Chapter 5

Curvilinear Motion

CHAPTER HIGHLIGHTS

- *Introduction*
- *Projectile motion*
- *Apparent weight in a lift*
- *Kinematics of rotation*
- *Angular Displacement and angular velocity*
- *Laws for rotary motion*
- *D'Alembert's principle for rotary motion*
- *Euler's equation of motion*
- *Simple harmonic motion and free vibrations*
- *Frequency of vibration of a spring mass system*

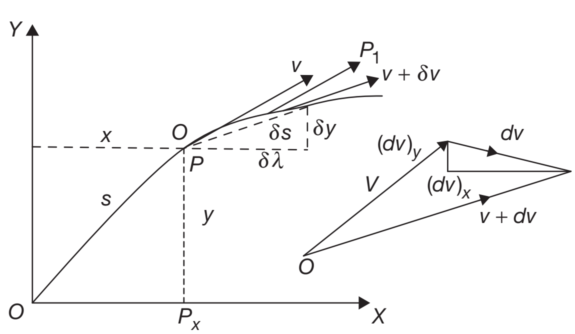
Introduction

Kinematics of Curvilinear Translation

Motion of a particle describing a curved path is called 'curvilinear motion'.

1. **Velocity and acceleration:** The curvilinear motion of a body 'P' may be imagined as the resultant of two rectilinear motions of its projections P_x and P_y on O_x and O_y axis.

   Velocity: Let us consider a body moving through a distance d_s from position P to P_1 along a curved path in time δ_t.



Consider PP_1 as a chord instead of an arc, we have:

$$V_{av} = \frac{\delta_s}{\delta_t}$$

Its projections on the x and y co-ordinates are:

$$(V_{av})_x = \frac{\delta_s}{\delta_t}\frac{\delta_x}{\delta_s} = \frac{\delta_x}{\delta_t}$$

$$(V_{av})_y = \frac{\delta_s}{\delta_t}\frac{\delta_y}{\delta_s} = \frac{\delta_y}{\delta_t}$$

Now, $\frac{\delta_x}{\delta_t}$ and $\frac{\delta_y}{\delta_t}$ are the average velocities of the projections P_x and P_y in the direction of their respective co-ordinates.

If d_t approaches zero, V_{av} becomes the instantaneous velocity. Instantaneous velocity at P, $V = \lim_{\delta t \to 0} \frac{\delta_s}{\delta_t} = \frac{d_s}{d_t}$ and its direction will be tangential to the path at position P.

Similarly, $$V_x = \frac{dx}{dt}$$

$$V_y = \frac{dy}{dt}$$

Total velocity $\boxed{V = \sqrt{V_x^{\,2} + V_y^{\,2}}}$

Acceleration: The average acceleration during the interval t $a_{av} = \dfrac{\delta_v}{\delta_t}$.

The direction will be same as that of the change of velocity δ_v.

The projections of a_{av} on x and y co-ordinates will be $\dfrac{\delta v_x}{\delta_t}$ and $\dfrac{\delta v_y}{dt}$.

When δ_t approaches zero, the instantaneous acceleration:

$$a = \lim_{\delta t \to 0} \frac{\delta v}{\delta_t} = \frac{dv}{dt}$$

$$a = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2 s}{dt^2}$$

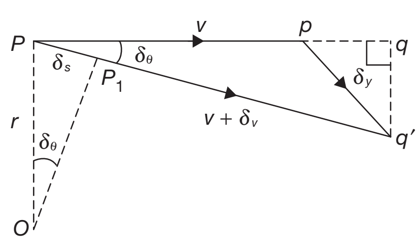
Similarly, the components of the instantaneous acceleration a are:

$$a_x = \frac{d^2 x}{dt^2}$$

$$a_y = \frac{d^2 y}{dt^2}$$

$$\boxed{a = \sqrt{a_x^2 + a_y^2}}$$

2. **Tangential and normal acceleration:** A particle moves on a curved path. From position P, it covers a distance δ_s to position P_1, in the time interval δ_t, such that at position P the instantaneous velocity is V, and at position P_1 it is $(V + \delta_V)$.



Resolving the acceleration into two components:

(a) Tangential to the path at the position P.

(b) Normal to the path at position P.

Let, r be the radius of the curved path PP_1, and δ_θ be the angle subtended at the centre O.

Let, θ be the angle included between the normals at P_1 and P.

From the given figure, it is observed that Pp = instantaneous velocity V at position P.

By resolving δ_V into two components (pq) in the direction tangential at P and qq' in the direction normal at P as shown in the figure.

Tangential acceleration:

$$a = \lim_{\delta_t \to 0} \frac{\text{tangential change in velocity}}{\delta_t} = \lim_{\delta_t \to 0} \frac{pq}{\delta_t}$$

From the triangle Pqq';

$pq = Pq - Pp = (V + \delta_v)\cos\delta_\theta - V = V + \delta_V - V = \delta_V$ ($\delta\theta$ being very small, $\cos\delta_\theta = 1$)

Then, $a_t = \lim_{\delta_t \to 0} \dfrac{\delta_v}{\delta_t} = \dfrac{dv}{dt}$

Now, normal acceleration: $a_n = \lim_{\delta_t \to 0} \dfrac{qq'}{\delta_t}$

$$qq' = pq \sin\delta\theta = (V + \delta_V)\delta\theta$$

($\delta\theta$ being small $\delta\theta = \delta\theta$ in radians)

$$= V\delta_v + \delta_v\delta\theta = v\delta\theta$$

($\delta\theta$ and δ_v being very small, their product will be negligible)

From the given figure, OPP_1:

$$\delta\theta = \frac{PP_1}{r} = \frac{\delta_s}{r}$$

$$qq' = \frac{V\delta_s}{r}$$

Substituting qq' in equation, we have:

$$a_n = \lim_{\delta_t \to 0} \frac{V\delta_s}{r\delta_t}$$

$$a_n = \frac{V}{r} \times \frac{ds}{dt},$$

But $\dfrac{ds}{dt} = V$

$$\therefore\ a_n = \frac{V^2}{r}$$

Normal acceleration is also known as 'centripetal acceleration'.

NOTE

During the motion of a particle along a curved path there is a change in the direction of its velocity from instant to instant with or without any change in magnitude. When both magnitude and direction of velocity change, the particle has the tangential and normal acceleration. When there is only change in the direction of velocity, the particle has only normal acceleration.

SOLVED EXAMPLE

Example 1

The equation of motion of a particle moving on a circular path, radius 400 m, is given by $S = 18t + 3t^2 + 2t^3$. Where S is the total distance covered from the starting point, in metres, till the position reached at the end of t seconds.

