a^2 = -1$ which is not possible as $a \in$ R. So $b \neq 0$.

$\therefore$ $a = 0$ and $b^2 = 1$

$\therefore$ $b = \pm 1$

$\therefore$ Square roots of -1 are $\pm i$. (as we expected since $i^2 = -1$)

Remember $i^2 = -1$.

Similarly the square roots of -4 are $\pm 2i$,

the square roots of -3 are $\pm\sqrt{3}i$.

(3) Let $z = a + ib$ be a square root of i.

$\therefore$ $(a + ib)^2 = i$

$\therefore$ $a^2 - b^2 + 2iab = i$

$\therefore$ $a^2 - b^2 = 0$ and $2ab = 1$

$\therefore$ $a = b$ or $a = -b$

But $a = -b$ gives $-2a^2 = 1$ using $2ab = 1$.

This is not possible.

$\therefore$ $a = b$ and $2a^2 = 1$

$\therefore$ $a = \pm\frac{1}{\sqrt{2}}$. Since $a = b$ we have $b = \pm\frac{1}{\sqrt{2}}$.

$\therefore$ Square roots of i are $\pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$.

(4) Let $z = -i$. From (3) above $a^2 - b^2 = 0$, $2ab = -1$

$\therefore$ $a = b$ or $a = -b$

If $a = b$, then $2a^2 = -1$ which is not possible.

$\therefore$ $a = -b$ and $2a^2 = 1$

$\therefore$ $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$ and $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

$\therefore$ The square roots of $-i$ are $\pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$.

2.9 Quadratic Equations having Complex Roots

We have studied quadratic equations and solved them in the set of real numbers when the value of discriminant is non-negative. i.e. when D $\geq$ 0. Now we can answer the unanswered question, 'What happens when D < 0 ?'

Now let us try to solve quadratic equation $ax^2 + bx + c = 0$, $a, b\ c \in R, a \neq 0$, where $D = b^2 - 4ac < 0$.

$$ax^2 + bx + c = \frac{1}{a}(a^2x^2 + abx + ac)$$

$$= \frac{1}{a}\left[\left(ax + \frac{b}{2}\right)^2 + ac - \frac{b^2}{4}\right]$$

$$= \frac{1}{a}\left[\left(ax + \frac{b}{2}\right)^2 + \frac{4ac - b^2}{4}\right]$$

If $ax^2 + bx + c = 0$, then $\left(ax + \frac{b}{2}\right)^2 = \frac{b^2 - 4ac}{4}$.

Now, $b^2 - 4ac < 0$

$\therefore$ Square root of $ax + \frac{b}{2}$ = Square root of $\frac{b^2 - 4ac}{4}$ that is $\frac{\pm i\sqrt{4ac - b^2}}{2}$.

$\therefore \quad ax + \frac{b}{2} = \frac{\pm i\sqrt{4ac - b^2}}{2}$

$\therefore \quad x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$ **($a \neq 0$)**

If D < 0, roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm i\sqrt{-D}}{2a}$.

Fundamental Theorem of Algebra :

Every polynomial equation having complex coefficients and degree $\geq$ 1 has at least one complex root.

Example 13 : Solve (1) $x^2 + 3 = 0$ (2) $2x^2 + x + 1 = 0$ (3) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Solution :

(1) $x^2 + 3 = 0$

$\therefore \quad x^2 = -3$

$\therefore \quad x = \pm\sqrt{3}\,i$

(2) Here, $a = 2$, $b = 1$, $c = 1$

$\therefore \quad b^2 - 4ac = 1 - 4 \cdot 2 \cdot 1 = -7 < 0$

Therefore, the solutions are given by $x = \frac{-b \pm i\sqrt{-D}}{2a} = \frac{-1 \pm \sqrt{7}i}{4}$

(3) Here, $a = \sqrt{3}$, $b = -\sqrt{2}$, $c = 3\sqrt{3}$

$\therefore \quad b^2 - 4ac = 2 - 4\sqrt{3} \cdot 3\sqrt{3} = 2 - 36 = -34 < 0$

Therefore, the solutions are given by $x = \frac{-b \pm i\sqrt{-D}}{2a} = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}} = \frac{1 \pm \sqrt{17}i}{\sqrt{6}}$

2.10 Cube Roots of Unity

Let z be a cube roots of unity.

Then, $z^3 = 1$

$\therefore \quad z^3 - 1 = 0$

$\therefore \quad (z-1)(z^2+z+1)=0$

$\therefore \quad z=1$ or $z^2+z+1=0$

$\therefore \quad z=1, \frac{-1\pm\sqrt{3}i}{2}$ **(a = 1, b = 1, c = 1, D = −3)**

Hence, the cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$.

Properties of Cube Roots of Unity :

(1) Each of the two non-real cube roots of unity is the square of each other.

Let $\omega=\frac{-1+\sqrt{3}i}{2}$. Then $\omega^2=\left(\frac{-1+\sqrt{3}i}{2}\right)^2=\frac{1}{4}(1-2\sqrt{3}i+3i^2)=\frac{-1-\sqrt{3}i}{2}$

Also, $(\omega^2)^2=\omega^4=\omega^3\omega=\omega$. Hence cube roots of unity are 1, ω, ω^2.

(2) We observe that sum of the cube roots of unity is 0. i.e. $1+\omega+\omega^2=0$

(3) It can easily verify that product of cube roots of unity is 1. i.e. $1\cdot\omega\cdot\omega^2=\omega^3=1$

(4) Representing 1, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$ in the Argand plane as A, B, C respectively then A is (1, 0), B is $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and C is $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$. Note that AB = BC = AC = $\sqrt{3}$. Thus A, B, C are the vertices of an equilateral triangle. (Figure 2.7)

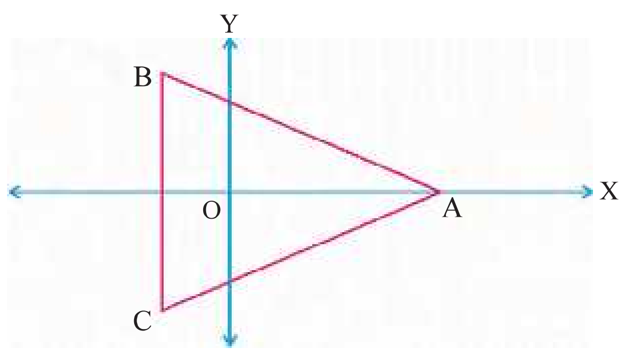

Figure 2.7

Exercise 2.3

1. Solve :

(1) $x^2+2=0$ (2) $x^2+x+1=0$ (3) $\sqrt{5}x^2+x+\sqrt{5}=0$

(4) $x^2+x+\frac{1}{\sqrt{2}}=0$ (5) $x^2+\frac{x}{\sqrt{2}}+1=0$ (6) $3x^2-4x+\frac{20}{3}=0$

2. Find the square roots of :

(1) $4+4\sqrt{3}i$ (2) $5-12i$ (3) $-48+14i$ (4) $3-4\sqrt{10}i$

(5) $\frac{1}{i}+\frac{1}{i^2}+\frac{1}{i^3}+\frac{1}{i^4}+\frac{1}{i^5}+\frac{1}{i^6}$ (6) $4i$ (7) $-16i$ (8) -25 (9) -10

3. When do we have $|z_1+z_2|=|z_1|+|z_2|$? Prove your contention.

4. Prove that in the Argand plane if P represents z and Q represents iz, then OP = OQ and $m\angle POQ=\frac{\pi}{2}$. State geometrical meaning.

5. Prove points representing z, iz, $-z$ and $-iz$ in Argand plane form a square.

6. What is the relation between representation of z and $\bar{z}$ in the Argand plane ?

*

Miscellaneous Problems :

Example 14 : Find all the complex numbers z satisfying the condition $\bar{z} = z^2$.

Solution : Let $z = x + iy$ be such that $\bar{z} = z^2$.

$\therefore \quad x - iy = (x^2 - y^2) + i(2xy)$

By definition of equality of complex numbers, we have $x = x^2 - y^2$ and $-y = 2xy$.

From the second result we have either $y = 0$ or $x = -\frac{1}{2}$.

Assume first $y = 0$. Then from $x = x^2 - y^2$, we have $x = x^2$

$\therefore \quad x = 0$ or $x = 1$ **($y = 0$)**

So in this case $z = 0$ or $z = 1$

Now, if $x = -\frac{1}{2}$, then $-\frac{1}{2} = \frac{1}{4} - y^2$ **($x = x^2 - y^2$)**

$\therefore \quad y = \pm\frac{\sqrt{3}}{2}$

So in the second case $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ or $z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Consequently, there are four complex numbers 0, 1, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ satisfying the equation $\bar{z} = z^2$.

Example 15 : Find real θ such that $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is real. Also find the number.

