Chapter 2

ALGEBRA

Learning Objectives

After studying this chapter, the students will be able to understand

- definition of partial fractions
- the various techniques to resolve into partial fractions
- different cases of permutation such as word formation
- different cases of combination such as word formation
- difference between permutation and combination
- the concept and principle of mathematical induction
- Binomial expansion techniques

Al-Khwarizmi

Introduction

Algebra is a major component of mathematics that is used to unify mathematics concepts. Algebra is built on experiences with numbers and operations along with geometry and data analysis. The word "algebra" is derived from the Arabic word "al-Jabr". The Arabic mathematician Al-Khwarizmi has traditionally been known as the "Father of algebra". Algebra is used to find the perimeter and area of any plane region, volume and CSA of solid.

2.1 Partial Fractions

Rational Expression

An expression of the form

$$\frac{p(x)}{q(x)} = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}, [q(x) \neq 0]$$

is called a rational algebraic expression.

If the degree of the numerator $p(x)$ is less than that of the denominator $q(x)$ then $\frac{p(x)}{q(x)}$ is called a proper rational expression. If the degree of $p(x)$ is greater than or equal to that of $q(x)$ then $\frac{p(x)}{q(x)}$ is called an improper rational expression.

An improper rational expression can be expressed as the sum of an integral function and a proper rational expression.

That is, $\frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$

For example,

$$\frac{x^2 + x + 1}{x^2 + 2x + 1} = 1 - \frac{x}{x^2 + 2x + 1}$$

Partial Fractions

We can express the sum or difference of two rational expressions $\frac{3}{x-1}$ and $\frac{2}{x-2}$ as a single rational expression.

i.e.,

$$\frac{3}{x-1}+\frac{2}{x-2} = \frac{5x-8}{(x-1)(x-2)}$$

$$\frac{5x-8}{(x-1)(x-2)} = \frac{3}{x-1}+\frac{2}{x-2}$$

Process of writing a single rational expression as a sum or difference of two or more simple rational expressions is called splitting up into *partial fractions*.

Generally if $p(x)$ and $q(x)$ are two rational integral algebraic functions of x and the fraction $\frac{p(x)}{q(x)}$ be expressed as the algebraic sum (or difference) of simpler fractions according to certain specified rules, then the rational expression $\frac{p(x)}{q(x)}$ is said to be resolved into partial fractions.

2.1.1 Denominator contains non-repeated Linear factors

In the rational expression $\frac{p(x)}{q(x)}$, if $q(x)$ is the product of non-repeated linear factors of the form $(ax+b)(cx+d)$, then $\frac{p(x)}{q(x)}$ can be resolved into partial fraction of the form $\frac{A}{ax+b}+\frac{B}{cx+d}$, where A and B are constants to be determined.

For example,

$\frac{3x+7}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$, the values of A and B are to be determined.

Example 2.1

Find the values of A and B if

$$\frac{1}{(x^2-1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

Solution

Let $\frac{1}{(x^2-1)} = \frac{A}{x-1}+\frac{B}{x+1}$

Multiplying both sides by $(x-1)(x+1)$, we get

$1= A(x+1)+B(x-1)$ ... (1)

Put x =1 in (1) we get, $1 = A(2)$

$\therefore A = \frac{1}{2}$

Put $x = -1$ in (1) we get, $1 = A(0)+B(-2)$

$\therefore B = -\frac{1}{2}$

Example 2.2

Resolve into partial fractions:

$$\frac{7x-1}{x^2-5x+6}$$

Solution

Write the denominator into the product of linear factors.

Here $x^2-5x+6 = (x-2)(x-3)$

Let $\frac{7x-1}{x^2-5x+6} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$ ...(1)

Multiplying both the sides of (1) by $(x-2)(x-3)$, we get

$7x-1 = A(x-3)+B(x-2)$ ...(2)

Put $x = 3$ in (2),we get

$21 - 1 = B(1)$

$\Rightarrow B = 20$

Put $x = 2$ in (2),we get

$14-1 = A(-1)$

$\Rightarrow A = -13$

Substituting the values of A and B in (1), we get

$$\frac{7x-1}{x^2-5x+6} = \frac{-13}{(x-2)} + \frac{20}{(x-3)}$$

DO YOU KNOW?

$$\frac{7x-1}{x^2-5x+6} =$$

$$\frac{1}{(x-2)}\left[\frac{(7x-1)}{(x-3)}\right]_{x=2} + \frac{1}{(x-3)}\left[\frac{(7x-1)}{(x-2)}\right]_{x=3}$$

$$= \frac{-13}{(x-2)} + \frac{20}{(x-3)}$$

Example 2.3

Resolve into partial fraction:

$$\frac{x+4}{(x^2-4)(x+1)}$$

Solution

Write the denominator into the product of linear factors

Here $(x^2-4)(x+1) = (x-2)(x+2)(x+1)$

$$\therefore \frac{x+4}{(x^2-4)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+1)} \quad ...(1)$$

Multiplying both the sides by $(x-2)(x+2)(x+1)$, we get

$$x+4 = A(x+2)(x+1) + B(x-2)(x+1) + C(x-2)(x+2) \quad ... (2)$$

Put $x = -2$ in (2), we get

$$-2+4 = A(0) + B(-4)(-1) + C(0)$$

$$\therefore \quad B = \frac{1}{2}$$

Put $x = 2$ in (2), we get

$$2+4 = A(4)(3) + B(0) + C(0)$$

$$\therefore \quad A = \frac{1}{2}$$

Put $x = -1$ in (2), we get

$$-1+4 = A(0) + B(0) + C(-3)(1)$$

$$\therefore \quad C = -1$$

Substituting the values of A, B and C in (1), we get

$$\frac{x+4}{(x^2-4)(x+1)} = \frac{1}{2(x-2)} + \frac{1}{2(x+2)} - \frac{1}{x+1}$$

2.1.2 Denominator contains Linear factors, repeated *n* times

In the rational fraction $\frac{p(x)}{q(x)}$, if $q(x)$ is of the form $(ax+b)^n$ [the linear factor is the product of $(ax+b)$, n times], then $\frac{p(x)}{q(x)}$ can be resolved into partial fraction of the form.

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

For example,

$$\frac{9x+7}{(x+4)(x+1)^2} = \frac{A}{(x+4)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

Example 2.4

Find the values of A, B and C if

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Solution

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \ ...(1)$$

Put $x = 1$ in (1)

$$1 = A(1+1)^2 + B(0) + C(0)$$

$$\therefore A = \frac{1}{4}$$

Put $x = -1$ in (1)

$$-1 = A(0) + B(0) + C(-1-1)$$

$$\therefore C = \frac{1}{2}$$

Equating the constant term on both sides of (1),we get

$$A - B - C = 0$$

$$\Rightarrow B = A - C$$

$$\therefore \quad B = -\frac{1}{4}$$

Example 2.5

Resolve into partial fraction

$$\frac{x+1}{(x-2)^2(x+3)}$$

Solution

Let $\frac{x+1}{(x-2)^2(x+3)}$

$$= \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+3)} \qquad ...(1)$$

Multiplying both the sides by $(x-2)^2(x+3)$, we get

$$x+1 = A(x-2)(x+3) + B(x+3) + C(x-2)^2 \qquad ...(2)$$

Put $x = 2$ in (2), we have

$$2+1 = A(0) + B(5) + C(0)$$

$$\therefore \quad B = \frac{3}{5}$$

Put $x = -3$ in (2), we have

$$-3+1 = A(0) + B(0) + C(-3-2)^2$$

$$\therefore \quad C = \frac{-2}{25}$$

Equating the coefficient of x^2 on both the sides of (2), we get

$$0 = A + C$$

$$A = -C$$

$$\therefore \quad A = \frac{2}{25}$$

Substituting the values of A, B and C in (1),

$$\frac{x+1}{(x-2)^2(x+3)} = \frac{2}{25(x-2)} + \frac{3}{5(x-2)^2} - \frac{2}{25(x+3)}$$