(i) The acceleration at the beginning is:
(A) 6 m/s^2 (B) 5 m/s^2
(C) 10 m/s^2 (D) 7 m/s^2

(ii) The time when the particle reaches its maximum velocity is:
(A) 0.5 seconds (B) 0.6 seconds
(C) 0.8 seconds (D) 0.95 seconds

(iii) The maximum velocity of the particle is:
(A) 19.58 m/s (B) 20.53 m/s
(C) 18.65 m/s (D) 13.5 m/s

Solution

(i) Given, $S = 18t + 3t^2 - 2t^3$

$$V = \frac{dS}{dt} = 18 + 6t - 6t^2$$

From the equation, $a = \dfrac{d^2s}{dt^2} = 6 - 12t$

At the beginning, when $t = 0$,

Acceleration:

$$a = 6 - 0 = 6 \text{ m/s}^2.$$

Hence, the correct answer is option (A).

(ii) For determining the condition for maximum velocity, we have:

$$\frac{d^2s}{dt^2} = 6 - 12t = 0 \ = 0.5 \text{ seconds}$$

Hence, the correct answer is option (A).

(iii) When $t = 0.5$ s,

$$V_{max} = 18 + 3 - 1.5 = 19.5 \text{ m/s}$$

Hence, the correct answer is Option (A).

Example 2

A particle moving along a curved path has the law of motion $V_x = 2t - 4$, $V_y = 3t^2 - 8t + 8$ where V_x and V_y are the rectangular components of the total velocity in the x and y co-ordinates. The co-ordinates of a point on the path at an instant when $t = 0$, are (4, –8). The equation of the path is:

(A) $x^2 + 3x - 2$ (B) $x^3 + 4x + 2$
(C) $x^{1/2} + 3x + 2$ (D) $x^{3/2} + 4x^{1/2} + 2$

Solution

$$V_x = 2t - 4$$
$$V_y = 3t^2 - 8t + 8$$

Integrating both sides, we have:

$$\int V_x\, dt = \int (2t - 4)\, dt$$

$$x = 2 \times \frac{t^2}{2} - 4t + C_1 = t^2 - 4t + C_1$$

$$\int V_y\, dt = \int (3t^2 - 8t + 8)\, dt$$

$$y = 3 \times \frac{t^3}{3} - 8 \times \frac{t^2}{2} + 8t + C_2 = t^3 - 4t^2 + 8t + C_2$$

where C_1 and C_2 are constants.

Given, $x = 4$, $y = -8$, when $t = 0$.

Substituting for x, y and t in equation $4 = 0 - 0 + C_1$;

$\therefore C_1 = 4$

$-8 = 0 - 0 + 0 + C_2$

$\therefore C_2 = -8$

Now, the equations of displacement are $x = t^2 - 4t + 4$ and $y = t^3 - 4t^2 + 8t - 8$

$$x = (t - 2)^2$$

$$x^{\frac{1}{2}} = t - 2$$

$$t = x^{1/2} + 2 \qquad (1)$$

$$y = t^3 - 4t^2 + 8t - 8 \qquad (2)$$

Substituting the value of t from Eq. (1) to Eq. (2), we get:

$$y = x^{3/2} + 4x^{1/2} + 2x$$

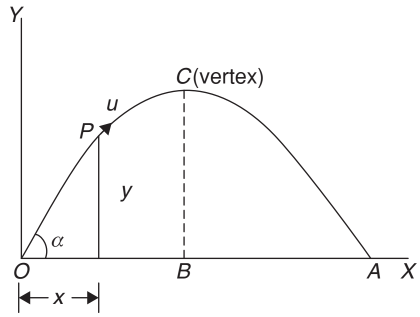
Hence, the correct answer is Option (D).

PROJECTILE MOTION

Definitions

1. **Projectiles:** *A* particle projected at a certain angle is called 'projectile'.
2. **Angle of projection:** Angle between the direction of projection and the horizontal plane through the point of projection is called the angle of projection. It is denoted by α.
3. **Trajectory:** The path traced out by a projectile is called the trajectory of the projectile.
4. **Velocity of projection (*u*):** The initial velocity of projectile is the velocity of projection.
5. **Time of flight (*T*):** The total time taken by a projectile is termed as the time of flight.
6. **Horizontal range (*R*):** It is the distance between the point of projection and the point where a trajectory meets the horizontal plane.

Equations of the Path of Projectile



P is the position occupied by a projectile after t seconds, and x and y are the two co-ordinates of P along the X-axis and Y-axis.

Along the X-axis, $u_x = u \cos\alpha$

Along the Y-axis, $u_y = u \sin\alpha$

The component u_x remains constant all throughout u_y retards due to the action of gravitational force.

We know, $S = vt$, for horizontal motion

$x = u \cos\alpha\, xt$

$$t = \frac{x}{u\cos\alpha}$$

$S = ut + \frac{1}{2}at^2$, for vertical motion.

Therefore, $y = u \sin\alpha t - \frac{1}{2}gt^2$

Substituting value of t, we can write:

$$y = u\sin\alpha\frac{x}{u\cos\alpha} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

This is the equation of the path of a projectile which represents a parabola.

Horizontal range:

$$\boxed{R = \frac{2u^2}{g}\sin\alpha\cos\alpha = \frac{u^2\sin 2\alpha}{g}}$$

Time of flight, $\boxed{T = \left[\frac{2u\sin\alpha}{g}\right]}$

Maximum height when the vertical component of the velocity is zero.

$$v_y = 0. \quad y_{\max} = \frac{u_y{}^2}{2g}$$

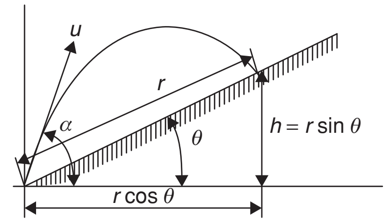
$$\boxed{y_{\max} = \frac{u^2\sin^2\alpha}{2g}}, \quad (\text{since, } u_y = u\sin\alpha)$$

Co-ordinates of vertex C:

$$\boxed{\frac{u^2\sin^2\alpha}{2g}, \frac{u^2\sin^2\alpha}{2g}}$$

Motion of a Projectile on an Inclined Plane

Consider the motion of projectile with an initial velocity u and making an angle α with the horizontal on an inclined plane of inclination θ, taking the coordinate axes x, y the expressions for the distance r and height h can be derived.



$$x = u(\cos\alpha)t = r\cos\theta$$

$$y = u(\sin\alpha)t - \frac{1}{2}gt^2 = h = r\sin\theta$$

By eliminating t, we get:

$$r\sin\theta = r\cos\theta\tan\alpha - \frac{gr^2\cos^2\theta}{2u^2\cos^2\alpha}.$$

$$\Rightarrow r = \frac{2u^2\cos^2\alpha}{g\cos\theta}(\tan\alpha - \tan\theta). \qquad (1)$$

∴ The distance r is given by Eq. (1), and thus the height 'h' and the distance on the horizontal plane can be found.

That is, $h = r\sin\theta$ and $x = r\cos\theta$

The maximum range possible on the inclined plane is found out by differentiation of Eq. (1) with respect to α and equating it to zero.

∴ $\tan 2\alpha = -\cot\theta$

∴ For maximum range the angle made by the velocity vector α should be equal to ($45° + \frac{\theta}{2}$) with the horizontal plane.