Solution : We have, $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} = \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \times \frac{1 + 2i\sin\theta}{1 + 2i\sin\theta}$

$$= \frac{3 + 6i\sin\theta + 2i\sin\theta + 4i^2\sin^2\theta}{1 + 4\sin^2\theta}$$

$$= \frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} + i\frac{8\sin\theta}{1 + 4\sin^2\theta}$$

If the given complex number is real, its imaginary part is zero.

Therefore, $\frac{8\sin\theta}{1 + 4\sin^2\theta} = 0$

$\therefore \quad \sin\theta = 0$

$\therefore \quad \theta = k\pi, k \in Z$

This number is $\frac{3 + 0}{1 - 0} = 3$

Exercise 2

1. Reduce : (1) $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$ (2) $\left(\frac{1}{1 - 4i} - \frac{2}{1 + i}\right)\left(\frac{3 - 4i}{5 + i}\right)$ to the standard form.

2. Find the modulus of $\frac{1 + i}{1 - i} - \frac{1 - i}{1 + i}$.

3. For any two complex numbers z_1 and z_2, prove that $Re(z_1z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$.

4. Find the value of $Re(f(z))$ and $Im(f(z))$ for $f(z) = \frac{1}{1-z}$ at $z = 7 + 2i$.

5. Show that the point set of the equation $|z - 1| = |z + i|$ represents a line through the origin whose slope is -1.

6. Prove that $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$.

7. If z_1 and z_2 are distinct complex numbers with $|z_2| = 1$, then find the value of $\left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right|$.

8. If $\frac{1}{\alpha + i\beta} + \frac{1}{a + ib} = 1$, where α, β, a and b real, express b in terms of α and β.

9. If $(x + iy)^3 = a + ib$, prove that $\frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2)$.

10. Solve : (1) $x^2 - 2x + \frac{3}{2} = 0$ (2) $27x^2 - 10x + 1 = 0$ (3) $21x^2 - 28x + 10 = 0$

11. If $z \in C$ and $|z| \le 2$, find the maximum and minimum values of $|z - 3|$.

12. For $z = 3 - 2i$ show that $z^2 - 6z + 13 = 0$. Hence obtain the value of $z^4 - 4z^3 + 6z^2 - 4z + 17$.

13. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

14. If $(x - iy)^2 = \frac{a - ib}{c - id}$, prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$

15. Find the value of z which satisfies the equation $|z| - z = 1 + 2i$.

16. If the complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then show that $z_1 + z_2 + z_3 = 0$.

17. Show that the area of the triangle in the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $\frac{1}{2}|z|^2$.

18. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, show that $|w| = 1 \Rightarrow z$ is real.

19. If $z = -5 + 4i$, show that $z^4 + 9z^3 + 35z^2 - z + 164 = 0$.

20. If $z = x + iy$, prove that $|x| + |y| \le \sqrt{2}|z|$.

21. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Solution of $|z - 4| < |z - 2|$ is given by ... ☐

(a) $Re(z) > 0$ (b) $Re(z) < 0$ (c) $Re(z) > 3$ (d) $Re(z) > 2$

(2) If $|z - 1|^2 = |z|^2 + 1$, then z lies on...... in the Argand diagram. ☐

(a) $x^2 + y^2 = 1$ (b) the imaginary axis

(c) the real axis (d) $2x + 3 = 0$

(3) If $|z + 4| \le 3$, then the maximum value of $|z + 1|$ is ... ☐

(a) 6 (b) 0 (c) 4 (d) 10

(4) The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is ... ☐

(a) $\frac{1}{i-1}$ (b) $\frac{-1}{i-1}$ (c) $\frac{1}{i+1}$ (d) $\frac{-1}{i+1}$

(5) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to ... ☐

(a) 1 (b) -1 (c) 0 (d) i^n

(6) The multiplicative inverse of $\frac{3+4i}{4-5i}$ is ...

(a) $-\frac{8}{25} + \frac{31}{25}i$ (b) $\frac{8}{25} - \frac{31}{25}i$ (c) $-\frac{8}{25} - \frac{31}{25}i$ (d) $\frac{8}{25} + \frac{31}{25}i$

(7) If $x + iy = \frac{u+iv}{u-iv}$, then $x^2 + y^2 =$ ....

(a) 1 (b) -1 (c) 0 (d) 2

(8) The smallest positive integer n for which $(1 + i)^{2n} = (1 - i)^{2n}$ is ...

(a) 4 (b) 8 (c) 2 (d) 12

(9) On the Argand plane the complex number $\frac{1+2i}{1-i}$ lies in the ...... quadrant.

(a) first (b) second (c) third (d) fourth

(10) $arg(-1) =$ ......

(a) 0 (b) π (c) $\frac{\pi}{2}$ (d) $-\pi$

(11) The complex numbers $sinx + icos2x$ and $cosx - isin2x$ are conjugate of each other, for ...

(a) $x = k\pi, k \in Z$ (b) $x = 0$

(c) $x = \left(k+\frac{1}{2}\right)\pi, k \in Z$ (d) no value of x

(12) If a complex number lies in the third quadrant, then its conjugate lies in the ...... quadrant.

(a) first (b) second (c) third (d) fourth

(13) The complex number with modulus 2 and argument $\frac{2\pi}{3}$ is ...

(a) $-1 + i\sqrt{3}$ (b) $-1 - i\sqrt{3}$ (c) $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ (d) $\frac{1}{2} - \frac{i\sqrt{3}}{2}$

(14) Argument of $1 - i\sqrt{3}$ is ...

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$

(15) If the cube roots of unity are $1, \omega, \omega^2$, then $1 + \omega + \omega^2 =$ ......

(a) 1 (b) 0 (c) -1 (d) ω

*

Summary

We studied following points in this chapter :

1. A number of the form $a + ib$, where a and b are real numbers, is called a complex number where $i^2 = -1$.

2. Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers.

$z_1 + z_2 = (a + c) + i(b + d)$, $z_1z_2 = (ac - bd) + i(ad + bc)$

3. $(a + ib)(a - ib) = a^2 + b^2$

4. Multiplicative inverse of a non-zero complex number $z = a + ib$ is $\frac{1}{z} = z^{-1} = \frac{a}{a^2+b^2} + \frac{-bi}{a^2+b^2}$.

5. $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

6. For complex number $z = a + bi$, its complex conjugate is $\bar{z} = a - bi$.

7. Modulus of a complex number $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$.

8. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa.

9. Square roots of $x + iy$ are $\begin{cases} \pm\left(\sqrt{\dfrac{|z|+x}{2}} + i\sqrt{\dfrac{|z|-x}{2}}\right), & y > 0 \\ \pm\left(\sqrt{\dfrac{|z|+x}{2}} - i\sqrt{\dfrac{|z|-x}{2}}\right), & y < 0 \end{cases}$

10. The cube roots of unity are 1, $\omega = \dfrac{-1+\sqrt{3}i}{2}$, $\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$

11. If $b^2 - 4ac < 0$, the solutions of $ax^2 + bx + c = 0$ where $a, b, c \in R$, $a \neq 0$ are $\dfrac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$.

— ❖ —

Brahmagupta was the first to use zero as a number. He gave rules to compute with zero. Negative numbers did not appear in *Brahmaphuta siddhanta* but in the Nine Chapters on the Mathematical Art (Jiu zhang suan-shu) around 200 BC. Brahmagupta's most famous work is his *Brahmasphutasiddhanta*.

Brahmagupta gave the solution of the general linear equation in chapter eighteen of *Brahmasphutasiddhanta*.

The difference between *rupas*, when inverted and divided by the difference of the unknowns, is the unknown in the equation. The *rupas* are [subtracted on the side] below that from which the square and the unknown are to be subtracted which is a solution equivalent to $x = \dfrac{e-c}{b-d}$, where *rupas* represents constants. He further gave two equivalent solutions to the general quadratic equation.

Diminish by the middle [number] the square root of the *rupas* multiplied by four times the square and increased by the square of the middle; divide the remainder by twice the square. the middle.

Whatever is the square root of the *rupas* multiplied by the square [and] increased by the square of half the unknown, diminish that by half the unknown [and] divide [the remainder] by its square. [The result is] the unknown which are, respectively, solutions equivalent to, $x = \dfrac{\sqrt{4ac+b^2} - b}{2a}$.

Brahmagupta then goes on to give the sum of the squares and cubes of the first n integers.

The sum of the squares is that [sum] multiplied by twice the [number of] step[s] increased by one [and] divided by three. The sum of the cubes is the square of that [sum] Piles of these with identical balls [can also be computed].

It is important to note here Brahmagupta found the result in terms of the sum of the first n integers.

He gives the sum of the squares of the first n natural numbers as $n(n + 1)(2n + 1)/6$ and the sum of the cubes of the first n natural numbers as $\left(\dfrac{n(n+1)}{2}\right)^2$.

