Chapter – 5

Binomial Theorem, Sequences and Series

Ex 5.1

Question 1.

Expand (i) $\left(2x^2 - \frac{3}{x}\right)^3$ **(ii)** $\left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$

Solution:

(i) $\left(2x^2 - \frac{3}{x}\right)^3 = {}^3C_0\,(2x^2)^3 + {}^3C_1\,(2x^2)^2\left(-\frac{3}{x}\right) + {}^3C_2\,(2x^2)^1\left(-\frac{3}{x}\right)^2 + {}^3C_3\left(-\frac{3}{x}\right)^3$

$${}^3C_0 = {}^3C_3 = 1\ ;\ {}^3C_1 = {}^3C_2 = 3$$

$$= 1(8)(x^6) + 3(4x^4)\left(-\frac{3}{x}\right) + 3(2x^2)\left(\frac{9}{x^2}\right) + 1\left(-\frac{27}{x^3}\right)$$

$$= 8x^6 - \frac{36x^4}{x} + \frac{54x^2}{x^2} - \frac{27}{x^3}$$

$$= 8x^6 - 36x^3 + 54 - \frac{27}{x^3}$$

(ii) Taking $2x^2$ as a and $3\sqrt{1-x^2}$ as b we have $(a-b)^4 + (a+b)^4$

Now $(a-b)^4 = {}^4C_0\,a^4 + {}^4C_1\,a^3(-b) + {}^4C_2\,(a^2)(-b)^2 + {}^4C_3(a)(-b)^3 + {}^4C_4(-b)^4$

$${}^4C_0 = 1 = {}^4C_4\ ;\ {}^4C_1 = 4 = {}^4C_3\ ;\ {}^4C_2 = \frac{4\times3}{2\times1} = 6$$

$$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

Similarly $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\therefore (a - b)^4 + (a + b)^4 = 2\,[a^4 + 6a^2b^2 + b^4]$

Substituting the value of a and b we get

$$2\left[(2x^2)^4 + 6(2x^2)^2\left(3\sqrt{1-x^2}\right)^2 + \left(3\sqrt{1-x^2}\right)^4\right]$$

$= 2[16x^8 + 6(4x^4)(9(1 - x^2)) + 81(1 - x^2)^2]$

$= 2[16x^8 + 216x^4(1 - x^2) + 81(1 - x^2)^2]$

$= 2[16x^8 + 216x^4 - 216x^6 + 81 + 81x^4 - 162x^2]$

$= 2[16x^8 – 216x^6 + 297x^4 – 162x^2 + 81]$
$= 32x^8 – 432x^6 + 594x^4 – 324x^2 + 162$

Question 2.
Compute
(i) 102^4
(ii) 99^4
(iii) 9^7

Solution:
(i) $102^4 = (100 + 2)^4 = (10^2 + 2)^4$

$$= {}^4C_0\,(10^2)^4 + {}^4C_1\,(10^2)^3(2) + {}^4C_2\,(10^2)^2\,(2)^2 + {}^4C_3\,(10^2)^1(2)^3 + {}^4C_4\,(2)^4$$

$${}^4C_0 = 1 = {}^4C_4\,;\ {}^4C_1 = 4 = {}^4C_3\,;\ {}^4C_2 = \frac{4 \times 3}{2 \times 1} = 6$$

$= 1(10^8) + 4(10^6)(2) + 6(10^4)(4) + 4(10^2)(8) + 16$
$= 100000000 + 8000000 + 240000 + 3200 + 16$
$= 108243216$

(ii) $994 = (100 – 1)^4 = (10^2 – 1)^4$

$$= {}^4C_0\,(10^2)^4 + {}^4C_1\,(10^2)^3(-1)^1 + {}^4C_2\,(10^2)^2\,(-1)^2 + {}^4C_3\,(10^2)^1\,(-1)^3 + {}^4C_4\,(-1)^4$$