Apparent Weight in a Lift

Consider a body of mass m kg which is carried by a lift moving downward. If a (m/s^2) is the acceleration of the lift, then the apparent weight of the body = $\left(1 - \frac{a}{g}\right)$N.

Example 3

Find the least initial velocity which a projectile may have, so that it may clear a wall of 3.6 m high and 6 m distant and strike the horizontal plane through the foot of the wall at a distance of 3.6 m beyond the wall. The point of projection positioned at the same level as the foot of the wall. (take $g = 9.81$ m/s^2)

(A) 10.2 m/s (B) 11 m/s
(C) 12 m/s (D) 13.5 m/s

Solution

Let u be the least initial velocity of the projectile, and α be the angle of projection with the horizontal plane.

Horizontal range of projectile: $R = 6 + 3.6 = 9.6$ m

$$R = \frac{2u^2 \sin\alpha \cos\alpha}{g}$$

$$9.6 = \frac{2u^2 \sin\alpha \cos\alpha}{g}$$

$$u^2 = \frac{9.6g}{2\sin\alpha \cos\alpha}$$

$$u^2 = \frac{4.8g \times \sec^2\alpha}{\tan\alpha} \qquad (1)$$

Equation for the path of projectile:

$$y_{max} = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$3.6 = 6\tan\alpha - \frac{6^2 g}{2u^2\cos^2\alpha}$$

substituting for u^2, we have:

$$3.6 = 6\tan\alpha - \frac{6^2\tan\alpha}{9.6}$$

$$3.6 = \tan\alpha\left[6 - \frac{6^2}{9.6}\right]$$

$$3.6 = 2.25\tan\alpha$$

$$\tan\alpha = \frac{3.6}{2.25} = 1.6$$

$$\alpha = 57.9°$$

From Eq. (1):

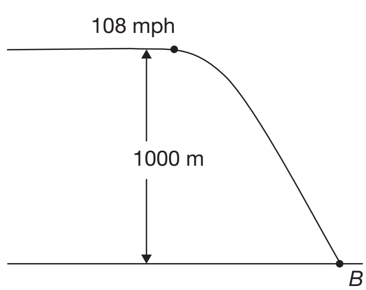
$$u^2 = \frac{4.8g \times \sec^2 57.9}{\tan 57.9} = \frac{4.8g \times 3.54}{1.594} = 104.57$$

$$u = 10.2 \text{ m/s}$$

Hence, the correct answer is option (A).

Example 4

An aeroplane is moving horizontally at 108 km/h at an altitude of 1000 m towards a target on the ground which is intended to be bombed.



(i) The distance from the target where the bomb must be released in order to hit the target, is
(A) 428.35 m
(B) 450.54 m
(C) 580.2 m
(D) 800 m

(ii) The velocity at which the bomb hits the target, is
(A) 143 m/s
(B) 148 m/s
(C) 150 m/s
(D) 161.2 m/s

Solution

(i) Let B be the point of target and A be the position of the aeroplane. The bomb is released from A to hit at point B. The horizontal component of the bomb velocity, which is uniform, is:

$$V = 108 \text{ km/h} = \frac{108 \times 1000}{60 \times 60} = 30 \text{ m/s}$$

Considering the vertical component of the bomb velocity at position A, $u = 0$, $g = 9.81$ m/s^2

$$S = \frac{1}{2}gt^2$$

Let t be the time required to hit point B, then

$$1000 = \frac{1}{2} \times 9.81 \times t^2$$

$$t^2 = \frac{2000}{9.81} = 203.87$$

$$t = 14.278 \text{ seconds}$$

Horizontal distance covered by the bomb, $S = Vt = 30 \times 14.278 = 428.35$ m, i.e., the bomb is released from plane when the horizontal distance is 428.35 from point B.

Hence, the correct answer is option (A).

(ii) Vertical component velocity at

$$B = u + gt = 0 + 9.81 \times 14.278$$
$$= 140.06 \text{ m/s}$$

Resultant velocity at

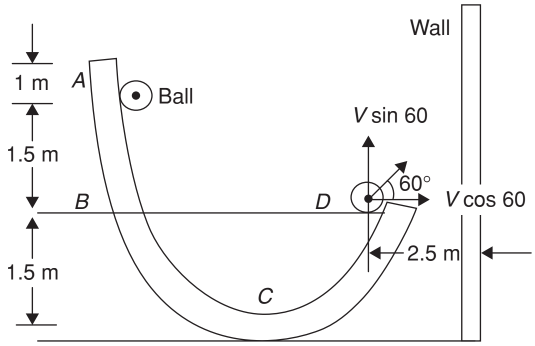
$$B = \sqrt{30^2 + 140.06^2}$$

$$= \sqrt{20518.8} = 143 \text{ m/s}$$

Hence, the correct answer is option (A).

Example 5

A ball weighing 10 N starts from position A, as shown in the figure, and slides down a frictionless chute under its own weight. After leaving the chute 1 at point D, the ball hits the wall as shown in the figure.



(i) The time interval of the ball's travel from the point D to the point of hit is
(A) 0.88 second (B) 0.92 second
(C) 0.733 second (D) 0.898 second

(ii) The distance on the wall above: the point D to the point of hit is
(A) 0.21 m (B) 0.158 m
(C) 0.32 m (D) 0.168 m

Solution

(i) The ball starts from point A. The vertical distance from A to C is equal to 3 m. Considering the motion of ball from A to C,

$$V^2 = 2as$$

Since initial velocity is zero, $a = g = 9.81\ \text{m/s}^2$

or $$V_C^2 = 2 \times 9.81 \times 3$$

$$V_C = 7.67\ \text{m/s},$$

This is the velocity of the ball at C.
The motion of the ball from C to D.
$V_D^2 = V_C^2 - 2as \quad 7.67^2 = 2 \times 9.81 \times 1.5 = 58.82 - 29.43 = 29.39$
$V_D = 5.42$ m/s
On reaching at point D, the horizontal component of the velocity of the ball

$$= V\cos 60^\circ = 5.42 \times \frac{1}{2} = 2.71\ \text{m/s}$$

Let t be the time taken by the ball to hit the wall from point D. Then,

$$t = \frac{2.5}{2.71} = 0.922\ \text{second}$$

Hence, the correct answer is option (B).

(ii) Finally, considering the vertical motion of the ball beyond the point D,

$$S = ut - \frac{1}{2}gt^2$$

Here, $u = V_D = 5.42$ m/s

$$= 5.42 \times 0.922 - \frac{1}{2} \times 9.81(0.922)^2$$

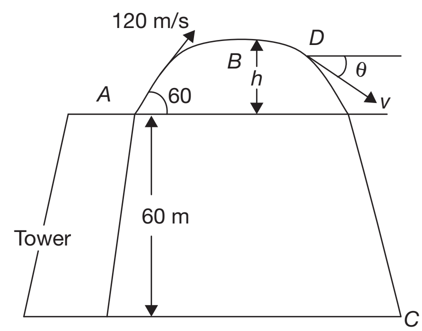
$$= 4.327 - 4.169 = 0.158\ \text{m}$$

Hence, the ball will hit the wall 0.158 m above the point D after 0.922 second.

