Maharashtra State Board
Mathematics and Statistics
Sample Question Paper - 1
Academic Year: 2024-2025

General Instructions: The question paper is divided into four sections.

1. **Section A**: Q.1 contains Eight multiple-choice types of questions, each carrying Two marks. Q.2 contains Four very short answer type questions, each carrying one mark.
2. **Section B**: Q.3 to Q.14 contains Twelve short answer type questions, each carrying Two marks. (Attempt any Eight)
3. **Section C**: Q.15 to Q.26 contain Twelve short answer type questions, each carrying Three marks. (Attempt any Eight)
4. **Section D**: Q. 27 to Q.34 contain Eight long answer type questions, each carrying Four marks. (Attempt any Five)
5. Use of Log table is allowed. Use of calculator is not allowed.
6. Figures to the right indicate full marks.
7. Use of graph paper is not necessary. Only rough sketch of graph is expected.
8. For each multiple-choice type question, it is mandatory to write the correct answer along with its alphabet. e.g., (a) .... /(b) .... /(c) .... /(d) .... ,etc. No mark(s) shall be given if ONLY the correct answer or the alphabet of the correct answer is written. Only the first attempt will be considered for evaluation.
9. Start answer to each section on a new page.

SECTION - A

Q1. Select and write the correct answer for the following multiple-choice type of questions:

1.1. Choose the correct option from the given alternatives :

Let f(x) and g(x) be differentiable for $0 \le x \le 1$ such that f(0) = 0, g(0), f(1) = 6. Let there exist a real number c in (0, 1) such that f'(c) = 2g'(c), then the value of g(1) must be ______.

1. 1
2. 3
3. 2.5
4. −1

Solution

Let f(x) and g(x) be differentiable for $0 \le x \le 1$ such that f(0) = 0, g(0), f(1) = 6. Let there exist a real number c in (0, 1) such that f'(c) = 2g'(c), then the value of g(1) must be **<u>3</u>**.

Explanation:

f'(c) = $\frac{f(b) - f(a)}{b - a}$ and g'(c) = $\frac{g(b) - g(a)}{b - a}$

$\therefore \frac{6 - 0}{1 - 0} = 2\left(\frac{g(1) - 0}{1 - 0}\right)$

$\frac{6}{1} = 2\frac{g(1)}{1}$

3 = g(1)

1.2. The maximum value of z = 5x + 3y subject to the constraints $3x + 5y \le 15$, $5x + 2y \le 10$, x, y ≥ 0 is ______.

1. 235
2. $\frac{235}{9}$
3. $\frac{235}{19}$
4. $\frac{235}{3}$

Solution

The maximum value of z = 5x + 3y subject to the constraints $3x + 5y \le 15$, $5x + 2y \le 10$,

x, y ≥ 0 is $\underline{\frac{235}{19}}$.

1.3. The area bounded by the curve $y = x^3$, the X-axis and the Lines x = −2 and x = 1 is ______.

1. –9 sq.units

2. $-\frac{15}{4}$ sq.units

3. $\frac{15}{4}$ sq.units

4. $\frac{17}{4}$ sq.units

Solution

The area bounded by the curve y = x^3, the X-axis and the Lines x = –2 and x = 1 is $\underline{\mathbf{\frac{15}{4} sq. units}}$.

1.4.

If $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \frac{k}{3}$, then k is equal to _____.

1. $\sqrt{2}\left(2\sqrt{2}-2\right)$

2. $\frac{\sqrt{2}}{3}\left(2-2\sqrt{2}\right)$

3. $\frac{2\sqrt{2}-2}{3}$

4. $4\sqrt{2}$

Solution

If $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \frac{k}{3}$, then k is equal to $\underline{\mathbf{4\sqrt{2}}}$.

1.5. Choose the correct option from the given alternatives :

$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cdot dx}{(1+\cos x)^2}$ = _____.

1. $\frac{4-\pi}{2}$

2. $\frac{\pi-4}{2}$

3. $4-\frac{\pi}{2}$

4. $\frac{4+\pi}{2}$

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cdot dx}{(1+\cos x)^2} = \underline{\mathbf{\frac{4-\pi}{2}}}.$$

Explanation:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x \cdot dx}{(1+\cos x)^2} = \int_0^{\frac{\pi}{2}} \frac{1-\cos^2 x}{(1+\cos x)^2} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1+\cos x)(1-\cos x)}{(1+\cos x)^2} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\cos x}{1+\cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \tan^2 \frac{x}{2} dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sec^2 \frac{x}{2} - 1\right) dx$$

$$= \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x$$

$$= 2\left[\tan \frac{x}{2} - x\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\tan \frac{\pi}{4} - \frac{\pi}{2}\right]$$

$$= 2 - \frac{\pi}{2}$$

$$= \frac{4-\pi}{2}$$

1.6. Choose the correct option from the given alternatives :

If x = –1 and x = 2 are the extreme points of y = αlogx + βx^2 + x\`, then ______.

1. $\alpha = -6, \beta = \frac{1}{2}$
2. $\alpha = -6, \beta = -\frac{1}{2}$
3. **$\alpha = 2, \beta = -\frac{1}{2}$**
4. $\alpha = 2, \beta = \frac{1}{2}$

Solution

If x = –1 and x = 2 are the extreme points of y = αlogx + βx^2 + x,

then $\boldsymbol{\alpha = 2, \beta = -\frac{1}{2}}$

Explanation:

[**Hint:** y = αlogx + βx^2 + x

$\therefore \frac{dy}{dx} = \frac{\alpha}{x} + \beta \times 2x + 1$

$= \frac{\alpha}{x} + 2\beta x + 1$

f(x) has extreme values at x = –1 and x = 2

∴ f'(–1) = 0 and f(2) = 0

$\therefore \alpha + 2\beta = 1$

and $\frac{\alpha}{2} + 4\beta = -1$

By solving these two equations, we get

$\alpha = 2, \beta = -\frac{1}{2}$].

1.7. Select the correct option from the given alternatives:

The principal solutions of equation $\sin\theta = -\frac{1}{2}$ are _____.

1. $\frac{5\pi}{6}, \frac{\pi}{6}$
2. **$\frac{7\pi}{6}, \frac{11\pi}{6}$**
3. $\frac{\pi}{6}, \frac{7\pi}{6}$
4. $\frac{7\pi}{6}, \frac{\pi}{3}$

Solution

The principal solutions of equation sin θ = $-\frac{1}{2}$ are $\underline{\mathbf{\frac{7\pi}{6}, \frac{11\pi}{6}}}$.