Example 2.6

Resolve into partial fraction

$$\frac{9}{(x-1)(x+2)^2}$$

Solution

$$\frac{9}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \quad \ldots (1)$$

Multiplying both sides by $(x-1)(x+2)^2$

$$\therefore 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad \ldots (2)$$

Put $x = -2$ in (2)

$$9 = A(0) + B(0) + C(-2-1)$$

$$\therefore \quad C = -3$$

Put $x = 1$ in (2)

$$9 = A(3)^2 + B(0) + C(0)$$

$$\therefore \quad A = 1$$

Equating the coefficient of x^2 on both the sides of (2), we get

$$0 = A + B$$

$$B = -A$$

$$\therefore \quad B = -1$$

Substituting the values of A, B and C in (1), we get

$$\frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

2.1.3 Denominator contains quadratic factor, which cannot be factorized into linear factors

In the rational fraction $\frac{p(x)}{q(x)}$, if one of the quadratic factor $q(x)$ is of the form ax^2+bx+c which cannot be factorized into linear factors, then $\frac{p(x)}{q(x)}$ can be resolved into partial fractions by taking the numerator of ax^2+bx+c of the form $Ax+B$.

Example 2.7

Resolve into partial fractions:

$$\frac{2x+1}{(x-1)(x^2+1)}$$

Solution

Here x^2+1 cannot be factorized into linear factors.

$$\text{Let } \frac{2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1}+\frac{Bx+c}{x^2+1} \quad \dots (1)$$

Multiplying both sides by $(x-1)(x^2+1)$

$$2x+1 = A(x^2+1)+(Bx+C)(x-1) \quad \dots (2)$$

Put $x = 1$ in (2)

$$2+1 = A(1+1)+0$$

$$\therefore A = \frac{3}{2}$$

Put $x = 0$ in (2)

$$0+1 = A(0+1)+(0+C)(-1)$$

$$1 = A - C$$

$$\therefore \; C = \frac{1}{2}$$

Equating the coefficient of x^2 on both the sides of (2), we get

$$A+B = 0$$

$$B = -A$$

$$\therefore \; B = \frac{-3}{2}$$

Substituting the values of A, B and C in (1), we get

$$\frac{2x+1}{(x-1)(x^2+1)} = \frac{\frac{3}{2}}{x-1}+\frac{\frac{-3}{2}x+\frac{1}{2}}{x^2+1}$$

$$= \frac{3}{2(x-1)}-\frac{3x-1}{2(x^2+1)}$$

Exercise 2.1

Resolve into partial fractions for the following:

1. $\frac{3x+7}{x^2-3x+2}$
2. $\frac{4x+1}{(x-2)(x+1)}$
3. $\frac{1}{(x-1)(x+2)^2}$
4. $\frac{1}{x^2-1}$
5. $\frac{x-2}{(x+2)(x-1)^2}$
6. $\frac{2x^2-5x-7}{(x-2)^3}$
7. $\frac{x^2-6x+2}{x^2(x+2)}$
8. $\frac{x^2-3}{(x+2)(x^2+1)}$
9. $\frac{x+2}{(x-1)(x+3)^2}$
10. $\frac{1}{(x^2+4)(x+1)}$

2.2 Permutations

2.2.1 Factorial

For any natural number n, 'n factorial' is defined as the product of the first n natural numbers and is denoted by $n!$ or $\lfloor n$.

For any natural number n,
$n! = n(n-1)(n-2) \; \dots \; 3\times2\times1$

For examples,

$5! = 5\times4\times3\times2\times1 = 120$

$4! = 4\times3\times2\times1 = 24$

$3! = 3\times2\times1 = 6$

$2! = 2\times1 = 2$

$1! = 1$

NOTE

$0! = 1$

For any natural number n

$$n! = n(n-1)(n-2) \; \dots \; 3\times2\times1$$

$$= n(n-1)!$$

$$= n(n-1)(n-2)!$$

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In particular,

$8! = 8\times(7\times6\times5\times4\times3\times2\times1)$

$= 8\times7!$

Example 2.8

Evaluate the following:

(i) $\frac{7!}{6!}$ (ii) $\frac{8!}{5!}$ (iii) $\frac{9!}{6!3!}$

Solution

(i) $\frac{7!}{6!} = \frac{7\times6!}{6!} = 7$

(ii) $\frac{8!}{5!} = \frac{8\times7\times6\times5!}{5!} = 336$

(iii) $\frac{9!}{6!3!} = \frac{9\times8\times7\times6!}{6!\times3!} = \frac{9\times8\times7}{3\times2\times1} = 84$

Example 2.9

Rewrite 7! in terms of 5!

Solution

$7! = 7\times6!$

$= 7\times6\times5! = 42\times5!$

Example 2.10

Find n, if $\frac{1}{9!}+\frac{1}{10!}=\frac{n}{11!}$

Solution

$$\frac{1}{9!}+\frac{1}{10!} = \frac{n}{11!}$$

$$\frac{1}{9!}+\frac{1}{10\times9!} = \frac{n}{11!}$$

$$\frac{1}{9!}\left[1+\frac{1}{10}\right] = \frac{n}{11!}$$

$$\frac{1}{9!}\times\frac{11}{10} = \frac{n}{11!}$$

$$n = \frac{11!\times11}{9!\times10}$$

$$= \frac{11!\times11}{10!}$$

$$= \frac{11\times10!\times11}{10!}$$

$$n = 121$$

2.2.2 Fundamental principle of counting

Multiplication principle of counting:

Consider the following situation in an auditorium which has three entrance doors and two exit doors. Our objective is to find the number of ways a person can enter the hall and then come out.

Assume that, P_1, P_2 and P_3 are the three entrance doors and S_1 and S_2 are the two exit doors. A person can enter the hall through any one of the doors P_1, P_2 or P_3 in 3 ways. After entering the hall, the person can come out through any of the two exit doors S_1 or S_2 in 2 ways.

Hence the total number of ways of entering the hall and coming out is

$3\times2 = 6$ ways.

These possibilities are explained in the given flow chart (Fig. 2.1).

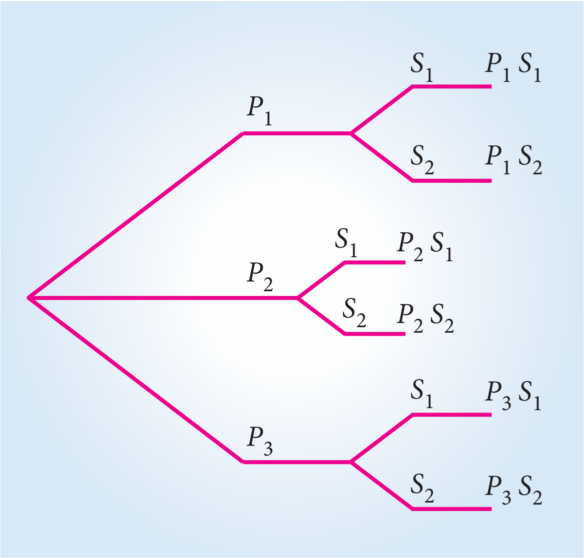

Fig. 2.1

The above problem can be solved by applying multiplication principle of counting.

Definition 2.1

There are two jobs. Of which one job can be completed in m ways, when it has completed in any one of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in $m \times n$ ways. This is called multiplication principle of counting.

Example 2.11

Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word "NOTE", where the repetition of the letters is not allowed.