Hence, the correct answer is option (B).

Example 6

From the top of a 60 m high tower, a bullet is fired at an angle of 60° with the horizontal plane. The initial velocity of the bullet is 120 m/s (as shown in the figure). Neglect air resistance.



(i) The maximum height the bullet would attain from the ground is:
(A) 528 m (B) 611 m
(C) 680 m (D) 720 m

(ii) The velocity of the bullet, 12 seconds after it is fired, is
(A) 55 m/s (B) 58 m/s
(C) 61 m/s (D) 80 m/s

Solution

(i) Height

$$h = \frac{u^2 \sin^2 \alpha}{2g}$$

$$= \frac{120 \times 120 \times (\sin 60)^2}{2 \times 9.81}$$

$$= \frac{120 \times 120 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}}{2 \times 9.81}$$

$$= \frac{10800}{2 \times 9.81} = 551\ \text{m}$$

Maximum height above the ground = 551 + 60 = 611 m.

Hence, the correct answer is Option (B).

(ii) Time of travel up to the highest point B is given by

$$t = \frac{u \sin \alpha}{g} = \frac{120 \times \sin 60}{9.81} = 10.6\,\text{s}.$$

Let D be the point reached by the bullet, 12 seconds after it is fired. Time taken by the bullet to reach point B from A (point from where it was fired) = 10.6 s.

So, time taken by the bullet to travel from point B to point D = 12 – 10.6 = 1.4 s.

Horizontal velocity at B, $V_H = 120 \cos 60° = 120 \times 0.5 = 60$ m/s

The vertical velocity after 1.4 s of travel from point B,

$$V_V = 0 + \frac{1}{2} \times 9.81 \times 1.4^2 = 9.62 \text{ m/s}$$

Velocity at point D:

$$V = \sqrt{V_H^2 + V_v^2}$$

$$= \sqrt{60^2 + 9.62^2} = 60.8 \text{ m/s}.$$

Hence, the correct answer is option (C).

Kinematics of Rotation

When a moving body follows a circular path it is known as circular motion. In circular motion, the centre of rotation is stationary.

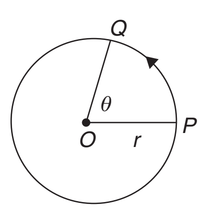
Angular Displacement and Angular Velocity

Angular displacement is defined as the change in angular position (usually referred as the angle θ) with respect to time.

Angular velocity is defined as the rate of change of angular displacement with respect to time. Let a body, moving along a circular path, be initially at point P and after time t seconds be at point Q.

Let $\angle POQ = \theta$

Then, angular displacement = $\angle POQ = \theta$.



Time taken = t

$$\text{Angular velocity} = \frac{\text{Angular displacement}}{\text{Time}} = \frac{\theta}{t}$$

Mathematically, it is expressed as $\frac{d\theta}{dt}$.

It is denoted by the symbol ω.

$$\boxed{\omega = \frac{d\theta}{dt}}$$

It is measured in radian/sec or rad/s.

Relation between Linear Velocity and Angular Velocity

Let V = Linear velocity = $\frac{\text{Linear displacement}}{\text{Time}}$

But, linear displacement = Arc $PQ = OP \times \theta = r\theta$

$$V = \frac{r \times \theta}{t} = r \times \text{Angular velocity}$$

$$\left(\because \frac{\theta}{t} = \text{Angular velocity} \right)$$

$$\boxed{V = r \times \omega}$$

Where, ω = Angular velocity.

Angular Acceleration

It is defined as the rate of change of angular velocity. It is measured in radians per sec^2 and written as rad/s^2. It is denoted by the α symbol.

α = Rate of change of angular velocity

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \left(\because \omega = \frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}.$$

$$\text{Also, } \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \frac{d\omega}{d\theta} \times \omega = \omega\frac{d\omega}{d\theta}$$

It has two components

Normal component = $\frac{V^2}{r} = \omega^2 r$, and tangential component = $\frac{dV}{dt} = r\frac{d\omega}{dt} = r\alpha$

If a is the linear acceleration, then:

$$\boxed{a = r\alpha}$$

Equations of Motion along a Circular Path

$$\boxed{\alpha = \frac{\omega - \omega_0}{t}}$$

$$\boxed{\begin{array}{l} \theta = \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 - \omega_0^2 = 2\alpha\theta \end{array}}$$

If N is the rpm:

$$\boxed{\begin{array}{l} \omega = \frac{2\pi N}{60} \text{ rad/s} \\ v = r\omega = \frac{2\pi N}{60} \times r = \frac{\pi DN}{60} \text{ m/s} \end{array}}$$

Where

ω_0 = Initial angular velocity in cycles/s

ω = Final angular velocity in cycles/s

t = Time (in seconds) during which angular velocity changes from ω_0 to ω

V = Linear speed in m/s

The rotational speed is N revolutions per minute or N rpm.

Example 7

A wheel rotates for 5 seconds with a constant acceleration and describes during the time 100 radians. It then rotates with a constant angular velocity and during the next 5 seconds, it describes 70 radians. The initial angular velocity and angular acceleration are:

(A) 15 rad/s, 2.5 rad/s^2 (B) 13 rad/s, 2 rad/s^2

(C) 15 rad/s, –2 rad/s^2 (D) 26 rad/s, –2.4 rad/s^2

Solution

Angular velocity

$$\omega = \frac{\theta}{t} = \frac{70}{5} = 14 \text{ rad/s}$$

α is constant angular acceleration and ω_0 be initial angular velocity.

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$100 = (\omega_0 5) + \frac{1}{2}\alpha \times 5^2$$

$$5\omega_0 + 12.5\alpha = 100 \quad (1)$$

$$\omega = \omega_0 + \alpha t$$

$$14 = \omega_0 + 5\alpha \quad (2)$$

Solving Eqs. (1) and (2):

$$\omega_0 = 26 \text{ rad/s}$$

$$\alpha = -2.4 \text{ rad/s}^2 \quad \text{(Retardation)}$$

Hence, the correct answer is option (D).

Example 8

A wheel rotating about a fixed axis at 20 rpm is uniformly accelerated for 80 seconds during which time it makes 60 revolutions.

(i) The angular velocity at the end of the time interval.

(ii) The time required for the speed to reach 100 rpm.

(A) 3.65 minutes (B) 2.14 minutes

(C) 1.85 minutes (D) 2.58 minutes

Solution

(i) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

ω_0 = initial angular velocity

$$\omega_0 = \frac{2\pi \times 20}{60} = 2.094$$

$$2\pi \times 60 = (2.094 \times 80) + \frac{1}{2}\alpha(80)^2$$

$$2\pi \times 60 = 167.52 + 3200\alpha$$

$$\alpha = \frac{2\pi 60 - 167.52}{3200}$$

$$= 0.065 \text{ rad/s}^2$$

Let ω be the angular velocity at the end of 80 seconds in rad/s. Then $\omega = \omega_0 + \alpha t$

$\omega = 2.094 + (0.065 \times 80) = 7.294$ rad/s

Hence, the correct answer is option (A).