Chapter 3

BINOMIAL THEOREM

The laws of nature are but the mathematical thoughts of God.

– Euclid

*

I like mathematics because it is not human and has nothing particular to do with this planet or with the whole accidental universe, because like Spinoza's God, it won't love us in return.

– Bertrand Russell

*

If there is God, he is a great mathematician.

– Paul Dirac

3.1 Introduction

In earlier classes, we have learnt about expansions like,

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and even $(a + b)^4$ as a product of $(a + b)^3$ with $(a + b)$

i.e. $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

However, the expansions of $(a + b)^5$, $(a + b)^6$, ... become difficult by using multiplication.

It is believed that in the eleventh century, Persian poet and mathematician **Omar Khayyám** gave the general formula for $(a + b)^n$, where n is a positive integer. This formula or expansion is called the **Binomial Theorem.**

Euclid (Fourth B.C.) a **Greek mathematician** gave a specific example of Binomial Expansion for $n = 2$. An **Indian mathematician Pingla** (Third Century B.C.) had given the idea about the higher order expansions. In the tenth century an **Indian mathematician Halayadha** was aware of general binomial theorem and Pascal's Triangle. Persian mathematician Al-Karaji and in 13th century Chinese mathematician Yang hui have also obtained such results.

The coefficients of the consecutive terms in the expansion of $(a + b)^n$, for $n = 1, 2, 3, ...$ can also be obtained from a row from triangular arrangement of numbers, known as **Pascal's Triangle** named after **French mathematician Blaise Pascal** (1623-1662).

Index	Coefficients
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
.	. . . .
.	. . . .
.	. . . .

In Pascal's Triangle first and last element of any row is 1, while the other elements are obtained by adding the numbers of the upper row which are at the beginning of the arrows.

Pascal's Triangle : The first row is 1 1

i.e. $\binom{1}{0}$ $\binom{1}{1}$

The second row is 1 2 1

Here the first and last entry is 1 and the middle term is obtained as sum of the two terms of 1st row, because $\binom{1}{0} + \binom{1}{1} = \binom{2}{1}$ $\left(\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}\right)$.

Similarly, the third row is 1 3 3 1, the first and last term is 1, the second term is obtained as the sum of 1st and 2nd term of 2nd row i.e. $1 + 2 = 3$ as $\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$ and 3rd term is obtained as the sum of 2nd and 3rd terms of 2nd row i.e. $2 + 1 = 3$ as $\binom{2}{1} + \binom{2}{2} = \binom{3}{2}$.

In the same manner, let us check 5th row in the light of above discussion.

4th row : 1 4 6 4 1

$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

the 5th row : 1 $(1 + 4)$ $(4 + 6)$ $(6 + 4)$ $(4 + 1)$ 1

1 5 10 10 5 1

i.e. $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

Here, $\binom{4}{0} + \binom{4}{1} = \binom{5}{1}$; $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$; $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$; $\binom{4}{3} + \binom{4}{4} = \binom{5}{4}$.

$$\left(\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}\right)$$

By using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $0 \le r \le n$, and also $\binom{n}{0} = 1 = \binom{n}{n}$, the Pascal's triangle can be written as,

Index					Coefficients				
1				$\binom{1}{0}$		$\binom{1}{1}$			
2			$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$		
3		$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$	
4	$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Observing above array, we can write the coefficients of the terms in the expansion of $(a+b)^n$, for any index n, without writing the earlier rows. For example, for index 7,

we have the coefficients of the terms as $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \binom{7}{4}, \binom{7}{5}, \binom{7}{6}, \binom{7}{7}$.

Now, we are in a position to write the binomial expansion of $(a+b)^n$ for any positive integral value of n.

3.2 Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 + ... + \binom{n}{r}a^{n-r}\cdot b^r + ... + \binom{n}{n}b^n, n \in \text{N}$$

We shall prove this theorem using the principle of mathematical induction.

Let, P(n) : $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 + ... + \binom{n}{r}a^{n-r}\cdot b^r + ... + \binom{n}{n}b^n, n \in \text{N}$

Let $n = 1$

L.H.S. $= (a+b)^1 = a + b$

R.H.S. $= \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}\cdot b = a + b$

$\therefore$ P(1) is true.

Let P(k) be true.

$$\therefore \quad (a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}\cdot b + \binom{k}{2}a^{k-2}\cdot b^2 + ...$$
$$+ \binom{k}{r-1}a^{k-(r-1)}\cdot b^{r-1} + \binom{k}{r}a^{k-r}\cdot b^r + ... + \binom{k}{k}b^k$$

Now, $(a+b)^{k+1} = (a+b)(a+b)^k$

$$= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}\cdot b + \binom{k}{2}a^{k-2}\cdot b^2 + ...\right.$$
$$\left. + \binom{k}{r-1}a^{k-(r-1)}\cdot b^{r-1} + \binom{k}{r}a^{k-r}\cdot b^r + ... + \binom{k}{k}b^k\right]$$

On multiplying both the factors and rearranging the terms, we get,

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k\cdot b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}\cdot b^2 + ...$$
$$+ \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k-(r-1)}\cdot b^r + ... + \binom{k}{k}b^{k+1}$$

Now, we know that; $\binom{n}{0} = 1 = \binom{n}{n}$ and $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, $1 \le r \le n$

$$\therefore \quad (a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^{(k+1)-1}\cdot b + \binom{k+1}{2}a^{(k+1)-2}\cdot b^2 + \ldots$$

$$+ \binom{k+1}{r}a^{(k+1)-r}\cdot b^r + \ldots + \binom{k+1}{k+1}b^{k+1}$$

$\therefore$ P$(k + 1)$ is true.

$\therefore$ By the principle of mathematical induction, P(n) is true, $\forall n \in$ N.

Some Corollaries :

(1) Substituting $a = 1$, $b = x$ in the binomial expansion of $(a + b)^n$, we have,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n, \ \forall n \in \text{N}$$

(2) Replacing b by $-b$, we obtain

$$(a-b)^n = \binom{n}{0}a^n - \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 - \binom{n}{3}a^{n-3}\cdot b^3 + \ldots +$$

$$(-1)^r\cdot\binom{n}{r}a^{n-r}\cdot b^r + \ldots + (-1)^n\cdot\binom{n}{n}b^n$$

(3) Taking $x = 1$ in (1), we get

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n}$$

$$\therefore \quad \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{r} + \ldots + \binom{n}{n} = 2^n$$

(4) Substituting $x = -1$ in (1), we have

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n\cdot\binom{n}{n} \qquad \textbf{(i)}$$

Also, $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$ **(ii)**

$\therefore$ Adding respective terms of (i) and (ii), we have,

$$2^n = 2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\right]$$

$$\therefore \quad \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + = 2^{n-1} \qquad \textbf{(iii)}$$

$$\therefore \quad \binom{n}{1} + \binom{n}{3} + \ldots = \binom{n}{0} + \binom{n}{2} + \ldots = 2^{n-1} \qquad \textbf{(From (i) and (iii)) (iv)}$$

Note : From the expansion of $(a + b)^n$, we observe the following points :

(1) There are $(n + 1)$ terms in the expansion.

(2) The index of 'a' in the first term is n and the index of 'a' decreases by 1 in the successive terms and simultaneously the index of b is zero in the first term and the index of b increases by 1 in the successive terms.

(3) Degree of each term (i.e. the sum of indices of a and b) is n, the index of $(a + b)$.

(4) The coefficients of the terms in order are $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$.

(5) As we know that, $\binom{n}{r} = \binom{n}{n-r}$, so the co-efficients of terms in the expansion are symmetrically situated successively from left or right i.e. $\binom{n}{0} = \binom{n}{n}$; $\binom{n}{1} = \binom{n}{n-1}$; $\binom{n}{2} = \binom{n}{n-2}$, ...