Solution

Let us allot four vacant places (boxes) to represent the four letters. N,O,T,E of the given word NOTE.

1	2	3	4

Fig. 2.2

The first place can be filled by any one of the 4 letters in 4 ways. Since repetition is not allowed, the number of ways of filling the second vacant place by any of the remaining 3 letters in 3 ways. In the similar way the third box can be filled in 2 ways and that of last box can be filled in 1 way. Hence by multiplication principle of counting, total number of words formed is $4 \times 3 \times 2 \times 1 = 24$ words.

Example 2.12

If each objective type questions having 4 choices, then find the total number of ways of answering the 4 questions.

Solution

Since each question can be answered in 4 ways, the total number of ways of answering 4 questions

$= 4 \times 4 \times 4 \times 4 = 256$ ways.

Example 2.13

How many 3 digits numbers can be formed if the repetition of digits is not allowed?

Solution

Let us allot 3 boxes to represent the digits of the 3 digit number

100^{th} place	10^{th} place	unit place
1	2	3

Fig. 2.3

Three digit number never begins with "0". $\therefore$ 100^{th} place can be filled with any one of the digits from 1 to 9 in 9 ways.

Since repetition is not allowed, 10^{th} place can be filled by the remaining 9 numbers (including 0) in 9 ways and the unit place can be filled by the remaining 8 numbers in 8 ways. Then by multiplication principle of counting, number of 3 digit numbers $= 9 \times 9 \times 8 = 648$.

2.2.3 Addition principle of counting

Let us consider the following situation in a class, there are 10 boys and 8 girls. The class teacher wants to select either a boy or a girl to represent the class in a function. Our objective is to find the number of ways that the teacher can select the student. Here the teacher has to perform either of the following two jobs.

One student can be selected in 10 ways among 10 boys.

One student can be selected in 8 ways among 8 girls

Hence the jobs can be performed in 10 + 8 = 18 ways

This problem can be solved by applying Addition principle of counting.

Definition 2.2

If there are two jobs, each of which can be performed independently in m and n ways respectively, then either of the two jobs can be performed in (m+n) ways. This is called addition principle of counting.

Example 2.14

There are 6 books on commerce and 5 books on accountancy in a book shop. In how many ways can a student purchase either a book on commerce or a book on accountancy?

Solution

Out of 6 commerce books, a book can be purchased in 6 ways.

Out of 5 accountancy books, a book can be purchased in 5 ways.

Then by addition principle of counting the total number of ways of purchasing any one book is 5+6=11 ways.

Exercise 2.2

1. Find x if $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$.
2. Evaluate $\frac{n!}{r!(n-r)!}$ when $n = 5$ and $r = 2$.
3. If $(n+2)! = 60[(n-1)!]$, find n
4. How many five digits telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 with no digit appears more than once?
5. How many numbers lesser than 1000 can be formed using the digits 5, 6, 7, 8 and 9 if no digit is repeated?

2.2.4 Permutation

Suppose we have a fruit salad with combination of "APPLES, GRAPES & BANANAS". We don't care what order the fruits are in. They could also be "bananas, grapes and apples" or "grapes, apples and bananas". Here the order of mixing is not important. Any how, we will have the same fruit salad.

But, consider a number lock with the number code 395. If we make more than three attempts wrongly, then it locked permanently. In that situation we do care about the order 395. The lock will not work if we input 3-5-9 or 9-5-3. The number lock will work if it is exactly 3-9-5. We have so many such situations in our practical life. So we should know the order of arrangement, called Permutation.

Definition 2.3

The number of arrangements that can be made out of n things taking r at a time is called the number of permutation of n things taking r at a time.

For example, the number of three digit numbers formed using the digits 1, 2, 3 taking all at a time is 6.

6 three digit numbers are 123, 132, 231, 213, 312, 321.

Notations: If $n \geq 1$ and $0 \leq r \leq n$, then the number of all permutations of n distinct things taken r at a time is denoted by

$$P(n,r) \text{ (or) } nP_r$$

We have the following theorem *without proof.*

Theorem: $np_r = \dfrac{n!}{(n-r)!}$

Example 2.15

Evaluate: $5P_3$ and $P(8, 5)$

Solution

$$5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60.$$

$$P(8,5) = 8P_5$$

$$= \frac{8!}{(8-5)!}$$

$$= \frac{8!}{3!}$$

$$= \frac{8\times7\times6\times5\times4\times3!}{3!} = 6720$$

Results

(i) $0! = 1$

(ii) $np_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$

(iii) $np_1 = \dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

(iv) $np_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

(v) $np_r = n(n-1)(n-2) \ldots [n-(r-1)]$

Example 2.16

Evaluate: (i) $8P_3$ (ii) $5P_4$

Solution

(i) $8P_3 = 8\times7\times6 = 336.$

(ii) $5P_4 = 5\times4\times3\times2 = 120.$

Example 2.17

In how many ways 7 pictures can be hung from 5 picture nails on a wall ?

Solution

The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is nothing but the number of permutation of 7 things taken 5 at a time

$$= 7P_5 = \frac{7!}{(7-5)!} = 2520 \text{ ways.}$$

Example 2.18

Find how many four letter words can be formed from the letters of the word "LOGARITHMS" (words are with or without meanings)

Solution

There are 10 letters in the word LOGARITHMS

Therefore $n = 10$

Since we have to find four letter words,

$r = 4$

Hence the required number of four letter words

$$= np_r = 10\,p_4$$

$$= 10\times9\times8\times7$$

$$= 5040.$$

Example 2.19

If $nP_r = 360$, find n and r.

Solution

$$nP_r = 360 = 36\times10$$

$$= 3\times3\times4\times5\times2$$

$$= 6\times5\times4\times3 = 6P_4$$

Therefore $n = 6$ and $r = 4$

Permutation of repeated things:

The number of permutation of n different things taken r at a time, when repetition is allowed is n^r.

Example 2.20

Using 9 digits from 1, 2, 3, … , 9 taking 3 digits at a time, how many 3 digits numbers can be formed when repetition is allowed?

Solution

Here, $n = 9$ and $r = 3$

$\therefore$ Number of 3 digit numbers = n^r

$= 9^3 = 729$ numbers

Permutations when all the objects are not distinct:

The number of permutations of n things taken all at a time, of which p things are of one kind and q things are of another kind, such that $p + q = n$ is $\frac{n!}{p!\,q!}$.

In general, the number of permutation of n objects taken all at a time, p_1 are of one kind, p_2 are of another kind, p_3 are of third kind, … p_k are of k^{th} kind such that $p_1 + p_2 + \ldots + p_k = n$ is $\frac{n!}{p_1!\,p_2!\ldots p_k!}$.

Example 2.21

How many distinct words can be formed using all the letters of the following words.

(i) MISSISSIPPI (ii) MATHEMATICS.

Solution

(i) There are 11 letters in the word MISSISSIPPI

In this word M occurs once

I occurs 4 times

S occurs 4 times

P occurs twice.

Therefore required number of permutation = $\frac{11!}{4!\,4!\,2!}$

(ii) In the word MATHEMATICS

There are 11 letters

Here M occurs twice

T occurs twice

A occurs twice

and the rest are all different. Therefore number of permutations = $\frac{11!}{2!\,2!\,2!}$

2.2.5 Circular permutation

In the last section, we have studied permutation of n different things taken all together is $n!$. Each permutation is a different arrangement of n things in a row or on a straight line. These are called Linear permutation. Now we consider the permutation of n things along a circle, called circular permutation.