$= 1(10^8) + 4(10^6)(-1) + 6\,(10^4)(1) + 4(\,10^4)(-1) + (-1)^4$
$= 100000000 – 4000000 + 60000 – 400 + 1$
$= 100060001 – 4000400 = 96059601$

(iii) $9^7 = (10 – 1)^7$

$$= {}^7C_0\,(10^7) + {}^7C_1\,(10^6)\,(-1)^1 + {}^7C_2\,(10)^5\,(-1)^2 + {}^7C_3\,(10)^4\,(-1)^3 + {}^7C_4\,(10)^3\,(-1)^4$$
$$+ {}^7C_5\,(10)^2\,(-1)^5 + {}^7C_6\,(10)^1\,(-1)^6 + {}^7C_7\,(-1)^7$$

$${}^7C_0 = 1 = {}^7C_7\,;\ {}^7C_1 = 7 = {}^7C_6\,;\ {}^7C_2 = \frac{7 \times 6}{2 \times 1} = 21 = {}^7C_5,\ {}^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35 = {}^7C_4$$

$= 1(10000000) + 7(1000000)(-1) + 21(100000)(1) + 35(10000)(-1) +$
$35(1000)(1) + 21(100)(-1) + 7(10)(1) + 1(-1)$
$= 10000000 – 7000000 + 2100000 – 350000 + 35000 – 2100 + 70 – 1$
$= 12135070 – 7352101 = 4782969$

Question 3.
Using binomial theorem, indicate which of the following two number is larger:

$(1.01)^{1000000}$, 10000.

Solution:

$(1.01)^{1000000} = (1 + 0.01)^{1000000}$

$$= {}^{1000000}C_0(1)^{1000000} + {}^{1000000}C_1(1)^{999999}(0.01)^1$$
$$+ {}^{1000000}C_2(1)^{999998}(0.01)^2 + {}^{1000000}C_3(1)^{999997}(0.01)^3 + \dots\dots\dots$$
$$= 1\,(1) + 1000000 \times \frac{1}{10^2} + \frac{1000000 \times 999999}{2} \times \frac{1}{10000} + \dots\dots\dots$$
$$= 1 + 10000 + 50 \times 999999 + \dots\dots$$

which is > 10000

So $(1.01)^{1000000} > 10000$ (i.e.) $(1.01)^{1000000}$ is larger

Question 4.

Find the coefficient of x^{15} in $\left(x^2 + \frac{1}{x^3}\right)^{10}$.

Solution:

General term $T_{r+1} = {}^{10}C_r\,(x^2)^{10-r}\left(\frac{1}{x^3}\right)^r$.

$$= {}^{10}C_r\,x^{20-2r}\,\frac{1}{x^{3r}} = {}^{10}C_r\,x^{20-2r}.\,x^{-3r}$$
$$= {}^{10}C_r\,x^{20-5r}$$

To find a coefficient of x^{15} we have to equate x power to 15

i.e. $20 - 5r = 15$

$20 - 15 = 5r \Rightarrow 5r = 5 \Rightarrow r = 5/5 = 1$

So the coefficient of x^{15} is ${}^{10}C_1 = 10$

Question 5.

Find the coefficient of x^6 and the coefficient of x^2 in $\left(x^2 - \frac{1}{x^3}\right)^6$.

Solution:

General term $T_{r+1} = {}^6C_r (x^2)^{6-r} \left(\frac{-1}{x^3}\right)^r$.

$$= {}^6C_r\, x^{12-2r} (-1)^r \frac{1}{x^{3r}}$$

$$= {}^6C_r (-1)^r x^{12-2r-3r} = {}^6C_r (-1)^r x^{12-5r}$$

To find coefficient of x^6

$12 - 5r = 6$

$12 - 6 = 5r \Rightarrow 5r = 6 \Rightarrow r = 6/5$ which is not an integer.

$\therefore$ There is no term involving x^6.