Example 1 : Expand : $\left(\frac{x}{2}+\frac{1}{x}\right)^5$, $x \neq 0$

Solution : Here $a = \frac{x}{2}$, $b = \frac{1}{x}$, $n = 5$

Substituting these values in the binomial theorem, we get,

$$\left(\frac{x}{2}+\frac{1}{x}\right)^5 = \binom{5}{0}\left(\frac{x}{2}\right)^5 + \binom{5}{1}\left(\frac{x}{2}\right)^4\left(\frac{1}{x}\right) + \binom{5}{2}\left(\frac{x}{2}\right)^3\left(\frac{1}{x}\right)^2 + \binom{5}{3}\left(\frac{x}{2}\right)^2\left(\frac{1}{x}\right)^3 + \binom{5}{4}\left(\frac{x}{2}\right)\left(\frac{1}{x}\right)^4 + \binom{5}{5}\left(\frac{1}{x}\right)^5$$

$$= 1\left(\frac{x^5}{32}\right) + 5\left(\frac{x^4}{16}\right)\left(\frac{1}{x}\right) + \frac{5\cdot4}{1\cdot2}\left(\frac{x^3}{8}\right)\left(\frac{1}{x^2}\right) + \frac{5\cdot4\cdot3}{1\cdot2\cdot3}\left(\frac{x^2}{4}\right)\left(\frac{1}{x^3}\right)$$

$$+ \frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}\left(\frac{x}{2}\right)\left(\frac{1}{x^4}\right) + 1\cdot\left(\frac{1}{x^5}\right)$$

$$= \frac{x^5}{32} + \frac{5}{16}x^3 + \frac{5}{4}x + \frac{5}{2x} + \frac{5}{2x^3} + \frac{1}{x^5}$$

Example 2 : Expand : $\left(2x-1+\frac{1}{x}\right)^4$, $x \neq 0$

Solution : Taking $a = 2x$, $b = 1 - \frac{1}{x}$, $n = 4$ in the corollary (2).

$$\left(2x-1+\frac{1}{x}\right)^4 = \left[2x-\left(1-\frac{1}{x}\right)\right]^4$$

$$= \binom{4}{0}(2x)^4 - \binom{4}{1}(2x)^3\left(1-\frac{1}{x}\right) + \binom{4}{2}(2x)^2\left(1-\frac{1}{x}\right)^2 - \binom{4}{3}(2x)\left(1-\frac{1}{x}\right)^3 + \binom{4}{4}\left(1-\frac{1}{x}\right)^4$$

$$= 16x^4 - 4(8x^3)\left(1-\frac{1}{x}\right) + \frac{4\cdot3}{1\cdot2}(4x^2)\left(1-\frac{2}{x}+\frac{1}{x^2}\right) - \frac{4\cdot3\cdot2}{1\cdot2\cdot3}(2x)\left(1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}\right)$$

$$+ \left[\binom{4}{0} - \binom{4}{1}\left(\frac{1}{x}\right) + \binom{4}{2}\left(\frac{1}{x^2}\right) - \binom{4}{3}\left(\frac{1}{x^3}\right) + \binom{4}{4}\left(\frac{1}{x^4}\right)\right]$$

$$= 16x^4 - 32x^3 + 32x^2 + 24x^2 - 48x + 24 - 8x + 24 - \frac{24}{x} + \frac{8}{x^2}$$

$$+ 1 - \frac{4}{x} + \frac{6}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}$$

$$= 16x^4 - 32x^3 + 56x^2 - 56x + 49 - \frac{28}{x} + \frac{14}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}$$

Example 3 : Evaluate $(0.99)^5$ using binomial theorem.

Solution :

$$(0.99)^5 = (1 - 0.01)^5$$

$$= \binom{5}{0} - \binom{5}{1}(0.01) + \binom{5}{2}(0.01)^2 - \binom{5}{3}(0.01)^3 + \binom{5}{4}(0.01)^4 - \binom{5}{5}(0.01)^5$$

$$= 1 - 5(0.01) + 10(0.0001) - 10(0.000001) + 5(0.00000001) - (0.0000000001)$$

$$= 0.9509900499$$

Example 4 : Which is smaller ? $(1.1)^{100000}$ or 100000

Solution : $(1.1)^{100000} = (1 + 0.1)^{100000}$

$$= \binom{100000}{0} + \binom{100000}{1}(0.1) + \text{Sum of some positive terms}$$

$$= 1 + 10000 + \text{Sum of positive terms}$$

$$> 10000$$

$\therefore$ 10000 is smaller out of $(1 \cdot 1)^{10000}$ and 10000.

Excercise 3.1

1. Expand the following :

(1) $\left(x^2 + \frac{1}{x}\right)^5, (x \neq 0)$ (2) $(1 - 2x)^4$ (3) $(3x - 2)^6$ (4) $\left(x - \frac{1}{2x}\right)^5, (x \neq 0)$

2. Expand : (1) $(1 + x + x^2)^4$ (2) $(1 - x + x^2)^3$

3. Evaluate by using binomial theorem :

(1) $(0.98)^4$ (2) $(99)^4$ (3) $(101)^6$

4. Using binomial theorem, indicate which one is larger ? $(1.01)^{10000}$ or 100

*

3.3 General and Middle Term

1. The expansion of $(a + b)^n$ contains $(n + 1)$ terms. If we consider $T_1, T_2, T_3, ..., T_{n+1}$ as the first, second, third, ... $(n + 1)$th terms respectively in the expansion of $(a + b)^n$, then $T_1 = \binom{n}{0}a^n$, $T_2 = \binom{n}{1}a^{n-1} \cdot b$, $T_3 = \binom{n}{2}a^{n-2} \cdot b^2$, ..., $T_{n+1} = \binom{n}{n}b^n$.

   We may take the **general term as $T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r, 0 \leq r \leq n$**

2. If in $(a + b)^n$; n is even, then $n + 1$ is odd. So the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

   So $\left(\frac{n}{2}+1\right)^{th}$ term $= \left(\frac{n+2}{2}\right)^{th}$ term is the **middle term.**

   For example, in the expansion of $(2x + y)^{10}$, the middle term is $\frac{10+2}{2}$ = 6th term. If n is odd, then $n + 1$ is even, so there are **two middle terms : $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.**

   For example, in the expansion of $(2x + y)^9$, the middle terms are $\frac{9+1}{2}$ = 5th term and $\frac{9+3}{2}$ = 6th term.

Example 5 : Find the fourth term in the expansion of $(3x - y)^7$.

Solution : Here, $a = 3x$, $b = -y$, $n = 7$

Now, $T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

To find T_4, we let $r = 3$ **$(r + 1 = 4)$**

$\therefore \quad T_4 = T_{3+1} = \binom{7}{3}(3x)^{7-3} \cdot (-y)^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}(81x^4)(-y)^3$

$= -2835x^4y^3$

Example 6 : Find the coefficient of x^{-2} in the expansion of $\left(x - \frac{1}{x^2}\right)^{16}$, $(x \neq 0)$

Solution : Here, $a = x$, $b = -\frac{1}{x^2}$, $n = 16$

$T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

$= \binom{16}{r}(x)^{16-r} \cdot \left(-\frac{1}{x^2}\right)^r = \binom{16}{r}(-1)^r \cdot x^{16-3r}$

For the index of x to be -2, we must have $16 - 3r = -2$ i.e. $r = 6$.

$\therefore \quad T_{6+1} = \binom{16}{6}(-1)^6 \cdot x^{16-3(6)}$

$\therefore \quad T_7 = \binom{16}{6} \cdot 1 \cdot x^{-2}$

$\therefore$ Coefficient of x^{-2} is $\binom{16}{6}$ or 8008.

Example 7 : Find the constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{11}$, if it exists. $(x \neq 0)$

Solution : Suppose the constant term (i.e. term in which index of x is zero) exists and it is $(r + 1)$th term.

Here, $a = 2x^2$, $b = -\frac{1}{x}$, $n = 11$

$T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

$= \binom{11}{r}(2x^2)^{11-r} \cdot \left(-\frac{1}{x}\right)^r = \binom{11}{r}(2)^{11-r} \cdot (-1)^r \cdot x^{22-3r}$

For the constant term, index of x is zero.

$\therefore \quad 22 - 3r = 0$

$\therefore \quad r = \frac{22}{3} \notin N$

$\therefore$ Our assumption is wrong.

$\therefore$ Constant term does not exist in the expansion.

Example 8 : Find the middle term / terms in the expansion of $\left(\frac{x}{2} + 3y\right)^9$.

Solution : As $n = 9$ is odd, so we have two middle terms namely,

$\frac{n+1}{2} = \frac{9+1}{2} = 5$th term and $\frac{n+3}{2} = \frac{9+3}{2} = 6$th term

Here, $a = \frac{x}{2}$, $b = 3y$, $n = 9$

$T_{r+1} = \binom{n}{r}a^{n-r} \cdot b^r$

$\therefore \quad T_5 = T_{4+1} = \binom{9}{4} \cdot \left(\frac{x}{2}\right)^{9-4} \cdot (3y)^4 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{x^5}{32}\right)(81y^4) = \frac{5103}{16}\ x^5y^4$

And $T_6 = \binom{9}{5} \cdot \left(\frac{x}{2}\right)^{9-5} \cdot (3y)^5$ **$(r + 1 = 6)$**

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\left(\frac{x^4}{16}\right)(243y^5) = \frac{15309}{8} x^4y^5$$

$\therefore$ Middle terms are $\frac{5103}{16} x^5y^4$ and $\frac{15309}{8} x^4y^5$.