1.8. Select the correct option from the given alternatives:

If cos pθ = cos qθ, p ≠ q, then ______.

1. **θ = $\frac{2n\pi}{p \pm q}$**

2. θ = 2nπ

3. θ = 2nπ ± p

4. θ = 2nπ ± q

Solution

If cos pθ = cos qθ, p ≠ q, then, θ = $\underline{\mathbf{\frac{2n\pi}{p \pm q}}}$

Explanation:

Given, cos pθ = cos qθ

cos pθ – cos qθ = 0

$2\sin\left(\frac{p\theta + q\theta}{2}\right)\sin\left(\frac{p\theta - q\theta}{2}\right) = 0$ ...(cos C – cos D)

$\sin\left(\frac{p\theta + q\theta}{2}\right) = 0$ or $\sin\left(\frac{p\theta - q\theta}{2}\right) = 0$

$\frac{p\theta + q\theta}{2} = n\pi$ or $\frac{p\theta - q\theta}{2} = n\pi$

$\theta = \frac{2n\pi}{(p+q)}$ or $\frac{2n\pi}{(p-q)} = \theta$

$\therefore \theta = \frac{2n\pi}{p \pm q}$

Q2. Answer the following questions:

2.1. Find the general solution of the following equation:

sinθ = $\frac{1}{2}$.

Solution

The general solution of sin θ = sin α is θ = nπ + $(-1)^n$α, n ∈ Z

Now,

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} \quad \ldots\left[\because \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

∴ the required general solution is $\theta = n\pi + (-1)^n \frac{\pi}{6}$, $n \in Z$.

2.2.

Evaluate: $\int_0^1 \frac{x^2-2}{x^2+1}.dx$

Solution

$$\int_0^1 \frac{x^2-2}{x^2+1}.dx$$

$$= \int_0^1 \frac{(x^2+1)-3}{x^2+1}.dx$$

$$= \int_0^1 \left(1-\frac{3}{x^2+1}\right).dx$$

$$= \int_0^1 1.dx - \int_0^1 \frac{3}{x^2+1}.dx$$

$$= [x]_0^1 - [3\tan^{-1}x]_0^1$$

$$= (1-0) - (3\tan^{-1}1 - 3\tan^{-1}0)$$

$$= 1 - 3\left(\frac{\pi}{4}\right) - 0$$

$$= 1 - \frac{3\pi}{4}.$$

2.3. Apply the given elementary transformation of the following matrix.

$A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, $R_1 \leftrightarrow R_2$

Solution

A = $\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$, we get,

A ~ $\begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$

2.4. State whether the following equation has a solution or not?

2sinθ = 3

Solution

2sinθ = 3

$\therefore$ sinθ = $\frac{3}{2}$

This is not possible because – 1 ≤ sinθ ≤ 1 for any θ.

∴ 2 sinθ = 3 does not have any solution.

SECTION – B

Attempt any EIGHT of the following questions:

Q3. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1/100. What is the probability that he will win a prize at least once.

Solution

Let X denote the number of times the person wins the lottery.

Then, X follows a binomial distribution with n = 50.

Let p be the probability of winning a prize

$\therefore$ p = $\frac{1}{100}$, q = $1 - \frac{1}{100} = \frac{99}{100}$

Hence, the distribution is given by

P(X = r) = ${}^{50}C_r\left(\frac{1}{100}\right)^r\left(\frac{99}{100}\right)^{50-r}$, r = 0, 1, 2,... 50

P(winning at least once) = P(X > 1)

$= 1 - P(X = 0)$

$$= 1 - \left(\frac{99}{100}\right)^{50}$$

Hence, probability of winning a prize at least once $= 1 - \left(\frac{99}{100}\right)^{50}$

Q4. Find k, the slope of one of the lines given by $kx^2 + 4xy - y^2 = 0$ exceeds the slope of the other by 8.

Solution

Comparing the equation $kx^2 + 4xy - y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, $a = k$, $2h = 4$, $b = -1$.

Let m_1 and m_2 be the slopes of the lines represented by $kx^2 + 4xy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{4}{-1} = 4 \quad \text{...(1)}$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{k}{-1} = -k \quad \text{...(2)}$$

We are given that $m_2 = m_1 + 8$

$4 - m_1 = m_1 + 8$ ...[By (1)]

$\therefore 2m_1 = -4$

$\therefore m_1 = -2$ ...(3)

Also, $m_1(m_1 + 8) = -k$ ...[By (2)]

$(-2)(-2 + 8) = -k$ ...[By (3)]

$\therefore (-2)(6) = -k$

$\therefore -12 = -k$

$\therefore k = 12$

Q5. Find the position vector of midpoint M joining the points L (7, –6, 12) and N (5, 4, –2).

Solution

The position vectors $\vec{l}$ and $\vec{n}$ of the points L(7, –6, 12) and N(5, 4, –2) are given by

$\vec{l} = 7\hat{i} - 6\hat{j} + 12\widehat{k}, \vec{n} = 5\hat{i} + 4\hat{j} - 2\widehat{k}$

If M($\vec{m}$) is the midpoint of LN, by midpoint formula,

$\vec{m} = \frac{\vec{l} + \vec{n}}{2}$

$= \frac{\left(7\hat{i} - 6\hat{j} + 12\widehat{k}\right) + \left(5\hat{i} + 4\hat{j} - 2\widehat{k}\right)}{2}$

$= \frac{1}{2}\left(12\hat{i} - 2\hat{j} + 10\widehat{k}\right) = 6\hat{i} - \hat{j} + 5\widehat{k}$

∴ Coordinates of M(6, –1, 5)

∴ Hence, position vector of M is $6\hat{i} - \hat{j} + 5\widehat{k}$ and the coordinates of M are (6, –1, 5).

Q6.

Find the vector equation of the lines passing through the point having position vector $\left(-\hat{i} - \hat{j} + 2\widehat{k}\right)$ and parallel to the line $\vec{r} = \left(\hat{i} + 2\hat{j} + 3\widehat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \widehat{k}\right)$.

Solution

Let A be point having position vector $\vec{a} = -\hat{i} - \hat{j} + 2\widehat{k}$.

The required Line is parallel to the line $\vec{r} = \left(\hat{i} + 2\hat{j} + 3\widehat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \widehat{k}\right)$

∴ It is parallel to the vector $\vec{b} = 3\hat{i} + 2\hat{j} + \widehat{k}$

The vector equation of the line passing through A$\left(\vec{a}\right)$ and parallel to $\vec{b}$ is r = $\vec{a} + \lambda\vec{b}$ where λ is a scalar.