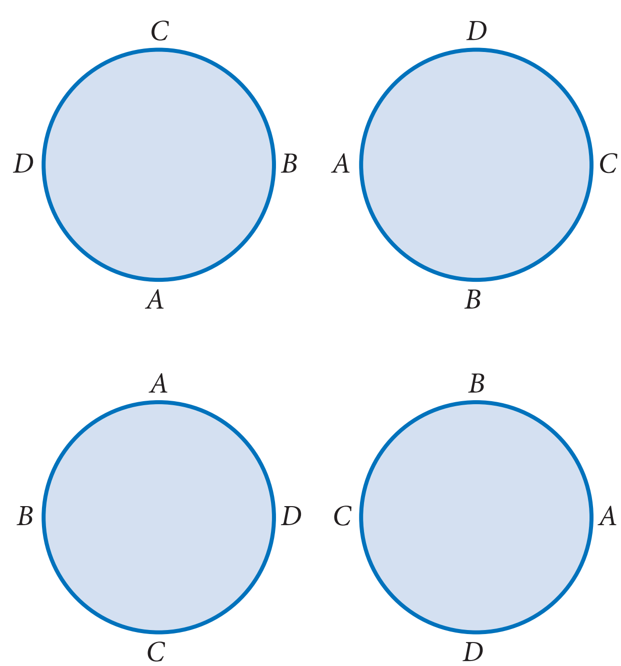

Fig. 2.4

Consider the four letters A, B, C, D and the number of row arrangement of these 4 letters can be done in 4! Ways. Of those 4! arrangements, the arrangements ABCD, BCDA, CDAB, DABC are one and the same when they are arranged along a circle.

So, the number of permutation of '4' things along a circle is $\frac{4!}{4} = 3!$

In general, circular permutation of n different things taken all at the time = $(n-1)!$

NOTE

If clock wise and anti-clockwise circular permutations are considered to be same (identical), the number of circular permutation of n objects taken all at a time is $\frac{(n-1)!}{2}$

Example 2.22

In how many ways 8 students can be arranged

(i) in a line (ii) along a circle

Solution

(i) Number of permutations of 8 students along a line = $8P_8 = 8!$

(ii) When the students are arranged along a circle, then the number of permutation is $(8-1)! = 7!$

Example 2.23

In how many ways 10 identical keys can be arranged in a ring?

Solution

Since keys are identical, both clock wise and anti-clockwise circular permutation are same.

The number of permutation is

$$\frac{(n-1)!}{2} = \frac{(10-1)!}{2} = \frac{9!}{2}$$

Example 2.24

Find the rank of the word 'RANK' in dictionary.

Solutions

Here maximum number of the word in dictionary formed by using the letters of the word 'RANK' is 4!

The letters of the word RANK in alphabetical order are A, K, N, R

DO YOU KNOW?

Rank of word in dictionary

*The **rank** of a word in a dictionary is to arrange the words in alphabetical order. In this dictionary the word may or may not be meaningful.*

Number of words starting with A =3! =6

Number of words starting with K =3! =6

Number of words starting with N =3! =6

Number of words starting with RAK =1! = 1

Number of words starting with RANK= 0! = 1

∴ Rank of the word RANK is

6 + 6 + 6 + 1+1= 20

Exercise 2.3

1. If $nP_4 = 12(nP_2)$, find n.
2. In how many ways 5 boys and 3 girls can be seated in a row, so that no two girls are together?
3. How many 6-digit telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each numbers

starts with 35 and no digit appear more than once?

4. Find the number of arrangements that can be made out of the letters of the word "ASSASSINATION"

5. (a) In how many ways can 8 identical beads be strung on a necklace?

   (b) In how many ways can 8 boys form a ring?

6. Find the rank of the word 'CHAT' in dictionary.

2.3 Combinations

A combination is a selection of items from a collection such that the order of selection does not matter. That is the act of combining the elements irrespective of their order. Let us suppose that in an interview two office assistants are to be selected out of 4 persons namely A, B, C and D. The selection may be of AB, AC, AD, BC, BD, CD (we cannot write BA, since AB and BA are of the same selection). Number of ways of selecting 2 persons out of 4 persons is 6. It is represented as $4C_2 = \frac{4\times3}{1\times2} = 6$.

The process of different selection without repetition is called Combination.

Definition 2.4

Combination is the selection of n things taken r at a time without repetition. Out of n things, r things can be selected in nc_r ways.

$$nC_r = \frac{n!}{r!(n-r)!},\ n \geq 1,\ 0 \leq r \leq n$$

Here $n \neq 0$ but r may be 0.

For example, out of 5 balls 3 balls can be selected in $5C_3$ ways.

$$5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!} = \left(\frac{5\times4\times3}{1\times2\times3}\right)$$
$$= 10 \text{ ways.}$$

Example 2.25

Find $8C_2$

Solution

$$8C_2 = \frac{8\times7}{2\times1} = 28$$

DO YOU KNOW?

Permutations are for lists (order matters) and combinations are for groups (order doesn't matter)

Properties

(i) $nC_0 = nc_n = 1$

(ii) $nC_1 = n$

(iii) $nC_2 = \frac{n(n-1)}{2!}$

(iv) $nC_x = nC_y$, then either $x = y$ or $x + y = n$

(v) $nC_r = nC_{n-r}$

(vi) $nC_r + nC_{r-1} = (n+1)C_r$

(vii) $nC_r = \frac{np_r}{r!}$

Example 2.26

If $nC_4 = nC_6$, find $12C_n$.

Solution

If $nC_x = nC_y$, then $x + y = n$.

Here $nC_4 = nC_6$

$\therefore n = 4 + 6 = 10$

$$12C_n = 12C_{10}$$
$$= 12C_2$$
$$= \frac{12\times11}{1\times2} = 66$$

Example 2.27

If $nP_r = 720$; $nC_r = 120$, find r

Solution

$$\text{We know that } nC_r = \frac{nP_r}{r!}$$

$$120 = \frac{720}{r!}$$

$$r! = \frac{720}{120} = 6 = 3\,!$$

$$\Longrightarrow r = 3$$

Example 2.28

If $15C_{3r} = 15C_{r+3}$, find r

Solution

$$15C_{3r} = 15C_{r+3}$$

Then by the property,

$nC_x = nC_y \Longrightarrow x + y = n$, we have

$$3r + r + 3 = 15$$

$$\Longrightarrow r = 3$$

Example 2.29

From a class of 32 students, 4 students are to be chosen for a competition. In how many ways can this be done?

Solution

$$\text{The number of combination} = 32C_4$$

$$= \frac{32!}{4!\,(32-4)!}$$

$$= \frac{32!}{4!\,(28)!}$$

Example 2.30

A question paper has two parts namely Part *A* and Part *B*. Each part contains 10 questions. If the student has to choose 8 from part *A* and 5 from part *B*, in how many ways can he choose the questions?

Solution

In part A, out of 10 questions 8 can be selected in $10C_8$ ways. In part B out of 10 questions 5 can be selected in $10C_5$. Therefore by multiplication principle the total number of selection is

$$10C_8 \times 10C_5 = 10C_2 \times 10C_5$$

$$= \frac{10 \times 9}{2 \times 1} \times \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 11340$$

Example 2.31

A Cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket-keepers. In how many different ways can a team be formed so that the team contains at least 3 bowlers and at least one wicket-keeper?

Solution

A Cricket team of 11 players can be formed in the following ways:

(i) 3 bowlers, 1 wicket keeper and 7 other players can be selected in

$$4C_3 \times 2C_1 \times 10C_7 \text{ ways}$$

$$4C_3 \times 2C_1 \times 10C_7 = 4C_1 \times 2C_1 \times 10C_3$$

$$= 960 \text{ ways}$$

(ii) 3 bowlers, 2 wicket keepers and 6 other players can be selected in

$$4C_1 \times 2C_2 \times 10C_6 \text{ ways}$$

$$4C_1 \times 2C_2 \times 10C_6 = 4C_1 \times 2C_2 \times 10C_4$$

$$= 840 \text{ ways}$$

(iii) 4 bowlers and 1 wicket keeper and 6 other players $4C_4 \times 2C_1 \times 10C_6$

$$4C_4 \times 2C_1 \times 10C_4 = 420 \text{ ways}$$

4 bowlers, 2 wicket keepers and 5 other players can be selected in

$4C_4 \times 2C_2 \times 10C_5$ ways

$4C_4 \times 2C_2 \times 10C_5 = 252$ ways

By addition principle of counting.