(ii) $7.294 = \dfrac{2\pi \times N}{60}$

$$N = 69.65 \text{ rpm}$$

$$\omega_1 = \omega_0 + \alpha t_1$$

Where $\omega_1 = \dfrac{2\pi \times 100}{60}$ rad/s

$$= 10.466 \text{ rad/s}$$

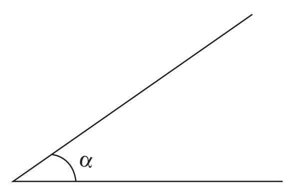
$$10.466 = 2.094 + 0.065 \times t_1$$

$$t_1 = \frac{8.372}{0.065} = 128.8 \text{ s} = 2.14 \text{ minutes.}$$

Hence, the correct answer is option (B).

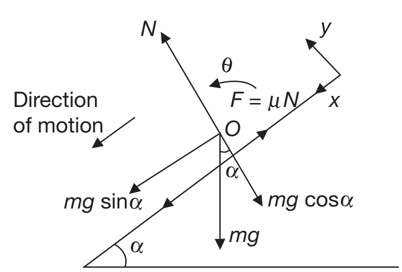
Example 9

As shown in the figure, a circular disc having mass m and radius r is rolling down a rough inclined plane. Find the acceleration of the centre of disc and limiting value of co-efficient of friction, assuming pure rolling.



Solution

For pure rolling motion, use the principle, Rolling motion = Translation of geometric centre + Rotation motion of any point on the circumference. In case of pure rolling, the disc will roll out the surface without slipping.



Let a be the acceleration of the centre of gravity 'O' in the disc.

Now, the equation of motion of the disc is: Translation motion of parallel to inclined plane is given by (Using D'Alembert's principle).

$ma = ma \sin\alpha - F = mg \sin\alpha - \mu N$

$= mg \sin\alpha - \mu\, mg \cos\alpha\ [N = mg \cos\alpha]$

$[F = \mu N = \mu\, mg \cos\alpha]$

$\Rightarrow a = g\,(\sin\alpha - \mu \cos\alpha)$

$\Rightarrow x = g\,(\sin\alpha - \mu \cos\alpha)$

Now, angular velovity of disc = θ,

angular acceleration = $\frac{d\theta}{dt} = \theta$ (or α)

Interia torque = $I\,\theta\,\frac{1}{2}\,mr^2\,\theta$

(where I = MOI of disc)

For rotation of the disc rotational Moment or torque; $M = I \times$ Angular acceleration.

Hence, $\frac{1}{2}\,mr^2\,\theta = Fr$

$\Rightarrow mr^2\,\theta = 2F$

$\Rightarrow r^2\,\theta = \frac{2F}{m}$

$\Rightarrow x = \frac{2F}{m}\ [a = ra \Rightarrow x = rq]$ (2)

From Eqs. (1) and (2), we have:

$\frac{2F}{m} = g \sin\alpha - \mu g \cos\alpha$

$\Rightarrow 2F = mg \sin\alpha - \mu g \cos\alpha = mg \sin\alpha - F$

$\Rightarrow 3F = mg \sin\alpha \Rightarrow F\ \frac{mg}{3}\ \sin\alpha$

$Ma = mx\ mg \sin\alpha - F = mg \sin\alpha - \frac{mg}{3}\ \sin\alpha$

Hence, $\frac{2}{3} g \sin\alpha$

$\Rightarrow x = \frac{2}{3}\,g \sin\alpha$

Now, $F \le \mu N$ (for limiting case) [μ = Coefficient of rolling friction]

$\Rightarrow \mu mg \cos\alpha \ge \frac{mg}{3} \sin\alpha$

$\Rightarrow \mu \ge \frac{1}{3} \tan\alpha.$

Curvilinear and Rotary Motion Kinetics

For a particle or a body moving in a curved path with particular emphasis to the circular path comes under this section.

In order to maintain the circular motion, an inward radial force called 'centripetal force' is acted upon the body, which is equal and opposite to the centrifugal force that is directed away from the centre of curvature. If r is the radius of the circular path, v is the linear velocity, ω is the angular velocity and t is the time, then:

Angular acceleration = $\frac{d\omega}{dt}$

Tangential acceleration = $r\frac{d\omega}{dt}$

Normal acceleration = $\frac{v^2}{r} = \omega^2 r$

Centripetal or centrifugal force $= \frac{W}{g} \times \frac{v^2}{r} = \frac{W}{g}\omega^2 r.$

Laws for Rotary Motion

First Law

It states that a body continues in its state of rest or of rotation about an axis with constant or uniform angular velocity unless it is compelled by an external torque to change that state.

Second Law

It states that the rate of change of angular momentum of a rotating body is proportional to the external torque applied on the body and takes place in the direction of the torque.

$I = Mk^2$,

Where

M = Mass of the body, and k = Radius of gyration

= Moment of inertia × Initial angular velocity

Initial angular momentum = $I\omega_0$

Final angular momentum = $I\omega$

Change of angular momentum = $I(\omega - \omega_0)$

Rate of change of angular momentum

$= \frac{\text{Change of angular momentum}}{\text{Time}}$

$I\,\frac{(\omega - \omega_0)}{t} = I\alpha \left[\because \alpha = \frac{\omega - \omega_0}{t} = \text{Angular acceleration}\right]$

From the second law of motion of rotation,

torque α is the rate of change of angular momentum.

$T = I\alpha$

$T = KI\alpha,$

where, K is a constant of proportionality. SI unit of torque is Nm.

Angular Momentum or Moment of Momentum

Moment of momentum of the body about $O = I\omega$, where the rigid body undergoes rotation about O.

Angular momentum is the moment of linear momentum.

Rotational Kinetic Energy

Rotational kinetic energy = $\frac{1}{2} I\omega^2$.

Angular Impulse or Impulsive Torque

Angular impulse or impulsive torque = $Id\omega$.

Work Done in Rotation

Work done in rotation = $T \times \theta$.

Kinetic Energy in Combined Motion

Kinetic energy due to translatory motion = $\frac{1}{2} mv^2$

Kinetic energy due to rotation = $\frac{1}{2} I\omega^2$

Kinetic energy due to combined motion

$$= \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

Conservation of Angular Momentum

The law of conservation of angular momentum states that the angular momentum of a body or a system will remain unaltered if the external torque acting on it is zero.

D'Alembert's Principle for Rotary Motion

D'Alembert's principle for rotary motion states that the sum of the external torques (also termed as active torques) acting on a system, due to external forces and the reversed active torques including the inertia torques (taken in the opposite direction of the angular momentum) is zero.

Suppose a disc of moment of inertia I rotates at an angular acceleration α under the influence of a torque T, acting in the clockwise direction. Inertia torque = I_α (acting in the anti-clockwise direction)

From D'Alembert's principle, $T - I_\alpha = 0$, the dynamic equation of equilibrium for a rotating system.
Rotation caused by a weight 'W' attached to one end of a string passing over a pulley of weight W_0.