∴ The required vector equation of the line is $\vec{r} = \left(-\hat{i} - \hat{j} + 2\widehat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \widehat{k}\right)$.

Q7. Find the derivative of the function y = f(x) using the derivative of the inverse function x = $f^{-1}(y)$ in the following:

y = $\sqrt{x}$

Solution

y = $\sqrt{x}$ ...(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1), we have

$y^2 = x$

$\therefore x = y^2$

$\therefore x = f^{-1}(y) = y^2$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}\left(y^2\right) = 2y$$

$= 2\sqrt{x}$ ...[By (1)]

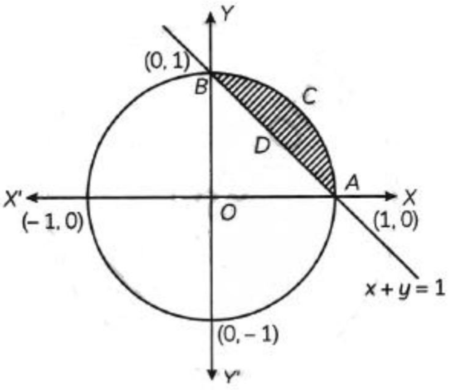
$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

$$= \frac{1}{2\sqrt{x}}.$$

Q8. Solve the following :

Find the area enclosed between the circle $x^2 + y^2 = 1$ and the line $x + y = 1$, lying in the first quadrant.

Solution



Required area = area of the region ACBDA

= (area of the region OACBO) – (area of the region OADBO) ...(1)

Now, area of the region OACBO

= area under the circle $x^2 + y^2 = 1$ between x = 0 and x = 1

$= \int_0^1 y.\,dx$, where $y^2 = 1 - x^2$,

i.e. $y = \sqrt{1 - x^2}$, as $y > 0$

$= \int_0^1 \sqrt{1 - x^2}.\,dx$

$= \left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$

$= \frac{1}{2}\sqrt{1 - 1} + \frac{1}{2}\sin^{-1} 1 - 0$

$= \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{4}$ ...(2)

Area of the region OADBO = area under the line x + y = 1 between x = 0 and x = 1

$= \int_0^1 y.\,dx$, where $y = 1 - x$

$= \int_0^1 (1 - x).\,dx$

$= \left[x - \frac{x^2}{2}\right]_0^1$

$= 1 - \frac{1}{2} - 0$

$= \frac{1}{2}$ ...(3)

Put the value of equation (2) and (3) in equation (1)

∴ Required area = $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units.

Q9. Find the co-factor of the element of the following matrix:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$$

Solution

Let A = $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

Here, $a_{11} = -1$, $M_{11} = 4$

$\therefore A_{11} = (-1)^{1+1}(4) = 4$

$a_{12} = 2$, $M_{12} = -3$

$\therefore A_{12} = (-1)^{1+2}(-3) = 3$

$a_{21} = -3$, $M_{21} = 2$

$\therefore A_{21} = (-1)^{2+1}(2) = -2$

$a_{22} = 4$, $M_{22} = -1$

$\therefore A_{22} = (-1)^{2+2}(-1) = -1$

Q10. Verify which of the following is p.d.f. of r.v. X:

f(x) = sin x, for $0 \leq x \leq \frac{\pi}{2}$

Solution

(a) f(x) = sin x ≥ 0 if $0 \leq x \leq \frac{\pi}{2}$

(b) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{-\infty}^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\infty} f(x)dx$

$= 0 + \int_{0}^{\frac{\pi}{2}} \sin x \, dx + 0$

$= [-\cos x]_{0}^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) + \cos 0 = 0 + 1 = 1$

Hence, f(x) is the p.d.f. of X.

Q11. Using the rule of negation write the negation of the following with justification.

p → (p ∨ ~ q)

Solution

The negation of p → (p ∨ ~ q) is ~ [p → (p ∨ ~ q)] ≡ p ∧ ~ (p ∨ ~ q) ....(Negation of implication)

$\equiv p \wedge [\sim p \wedge \sim (\sim q)]$ .....(Negation of disjunction)

$\equiv p \wedge (\sim p \wedge q)$ ....(Negation of negation)

Q12. Find the Cartesian equation of the plane passing through A(–1, 2, 3), the direction ratios of whose normal are 0, 2, 5.

Solution

The Cartesian equation of the plane passing through (x_1, y_1, z_1), the direction ratios of whose normal are a, b, c, is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

∴ The cartesian equation of the required plane is $0(x + 1) + 2(y - 2) + 5(z - 3) = 0$

i.e. $0 + 2y - 4 + 5z - 15 = 0$

i.e. $2y + 5z = 19$.

Q13. Solve graphically: 2x – 3 ≥ 0

Solution

Consider the line whose equation is $2x - 3 \geq 0$, i.e. $x = \frac{3}{2}$

This represents a line parallel to Y-axis passing through the point $\left(\frac{3}{2}, 0\right)$.

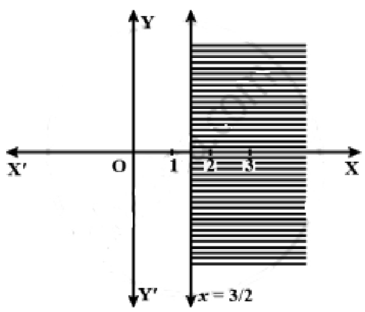
Draw the line $x = \frac{3}{2}$.

To find the solution set, we have to check the position of the origin (0, 0).

When $x = 0$, $2x - 3 = 2 \times 0 - 3 = -3 \not\geq 0$

∴ The coordinates of the origin does not satisfy the given inequality.

∴ The solution set consists of the line $x = \frac{3}{2}$ and the non-origin side of the line which is shaded in the graph.



Q14. Find the condition that the line 4x + 5y = 0 coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$

Solution

The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$

Given that 4x + 5y = 0 is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$

The slope of the line 4x + 5y = 0 is $-\frac{4}{5}$

$\therefore m = -\frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$

$$\therefore b\left(-\frac{4}{5}\right)^2 + 2h\left(-\frac{4}{5}\right) + a = 0$$

$$\therefore \frac{16b}{25} - \frac{8h}{5} + a = 0$$

$\therefore 16b - 40h + 25a = 0$

$\therefore 25a + 16b - 40h = 0$

This is the required condition.

SECTION – C

Attempt any EIGHT of the following questions:

Q15. In a large school, 80% of the pupil like Mathematics. A visitor to the school asks each of 4 pupils, chosen at random, whether they like Mathematics.