Total number of ways

$= 960 + 840 + 420 + 252$

$= 2472$

Example 2.32

If $4(nC_2) = (n+2)C_3$, find n

Solution

$$
\begin{aligned}
4(nC_2) &= (n+2)C_3 \\
4\frac{n(n-1)}{1\times 2} &= \frac{(n+2)(n+1)(n)}{1\times 2\times 3} \\
12(n-1) &= (n+2)\ (n+1) \\
12(n-1) &= (n^2+3n+2) \\
n^2-9n+14 &= 0 \\
(n-2)(n-7) &= 0 \Rightarrow n=2, n=7
\end{aligned}
$$

Example 2.33

If $(n+2)C_n = 45$, find n

Solution

$$
\begin{aligned}
(n+2)C_n &= 45 \\
(n+2)C_{n+2-n} &= 45 \\
(n+2)C_2 &= 45 \\
\frac{(n+2)(n+1)}{2} &= 45 \\
n^2+3n-88 &= 0 \\
(n+11)(n-8) &= 0 \\
n &= -11, 8
\end{aligned}
$$

$n = -11$ is not possible

$\therefore n = 8$

Exercise 2.4

1. If $nP_r = 1680$ and $nC_r = 70$, find n and r.
2. Verify that $8C_4 + 8C_3 = 9C_4$.
3. How many chords can be drawn through 21 points on a circle?
4. How many triangles can be formed by joining the vertices of a hexagon?
5. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?
6. If four dice are rolled, find the number of possible outcomes in which atleast one die shows 2.
7. There are 18 guests at a dinner party. They have to sit 9 guests on either side of a long table, three particular persons decide to sit on one particular side and two others on the other side. In how many ways can the guests to be seated?
8. If a polygon has 44 diagonals, find the number of its sides.
9. In how many ways can a cricket team of 11 players be chosen out of a batch of 15 players?
   (i) There is no restriction on the selection.
   (ii) A particular player is always chosen.
   (iii) A particular player is never chosen.
10. A Committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done when
    (i) atleast two ladies are included
    (ii) atmost two ladies are included

2.4 Mathematical Induction

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

Mathematical Induction is used in the branches of Algebra, Geometry and Analysis where it turns out to be necessary to prove some truths of propositions.

The principle of mathematical induction:

Let $P(n)$ be a given statement for $n \in N$.

(i) Initial step: Let the statement is true for $n = 1$ i.e., $P(1)$ is true and

(ii) Inductive step: If the statement is true for $n = k$ (where k is a particular but arbitrary natural number) then the statement is true for $n = k + 1$. i.e., truth of $P(k)$ implies the truth of $P(k+1)$. Then $P(n)$ is true for all natural numbers n.

Example 2.34

Using mathematical induction method, Prove that

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}, \ n \in N.$$

Solution

Let the given statement $P(n)$ be defined as

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} \text{ for } n \in N$$

Step 1: Put $n = 1$

LHS $P(1) = 1$

RHS $P(1) = \frac{1(1+1)}{2} = 1$

LHS = RHS for $n = 1$

$\therefore$ P(1) is true

Step 2: Let us assume that the statement is true for $n = k$.

i.e., $P(k)$ is true

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \text{ is true}$$

Step 3: To prove that $P(k + 1)$ is true

$$P(k+1) = 1 + 2 + 3 + \ldots + k + (k+1)$$
$$= P(k) + k + 1$$
$$= \frac{k(k+1)}{2} + k + 1$$
$$= \frac{k(k+1)+2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ $P(k+1)$ is true

Thus if $P(k)$ is true, then $P(k+1)$ is also true.

$\therefore$ $P(n)$ is true for all $n \in N$

Hence $1+2+3+\ldots+n = \frac{n(n+1)}{2}, n \in N$

Example 2.35

By the principle of Mathematical Induction, prove that

$1 + 3 + 5 + \ldots + (2n-1) = n^2$, for all $n \in N$.

Solution

Let $P(n)$ denote the statement

$1 + 3 + 5 + \ldots + (2n-1) = n^2$

Put $n = 1$

LHS = 1

RHS = 1^2

= 1

LHS = RHS

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true.

i.e., $1 + 3 + 5 + \ldots + (2k-1) = k^2$

To prove: $P(k+1)$ is true.

$\therefore$ 1 + 3 + 5 + … + (2k–1) + (2k+1)

$$= P(k)+(2k+1)$$
$$= k^2+2k+1$$
$$= (k+1)^2$$

$P(k+1)$ is true whenever $P(k)$ is true.

$\therefore P(n)$ is true for all $n \in N$.

Example 2.36

By Mathematical Induction, prove that $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \in N$.

Solution

Let $P(n)$ denote the statement:

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Put $n = 1$

$$\therefore \text{LHS} = 1^2 = 1$$

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = 1$$

$$\text{LHS} = \text{RHS}$$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$$P(k) = 1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now,

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = p(k)+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore P(k+1)$ is true whenever $P(k)$ is true

$\therefore P(n)$ is true for all $n \in N$.

Example 2.37

Show by the principle of mathematical induction that $2^{3n}-1$ is a divisible by 7, for all $n \in N$.

Solution

Let the given statement $P(n)$ be defined as $P(n) = 2^{3n}-1$

Step 1: Put $n = 1$

$\therefore P(1) = 2^3-1$

$= 7$ is divisible by 7

i.e., $P(1)$ is true.

Step 2: Let us assume that the statement is true for $n = k$ i.e., $P(k)$ is true.

We assume $2^{3k}-1$ is divisible by 7

$\Rightarrow 2^{3k}-1 = 7m$

Step 3: To prove that $P(k+1)$ is true

$$P(k+1) = 2^{3(k+1)}-1$$
$$= 2^{(3k+3)}-1$$
$$= 2^{3k}\cdot 2^3-1$$
$$= 2^{3k}\cdot 8-1$$
$$= 2^{3k}\cdot(7+1)-1$$
$$= 2^{3k}\cdot 7+2^{3k}-1$$
$$= 2^{3k}\cdot 7+7m = 7(2^{3k}+m)$$

which is divisible by 7

$P(k+1)$ is true whenever $p(k)$ is true

$\therefore P(n)$ is true for all $n \in N$.

Hence the proof.

Example 2.38

By the principle of mathematical induction prove that $n^2 + n$ is an even number, for all $n \in N$.

Solution

Let $P(n)$ denote the statement that, "$n^2 + n$ is an even number".

Put $n = 1$

$\therefore 1^2 + 1 = 1 + 1 = 2$, an even number.

Let us assume that $P(k)$ is true.

$\therefore$ $k^2 + k$ is an even number is true

$\therefore$ Take $k^2 + k = 2m$ ...(1)

To prove $P(k + 1)$ is true

$$\therefore \ (k+1)^2 + (k+1)$$
$$= k^2 + 2k + 1 + k + 1$$
$$= k^2 + k + 2k + 2$$
$$= 2m + 2(k + 1) \text{ by (1)}$$
$$= 2(m + k + 1) \text{ (a multiple of 2)}$$

$\therefore$ $(k + 1)^2 + (k + 1)$ is an even number

$\therefore$ $P(k + 1)$ is true whenever $P(k)$ is true.