From D'Alembert's principle, it can be shown that

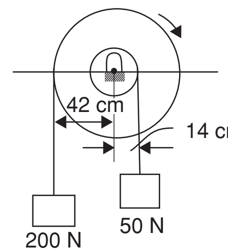
$$\boxed{a = \frac{gW}{\left(W + \frac{W_0}{2}\right)}}$$, when the pulley is considered as a disc.

Rotation caused due to two weights W_1 and W_2 attached to the two ends of a string which passes over a rough pulley of weight W_0.

$$\boxed{a = \frac{g(W_1 - W_2)}{\left(W_1 + W_2 + \frac{W_0}{2}\right)}}$$

Example 10

In a pulley system, as shown in the figure, the pulley weighs 20 N and its radius of gyration is 40 cm. A 200 N weight is attached to the end of a string and a 50 N is attached to the end of the other string as shown in the figure.



(i) The torque to be applied to the shaft to raise the 200 N weight at an acceleration of 1.5 m/s^2, is

(A) 6812 N-cm (B) 9136 N-cm
(C) 700 N-cm (D) 7832 N-cm.

(ii) The tensions in the strings are

(A) 170.4 N, 35.6 N (B) 180 N, 40 N
(C) 190.2 N 35 N (D) 180.6 N, 42.34 N

Solution

(i) Moment of inertia of the pulley

$$I = \frac{W}{g} k^2$$

$$I = \frac{20}{981} \times (40)^2 \text{ Ncm}^2 = 32.62 \text{ N cm}^2$$

T_1 = Torque produced by 200 N
= 200 × 42 = 8400 N-cm
T_2 = Torque developed by 50 N = 50 × 14
= 700 N-cm
Inertia torque due to angular rotation of the pulley with angular acceleration:

$$\alpha = I\alpha$$
$$= 32.62\,\alpha \text{ N-cm.}$$

Torque due to inertia force on

$$200 \text{ N} = (ma)\, r = \frac{200}{981} r\alpha r = \frac{200}{981} \times \alpha \times (42)^2$$
$$= 359.63\alpha \text{ N-cm}$$

Torque due to inertia force on 50 N

$$= \frac{50}{981} \times \alpha \times 14^2 = 9.99\,\alpha \text{ N-cm}$$

Let T be the torque applied to the shaft.
For dynamic equilibrium, $\sum T = 0$
$T + 700 = 8400 + 32.62\alpha + 359.63\alpha + 9.99\alpha$

$$T = 8400 + 312.33 = 9136 \text{ N-cm},$$

since $\alpha = \frac{150}{42} = 3.57 \text{ rad/s}^2$.

Hence, the correct answer is option (B).

(ii) Let F_1 and F_2 be the tensions in the strings. Applying D'Almbert's principle for linear motion, we get:

$$F_1 - 200 - \frac{200}{9.8} \times 1.5 = 0$$

$$F_2 + 50 - F_2 = \frac{50}{9.8} \times 1.5$$

$$F_1 = 200 + \frac{200}{9.8} \times 1.5 = 200 + 22.96$$

$$= 180.6 \text{ N}$$

$$F_2 = \frac{50 \times 9.8 - 50 \times 1.5}{9.8} = 42.34 \text{ N}$$

Hence, the correct answer is option (D).

Euler's Equation of Motion

According to Newton's second law of motion, when a force F is applied on a particle:

$$F = m \times a$$

$$= m \times \frac{dv}{dt}$$

Where

m = Mass of the particle

v = Velocity

a = Acceleration

The above is equivalent to $F = \frac{d}{dt}(mv)$ where mv = momentum of the particle Newton's law deals with particles. When a force is applied to a rigid body, the same law can be applied, treating the whole mass of the body concentrated at the centre of mass of the body.

A rigid body can have rotational translation as well as a linear translation. So, a particle in the body can be subjected to angular momentum as well as linear momentum.

Angular momentum of a particle:

L = Moment of momentum = $r \times mv$, where r = Distance from centre of mass.

Torque or moment:

$$M = \text{Force} \times \text{Distance from axis of rotation}$$

$$= F \times r$$

$$= \frac{d}{dt}(mvr) = \frac{d}{dt}(L)$$

= Rate of change of angular momentum

Therefore,

1. Sum total of external forces acting on a body:

$$\Sigma F = \frac{d}{dt}(mv)$$

= Rate of change of linear momentum.

2. Sum total of external momentum acting on a body:

$$\Sigma M = \frac{d}{dt}(L) = \dot{L}$$

also $$M = \frac{d}{dt}(mvr)$$

$$= \frac{d}{dt}(m\omega r^2)$$

$$\sum M = I\frac{d\omega}{dt} = I\dot{\omega} = I\alpha.$$

Described above are the bases for Euler's equations.

Angular momentum L can be considered as a vector which has a local variation within a local reference frame as well as a variation due to its rotation about a global reference frame with an angular velocity (say, Ω)

Then, $$\Sigma M = \dot{L}$$

$$= \dot{L}_{x,y,z} + \Omega \times L$$

When the body is fixed to the local reference frame, $\Omega = \omega$, the angular velocity of the body.

The above equation can be stated as:

$$\sum M = I\dot{\omega} + \omega \times (I \cdot \omega)$$

In three-dimensional, principal orthogonal coordinates, individual components of the equation are:

$$\Sigma M_x = \dot{L}_x - L_y\omega_z + L_z\omega_y$$

$$\Sigma M_y = \dot{L}_y - L_z\omega_x + L_x\omega_z$$

$$\Sigma M_z = \dot{L}_z - L_x\omega_y + L_y\omega_x$$

If the coordinate axes are chosen such that they coincide with principal axes, the angular momentum terms can be related to the principal moments of inertia and the following expressions can be obtained:

$$\Sigma M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z$$

$$\Sigma M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_x\omega_y$$

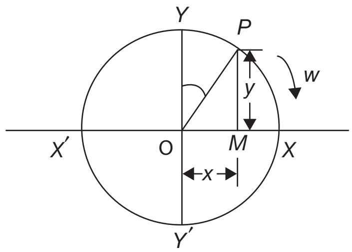
$$\Sigma M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y$$

The above equations are known as Euler's equation of motion and find application in rigid body dynamics.

Simple Harmonic Motion and Free Vibrations

Simple harmonic motion: It is defined as the type of motion in which the acceleration of the body in its path of motion, varies directly as its displacement from the equilibrium position and is directed towards the equilibrium point.

Oscillation, Amplitude, Frequency and Period



In the given figure, when a particle P is describing a circular path, M being the projection of P, it describes a simple harmonic motion.

The motion of M from X to X' and back to X is called an oscillation or simple harmonic motion.

$OX = OX'$ is the amplitude

This amplitude is the distance between the centre of simple harmonic motion and the point where the velocity is zero.

The period of one complete oscillation is the period of simple harmonic motion.

Thus, the period of simple harmonic motion is the time in which M describes 2π radians at ω rad/s.