Example 9 : Obtain the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}} + \sqrt{\frac{3}{2x^2}}\right)^{12}$. **$(x > 0)$**

Solution : Here, $T_{r+1} = \binom{12}{r}\left(\sqrt{\frac{x}{3}}\right)^{12-r} \cdot \left(\sqrt{\frac{3}{2x^2}}\right)^r$

$$= \binom{12}{r} \cdot \frac{\sqrt{3}^r}{(\sqrt{3})^{12-r}} \times \frac{1}{(\sqrt{2})^r} \cdot x^{6 - \frac{r}{2} - r}$$

$$= \binom{12}{r} \cdot \frac{1}{(\sqrt{3})^{12-2r}} \times \frac{1}{(\sqrt{2})^r} \times x^{6 - \frac{3r}{2}}$$

For the term independent of x, we let $6 - \frac{3r}{2} = 0$

$\therefore \quad r = 4$

$$T_5 = \binom{12}{4} \cdot \frac{1}{(\sqrt{3})^4} \cdot \frac{1}{(\sqrt{2})^4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1}{36} = \frac{55}{4}$$

Exercise 3.2

1. Find the coefficient of : (1) x^6 in $(x + 2)^9$ (2) x^{32} in $\left(x^4 - \frac{1}{x^3}\right)^{15}$, $(x \neq 0)$

2. Find the constant term in the expansion of :

(1) $\left(\frac{3}{x^2} + \frac{\sqrt{x}}{3}\right)^{10}$, $(x > 0)$ (2) $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$, $(x \neq 0)$

3. The coefficients of x^7 and x^8 in the expansion of $\left(2 + \frac{x}{3}\right)^n$ are equal, find n.

4. Find the middle term or terms in the expansion of :

(1) $\left(2 - \frac{x^3}{3}\right)^7$ (2) $\left(\frac{x}{2} + 3y\right)^8$ (3) $\left(\frac{3}{2x} - \frac{2x^2}{3}\right)^{20}$, $(x \neq 0)$ (4) $(3x + 2y)^5$

5. If the coefficient of x^3 in the expansion of $(1 + x)^n$ is 20, find n.

6. If the coefficients of fifth, sixth and seventh terms in the expansion of $(1 + x)^n$ are in arithmetic progression, find n.

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Miscellaneous Problems :

Example 10 : Find the coefficient of x^3 in the expansion of the product $(1 - x)^{15} \cdot (1 + 3x)^4$.

Solution : Applying binomial theorem to get $(1 - x)^{15}$ and $(1 + 3x)^4$, we have

$(1-x)^{15} = \binom{15}{0} - \binom{15}{1}x + \binom{15}{2}x^2 - \binom{15}{3}x^3 + \ldots - \binom{15}{15}x^{15}$ and

$(1+3x)^4 = (3x+1)^4 = \binom{4}{0}(3x)^4 + \binom{4}{1}(3x)^3 + \binom{4}{2}(3x)^2 + \binom{4}{3}(3x) + \binom{4}{4}\cdot 1$

Now, we want to find the coefficient of x^3 in the product $(1-x)^{15} \cdot (1+3x)^4$, we shall simply collect the terms containing x^3 from the product, without finding complete product.

They are, $\binom{15}{0}\cdot\binom{4}{1}(27x^3) - \binom{15}{1}x\cdot\binom{4}{2}(9x^2) + \binom{15}{2}x^2\cdot\binom{4}{3}(3x) - \binom{15}{3}x^3\cdot\binom{4}{4}$

$= 1\cdot 4\cdot 27x^3 - 15\cdot x\cdot \frac{4\cdot 3}{1\cdot 2}\cdot 9x^2 + \frac{15\cdot 14}{1\cdot 2}x^2\cdot 4\cdot 3x - \frac{15\cdot 14\cdot 13}{1\cdot 2\cdot 3}x^3\cdot 1$

$= (108 - 810 + 1260 - 455)x^3 = 103x^3$

$\therefore$ Coefficient of x^3 in $(1-x)^{15}\cdot(1+3x)^4$ is 103.

Example 11 : If the middle term in the expansion of $\left(\frac{x}{3}+3\right)^{10}$ is 8064, find x.

Solution : Here $n = 10$

$\therefore$ n is even, so middle term is $\frac{n+2}{2} = \frac{10+2}{2} = 6$th term

$\therefore$ $T_6 = T_{5+1} = \binom{10}{5}\cdot\left(\frac{x}{3}\right)^{10-5}\cdot(3)^5$

$\therefore$ $8064 = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{x^5}{3^5}\cdot 3^5$

$\therefore$ $\frac{8064}{252} = x^5$

$\therefore$ $x^5 = 32 = 2^5$

$\therefore$ $x = 2$

Example 12 : Prove that $(3+\sqrt{8})^5 + (3-\sqrt{8})^5 = 6726$. Hence deduce that, $6725 < (3+\sqrt{8})^5 < 6726$. Hence obtain $[(3+\sqrt{8})^5]$.

Solution : $(3+\sqrt{8})^5 = \binom{5}{0}(3)^5 + \binom{5}{1}(3)^4(\sqrt{8}) + \binom{5}{2}(3)^3(\sqrt{8})^2 + \binom{5}{3}(3)^2(\sqrt{8})^3$

$+ \binom{5}{4}(3)(\sqrt{8})^4 + \binom{5}{5}(\sqrt{8})^5$ **(i)**

$(3-\sqrt{8})^5 = \binom{5}{0}(3)^5 - \binom{5}{1}(3)^4(\sqrt{8}) + \binom{5}{2}(3)^3(\sqrt{8})^2 - \binom{5}{3}(3)^2(\sqrt{8})^3$

$+ \binom{5}{4}(3)(\sqrt{8})^4 - \binom{5}{5}(\sqrt{8})^5$ **(ii)**

Adding (i) and (ii), we have

$(3+\sqrt{8})^5 + (3-\sqrt{8})^5 = 2\left[\binom{5}{0}(3)^5 + \binom{5}{2}(3)^3(\sqrt{8})^2 + \binom{5}{4}(3)(\sqrt{8})^4\right]$

$= 2\left[1\cdot 243 + \frac{5\cdot 4}{1\cdot 2}\cdot 27\cdot 8 + 5\cdot 3\cdot 64\right]$ $\left(\binom{5}{4}=\binom{5}{1}\right)$

$= 2[243 + 2160 + 960]$

$= 2[3363]$

$= 6726$

Now, $(3 + \sqrt{8})(3 - \sqrt{8}) = 9 - 8 = 1$ and $(3 + \sqrt{8}) > 0$. Hence $3 - \sqrt{8} > 0$.

Also $(3 + \sqrt{8}) > 1$

$\therefore \quad 3 - \sqrt{8} < 1$

$\therefore \quad 0 < 3 - \sqrt{8} < 1$

$\therefore \quad 0 < (3 - \sqrt{8})^5 < 1$

$\therefore \quad (3 + \sqrt{8})^5 < (3 + \sqrt{8})^5 + (3 - \sqrt{8})^5 = 6726 < (3 + \sqrt{8})^5 + 1$

$\therefore \quad (3 + \sqrt{8})^5 < 6726$ and $6726 < (3 + \sqrt{8})^5 + 1$

$\therefore \quad 6725 < (3 + \sqrt{8})^5 < 6726$

$\therefore$ According to definition of integer part, $[(3 + \sqrt{8})^5] = 6725$

Example 13 : The sum of the coefficients of powers of x in the first three terms in the expansion of $\left(x^2 - \frac{2}{x}\right)^n$ $(x \neq 0)$ is 127, find n. $(n \in \mathbb{N})$

Solution : In the expansion of $\left(x^2 - \frac{2}{x}\right)^n$, the first three terms are $\binom{n}{0}(x^2)^n$, $\binom{n}{1}(x^2)^{n-1}\cdot\left(\frac{-2}{x}\right)$ and $\binom{n}{2}(x^2)^{n-2}\cdot\left(\frac{-2}{x}\right)^2$. As the sum of the coefficients of these terms is 127, we have,

$$\binom{n}{0} - \binom{n}{1}2 + \binom{n}{2}\cdot 4 = 127$$

$\therefore \quad 1 - 2n + \frac{4n(n-1)}{2} = 127$

$\therefore \quad 1 - 2n + 2n(n-1) = 127$

$\therefore \quad 1 - 2n + 2n^2 - 2n - 127 = 0$

$\therefore \quad 2n^2 - 4n - 126 = 0$

$\therefore \quad n^2 - 2n - 63 = 0$

$\therefore \quad (n - 9)(n + 7) = 0$

$\therefore \quad n = 9$ or $n = -7$ But $-7 \notin \mathbb{N}$

$\therefore \quad n = 9$

Example 14 : Use the binomial theorem to show that dividing $8^n - 7n$ by 49 leaves the remainder 1.