$P(n)$ is true for $n \in N$.

Exercise 2.5

By the principle of mathematical induction, prove the following

1. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in N$.
2. $1.2 + 2.3 + 3.4 + \ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$, for all $n \in N$.
3. $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$, for all $n \in N$.
4. $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$, for all $n \in N$.
5. $3^{2n} - 1$ is a divisible by 8, for all $n \in N$.
6. $a^n - b^n$ is divisible by $a - b$, for all $n \in N$.
7. $5^{2n} - 1$ is divisible by 24, for all $n \in N$.
8. $n(n + 1)(n+2)$ is divisible by 6, for all $n \in N$.
9. $2^n > n$, for all $n \in N$.

2.5 Binomial Theorem

An algebraic expression of sum or the difference of two terms is called a binomial. For example $(x+y)$, $(5a-2b)$, $\left(x+\frac{1}{y}\right)$, $\left(p+\frac{5}{p}\right)$, $\left(\frac{7}{4}+\frac{1}{y^2}\right)$ etc., are binomials. The Binomial theorem or Binomial Expression is a result of expanding the powers of binomials. It is possible to expand $(x + y)^n$ into a sum involving terms of the form ax^by^c, exponents b and c are non-negative integers with $b + c = n$, the coefficient 'a' of each term is a positive integer called binomial coefficient.

Expansion of $(x + a)^2$ was given by Greek mathematician Euclid on 4th century and there is an evidence that the binomial theorem for cubes i.e., expansion of $(x+a)^3$ was known by 6th century in India. The term Binomial coefficient was first introduced by Michael Stifle in 1544. Blaise Pascal (19 June 1623 to 19 August 1662) a French Mathematician, Physicist, inventor, writer and catholic theologian. In his Treatise on Arithmetical triangle of 1653 described the convenient tabular presentation for Binomial coefficient now called Pascal's triangle. Sir Issac Newton generalized the Binomial theorem and made it valid for any rational exponent.

Now we study the **Binomial theorem for $(x + a)^n$**

Theorem(without proof)

If x and 'a' are real numbers, then for all $n \in N$

$$(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1$$
$$nC_2 x^{n-2} a^2 + ... + nC_r x^{n-r} a^r + ...$$
$$+ nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n = \sum_{r=0}^{n} nC_r x^{n-r} a^r$$

NOTE

When $n = 0$, $(x+a)^0 = 1$

When $n = 1$,

$(x+a) = 1C_0 x + 1C_1 a = x + a$

When $n = 2$,

$(x+a)^2 = 2C_0 x^2 + 2C_1 x^a + 2C_2 a^2$

$= x^2 + 2xa + a^2$

When $n = 3$,

$(x+a)^3 = 3C_0 x^3 + 3C_1 x^2 a + 3C_2 x a^2 + 3C_3 a^3$

$= x^3 + 3x^2 a + 3xa^2 + a^3$ and so on.

Given below is the Pascal's Triangle showing the co- efficient of various terms in Binomial expansion of $(x+a)^n$

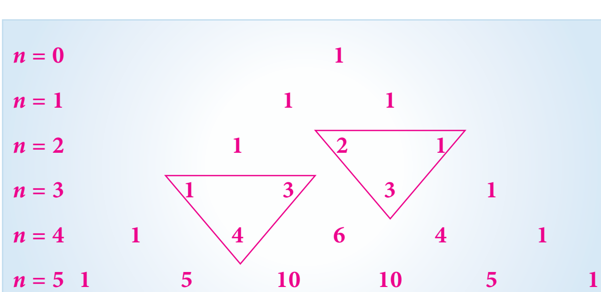

Fig. 2.5

NOTE

(i) Number of terms in the expansion of $(x + a)^n$ is n+1

(ii) Sum of the indices of x and a in each term in the expansion is n

(iii) $nC_0, nC_1, nC_2, nC_3, ... nC_r, ... nC_n$ are also represented as $C_0, C_1, C_2, C_3, ..., C_r, ..., C_n$, are called Binomial co-efficients.

(iv) General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$

(v) Since $nC_r = nC_{n-r}$, for $r = 0, 1, 2, \ldots, n$ in the expansion of $(x + a)^n$, binomial co-efficients of terms equidistant from the beginning and from the end of the expansion are equal.

(vi) Sum of the co-efficients in the expansion of $(1+x)^n$ is equal of 2^n

(vii) In the expansion of $(1+x)^n$, the sum of the co-efficients of odd terms = the sum of the co-efficients of the even terms $=2^{n-1}$

Middle term of $(x + a)^n$

Case (i) If n is even, then the number of terms $n + 1$ is odd, then there is only one middle term, given by $t_{\frac{n}{2}+1}$

Case (ii) If n is odd, then the number of terms $(n+1)$ is even. Therefore, we have two middle terms given by $t_{\frac{n+1}{2}}$ and $t_{\frac{n+3}{2}}$

Sometimes we need a **particular term** in the expansion of $(x + a)^n$. For this we first write the general term t_{r+1}. The value of r can be obtained by taking the term t_{r+1} as the required term. To find the **term independent of *x*** (term without *x*), equate the power of *x* in t_{r+1} to zero, we get the value of '*r*'. By substituting the value of *r* in t_{r+1}, we get the term independent of *x*.

NOTE

$(x+a)^n = nC_0x^na^0 + nC_1x^{n-1}a^1 + nC_2x^{n-2}a^2 + ... + nC_rx^{n-r}a^r + ... + a^n \ ... (1)$

(i) To find $(x-a)^n$, replace by $-a$

$(x-a)^n = nC_0x^na^0 + nC_1x^{n-1}(-a) + nC_2x^{n-2}(-a)^2 + ... + nC_rx^{n-r}(-a)^r + ... + (-a)^n$

$= nC_0x^na^0 - nC_1x^{n-1}(a) + nC_2x^{n-2}(a)^2 + ... + (-1)^r nC_rx^{n-r}(a)^r + ... + (-1)^n(a)^n \ ...(2)$

Note that we have the signs alternatively.

(ii) If we put $a = 1$ in (1), we get

$(1+x)^n = 1 + nC_1x + nC_2x^2 + ... + nC_rx^r + ... + nC_nx^n \ ...(3)$

(iii) If we replace $xby - x$ in (3), we get

$(1-x)^n = 1 - nC_1x + nC_2x^2 - ... + nC_r(-1)^rx^r + ... + nC_n(-1)^n x^n$

Example 2.39

Expand $(2x + 3y)^5$ using binomial theorem.

Solution

$$(x+a)^n = nC_0x^na^0 + nC_1x^{n-1}a^1 + nC_2x^{n-2}a^2 + ... + nC_rx^{n-r}a^r + ... + nC_{n-1}x^1a^{n-1} + nC_nx^0a^n$$

$$(2x+3y)^5 = 5C_0(2x)^5 + 5C_1(2x)^4(3y) + 5C_2(2x)^3(3y)^2 + 5C_3(2x)^2(3y)^3 + 5C_4(2x)(3y)^4 + 5C_5(3y)^5$$

$$= 32x^5 + 5(16)x^4(3y) + \frac{5\times4}{2\times1}(8x^3)(9y^2) + 5(2x)81y^4 + 243y^5$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