Find the probability that the visitor obtains answer yes from at least 2 pupils:

a. **when the number of pupils questioned remains at 4.**
b. **when the number of pupils questioned is increased to 8.**

Solution

Let X = number of pupils like Mathematics.

p = probability that pupils like Mathematics

$\therefore$ p = 80% = $\frac{80}{100} = \frac{4}{5}$

and q = 1 – p = $1 - \frac{4}{5} = \frac{1}{5}$

Given: n = 4

$\therefore$ X ~ B $\left(4, \frac{4}{5}\right)$

The p.m.f. of X is given by P(X = x) = ${}^nC_x\, p^x\, q^{n-x}$

i.e. p(x) = ${}^4C_x \left(\frac{4}{5}\right)^x \left(\frac{1}{5}\right)^{4-x}$ x = 0, 1, 2, 3, 4

a. P(visitor obtains the answer yes from at least 2 pupils when the number of pupils questioned remains at 4) = P(X ≥ 2)

= P(X = 2) + P(X = 3) + P(X = 4)

$$= {}^4C_2 \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^{4-2} + {}^4C_3 \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^{4-3} + {}^4C_4 \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right)^{4-4}$$

$$= \frac{4 \times 3}{1 \times 2} \times \frac{16}{5^2} \times \frac{1}{5^2} + 4 \times \frac{64}{5^3} \times \frac{1}{5} + 1 \times \frac{256}{5^4}$$

$$= \frac{96}{5^4} + \frac{256}{5^4} + \frac{256}{5^4}$$

$$= (96 + 256 + 256)\frac{1}{5^4}$$

$$= \frac{608}{5^4} = \frac{608}{625}$$

b. P(the visitor obtains the answer yes from at least 2 pupils when number of pupils questioned is increased to 8)

= P(X ≥ 2)

= 1 – P(X < 2)

= 1 – [P(X = 0) + P(X = 1)]

$$= 1 - \left[{}^8C_0 \left(\frac{4}{5}\right)^0 \left(\frac{1}{5}\right)^{8-0} + {}^8C_1 \left(\frac{4}{5}\right)^1 \left(\frac{1}{5}\right)^{8-1}\right]$$

$= 1 - \left[1(1)\left(\frac{1}{5}\right)^8 + (8)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^7\right]$

$= 1 - \left[\frac{1}{5^8} + \frac{32}{5^8}\right]$

$= 1 - \frac{33}{5^8}.$

Q16. Differentiate the following w.r.t. x: $x^e + x^x + e^x + e^e$

Solution

Let $y = x^e + x^x + e^x + e^e$

Let $u = x^x$

Then $\log u = \log x^x = x\log x$

Differentiating both sides w.r.t. x, we get

$\frac{1}{u}.\frac{du}{dx} = \frac{d}{dx}(x\log x)$

$= x\frac{d}{dx}(\log x) + (\log x).\frac{d}{dx}(x)$

$= x \times \frac{1}{x} + (\log x)(1)$

$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$ ...(1)

Now, $y = x^e + u + e^x + e^e$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^e) + \frac{du}{dx} + \frac{d}{dx}(e^x) + \frac{d}{dx}(e^e)$

$= ex^{e-1} + x^x(1 + \log x) + e^x + 0$ ...[By (1)]

$= ex^{e-1} + x^x(1 + \log x) + e^x$

$= ex^{e-1} + e^x + x^x(1 + \log x).$

Q17.

Find $\frac{dy}{dx}$, if $x^3 + x^3y + xy^2 + y^3 = 81$

Solution

$x^3 + x^3y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t x. we get

$$3x^2 + x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) + x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) + 3y^2\frac{dy}{dx} = 0$$

$$\therefore 3x^2 + x^2\frac{dy}{dx} + y \times 2x + x \times 2y\frac{dy}{dx} + y^2 \times 1 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore 3x^2 + x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore (x^2 + 2xy + 3y^2)\frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}.$$

Q18. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Solution 1

It is given that out of 30 bulbs, 6 are defective.

⇒ Number of non-defective bulbs = 30 − 6 = 24

4 bulbs are drawn from the lot with replacement.

Let X be the random variable that denotes the number of defective bulbs in the selected bulbs.

∴ X can take the value 0, 1, 2, 3, 4.

∴ P (X = 0) = P (4 non-defective and 0 defective)

$$= {}^4C_0 \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} = \frac{256}{625}$$

P (X = 1) = P (3 non-defective and 1defective)

$$= {}^4C_1 \times \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^3 = \frac{256}{625}$$

P (X = 2) = P (2 non-defective and 2defective)

$$= {}^4C_2 \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^2 = \frac{96}{625}$$

P (X = 3) = P (1 non-defective and 3defective)

$$= {}^4C_3 \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right) = \frac{16}{625}$$

P (X = 4) = P (0 non-defective and 4defective)

$$= {}^4C_4 \times \left(\frac{1}{5}\right)^4 \times \left(\frac{4}{5}\right)^0 = \frac{1}{625}$$

Therefore, the required probability distribution is as follows.

X = x	0	1	2	3	4
P (X = x)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

Solution 2

Here, the number of defective bulbs is the random variable.

Let the number of defective bulbs be denoted by X.

∴ X can take the value 0, 1, 2, 3, 4.

Since the draws are done with replacement, therefore the four draws are independent experiments.

Total number of bulbs is 30 which include 6 defectives.

∴ P (X = 0) = P (0) = P (all 4 non-defective bulbs)

$$= \frac{24}{30} \times \frac{24}{30} \times \frac{24}{30} \times \frac{24}{30} = \frac{256}{625}$$

P (X = 1) = P (1) = P (1 defective and 3 non-defective bulbs)

$$= \frac{6}{30} \times \frac{24}{30} \times \frac{24}{30} \times \frac{24}{30} + \frac{24}{30} \times \frac{24}{30} \times \frac{6}{30} \times \frac{24}{30} +$$

$$\frac{24}{30} \times \frac{24}{30} \times \frac{6}{30} \times \frac{24}{30} + \frac{24}{30} \times \frac{24}{30} \times \frac{24}{30} \times \frac{24}{30} \times \frac{6}{30} = \frac{256}{625}$$

P (X = 2) = P(2) = P (2 defective and 2 non-defective)

$$= \frac{6}{30} \times \frac{6}{30} \times \frac{24}{30} \times \frac{24}{30} + \frac{24}{30} \times \frac{6}{30} \times \frac{6}{30} \times \frac{24}{30} +$$

$$\frac{24}{30} \times \frac{24}{30} \times \frac{6}{30} \times \frac{6}{30} + \frac{6}{30} \times \frac{24}{30} \times \frac{6}{30} \times \frac{24}{30} +$$

$$\frac{6}{30} \times \frac{24}{30} \times \frac{24}{30} \times \frac{6}{30} + \frac{24}{30} \times \frac{6}{30} \times \frac{24}{30} \times \frac{6}{30} = \frac{96}{625}$$