Solution : $8^n = (1 + 7)^n$

$$= 1 + \binom{n}{1}7 + \binom{n}{2}7^2 + \binom{n}{3}7^3 + \dots + \binom{n}{n}7^n$$

$$= 1 + 7n + 7^2\left[\binom{n}{2} + \binom{n}{3}7 + \dots + \binom{n}{n}7^{n-2}\right]$$

$\therefore \quad 8^n - 7n = 1 + 49m$, where $m = \left[\binom{n}{2} + \binom{n}{3}7 + \dots + \binom{n}{n}7^{n-2}\right] \in \mathbb{N}$

$\therefore$ Dividing $8^n - 7n$ by 49 leaves the remainder 1.

Example 15 : Prove that : $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2}$, $\forall n \in N$

Solution : [Motivation : See the R.H.S. $= \frac{(2n)!}{(n!)(n!)} = \frac{(2n)!}{(2n-n)!n!} = \binom{2n}{n}$,

which is the coefficient of x^n in the expansion of $(1 + x)^{2n}$.]

$(1 + x)^{2n} = (1 + x)^n (x + 1)^n$

$$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right] \times$$

$$\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + ... + \binom{n}{n-1}x + \binom{n}{n}\right]$$

Now coefficient of x^n in the expansion of $(1 + x)^{2n}$ is $\binom{2n}{n}$ and

coefficient of x^n in R.H.S. $= \binom{n}{0}^2 + \binom{n}{1}^2 + ... + \binom{n}{n}^2$ **(Taking product term wise)**

$\therefore \quad \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2 = \binom{2n}{n}$

$$= \frac{(2n)!}{n! \cdot n!} = \frac{(2n)!}{(n!)^2}$$

Example 16 : Prove that : $\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + ... + \binom{n}{n-1}\binom{n}{n} = \frac{(2n)!}{(n-1)!(n+1)!}$, $\forall n \in N$

Solution : [Motivation : See R.H.S. $= \frac{(2n)!}{[2n-(n-1)]!(n-1)!} = \binom{2n}{n-1}$. It is the coefficient of x^{n-1} in the expansion of $(1 + x)^{2n}$.]

$(1 + x)^{2n} = (1 + x)^n (x + 1)^n$

$$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right] \times$$

$$\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} + ... + \binom{n}{n}\right]$$

Now coefficient of x^{n-1} in $(1 + x)^{2n}$ is $\binom{2n}{n-1}$ and

the coefficient of x^{n-1} in R.H.S. is $\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{3} + ... + \binom{n}{n-1}\binom{n}{n}$

$\therefore \quad \binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + ... + \binom{n}{n-1}\binom{n}{n} = \binom{2n}{n-1} = \frac{(2n)!}{(n+1)!(n-1)!}$

Example 17 : Prove that : $\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + ... + (2n + 1)\binom{n}{n} = (n + 1)2^n$, $\forall n \in N$

Solution : Let $\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + ... + (2n - 1)\binom{n}{n-1} + (2n + 1)\binom{n}{n} = S$ **(i)**

Using $\binom{n}{r} = \binom{n}{n-r}$ and taking terms in the reverse order, we have

$(2n + 1)\binom{n}{0} + (2n - 1)\binom{n}{1} + (2n - 3)\binom{n}{2} + ... + 5\binom{n}{n-2} + 3\binom{n}{n-1} + \binom{n}{n} = S$ **(ii)**

Adding corresponding terms of (i) and (ii), we have

$\therefore \quad (1 + (2n + 1))\binom{n}{0} + (3 + (2n - 1))\binom{n}{1} + (5 + (2n - 3))\binom{n}{2} + ... +$

$$((2n - 3) + 5)\binom{n}{n-2} + ((2n - 1) + 3)\binom{n}{n-1} + ((2n + 1) + 1)\binom{n}{n} = 2S$$

$\therefore \quad (2n+2)\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n}\right] = 2S$

$\therefore \quad 2(n+1)\cdot 2^n = 2S$

$\therefore \quad S = (n+1)2^n$

So, $\binom{n}{0}+3\binom{n}{1}+5\binom{n}{2}+...+(2n+1)\binom{n}{n} = (n+1)2^n$

Example 18 : If in the expansion of $(x-2y)^n$, the sum of fifth and sixth term is zero then find the value of $\frac{x}{y}$. If $n = 8$ then find $\frac{x}{y}$.

Solution : Here, $T_5 = \binom{n}{4}\cdot x^{n-4}\cdot(-2y)^4$ and $T_6 = \binom{n}{5}\cdot x^{n-5}\cdot(-2y)^5$

Now, $T_5 + T_6 = 0$. So, $T_5 = -T_6$

$\therefore \quad \binom{n}{4}\cdot(-2)^4\cdot x^{n-4}\cdot y^4 = -\binom{n}{5}\cdot(-2)^5\cdot x^{n-5}\cdot y^5$

$\therefore \quad \frac{n!}{4!(n-4)!}\cdot 16\cdot x^{n-4}\cdot y^4 = -\frac{n!}{5!(n-5)!}\cdot(-32)\cdot x^{n-5}\cdot y^5$

$\therefore \quad \frac{x^{n-4}\cdot y^4}{x^{n-5}\cdot y^5} = \frac{n!}{5!(n-5)!}\times\frac{4!(n-4)!}{n!}\times\frac{32}{16}$

$\therefore \quad \frac{x}{y} = \frac{4!(n-4)(n-5)!}{5\cdot 4!(n-5)!}\times 2$

$\therefore \quad \frac{x}{y} = \frac{n-4}{5}\times 2$

Taking $n = 8$, we have

$\frac{x}{y} = \frac{8}{5}$

Example 19 : Obtain the sum of the last thirty coefficients in the expansion of $(1+x)^{59}$.

Solution : There are 60 terms in the expansion of $(1+x)^{59}$.

$\therefore$ Sum of the coefficients of last thirty terms is,

$S = \binom{59}{30}+\binom{59}{31}+\binom{59}{32}+...+\binom{59}{58}+\binom{59}{59}$ **(first 30 coefficients $\binom{59}{0}, \binom{59}{1}, ..., \binom{59}{29}$)** **(i)**

i.e. $S = \binom{59}{29}+\binom{59}{28}+\binom{59}{27}+...+\binom{59}{1}+\binom{59}{0}$ **(Using $\binom{n}{r} = \binom{n}{n-r}$)** **(ii)**

$\therefore \quad 2S = \binom{59}{0}+\binom{59}{1}+...+\binom{59}{59}$ **(adding respective sides of (i) and (ii))**

$\therefore \quad S = \frac{2^{59}}{2} = 2^{58}$

Exercise 3

1. Obtain the ratio of the coefficients of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$.
2. If the coefficients of $(r-2)$th and $(2r-5)$th terms in the expansion of $(1+x)^{36}$ are equal, find r.
3. Find x, y and n in the expansion of $(x+y)^n$, if the first three terms in the expansion are 64, 960 and 6000.
4. The 2nd, 3rd and 4th terms in the expansion of $(a+b)^n$ are 240, 720 and 1080, find a, b and n.

5. Prove that $(2 + \sqrt{3})^7 + (2 - \sqrt{3})^7 = 10084$.

Hence deduce that, $10083 < (2 + \sqrt{3})^7 < 10084$.

6. Find n, if the ratio of the fourth term to the fourth term from the end in the expansion of $\left(\sqrt[5]{2} + \frac{1}{\sqrt[5]{3}}\right)^n$ is 6 : 1.

7. Find the coefficient of x^4 in the expansion of $(1 - x)^{12} \cdot (1 + 2x)^6$.

8. The sum of the coefficients of the first three terms in the expansion of $\left(x^2 - \frac{3}{x}\right)^n$ $(x \neq 0)$ is 376, find the coefficient of x^8.