$T = \dfrac{2\pi}{\omega}$, where T is the time period in seconds.

Velocity and Acceleration

The simple harmonic displacement,

$X = r \sin \omega t$

$v = \omega\sqrt{r^2 - x^2}$

$$\text{Acceleration} = \frac{d^2x}{dt^2} = -\omega^2 r \sin \omega t$$

$a = -\omega^2 x$

$$\text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{a}{x}}.$$

Frequency of Vibration of a Spring Mass System

Consider a helical spring subjected to a load W. The static equilibrium position is 00. Let S be the stiffness of the spring which is defined as force required to cause one unit extension. If the weight is displaced and stretched to position 1–1′ by an amount 'y', as shown in the figure, then the acceleration at which the load springs back:

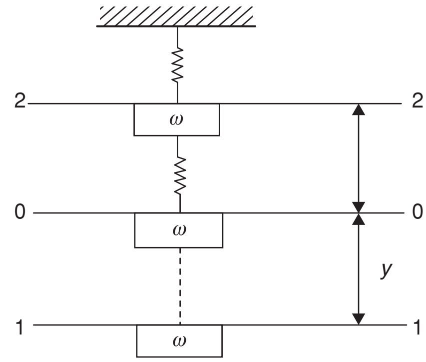
$$\frac{w}{g} a = -sy$$

$$\therefore a = \frac{s \times g}{-W} \cdot y$$

This is of the form, $a = -\omega_n^2 y$

where $\omega_n^2 = \dfrac{sg}{w} = \dfrac{g}{\delta}$, δ being $\dfrac{\omega}{s}$

Frequency, $f = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\delta}}$.



Oscillations of a Simple Pendulum

Period of oscillation, $T = \dfrac{1}{f} = 2\pi\sqrt{\dfrac{l}{g}}$ (for 2 beats) l = Length of pendulum. Half of an oscillation is called a beat or swing. A pendulum executing one half oscillations per second is called seconds pendulum. Time of one beat or swing $= \pi\sqrt{\dfrac{l}{g}} = \dfrac{T}{2}$. For n number of beats, time $= n\pi\sqrt{\dfrac{l}{g}}$. For a compound pendulum:

$$T = 2\pi\sqrt{\frac{K_G^2 + h^2}{gh}}$$

where, h is the distance between the point of suspension and centre of gravity. Where, k_G = radius of gyration about O, the centre of suspension. A compound pendulum is a rigid body free to oscillate about a smooth horizontal axis passing through it.

A simple pendulum whose period of oscillation is the same as that of a compound pendulum is called a simple equivalent pendulum.

$$L = \frac{k_G^2}{h} + h.$$

Example 11

A body performing simple harmonic motion has a velocity 12 m/s when the displacement is 50 mm and 3 m/s when the displacement is 200 mm, the displacement being measured from the mean position.

(i) Calculate the frequency of the motion.

(A) 35 cycles/s (B) 40.5 cycles/s
(C) 31.8 cycles/s (D) 35.5 cycles/s

(ii) What is the acceleration when the displacement is 75 mm?

(A) 15 m/s^2 (B) 16.5 m/s^2

(C) 13.8 m/s^2 (D) 15.6 m/s^2

Solution

(i) In simple harmonic motion:

$V^2 = \omega^2(r^2 - x^2)$

V = Velocity, r = Amplitude

x = Distance from mid-positions

$x_1 = 50$ mm, x_2 200 mm

$V_1 = 12$ m/s $V_2 = 3$ m/s

$$12^2 = \omega^2\left[r^2 - \left(\frac{50}{1000}\right)^2\right]$$

$$3^2 = \omega^2\left[r^2 - \left(\frac{200}{1000}\right)^2\right] \quad (1)$$

By dividing, we get:

$$\frac{144}{9} = \frac{r^2 - \frac{1}{400}}{r^2 - \frac{4}{100}}$$

$$16 = \frac{r^2 - \frac{1}{400}}{r^2 - \frac{4}{100}}$$

$$16r^2 - \frac{16 \times 4}{100} = r^2 - \frac{1}{400}$$

$$15r^2 = \frac{16 \times 2}{50} - \frac{1}{400}$$

$$15r^2 = \frac{2 \times 64 \times 4}{400} - \frac{1}{400} = \frac{511}{400}$$

$$r^2 = \frac{511}{400 \times 15} = 0.085$$

$r = 0.29\ m = 290$ mm.

Putting the value of r^2 in Eq. (1), we get: $9 = \omega^2$ [0.085 – 0.04]

$$\omega^2 = \frac{9}{0.045};\ \omega = 200 \text{ rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{200}{2\pi} = 31.83 \text{ cycles/s.}$$

Hence, the correct answer is Option (C).

(ii) If 'a' be the acceleration when displacement $x = 75$ mm,

$$a = \omega^2 x = \left(\frac{9}{0.045} \times \frac{75}{1000}\right) = 15 \text{ m/s}^2.$$

Hence, the correct answer is option (A).

Example 12

The number of seconds a clock would lose per day, if the length were increased in the ratio 800 : 801 is:

(A) 48 s (B) 54 s

(C) 50 s (D) 60 s

Solution

Given, $l = 800$ units

$l + dl = 801$ units

$dl = 1$ unit

$$\frac{dl}{l} = \frac{1}{800}$$

$$\frac{dn}{n} = \frac{-dl}{2l} = \frac{1}{1600}$$

$$dn = -\frac{n}{1600} = -\frac{86400}{1600} = -54$$

where n = 86400, as a seconds pendulum will beat 86400 times/day. The clock will loose 54 seconds in a day.

Hence, the correct answer is option (B).

Super Elevation

Whenever a roadways (or railways) is laid on a curved path, its outer edge is always made higher than the inner edge to keep the moving vehicles in equilibrium state. The amount by which the outer edge is raised is known as 'cant' or 'super elevation'. In case of roadways, the process of providing super elevation is known as banking of the road. In general practice, to define super elevation in roadways, is to mention the angle of inclination (also known as angle of banking) of the road surface, so that:

$$\tan\theta = \frac{v2}{gr}$$

where v = Velocity of the vehicle, and r = Radius of circular path.

In case of railways, the general practice is to define the super elevation is to mention the difference of levels between the two rails. in such a case, super elevation is given by:

$$S = \frac{Gv^2}{gr},$$

where G = gauge of the track.

Exercises

Direction for questions 1 and 2:

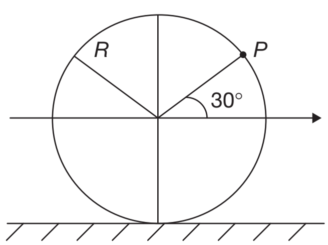
A thin circular ring of mass 200 kg and radius 2 m resting flat on a smooth surface is subjected to a sudden application of a force of 300 N at a point of its periphery.