P (X = 3) = P (3) = P (3 defectives and 1 non-defective)

$$= \frac{6}{30} \times \frac{6}{30} \times \frac{6}{30} \times \frac{24}{30} + \frac{6}{30} \times \frac{6}{30} \times \star \frac{24}{30} \times \frac{6}{30} +$$

$$\frac{6}{30} \times \frac{24}{30} \times \frac{6}{30} \times \frac{6}{30} + \frac{24}{30} \times \frac{6}{30} \times \frac{6}{30} \times \frac{6}{30} = \frac{16}{625}$$

P (X = 4) = P (4) = P (all 4 defectives)

$$= \frac{6}{30} \times \frac{6}{30} \times \frac{6}{30} \times \frac{6}{30} = \frac{1}{625}$$

∴ The required probability distribution is

X = x	0	1	2	3	4
P (X = x)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

Solution 3

Let X denotes the number of defective bulbs.

∴ Possible values of X are 0, 1, 2, 3, 4.

Let P(getting a defective bulb) = p = $\frac{6}{30} = \frac{1}{5}$

∴ q = 1 – p = $1 - \frac{1}{5} = \frac{4}{5}$

∴ P(X = 0) = P(no defective bulb)

= qqqq

$= q^4$

$= \left(\frac{4}{5}\right)^4$

$= \frac{256}{625}$

P(X = 1) = P(one defective bulb)

= qqqp + qqpq + qpqq + pqqq

$= 4pq^3$

$= 4 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3$

$= \frac{256}{625}$

P(X = 2) = P(two defective bulbs)

= ppqq + pqqp + qqpp + pqpq + qpqp + qppq

$= 6p^2q^2$

$= 6\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2$

$= \frac{96}{625}$

P(X = 3) = P(three defective bulbs)

= pppq + ppqp + pqpp + qppp

$= 4qp^3$

$= 4\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3$

$= \frac{16}{625}$

P(X = 4) = P(four defective bulbs)

= pppp

$= p^4$

$= \left(\frac{1}{5}\right)^4$

$= \frac{1}{625}$

∴ Probability distribution of X is as follows:

X	0	1	2	3	4
P(X = x)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

Q19.

Evaluate the following integrals: $\int \frac{2x-7}{\sqrt{4x-1}}.dx$

Solution

$$\int \frac{2x-7}{\sqrt{4x-1}}.dx$$

$$= \frac{1}{2}\int \frac{2(2x-7)}{\sqrt{4x-1}}.dx$$

$$= \frac{1}{2}\int \frac{(4x-1)-13}{\sqrt{4x-1}}.dx$$

$$= \frac{1}{2}\int \left(\frac{(4x-1)}{\sqrt{4x-1}} - \frac{13}{\sqrt{4x-1}}\right).dx$$

$$= \frac{1}{2}\int (4x-1)^{\frac{1}{2}}.dx - \frac{13}{2}\int (4x-1)^{-\frac{1}{2}}.dx$$

$$= \frac{1}{2}\frac{(4x-1)^{\frac{3}{2}}}{(4)\left(\frac{3}{2}\right)} - \frac{13}{2}.\frac{(4x-1)^{\frac{1}{2}}}{(4)\left(\frac{1}{2}\right)} + c$$

$$= \frac{1}{12}(4x-1)^{\frac{3}{2}} - \frac{13}{4}\sqrt{4x-1} + c$$

Q20. It is observed that it rains on 12 days out of 30 days. Find the probability that it it will rain at least 2 days of given week.

Solution

Let X = number of days it rains in a week.

p = probability that it rains

$\therefore$ p = $\frac{12}{30} = \frac{2}{5}$

and q = 1 – p = $1 - \frac{2}{5} = \frac{3}{5}$

Given: n = 7

$\therefore$ X – B$\left(7, \frac{2}{5}\right)$

The p.m.f. of x is given by

P(X = x) = ${}^{n}C_x\, p^x\, q^{n-x}$

i.e. p(x) = ${}^{7}C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{7-x}$ x = 0, 1, 2, ..., 7

P(it will rain on at least 2 days of given week)

= P(X ≥ 2) = 1 – P(X < 2)

= 1 – [P(X = 0) + P(X = 1)]

$$= 1 - \left[{}^{7}C_0 \left(\frac{2}{5}\right)^0 \left(\frac{3}{5}\right)^{7-0} + {}^{7}C_1 \left(\frac{2}{5}\right)^1 \left(\frac{3}{5}\right)^{7-1}\right]$$

$$= 1 - \left[1(1)\left(\frac{3}{7}\right)^7 + 7\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^6\right]$$

$$= 1 - \left[\frac{3}{5} + \frac{14}{5}\right]\left(\frac{3}{5}\right)^6$$

$$= 1 - \left(\frac{17}{5}\right)\left(\frac{729}{(5)^6}\right) = 1 - \frac{12393}{5^7}$$

$$= 1 - \frac{12393}{78125} = 1 - 0.1586$$

= 0.8414

Hence, the probability that it rains at least 2 days of given week

= $1 - \frac{12393}{5^7}$ OR 0.8414

Q21. Find the inverse of the following matrix by the adjoint method.

$$\begin{bmatrix}1 & 0 & 0\\3 & 3 & 0\\5 & 2 & -1\end{bmatrix}$$

Solution

Let $A = \begin{bmatrix}1 & 0 & 0\\3 & 3 & 0\\5 & 2 & -1\end{bmatrix}$

$\therefore |A| = \begin{vmatrix}1 & 0 & 0\\3 & 3 & 0\\5 & 2 & -1\end{vmatrix}$

$= 1(-3 - 0) - 0 + 0$

$= -3 \neq 0$

$\therefore A^{-1}$ exist

First, we have to find the co-factor matrix

$= [A_{ij}]_{3\times3}$, where $A_{ij} = (-1)^{i+j}M_{ij}$

Now, $A_{11} = (-1)^{1+1}M_{11} = 1\begin{vmatrix}3 & 0\\2 & -1\end{vmatrix} = 1(-3 - 0) = -3$

$A_{12} = (-1)^{1+2}M_{12} = -1\begin{vmatrix}3 & 0\\5 & -1\end{vmatrix} = -1(-3 - 0) = 3$