9. Using the binomial theorem, show that $3^{2n} - 8n - 1$ is divisible by 64, for $n \in N$.

10. Prove the following identities : $(\forall n \in N)$

(1) $\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + ... + (n + 1)\binom{n}{n} = (n + 2) \cdot 2^{n-1}$

(2) $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + ... + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

11. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If the coefficients of 5th and 19th terms in the expansion of $(1 + x)^n$ are equal, then n = ... ☐

(a) 18 (b) 24 (c) 22 (d) 20

(2) If the coefficients of $(r - 6)$th and $(2r - 2)$th terms in the expansion of $(1 + x)^{32}$ are equal, then r = ... ☐

(a) -2 (b) 14 (c) 34 (d) 20

(3) The coefficient of x^{21} in the expansion of $(x + x^2)^{20}$ is ...... ☐

(a) $\binom{20}{1}$ (b) $\binom{20}{0}$ (c) $\binom{20}{2}$ (d) $\binom{20}{12}$

(4) The number of terms in the expansion of $(2x + 3y + 4z)^5$ of type $x^a \cdot y^b \cdot z^c$ is ...... ☐

(a) 10 (b) 15 (c) 21 (d) 42

(5) If $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4 = x + y\sqrt{3}$, then y = ... ☐

(a) 0 (b) 56 (c) 112 (d) 97

(6) If T_{r-1} is the middle term of $(a + b)^{10}$, then r = ... ☐

(a) 6 (b) 5 (c) 7 (d) 8

(7) Constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$, $(x \neq 0)$ is ...... ☐

(a) 7920 (b) 495 (c) -7920 (d) -495

(8) $\binom{n-1}{1} + \binom{n-1}{2} + ... + \binom{n-1}{n-1}$ = ... $(n > 1)$ ☐

(a) 2^n (b) 2^{n-1} (c) $2^n - 1$ (d) $2^{n-1} - 1$

(9) Middle term in the expansion of $\left(2x+\frac{1}{2x}\right)^8$ is ...... $(x \neq 0)$

(a) $\binom{8}{4}$ (b) $\binom{8}{4}(2x)$ (c) $\binom{8}{4}\left(\frac{1}{2x}\right)$ (d) $\binom{8}{4}(2)$

(10) Sum of the coefficients of $x^{13}y^2$ and x^2y^{13} in the expansion of $(x + y)^{15}$ is ....

(a) $\binom{15}{2}$ (b) $2\binom{15}{13}$ (c) $\binom{15}{3}$ (d) $2\binom{15}{3}$

*

Summary

We studied following points in this chapter :

1. The Binomial Expansion for $n \in$ N is given by the binomial theorem as

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\cdot b + \binom{n}{2}a^{n-2}\cdot b^2 + ... + \binom{n}{r}a^{n-r}\cdot b^r + ... + \binom{n}{n}b^n,\ n \in \text{N}$$

2. The coefficients of binomial theorem are arranged in an array, known as Pascal's Triangle.

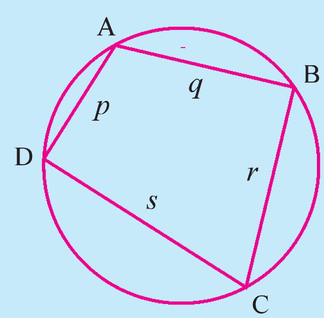
3. The general term of the expansion $(a + b)^n$ is $T_{r+1} = \binom{n}{r}a^{n-r}\cdot b^r$.

4. The middle term in the expansion of $(a+b)^n$ is $\left(\frac{n}{2}+1\right)$ or $\left(\frac{n+2}{2}\right)^{th}$ term, if n is even and $\left(\frac{n+1}{2}\right)^{th}$ as well as $\left(\frac{n+3}{2}\right)^{th}$ terms are the middle terms, if n is odd.

—❖—

Brahmagupta's formula

Brahmagupta's most famous result in geometry is his formula for cyclic quadrilaterals. Given the lengths of the sides of any cyclic quadrilateral, Brahmagupta gave an approximate and an exact formula for the figure's area.



The approximate area is the product of the halves of the sums of the opposite sides of a quadrilateral. The accurate [area] is the square root the product of the half of the sum of the sides diminished by [each] side of the quadrilateral.

So given the lengths p, q, r and s of sides of a cyclic quadrilateral, the approximate area is $\left(\frac{p+r}{2}\right)\left(\frac{q+s}{2}\right)$ while, letting $t = \frac{p+q+r+s}{2}$, the exact area is

$$\sqrt{(t-p)(t-q)(t-r)(t-s)}$$

Heron's formula is a special case of this formula and it can be derived by setting one of the sides equal to zero.

Chapter 4

ADDITION FORMULAE AND FACTOR FORMULAE

Music is the pleasure the human mind experiences from counting without being aware that it is counting.

– Gottfried Leibnitz

4.1 Introduction

We have studied the fundamental ideas and properties of trigonometric functions. Now, we will see how to express values of trigonometric functions with variables $\alpha + \beta$ and $\alpha - \beta$ in terms of values of trigonometric functions with variables α and β, where α and β are real numbers. These formulae are known as addition formulae. With the help of these formulae, we will derive factor formulae and study their uses.

If $f(x) = ax$, $x \in R$ is a linear function, then

$$f(x - y) = a(x - y) = ax - ay = f(x) - f(y)$$

Thus, $f(x - y) = f(x) - f(y)$

Now, consider the trigonometric function $f(x) = cosx$, $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{6}$.

For these values of α and β, $\alpha - \beta = \frac{\pi}{3} - \frac{\pi}{6}$. So $cos(\alpha - \beta) = cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

But $cos\alpha - cos\beta = cos\frac{\pi}{3} - cos\frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} \neq \frac{\sqrt{3}}{2}$

Thus, $cos(\alpha - \beta) \neq cos\alpha - cos\beta$

Thus, what is true for a linear function may not be true for trigonometric functions. Similarly other results can also be quoted. Now, we will obtain the formula of $cos(\alpha - \beta)$ using $cos\alpha$, $cos\beta$, $sin\alpha$, $sin\beta$.

4.2 The Addition Formulae

We shall first prove a formula for $cos(\alpha - \beta)$ and $cos(\alpha + \beta)$.

Let us see the expression for $cos(\alpha - \beta)$.

Theorem 1 : For $\alpha, \beta \in R$

(1) $cos(\alpha - \beta) = cos\alpha \; cos\beta + sin\alpha \; sin\beta$

(2) $cos(\alpha + \beta) = cos\alpha \; cos\beta - sin\alpha \; sin\beta$

Proof : Case (1) : Let $\alpha, \beta \in [0, 2\pi)$.

We have three possibilities for α and β by law of trichotomy.

They are (i) $\alpha > \beta$ (ii) $\alpha = \beta$ (iii) $\alpha < \beta$

(i) $\alpha > \beta$

Suppose the trigonometric points on the unit circle corresponding to α, β and $\alpha - \beta$ are P, Q and R respectively.

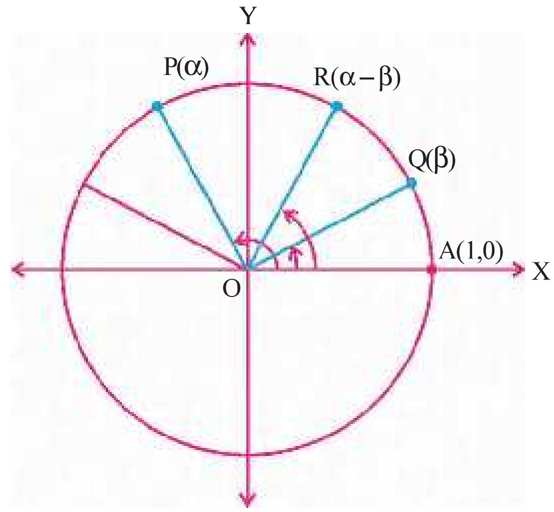


Figure 4.1

$\therefore$ By definition, $P(\alpha) = (cos\alpha, sin\alpha)$,

$Q(\beta) = (cos\beta, sin\beta)$ and $R(\alpha - \beta) = (cos(\alpha - \beta), sin(\alpha - \beta))$.

Also A is (1, 0).

As shown in figure we have $l(\overset{\frown}{AP}) = \alpha$, $l(\overset{\frown}{AQ}) = \beta$ and $l(\overset{\frown}{AR}) = \alpha - \beta$.

As $\beta < \alpha$ and $Q \in \overset{\frown}{AP}$,

$l(\overset{\frown}{PQ}) = l(\overset{\frown}{AP}) - l(\overset{\frown}{AQ})$

$\therefore$ $l(\overset{\frown}{PQ}) = \alpha - \beta = l(\overset{\frown}{AR})$

$\therefore$ $\overset{\frown}{PQ} \cong \overset{\frown}{AR}$

Chords corresponding to congruent arcs of the same circle are congruent.