To find coefficient of x^2

$12 - 5r = 2$

$5r = 12 - 2 = 10 \Rightarrow r = 2$

So coefficient of x^2 is ${}^6C_2 (-1)^2 = \frac{6 \times 5}{2 \times 1} (1) = 15$

Question 6.

Find the coefficient of x^4 in the expansion of $(1 + x^3)^{50} \left(x^2 + \frac{1}{x}\right)^5$.

Solution:

$$\left(x^2 + \frac{1}{x}\right)^5 = {}^5C_0 (x^2)^5 + {}^5C_1 (x^2)^4 \left(\frac{1}{x}\right) + {}^5C_2 (x^2)^3 \left(\frac{1}{x}\right)^2$$

$$+ {}^5C_3 (x^2)^2 \left(\frac{1}{x}\right)^3 + {}^5C_4 (x^2) \left(\frac{1}{x}\right)^4 + {}^5C_5 \left(\frac{1}{x}\right)^5$$

$${}^5C_0 = 1 = {}^5C_5 \;;\; {}^5C_1 = 5 = {}^5C_4 \;;\; {}^5C_2 = \frac{5 \times 4}{2 \times 1} = 10 = {}^5C_3$$

$$= x^{10} + 5(x^8)\left(\frac{1}{x}\right) + 10(x^6)\left(\frac{1}{x^2}\right) + 10(x^4)\left(\frac{1}{x^3}\right) + 5(x^2)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$

$$= x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}$$

$$(1+x^3)^{50} = 50C_0 (1)^{50} (x^3)^0 + 50C_1 (1)^{49} (x^3)^1 + 50C_2 (1)^{48} (x^3)^2 + 50C_3 (1)^{47} (x^3)^3$$

$$+ \ldots\, 50C_{50} (1)^{\circ} (x^3)^{50}$$

$$50C_0 = 1 = 50C_{50},\; 50C_1 = 50,\; 50C_2 = 1225,\; {}^{50}C_3 = 19600$$

$$= (1) + (50)x^3 + 1225x^6\; 7600x^9 \ldots\, x^{150}$$

$$(1+x^3)^{50}(x^2+\frac{1}{x})^5 = 1+50x^3+1225x^6+19600\,x^9 \dots x^{150}) \times$$
$$(x^{10}+5x^7+10x^4+10x+\frac{5}{x^2}+\frac{1}{x^5})$$

when multiplying these terms, we get x^4 terms

$$= (1\times 10x^4)+(50x^3\times 10x)+(1225x^6\times\frac{5}{x^2})+(19600x^9\times\frac{1}{x^5})$$
$$= 10x^4+500x^4+6125x^4+19600x^4$$
$$= 26325x^4$$

∴ The co-eff of x^4 is 26325

Question 7.

Find the constant term of $\left(2x^3-\frac{1}{3x^2}\right)^5$.

Solution:

General term $T_{r+1} = {}^5C_r\,(2x^3)^{5-r}\left(\frac{-1}{3x^2}\right)^r$

$$= {}^5C_r\,2^{5-r}\,x^{15-3r}\,(-1)^r\,\frac{1}{3^r}\,\frac{1}{x^{2r}}$$
$$= {}^5C_r\,\frac{2^{5-r}}{3^r}\,(-1)^r\,x^{15-3r-2r}$$
$$= {}^5C_r\,(-1)^r\,\frac{2^{5-r}}{3^r}\,x^{15-5r}$$

To find the constant term

$$15-5r=0 \Rightarrow 5r=15 \Rightarrow r=3$$

∴ Constant term $= {}^5C_3\,(-1)^3\,\frac{2^{5-3}}{3^3}$

$$= \frac{5\times4\times3}{3\times2\times1}\,(-1)\,\frac{(2^2)}{3^3} = \frac{10(-1)(4)}{27} = \frac{-40}{27}$$

Question 8.

Find the last two digits of the number 3^{600}.