$A_{13} = (-1)^{1+3}M_{13} = 1\begin{vmatrix}3 & 3\\5 & 2\end{vmatrix} = 1(6 - 15) = -9$

$A_{21} = (-1)^{2+1}M_{21} = -1\begin{vmatrix}0 & 0\\2 & -1\end{vmatrix} = -1(0 - 0) = 0$

$A_{22} = (-1)^{2+2}M_{22} = 1\begin{vmatrix}1 & 0\\5 & -1\end{vmatrix} = 1(-1 - 0) = -1$

$A_{23} = (-1)^{2+3}M_{23} = -1\begin{vmatrix}1 & 0\\5 & 2\end{vmatrix} = -1(2 - 0) = -2$

$A_{31} = (-1)^{3+1}M_{31} = 1\begin{vmatrix}0 & 0\\3 & 0\end{vmatrix} = 1(0 - 0) = 0$

$A_{32} = (-1)^{3+2}M_{32} = -1\begin{vmatrix}1 & 0\\3 & 0\end{vmatrix} = -1(0 - 0) = 0$

$A_{33} = (-1)^{3+3}M_{33} = 1\begin{vmatrix}1 & 0\\3 & 3\end{vmatrix} = 1(3 - 0) = 3$

∴ The co-factor matrix

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore \text{adj } A = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

$$= -\frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 9 & 2 & -3 \end{bmatrix}$$

Q22. Integrate the following functions w.r.t.x:

$\cos^8 x \cot x$

Solution

$$\text{Let } I = \int \cos^8 x \cot x dx$$

$$= \int \cos^8 x . \frac{\cos x}{\sin x} . dx$$

Put sinx = t

∴ cosx dx = dt

$\cos^8 x = (\cos^2 x)^4$

$= (1 - \sin^2 x)^4$

$= (1 - t^2)^4$

$= 1 - 4t^2 + 6t^4 - 4t^6 + t^8$

$$I = \int \frac{1 - 4t^2 + 6t^4 - 4t^6 + t^8}{t} dt$$

$$= \int \left[\frac{1}{t} - 4t + 6t^3 - 4t^5 + t^7 \right] dt$$

$$= \int \frac{1}{t}dx - 4\int t dt + 6\int t^3 dt - 4\int t^5 dt + \int t^7 dt$$

$$= \log|t| - 4\left(\frac{t^2}{2}\right) + 6\left(\frac{t^4}{4}\right) - 4\left(\frac{t^6}{6}\right) + \frac{t^8}{8} + c$$

$$= \log|\sin x| - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + \frac{\sin^8 x}{8} + c.$$

Q23. Integrate the following functions w.r.t.x:

$$\frac{2\sin x\cos x}{3\cos^2 x + 4\sin^2 x}$$

Solution

Let I = $\int \frac{2\sin x\cos x}{3\cos^2 x + 4\sin^2 x}.dx$

Put $3\cos^2 x + 4\sin^2 x = t$

$\therefore \left[3(2\cos x)\frac{d}{dx}(\cos x) + 4(2\sin x)\frac{d}{dx}(\sin x)\right]dx$ = dt

∴ [−6 cosx sinx + 8 sinx cosx]dx = dt

∴ 2 sinx cosx dx = dt

Then I = $\int \frac{dt}{t}$ = log|t| + c

= log|3cos^{2}x + 4sin^{2}x| + c

Q24.

Find two unit vectors each of which is perpendicular to both $\overline{u}$ and $\overline{v}$ where $\overline{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\overline{v} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution

Let $\overline{u} = 2\hat{i} + \hat{j} - 2\hat{k}$,

$\overline{v} = \hat{i} + 2\hat{j} - 2\hat{k}$

Then $\overline{u} \times \overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \widehat{k} \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

= $(-2-(-4))\hat{i} - (-4-(-2))\hat{j} + (4-1)\widehat{k}$

= $(-2+4)\hat{i} - (-4+2)\hat{j} + 3\widehat{k}$

= $2\hat{i} + 2\hat{j} + 3\widehat{k}$

$\therefore |\overline{u} \times \overline{v}| = \sqrt{(2)^2 + (2)^2 + (3)^2}$

= $\sqrt{4+4+9}$

= $\sqrt{17}$

= $\pm \frac{\overline{u} \times \overline{v}}{|\overline{u} \times \overline{v}|} = \pm \frac{2\hat{i} + 2\hat{j} + 3\widehat{k}}{\sqrt{17}}$

= $\pm\left(\frac{2}{\sqrt{17}}\hat{i} + \frac{2}{\sqrt{17}}\hat{j} + \frac{3}{\sqrt{17}}\widehat{k}\right)$

Q25.

Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\widehat{k}$ and $-5\hat{i} + 2\hat{j} - 5\widehat{k}$ in the ratio 3:2 is internally.

Solution

It is given that the points P and Q have position vectors $\overline{p} = 2\hat{i} - \hat{j} + 3\widehat{k}$ and $\overline{q} = -5\hat{i} + 2\hat{j} - 5\widehat{k}$ respectively.

If R($\overline{r}$) divides the line segment PQ internally in the ratio 3:2, by section formula for internal division,

$\overline{r} = \frac{3q + 2p}{3+2}$

= $\frac{3\left(-5\hat{i} + 2\hat{j} - 5\widehat{k}\right) + 2\left(2\hat{i} - \hat{j} + 3\widehat{k}\right)}{5}$

$= \dfrac{-11\hat{i} + 4\hat{j} - 9\hat{k}}{5}$

$= \dfrac{1}{5}\left(-11\hat{i} + 4\hat{j} - 9\hat{k}\right)$

$\therefore$ Coordinates of R = $\left(-\dfrac{11}{5}, \dfrac{4}{5}, -\dfrac{9}{5}\right)$

Hence, the position vector of R is $\dfrac{1}{5}\left(-11\hat{i} + 4\hat{j} - 9\hat{k}\right)$

and the coordinates of R are $\left(-\dfrac{11}{5}, \dfrac{4}{5}, -\dfrac{9}{5}\right)$

Q26. Evaluate the following integrals as limit of a sum:

$$\int_0^2 (3x^2 - 1) \cdot dx$$

Solution

Let f(x) = $3x^2 - 1$, for $0 \le x \le 2$.

Divide the closed interval [0, 2] into n subintervals each of length h at the points.