$\therefore$ $PQ = AR$

$\therefore$ $PQ^2 = AR^2$

Now using distance formula,

$PQ^2 = (cos\alpha - cos\beta)^2 + (sin\alpha - sin\beta)^2$

$= cos^2\alpha - 2cos\alpha \; cos\beta + cos^2\beta + sin^2\alpha - 2sin\alpha \; sin\beta + sin^2\beta$

$= cos^2\alpha + sin^2\alpha + cos^2\beta + sin^2\beta - 2cos\alpha \; cos\beta - 2sin\alpha \; sin\beta$

$= 2 - 2(cos\alpha \; cos\beta + sin\alpha \; sin\beta)$

$AR^2 = (1 - cos(\alpha - \beta))^2 + (0 - sin(\alpha - \beta))^2$

$= 1 - 2cos(\alpha - \beta) + cos^2(\alpha - \beta) + sin^2(\alpha - \beta)$

$= 2 - 2cos(\alpha - \beta)$

But $AR^2 = PQ^2$

$\therefore$ $2 - 2cos(\alpha - \beta) = 2 - 2(cos\alpha \; cos\beta + sin\alpha \; sin\beta)$

$\therefore$ $-2cos(\alpha - \beta) = -2(cos\alpha \; cos\beta + sin\alpha \; sin\beta)$

$\therefore$ $cos(\alpha - \beta) = cos\alpha \; cos\beta + sin\alpha \; sin\beta$

(ii) Suppose $\alpha = \beta$

Then, L.H.S. $= cos(\alpha - \beta) = cos(\alpha - \alpha) = cos0 = 1$

R.H.S. $= cos\alpha\ cos\beta + sin\alpha\ sin\beta$

$= cos\alpha\ cos\alpha + sin\alpha\ sin\alpha$

$= cos^2\alpha + sin^2\alpha = 1$

$\therefore$ L.H.S. = R.H.S.

(iii) Suppose $\alpha < \beta$

Then, $\alpha - \beta = -(\beta - \alpha)$

$\therefore\ cos(\alpha - \beta) = cos(-(\beta - \alpha))$

$= cos(\beta - \alpha)$ **(*cosine* is an even function.)**

$= cos\beta\ cos\alpha + sin\beta\ sin\alpha$ **($\beta > \alpha$)**

$\therefore\ cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

Case (2) : $\alpha, \beta \in R$

For the given $\alpha, \beta \in R$, we can find $\alpha_1, \beta_1 \in [0, 2\pi)$,

such that $\alpha = 2m\pi + \alpha_1$ and $\beta = 2n\pi + \beta_1$, $m, n \in Z$

$\therefore\ \alpha - \beta = 2m\pi + \alpha_1 - (2n\pi + \beta_1)$

$= 2(m - n)\pi + \alpha_1 - \beta_1$

$= 2k\pi + \alpha_1 - \beta_1$, where $k = m - n \in Z$

As *sin* and *cos* are periodic functions whose principal period is 2π

$cos\alpha = cos\alpha_1$, $cos\beta = cos\beta_1$ and $cos(\alpha - \beta) = cos(\alpha_1 - \beta_1)$

Thus, $cos(\alpha - \beta) = cos(\alpha_1 - \beta_1)$

$= cos\alpha_1\ cos\beta_1 + sin\alpha_1\ sin\beta_1$ **(Case (1))**

$= cos\alpha\ cos\beta + sin\alpha\ sin\beta$

$\therefore\ cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

From case (1) and case (2) we see that for all $\alpha, \beta \in R$

$$cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$$

(2) We have, $cos(\alpha + \beta) = cos(\alpha - (-\beta))$

$= cos\alpha\ cos(-\beta) + sin\alpha\ sin(-\beta)$

$= cos\alpha\ cos\beta - sin\alpha\ sin\beta$ **($cos(-\beta) = cos\beta$, $sin(-\beta) = -sin\beta$)**

$$\therefore\ cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta$$

Corollay 1 : (1) $cos\left(\frac{\pi}{2} - \theta\right) = sin\theta$ (2) $sin\left(\frac{\pi}{2} - \theta\right) = cos\theta$

Proof : (1) We know that for all $\alpha, \beta \in R$,

$cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

We substitute $\alpha = \frac{\pi}{2}$ and $\beta = \theta$ in the above identity. We get,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$
$$= 0 \cdot \cos\theta + 1 \cdot \sin\theta$$
$$= \sin\theta$$

$\therefore$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

(2) If we replace θ by $\frac{\pi}{2} - \theta$ in $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, we get

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \sin\left(\frac{\pi}{2} - \theta\right)$$

$\therefore$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\therefore$ $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Theorem 2 : (1) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(2) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Proof : (1) $\sin(\alpha - \beta) = \cos\left[\frac{\pi}{2} - (\alpha - \beta)\right]$ $\quad\left(\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right)$

$$= \cos\left[\left(\frac{\pi}{2} - \alpha\right) + \beta\right]$$
$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$\therefore$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(2) $\sin(\alpha + \beta) = \sin[\alpha - (-\beta)]$

$$= \sin\alpha \cdot \cos(-\beta) - \cos\alpha \cdot \sin(-\beta)$$
$$= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \quad (\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta)$$

$\therefore$ $\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$

4.3 Other Formulae for Allied Numbers

We have seen from theorems 1 and 2 that for all real numbers α and β.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ **(i)**

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ **(ii)**

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ **(iii)**

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ **(iv)**

We have also seen that for all $\theta \in R$,

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$$\therefore \quad \tan\left(\frac{\pi}{2} - \theta\right) = \left(\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)}\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Putting $\alpha = \frac{\pi}{2}$ and $\beta = \theta$ in (iv) and (ii) respectively, we get

$$sin\left(\frac{\pi}{2}+\theta\right) = sin\frac{\pi}{2}\,cos\theta + cos\frac{\pi}{2}\,sin\theta = 1 \cdot cos\theta + 0 \cdot sin\theta = cos\theta$$

$$\therefore \quad sin\left(\frac{\pi}{2}+\theta\right) = cos\theta$$

$$cos\left(\frac{\pi}{2}+\theta\right) = cos\frac{\pi}{2}\,cos\theta - sin\frac{\pi}{2}\,sin\theta = 0 \cdot cos\theta - 1 \cdot sin\theta = -sin\theta$$

$$\therefore \quad cos\left(\frac{\pi}{2}+\theta\right) = -sin\theta$$

and hence, $tan\left(\frac{\pi}{2}+\theta\right) = -cot\theta$

Similarly putting, $\alpha = \frac{3\pi}{2}$ and $\beta = \theta$ in (i) to (iv), we get

$$sin\left(\frac{3\pi}{2}-\theta\right) = -cos\theta,\ cos\left(\frac{3\pi}{2}-\theta\right) = -sin\theta$$

$$\therefore \quad tan\left(\frac{3\pi}{2}-\theta\right) = cot\theta$$

Similarly, $sin\left(\frac{3\pi}{2}+\theta\right) = -cos\theta,\ cos\left(\frac{3\pi}{2}+\theta\right) = sin\theta$

$$\therefore \quad tan\left(\frac{3\pi}{2}+\theta\right) = -cot\theta$$

Again putting $\alpha = \pi,\ \beta = \theta$ and $\alpha = 2\pi,\ \beta = \theta$ in (i) to (iv), we can prove the following :

$sin(\pi - \theta) = sin\theta,\ cos(\pi - \theta) = -cos\theta,\ tan(\pi - \theta) = -tan\theta$

$sin(\pi + \theta) = -sin\theta,\ cos(\pi + \theta) = -cos\theta,\ tan(\pi + \theta) = tan\theta$

$sin(2\pi - \theta) = -sin\theta,\ cos(2\pi - \theta) = cos\theta,\ tan(2\pi - \theta) = -tan\theta$

$sin(2\pi + \theta) = sin\theta,\ cos(2\pi + \theta) = cos\theta,\ tan(2\pi + \theta) = tan\theta$

We will be using these formulae frequently for solving examples, so it would be very useful to remember them. As an aid to memory, remember the following.

First of all, it is enough to consider values of trigonometric functions $sin\alpha$, $cos\alpha$ etc. where $0 \le \alpha < 2\pi$, because if $\theta \in R$ then $\theta = 2n\pi + \alpha$, $0 \le \alpha < 2\pi$. We let $0 < \beta < \frac{\pi}{2}$. Then typical real numbers $\frac{\pi}{2} - \beta$, $\frac{\pi}{2} + \beta$, $\frac{3\pi}{2} - \beta$ and $\frac{3\pi}{2} + \beta$ correspond to the trigonometric points which lie in the I, II, III, IV quadrants respectively.

$\frac{\pi}{2} + \beta$	$\frac{\pi}{2} - \beta$
$\frac{3\pi}{2} - \beta$	$\frac{3\pi}{2} + \beta$

Figure 4.2

From figure 4.2 for any real value, trigonometric function change as under, *sin* $\to$ *cos*, *cos* $\to$ *sin*, *tan* $\to$ *cot*, *cot* $\to$ *tan*, *sec* $\to$ *cosec*, *cosec* $\to$ *sec*.

$P\left(\frac{\pi}{2}+\beta\right)$ is in second quadrant.

In the second quadrant $sin\left(\frac{\pi}{2}+\beta\right) > 0$.

Note : Choice of sign is according to the original function on the left.

$$\therefore \quad sin\left(\frac{\pi}{2}+\beta\right) = cos\beta$$

$P\left(\frac{3\pi}{2}-\beta\right)$ is in the third quadrant and in the third quadrant $cos\left(\frac{3\pi}{2}-\beta\right)$ is $-ve$.