1. The angular acceleration is
   (A) 0.75 rad/s^2 (B) 1.5 rad/s^2
   (C) 2 rad/s^2 (D) 2.5 rad/s^2

2. The acceleration of mass centre is
   (A) 1 m/s^2 (B) 1.5 m/s^2
   (C) 2 m/s^2 (D) 3 m/s^2

3. A carpet of mass m made of an inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough horizontal floor. When a small push, of negligible force, is given to the carpet, it starts unrolling without sliding on the floor. The horizontal velocity of the axis of the cylindrical part of the carpet is $\sqrt{\frac{63}{3}gR}$ when the radius of the carpet reduces to
   (A) $\frac{3R}{4}$ (B) $R/4$
   (C) $R/2$ (D) $R/5$

4. A circular disc of radius 'R' rolls without slipping at a velocity 'V'. The magnitude of the velocity at point P(see figure) is

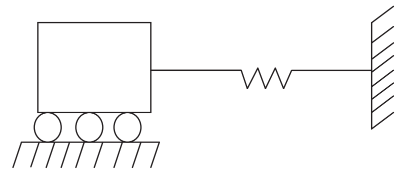


   (A) $\sqrt{3}\,V$ (B) $\frac{\sqrt{3}}{2}V$
   (C) $\frac{V}{2}$ (D) $\frac{2}{\sqrt{3}}V$

5. A stone is projected horizontally from a cliff at 10 m/s and lands on the ground below at 20 m from the base of the cliff. Find the height 'h' of the cliff. Use $g = 10 \text{ m/s}^2$.
   (A) 18 m (B) 20 m
   (C) 22 m (D) 24 m

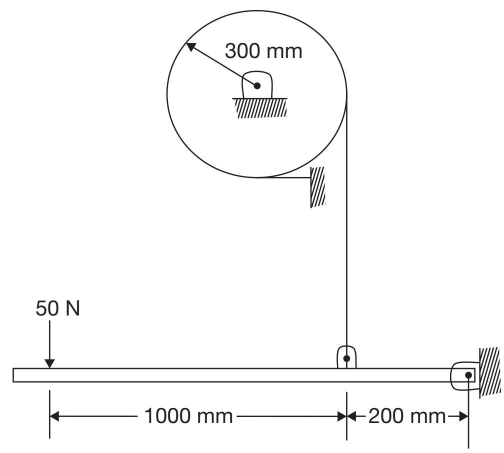
6. Two cars are going with constant speeds, round concentric circles of radii r_1 and r_2 and take the same time to complete their circular paths. Their speeds will correspond to the ratio
   (A) 1 : 1 (B) $\frac{r_1}{r_2}$
   (C) $\left(\frac{r_1}{r_2}\right)^2$ (D) $\frac{r_2}{r_1}$

7. A truck weighing 150 kN and traveling at 2 m/s impacts with a buffer spring, which compresses 1.25 cm per 10 kN. The maximum compression of the spring is



   (A) 20 cm (B) 22.85 cm
   (C) 27.65 cm (D) 30 cm

Direction for questions 8 and 9:



A band brake is used to control the speed of a flywheel as shown in figure. The coefficient of friction between the band and flywheel is 0.3. Radius of the flywheel is 300 mm. A force of 50 N is applied at the end of the lever as shown in the figure.

8. Torque applied on the flywheel when it is rotating clockwise is
   (A) 262 Nm (B) 280 Nm
   (C) 315 Nm (D) 326 Nm

9. Torque applied on the flywheel when it is rotating counter clockwise is
   (A) 94 Nm (B) 82 Nm
   (C) 76 Nm (D) 68 Nm

10. A wheel at rest is accelerated uniformly from rest to 3000 rpm in 30 seconds. Its angular acceleration is
   (A) 6.624 rad/s (B) 8.368 rad/s
   (C) 10.472 rad/s (D) 14.376 rad/s

11. If a projectile motion with usual notations is expressed as

$$y = xP - \frac{gx^2}{2u^2Q^2} \quad (\alpha = \text{Angle of projection}), \text{ then } `P'$$

and 'Q' are

(A) $\tan\alpha$ and $\cos^2\alpha$ (B) $\tan\alpha$ and $\cos\alpha$
(C) $\tan\alpha$ and $\sec\alpha$ (D) $\tan\alpha$ and $\sec^2\alpha$

12. A hill has the shape of a right circular cone with vertex angle 60°. A particle is projected from the base of the hill such that it grazes the vertex and falls at the base of the hill just opposite to the starting point.

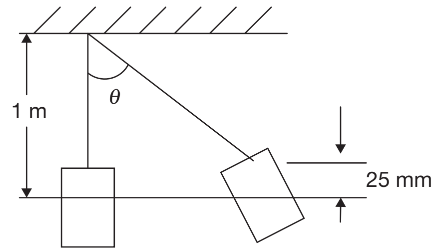
The angle of projection measured from horizontal is

(A) 73.9° (B) 69.8°
(C) 64.4° (D) 61.7°

13. A stone of mass 1 kg is tied to a string of 1 m length and whirled in a horizontal circle at a constant angular speed of 5 rad/s. The tension (in N) in the string will be

(A) 5 (B) 10
(C) 25 (D) None of these

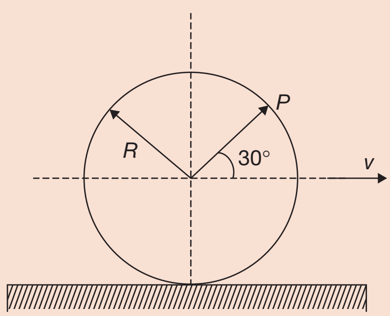
14. A 0.05 N bullet was fired horizontally into a 50 N sand bag suspended on a rope 1 m long as shown in the figure. It was found that the bag with the bullet embedded in it swung to a height of 25 mm. Determine the speed of the bullet as it entered the bag.



(A) 700.7 m/s (B) 800.2 m/s
(C) 900.2 m/s (D) 920.7 m/s

Previous Years' Questions

1. A circular disk of radius R rolls without slipping at a velocity v. The magnitude of the velocity at point P (see figure) is **[GATE, 2008]**



(A) $\sqrt{3}\,v$ (B) $\sqrt{3}\,v/2$
(C) $v/2$ (D) $2v/\sqrt{3}$

2. An annular disc has a mass m, inner radius R and outer radius $2R$. The disc rolls on a flat surface without slipping. If the velocity of the centre of mass is v, the kinetic energy of the disc is **[GATE, 2014]**

(A) $\frac{9}{16}mv^2$ (B) $\frac{11}{16}mv^2$
(C) $\frac{13}{16}mv^2$ (D) $\frac{15}{16}mv^2$

3. Consider a steel (Young's modulus, E = 200 GPa) column hinged on both sides. Its height is 1.0 m and cross-section is 10 mm × 20 mm. The lowest Euler critical buckling load (in N) is ______. **[GATE, 2015]**

Answer Keys

Exercises

1. A **2.** B **3.** B **4.** A **5.** B **6.** B **7.** C **8.** B **9.** D **10.** C
11. B **12.** A **13.** C **14.** A

Previous Years' Questions

1. A **2.** C **3.** 3285 to 3295