0, 0 + h, 0 + 2h, ..., 0 + rh, .., 0 + nh = 2

i.e. 0, h, 2h, ..., rh, ..., nh = 2

$\therefore$ h = $\dfrac{2}{n}$ and as $n \to \infty, h \to 0$

Here, a = 0

$\therefore$ f(a + rh) = f(0 + rh) = f(rh) = $3(rh)^2 - 1 = 3r^2h^2 - 1$

$$\because \int_a^b f(x) \cdot dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(a + rh) \cdot h$$

$$= \int_0^2 (3x^2 - 1) \cdot dx = \lim_{n \to \infty} \sum_{r=1}^{n} (3r^2h^2 - 1) \cdot h$$

= $\lim_{n\to\infty}\sum_{r=1}^{n}(3r^2\times\frac{4}{n^2}-1)\cdot\frac{2}{n}\ldots[\because h=\frac{2}{n}]$

= $\lim_{n\to\infty}\sum_{r=1}^{n}(\frac{24r^2}{n^3}-\frac{2}{n})$

= $\lim_{n\to\infty}[\frac{24}{n^3}\sum_{r=1}^{n}r^2-\frac{2}{n}\sum_{r=1}^{n}1]$

= $\lim_{n\to\infty}[\frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{2}{n}\cdot n]$

= $\lim_{n\to\infty}[4\cdot(\frac{n+1}{n})(\frac{2n+1}{n})-2]$

= $\lim_{n\to\infty}[4(1+\frac{1}{n})(2+\frac{1}{n})-2]$

= $4(1+0)(2+0)-2\ldots[\because \lim_{n\to\infty}\frac{1}{n}=0]$

= 8 – 2

= 6

SECTION – D

Attempt any FIVE of the following questions:

Q27. Evaluate the following integrals:

$$\int\frac{7x+3}{\sqrt{3+2x-x^2}}.dx$$

Solution

Let I = $\int\frac{7x+3}{\sqrt{3+2x-x^2}}.dx$

Let 7x + 3 = $A\left[\frac{d}{dx}(3+2x-x^2)\right]+B$

= A(2 – 2x) + B

∴ 7x + 3 = 2Ax + (2A + B)

Comparing the coefficient of x and constant on both the sides, we get

– 2A = 7 and 2A + B = 3

$\therefore$ A = $\frac{-7}{2}$ and $2\left(-\frac{7}{2}\right) +$ B = 3

$\therefore$ B = 10

$\therefore$ 7x + 3 = $\frac{-7}{2}(2-2x)+10$

$\therefore$ I = $\int \frac{\frac{-7}{2}(2-2x)+10}{\sqrt{3+2x-x^2}}.dx$

= $\frac{-7}{2}\int \frac{(2-2x)}{\sqrt{3+2x-x^2}}.dx + 10\int \frac{1}{\sqrt{3+2x-x^2}}x$

= $\frac{-7}{2}I_1 + 10I_2$

In I_1, put $3 + 2x - x^2 = t$

$\therefore$ (2 – 2x) dx = dt

$\therefore$ I_1 = $\int \frac{1}{\sqrt{t}}dt$

= $\int t^{-\frac{1}{2}}dt$

= $\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c_1$

= $2\sqrt{3+2x-x^2} + c_1$

I_2 = $\int \frac{1}{\sqrt{3-(x^2-2x+1)+1}}.dx$

= $\int \frac{1}{\sqrt{(2)^2-(x-1)^2}}.dx$

= $\sin^{-1}\left(\frac{x-1}{2}\right) + c_2$

$\therefore$ I = $-7\sqrt{3+2x-x^2} + 10\sin^{-1}\left(\frac{x-1}{2}\right) + c$, where $c = c_1 + c_2$.

Q28. Find the vector and Cartesian equations of the line passing through the point (–1, –1, 2) and parallel to the line 2x − 2 = 3y + 1 = 6z − 2.

Solution

Let $\bar{a}$ be the position vector of the point A(–1, –1, 2) w.r.t. the origin.

Then $\bar{a} = -\hat{i} - \hat{j} + 2\hat{k}$

The equation of given line is 2x – 2 = 3y + 1 = 6z – 2

$\therefore$ 2(x – 1) = $3\left(y + \frac{1}{3}\right) = 6\left(z - \frac{1}{3}\right)$

$\therefore \frac{x-1}{\left(\frac{1}{2}\right)} = \frac{y+\frac{1}{3}}{\left(\frac{1}{3}\right)} = \frac{z-\frac{1}{3}}{\left(\frac{1}{6}\right)}$

The direction ratios of this line are

$\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ i.e. 3, 2, 1

Let $\bar{b}$ be the vector parallel to this line.

Then $\bar{b} = 3\hat{i} + 2\hat{j} + \hat{k}$

The vector equation of the line passing through A($\bar{a}$)and parallel to $\bar{b}$ is

$\bar{r} = \bar{a} + \lambda\bar{b}$, where λ is a scalar,

$\therefore$ the vector equation of the required line is

$$\bar{r} = \left(-\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \hat{k}\right).$$

The line passes through (–1, – 1, 2) and has direction ratios 3, 2, 1

$\therefore$ the cartesian equations of the line are

$$\frac{x-(-1)}{3} = \frac{y-(-1)}{2} = \frac{z-2}{1}$$

i.e. $\frac{x+1}{3} = \frac{y+1}{2} = \frac{z-2}{1}$.

Q29. Using the truth table prove the following logical equivalence.

$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

Solution

1	2	3	4	5	6	7	8
p	q	r	$p \vee q$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The entries in columns 5 and 8 are identical.

$\therefore (p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

Q30. Solve the following : An open box with a square base is to be made out of given quantity of sheet of area a^2.

Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

Solution

Let x be the side of square base and h be the height of the box.

Then $x^2 + 4xh = a^2$

$\therefore$ h = $\dfrac{a^2 - x^2}{4x}$ ...(1)

Let V be the volume of the box.