Example 2.40

Using binomial theorem, expand $\left(x^2 + \frac{1}{x^2}\right)^4$

Solution

$$\left(x^2+\frac{1}{x^2}\right)^4 = (x^2)^4 + 4C_1(x^2)^3\frac{1}{x^2} + 4C_2(x^2)^2\left(\frac{1}{x^2}\right)^2 + 4C_3(x^2)\left(\frac{1}{x^2}\right)^3 + 4C_4\left(\frac{1}{x^2}\right)^4$$

$$= x^8 + 4x^4 + 6 + \frac{4}{x^2} + \frac{4}{x^4} + \frac{1}{x^8}$$

Example 2.41

Using binomial theorem, evaluate $(101)^5$

Solution

$$(101)^5 = (100+1)^5$$

$$= (100)^5 + 5C_1(100)^4 + 5C_2(100)^3 + 5C_3(100)^2 + 5C_4(100) + 5C_5$$

$$= 10000000000 + 5(100000000) + 10(1000000) + 10(10000) + 5(100) + 1$$

$$= 10000000000 + 500000000 + 10000000 + 100000 + 500 + 1$$

$$= 10510100501$$

Example 2.42

Find the 5th term in the expansion of $\left(x-\frac{3}{x^2}\right)^{10}$

Solution

General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$ ...(1)

To find the 5th term of $\left(x-\frac{3}{x^2}\right)^{10}$

For this take $r = 4$

Then,

$$t_{4+1} = t_5 = 10C_4 (x)^6 \left(-\frac{3}{x^2}\right)^4$$

(Here $n = 10$, $x = x$, $a = -\frac{3}{x^2}$)

$$= 10C_4 (x)^6 \frac{3^4}{x^8}$$

$$= \frac{17010}{x^2}$$

Example 2.43

Find the middle term in the expansion of $\left(x^2-\frac{2}{x}\right)^{10}$

Solution

Compare $\left(x^2-\frac{2}{x}\right)^{10}$ with $(x+a)^n$

$n = 10$, $x = x^2$, $a = -\frac{2}{x}$

Since $n = 10$, we have 11 terms (odd)

$\therefore$ 6th term is the middle term.

The general term is

$$t_{r+1} = nC_r x^{n-r} a^r \quad ...(1)$$

To get t_6, put $r = 5$

$$t_{5+1} = t_6 = 10C_5 (x^2)^5 \left(\frac{-2}{x}\right)^5$$

$$= 10C_5 x^{10} \frac{(-2)^5}{x^5}$$

$$= -8064x^5$$

Example 2.44

Find the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^9$

Solution

Compare $\left(\frac{x}{3}+9y\right)^9$ with $(x + a)^n$

Since $n = 9$,we have 10 terms(even)

$\therefore$ There are two middle terms namely $t_{\frac{n+1}{2}}$, $t_{\frac{n+3}{2}}$ i.e., $t_{\frac{9+1}{2}}$, $t_{\frac{9+3}{2}}$

General term in the expansion of $(x + a)^n$ is

$$t_{r+1} = nC_r x^{n-r} a^r \quad ... (1)$$

Here t_5 and t_6 are middle terms.

put $r = 4$ in (1),

$$t_{4+1} = t_5 = 9C_4 \left(\frac{x}{3}\right)^5 \cdot (9y)^4$$

$$= 9C_4 \frac{x^5}{3^5} \cdot 9^4 y^4$$

$$= \frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1} \cdot \frac{x^5}{3^5} \cdot 9^4 y^4$$

$$= 3402\, x^5 y^4$$

put $r = 5$ in (1),

$$t_{5+1} = t_6 = 9C_5 \left(\frac{x}{3}\right)^4 \cdot (9y)^5$$

$$= 9C_5 \frac{x^4}{3^4} \cdot 9^5 y^5$$

$$= 91854\, x^4 y^5$$

Example 2.45

Find the Coefficient of x^{10} in the binomial expansion of $\left(2x^2-\frac{3}{x}\right)^{11}$

Solution

General term of $(x + a)^n$ is

$$t_{r+1} = nC_r x^{n-r} a^r \quad (1)$$

Compare $\left(2x^2-\frac{3}{x}\right)^{11}$ with $(x + a)^n$

$$t_{r+1} = 11C_r(2x^2)^{11-r}\left(\frac{-3}{x}\right)^r$$

$$= 11C_r 2^{11-r}x^{2(11-r)}(-3)^r\left(\frac{1}{x}\right)^r$$

$$= 11C_r 2^{11-r}(-1)^r\cdot 3^r x^{22-3r}$$

To find the co-efficient of x^{10}, take $22-3r = 10$

$r = 4$. Put $r = 4$ in (1)

$$t_5 = 11C_4 2^{11-4}3^4x^{10} = 11C_4\cdot 2^7 3^4 x^{10}$$

$\therefore$ Co-efficient of x^{10} is $11C_4 2^7 3^4$.

Example 2.46

Find the term independent of x in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$.

Solution

General term in the expansion of $(x + a)^n$ is $t_{r+1} = nC_r x^{n-r}a^r$

Compare $\left(2x+\frac{1}{3x^2}\right)^9$ with $(x + a)^n$

$$\therefore\ t_{r+1} = 9C_r(2x)^{9-r}\left(\frac{1}{3x^2}\right)^r$$

$$= 9C_r 2^{9-r}x^{9-r}\frac{1}{3^r}x^{-2r}$$

$$t_{r+1} = 9C_r\frac{2^{9-r}}{3^r}\cdot x^{9-3r} \quad \ldots (1)$$

To get the term independent of x, equating the power of x as 0

$9 - 3r = 0$

$\therefore\ r = 3$

Put $r = 3$ in (1)

$$t_{3+1} = 9C_3\frac{2^{9-3}}{3^5}\cdot x^0$$

$$= 9C_3\frac{2^6}{3^5}$$

$$= 1792$$

Exercise 2.6

1. Expand the following by using binomial theorem
   (i) $(2a-3b)^4$ (ii) $\left(x+\frac{1}{y}\right)^7$
   (iii) $\left(x+\frac{1}{x^2}\right)^6$
2. Evaluate the following using binomial theorem:
   (i) $(101)^4$ (ii) $(999)^5$
3. Find the 5th term in the expansion of $(x-2y)^{13}$.
4. Find the middle terms in the expansion of
   (i) $\left(x+\frac{1}{x}\right)^{11}$ (ii) $\left(3x+\frac{x^2}{2}\right)^8$
   (iii) $\left(2x^2-\frac{3}{x^3}\right)^{10}$
5. Find the term independent of x in the expansion of
   (i) $\left(x^2-\frac{2}{3x}\right)^9$ (ii) $\left(x-\frac{2}{x^2}\right)^{15}$
   (iii) $\left(2x^2+\frac{1}{x}\right)^{12}$
6. Prove that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1\cdot 3\cdot 5\ldots(2n-1)2^n}{n!}$.
7. Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5\ldots(2n-1)2^n x^n}{n!}$

Exercise 2.7

Choose the correct answer

1. If $nC_3 = nC_2$ then the value of nc_4 is
   (a) 2 (b) 3 (c) 4 (d) 5
2. The value of n, when $np_2 = 20$ is
   (a) 3 (b) 6 (c) 5 (d) 4