Solution:

$3^{600} = 3^{2\times300} = (9)300 = (10-1)^{300}$

$$= {}^{300}C_0\,(10)^{300}(-1)^0 - {}^{300}C_1\,(10)^{299}(-1)^1 + {}^{300}C_2\,(10)^{298}(-1)^2$$
$$+ {}^{300}C_3\,(10)^{297}(-1)^3 + \ldots\ldots + {}^{300}C_{299}\,(10)^1(-1)^{299} + {}^{300}C_{300}(-1)^{300}$$
$$= 10^{300} - 300 \times 10^{299} + \ldots\ldots - 300(10) + 1\,(1)$$

All the terms except last term are ÷ by 100. So the last two digits will be 01.

Question 9.
If n is a positive integer, show that, $9^{n+1} - 8n - 9$ is always divisible by 64.

Solution:

$$9^{n+1} = (1+8)^{n+1} = {}^{(n+1)}C_0\,(1) + {}^{(n+1)}C_1\,(1)^n\,(8)^1 + {}^{(n+1)}C_2\,(8)^2$$
$$+ {}^{(n+1)}C_3\,(8)^3 + \ldots\ldots$$
$$= 1 + (n+1)\,8 + \frac{(n+1)\,n}{2!}\,(8^2) + \ldots\ldots$$
$$(i.e.)\ 9^{n+1} = 1 + 8n + 8 + \frac{(n+1)\,n}{2!}\,(8^2) + \frac{(n+1)\,(n)\,(n-1)}{3!}\,(8^3) + \ldots\ldots$$

∴ $9^{n+1} - 8n - 9 = 64$ [an integer]
⇒ $9^{n+1} - 8n - 9$ is divisible by 64

Question 10.
If n is an odd positive integer, prove that the coefficients of the middle terms in the expansion of $(x + y)^n$ are equal.

Solution:
Given n is odd. So let n = 2n + 1, where n is an integer.
The expansion $(x + y)^n$ has n + 1 terms.
= 2n + 1 + 1 = 2(n + 1) terms which is an even number.

So the middle term are $\frac{t_2(n+1)}{2} = t_{n+1}$ $t_{2(n+1)} = t_{n+1}$ and $t_{n+1+1} = t_{n+2}$

(*i.e.*) The middle terms are t_{n+1} and t_{n+2}

$$t_{n+1} = {}^{2n+1}C_n \text{ and } t_{n+2} = t_{n+1+1} = {}^{2n+1}C_{n+1}$$

Now $n + n + 1 = 2n + 1$

$$\Rightarrow {}^{2n+1}C_n = {}^{2n+1}C_{n+1}$$

⇒ The coefficient of the middle terms in $(x + y)^n$ are equal.

Question 11.
If n is a positive integer and r is a non – negative integer, prove that the coefficients of x^r and x^{n-r} in the expansion of $(1 + x)^n$ are equal.

Solution:

In $(1+x)^n$ the general term is $t_{r+1} = {}^nC_r x^r$

$\therefore$ Coefficient of x^r is ${}^nC_r$ and coefficient of x^{n-r} is ${}^nC_{n-r}$.

Now ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ ....(1)

And ${}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-\overline{n-r})!(n-n+r)}$

$= \dfrac{n!}{(n-r)!r!}$ ....(2)

(1) = (2) (i.e.) ${}^nC_r = {}^nC_{n-r}$

$\Rightarrow$ The coefficient of x^r and x^{n-r} are equal.