Then V = x^2h

$\therefore$ V = $x^2\left(\dfrac{a^2 - x^2}{4x}\right)$ ...[By (1)]

$\therefore$ V = $\dfrac{1}{4}\left(a^2x - x^3\right)$ ...(2)

$\therefore \dfrac{dV}{dx} = \dfrac{1}{4}\dfrac{d}{dx}\left(a^2x - x^3\right)$

= $\dfrac{1}{4}\left(a^2 \times 1 - 3x^2\right)$

= $\dfrac{1}{4}\left(a^2 - 3x^2\right)$

and

$\dfrac{d^2V}{dx^2} = \dfrac{1}{4}.\dfrac{d}{dx}\left(a^2 - 3x^2\right)$

= $\dfrac{1}{4}(0 - 3 \times 2x)$

= $-\dfrac{3}{2}x$

Now, $\dfrac{dV}{dx} = 0$ gives $\dfrac{1}{4}\left(a^2 - 3x^2\right)$ = 0

$\therefore$ $a^2 - 3x^2 = 0$

$\therefore$ $3x^2 = a^2$

$\therefore$ $x^2 = \dfrac{a^2}{3}$

$\therefore$ x = $\dfrac{a}{\sqrt{3}}$ ...[$\because$ x > 0]

and

$\left(\dfrac{d^2V}{dx^2}\right)_{\text{at } x=\frac{a}{\sqrt{3}}}$

= $-\dfrac{3}{2} \times \dfrac{a}{\sqrt{3}}$

= $-\frac{\sqrt{3}}{2}a < 0$

$\therefore$ V is maximum when x = $\frac{a}{\sqrt{3}}$

From (2), maximum volume = $\left[\frac{1}{4}\left(a^2x - x^3\right)\right]_{\text{at } x=\frac{a}{\sqrt{3}}}$

= $\frac{1}{4}\left(a^2 \times \frac{a}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right)$

= $\frac{1}{4}\left(\frac{2a^3}{3\sqrt{3}}\right)$

= $\frac{a^3}{6\sqrt{3}}$

Hence, the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$ cu. unit.

Q31.

If $\bar{a} = \hat{i} - 2\hat{j}$, $\bar{b} = \hat{i} + 2\hat{j}, \bar{c} = 2\hat{i} + \hat{j} - 2\widehat{k}$,

then find (i) $\bar{a} \times \left(\bar{b} \times \bar{c}\right)$ (ii) $\left(\bar{a} \times \bar{b}\right) \times \bar{c}$.

Are the results same? Justify.

Solution

(i) $\bar{a} \times \left(\bar{b} \times \bar{c}\right)$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \widehat{k} \\ 1 & 2 & 0 \\ 2 & 1 & -2 \end{vmatrix}$$

= $(-4 - 0)\hat{i} - (-2 - 0)\hat{j} + (1 - 4)\widehat{k}$

= $-4\hat{i} + 2\hat{j} - 3\widehat{k}$

$$\therefore \bar{a} \times \left(\bar{b} \times \bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \widehat{k} \\ 1 & -2 & 0 \\ -4 & 2 & -3 \end{vmatrix}$$

$= (6-0)\hat{i} - (-3-0)\hat{j} + (2-8)\widehat{k}$

$= 6\hat{i} + 3\hat{j} - 6\widehat{k}$

(ii) $\left(\bar{a} \times \bar{b}\right) \times \bar{c}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \widehat{k} \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$= (0-0)\hat{i} - (0-0)\hat{j} + (2-(-2))\widehat{k}$

$= 4\widehat{k}$

$$\therefore \left(\bar{a} \times \bar{b}\right) \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \widehat{k} \\ 0 & 0 & 4 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (0-4)\hat{i} - (0-8)\hat{j} + (0-0)\widehat{k}$

$= -4\hat{i} + 8\hat{j}$

$\bar{a} \times \left(\bar{b} \times \bar{c}\right) \neq \left(\bar{a} \times \bar{b}\right) \times \bar{c}$

Q32. Solve the following differential equation:

$(x^2 + y^2)dx - 2xy\,dy = 0$

Solution

$(x^2 + y^2)dx - 2xy\,dy = 0$

$\therefore 2xy\,dy = (x^2 + y^2)dx$

$$\therefore \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad ....(1)$$

Put $y = vx$

$$\therefore \frac{dy}{dx} = v + \frac{xdv}{dx}$$

$$\therefore \text{(1) becomes, } v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x(vx)}$$

$$\therefore v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\therefore \frac{2v}{1-v^2}dv = \frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$$

$$-\int \frac{-2v}{1-v^2}dv = \int \frac{1}{x}dx$$

$\therefore$ - log |1 - v^2| = log x + log c_1 ....

$$\left[\because \frac{d}{dv}\left(1-v^2\right) = -2v \text{ and } \int \frac{f\prime(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$$\therefore \log\left|\frac{1}{1-v^2}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{1}{1-\left(\frac{y^2}{x^2}\right)}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{x^2}{x^2-y^2}\right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2-y^2} = c_1 x$$

$$\therefore x^2 - y^2 = \frac{1}{c_1}x$$

$$\therefore x^2 - y^2 = cx, \text{ where } c = \frac{1}{c_1}$$

This is the general solution.

Q33. Find the second order derivatives of the following : e^{4x}. cos 5x

Solution

Let y = e^{4x}. cos 5x

Then $\frac{dy}{dx} = \frac{d}{dx}\left(e^{4x}. \cos 5x\right)$

= $e^{4x}. \frac{d}{dx}(\cos 5x) + \cos 5x. \frac{d}{dx}\left(e^{4x}\right)$

= $e^{4x}. (-\sin 5x). \frac{d}{dx}(5x) + \cos 5x \times e^{4x}. \frac{d}{dx}(4x)$

= – e^{4x} . sin 5x × 5 + e^{4x} cos 5x × 4

= e^{4x} (4 cos 5x – 5 sin 5x)

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{4x}(4\cos 5x - 5\sin 5x)\right]$

$= e^{4x}\frac{d}{dx}(4\cos 5x - 5\sin 5x) +$

$(4\cos 5x - 5\sin 5x). \frac{d}{dx}\left(e^{4x}\right)$

$= e^{4x}\left[4(-\sin 5x). \frac{d}{dx}(5x) - 5\cos 5x. \frac{d}{dx}(5x)\right] +$

$(4\cos 5x - 5\sin 5x) \times e^{4x}. \frac{d}{dx}(4x)$

= e^{4x} [– 4 sin 5x × 5 – 5 cos 5x × 5] + (4 cos 5x – 5 sin 5x) e^{4x} × 4

= e^{4x}(– 20 sin 5x – 25 cos 5x + 16 cos 5x – 20 sin 5x)

= e^{4x} (– 9 cos 5x – 40 sin 5x)

= – e^{4x} (9 cos 5x + 40 sin 5x)

Q34. If | x | < 1, then prove that

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Solution

Let $\tan^{-1}x = y$

Then, $x = \tan y$

Now, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan y}{1-\tan^2 y}\right)$

$= \tan^{-1}(\tan 2y)$

$= 2y$

$= 2\tan^{-1}x$ ......(1)

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan y}{1+\tan^2 y}\right)$

$= \sin^{-1}(\sin 2y)$

$= 2y$

$= 2\tan^{-1}x$ ......(2)

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)$

$= \cos^{-1}(\cos 2y)$

$= 2y$

$= 2\tan^{-1}x$ ......(3)

From (1), (2) and (3), we get

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$