3. The number of ways selecting 4 players out of 5 is

(a) 4! (b) 20 (c) 25 (d) 5

4. If $nP_r = 720(nC_r)$, then r is equal to

(a) 4 (b) 5 (c) 6 (d) 7

5. The possible out comes when a coin is tossed five times

(a) 2^5 (b) 5^2 (c) 10 (d) $\frac{5}{2}$

6. The number of diagonals in a polygon of n sides is equal to

(a) nC_2 (b) $nC_2 - 2$

(c) $nC_2 - n$ (d) $nC_2 - 1$

7. The greatest positive integer which divide $n(n+1)(n+2)(n+3)$ for all $n \in N$ is

(a) 2 (b) 6 (c) 20 (d) 24

8. If n is a positive integer, then the number of terms in the expansion of $(x+a)^n$ is

(a) n (b) $n + 1$

(c) $n - 1$ (d) $2n$

9. For all $n > 0$, $nC_1 + nC_2 + nC_3 + \ldots + nC_n$ is equal to

(a) 2^n (b) $2^n - 1$

(c) n^2 (d) $n^2 - 1$

10. The term containing x^3 in the expansion of $(x-2y)^7$ is

(a) 3rd (b) 4th (c) 5th (d) 6th

11. The middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is

(a) $10C_4\left(\frac{1}{x}\right)$ (b) $10C_5$

(c) $10C_6$ (d) $10C_7x^4$

12. The constant term in the expansion of $\left(x+\frac{2}{x}\right)^6$ is

(a) 156 (b) 165

(c) 162 (d) 160

13. The last term in the expansion of $(3+\sqrt{2})^8$ is

(a) 81 (b) 16

(c) $8\sqrt{2}$ (d) $27\sqrt{3}$

14. If $\frac{kx}{(x+4)(2x-1)} = \frac{4}{x+4} + \frac{1}{2x-1}$ then k is equal to

(a) 9 (b) 11 (c) 5 (d) 7

15. The number of 3 letter words that can be formed from the letters of the word number when the repetition is allowed are

(a) 206 (b) 133 (c) 216 (d) 300

16. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is

(a) 18 (b) 12 (c) 9 (d) 6

17. There are 10 true or false questions in an examination. Then these questions can be answered in

(a) 240 ways (b) 120 ways

(c) 1024 ways (d) 100 ways

18. The value of $(5C_0+5C_1)+(5C_1+5C_2)+(5C_2+5C_3)+(5C_3+5C_4)+(5C_4+5C_5)$ is

(a) $2^6 - 2$ (b) $2^5 - 1$

(c) 2^8 (d) 2^7

19. The total number of 9 digit number which have all different digit is

(a) 10! (b) 9!

(c) $9 \times 9!$ (d) $10 \times 10!$

20. The number of ways to arrange the letters of the word " CHEESE "

(a) 120 (b) 240 (c) 720 (d) 6

21. Thirteen guests has participated in a dinner. The number of handshakes happened in the dinner is

(a) 715 (b) 78

(c) 286 (d) 13

22. Number of words with or without meaning that can be formed using letters of the word "EQUATION" , with no repetition of letters is

(a) 7! (b) 3! (c) 8! (d) 5!

23. Sum of Binomial co-efficient in a particular expansion is 256, then number of terms in the expansion is

(a) 8 (b) 7

(c) 6 (d) 9

24. The number of permutation of n different things taken r at a time, when the repetition is allowed is

(a) r^n (b) n^r

(c) $\frac{n!}{(n-r)!}$ (d) $\frac{n!}{(n+r)!}$

25. Sum of the binomial coefficients is

(a) 2^n (b) n^2

(c) $2n$ (d) $n+17$

Miscellaneous Problems

1. Resolve into Partial Fractions : $\frac{5x+7}{(x-1)(x+3)}$

2. Resolve into Partial Fractions: $\frac{x-4}{x^2-3x+2}$

3. Decompose into Partial Fractions: $\frac{6x^2-14x-27}{(x+2)(x-3)^2}$

4. Decompose into Partial Fractions: $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)}$

5. Evaluate the following.

(i) $\frac{3!\times 0!+0!}{2!}$ (ii) $\frac{3!+1!}{(2^2)!}$

(iii) $\frac{(3!)!\times 2!}{5!}$

6. How many code symbols can be formed using 5 out of 6 letters A, B, C, D, E, F so that the letters a) cannot be repeated b) can be repeated c) cannot be repeated but must begin with E d) cannot be repeated but end with CAB.

7. From 20 raffle tickets in a hat, four tickets are to be selected in order. The holder of the first ticket wins a car, the second a motor cycle, the third a bicycle and the fourth a skateboard. In how many different ways can these prizes be awarded?

8. In how many different ways, 2 Mathematics, 2 Economics and 2 History books can be selected from 9 Mathematics, 8 Economics and 7 History books?

9. Let there be 3 red, 2 yellow and 2 green signal flags. How many different signals are possible if we wish to make signals by arranging all of them vertically on a staff?

10. Find the Co-efficient of x^{11} in the expansion of $\left(x+\frac{2}{x^2}\right)^{17}$

Summary

- For any natural number n, n factorial is the product of the first n natural numbers and is denoted by $n!$ or $\underline{|n}$.
- For any natural number n, $n! = n(n-1)\,(n-2)\ldots 3\times2\times1$
- $0!=1$
- If there are two jobs, each of which can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m+n)$ ways.
- The number of arrangements that can be made out of n things taking r at a time is called the number of permutation of n things taking r at a time.
- $np_r = \dfrac{n!}{(n-r)!}$
- $np_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$
- $np_1 = \dfrac{n!}{(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$
- $np_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$
- $np_r = n(n-1)(n-2)\ldots[n-(r-1)]$
- The number of permutation of n different things taken r at a time, when repetition is allowed is n^r.
- The number of permutations of n things taken all at a time, of which p things are of one kind and q things are of another kind, such that $p+q = n$ is $\dfrac{n!}{p!\,q!}$
- Circular permutation of n different things taken all at a time = $(n-1)!$
- The number of circular permutation of n identical objects taken all at a time is $\dfrac{(n-1)!}{2}$
- Combination is the selection of n things taken r at a time without repetition.
- Out of n things, r things can be selected in nc_r ways.
- $nC_r = \dfrac{n!}{r!(n-r)!},\ 0 \le r \le n$
- $nC_0 = nC_n = 1.$
- $nC_1 = n.$
- $nC_2 = \dfrac{n(n-1)}{2!}$
- $nC_x = nC_y$, then either $x = y$ or $x + y = n$.

- $nC_r = nC_{n-r}$
- $nC_r + nC_{r-1} = (n+1)C_r$
- $(x+a)^n = nC_o x^n a^o + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \ldots + nC_r x^{n-r} a^r + \ldots$

  $+ nC_n x^0 a^n = \sum_{r=0}^{n} nC_r x^{n-r} a^r$
- Number of terms in the expansion of $(x+a)^n$ is $n+1$
- Sum of the indices of x and a in each term in the expansion of $(x+a)^n$ is n
- $nC_0, nC_1, nC_2, nC_3 \ldots\ldots nC_r \ldots nC_n$ are also represented as $C_0, C_1, C_2, C_3 \ldots C_r \ldots C_n$, called Binomial co-efficients.
- Since $nC_r = nC_{n-r}$, for $r = 0, 1, 2, \ldots n$ in the expansion of $(x+a)^n$, Binomial co -efficients which are equidistant from either end are equal
- $nC_0 = nC_n, nC_1 = nC_{n-1}, nC_2 = nC_{n-2}$
- Sum of the co-efficients is equal of 2^n
- Sum of the co-efficients of odd terms = sum of the co-efficients of even terms = 2^{n-1}
- General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$

GLOSSARY (கலைச்சொற்கள்)	
Binomial	ஈருறுப்பு
Circular Permutation	வட்ட வரிசை மாற்றம்
Coefficient	கெழு
Combination	சேர்வு
Factorial	காரணீயப் பெருக்கம்
Independent term	சாரா உறுப்பு
Linear factor	ஒரு படிக்காரணி
Mathematical Induction	கணிதத் தொகுத்தறிதல்
Middle term	மைய உறுப்பு
Multiplication Principle of counting	எண்ணுதலின் பெருக்கல் கொள்கை
Partial fractions	பகுதிப் பின்னங்கள்
Pascal's Triangle	பாஸ்கலின் முக்கோணம்
Permutations	வரிசை மாற்றங்கள்
Principle of counting	எண்ணுதலின் கொள்கை
Rational Expression	விகிதமுறு கோவை