Question 12.
If a and b are distinct Integers, prove that a – b is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a - b + b)^n$ and expand]

Solution:

$a = a - b + b$

So, $a^n = [a-b+b]^n = [(a-b)+b]^n$

$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b^1 + {}^nC_2(a-b)^{n-2}b^2 + \ldots\ldots + {}^nC_{n-1}(a-b)\,b^{n-1} + {}^nC_n(b^n)$

$\Rightarrow a^n - b^n = (a-b)^n + {}^nC_1(a-b)^{n-1}b + {}^nC_2(a-b)^{n-2}b^2 + \ldots\ldots + {}^nC_{n-1}(a-b)\,b^{n-1}$

$= (a-b)\,[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + {}^nC_2(a-b)^{n-3}b^2 + \ldots\ldots + {}^nC_{n-1}\,b^{n-1}]$

= (a – b)[a^n integer]
$\Rightarrow a^n - b^n$ is divisible by (a – b)

Question 13.
In the binomial expansion of $(a + b)a^n$, the coefficients of the 4^{th} and 13^{th} terms are equal to each other, find n.
Solution:

In the expansion of $(a + b)^n$,
The general term is $T_{r+1} = nC_r \,.\, a^{n-r} \,.\, b^r$ ……….. (1)
To find the coefficient of 4th term, Put r = 3 in equation (1)
$\therefore T_{3+1} = nC_3\, a^{n-3} \,.\, b^3$
To find the coefficient of 13th term , Put r = 12 in equation (1)
$\therefore T_{12+1} = nC_{12}\, a^{n-12} \,.\, b^{12}$
Given $nC_3 = nC_{12}$
$nC_x = nC_y \Rightarrow x = y$ or $x + y = n$
$\therefore 3 + 12 = n \Rightarrow n = 15$

Question 14.
If the binomial coefficients of three consecutive terms in the expansion of $(a + x)^n$ are in the ratio 1 : 7 : 42, then find n.

Solution:
In $(a + x)^n$ general term is $t_{r+1} = {}^nC_r$
So, the coefficient of t_{r+1} is ${}^nC_r$
We are given that the coefficients of three consecutive terms are in the ratio 1 : 7 : 42.

$$\Rightarrow {}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42$$

$$(i.e.)\ \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7} \qquad \ldots(1)$$

$$\text{and } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{7}{42} = \frac{1}{6} \qquad \ldots(2)$$

$$(1) \Rightarrow \frac{n!}{(r-1)!(n-\overline{r-1})!} \Big/ \frac{n!}{r!(n-r)!} = \frac{1}{7}$$

$$(i.e.)\ \frac{\cancel{n}!}{(r-1)!(n+1-r)!} \times \frac{r!(n-r)!}{\cancel{n}!} = \frac{1}{7}$$

$$\Rightarrow \frac{r}{n+1-r} = \frac{1}{7} \Rightarrow 7r = n+1-r \Rightarrow 8r - n = 1 \rightarrow (A)$$

$$(2) \Rightarrow \frac{n!}{r!(n-r)!} \Big/ \frac{n!}{(r+1)!(n-\overline{r+1})![n-r-1]} = \frac{1}{6}$$

$$(i.e.)\ \frac{\cancel{n}!}{r!(n-r)!} \times \frac{(r+1)!(n-r-1)!}{\cancel{n}!} = \frac{1}{6}$$

$$\frac{(r+1)}{n-r} = \frac{1}{6}$$

$$n - r = 6r + 6$$

$$n - 7r = 6 \qquad \rightarrow (B)$$

Solving (A) and (B)

$$-n + 8r = 1 \qquad \rightarrow (A)$$

$$n - 7r = 6 \qquad \rightarrow (B)$$

$$(A) + (B) \Rightarrow \qquad r = 7$$

Substituting $r = 7$ in (B)

$$n = 6 + 7 \times 7$$

$$n = 6 + 49 = 55$$

Question 15.
In the binomial coefficients of $(1 + x)^n$, the coefficients of the 5th, 6th and 7 terms are in AP. Find all values of n.

Solution:
Coefficients of T_5, T_6, T_7, are in A.P.

$\therefore\ ^nC_4, ^nC_5, ^nC_6,$ are in A.P.

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{4.3.2.1}, \frac{n(n-1)(n-2)(n-3)(n-4)}{5.4.3.2.1},$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6.5.4.3.2.1} \text{ are in A.P.}$$

Multiplying each term by $\frac{4.3.2.1}{n(n-1)(n-2)(n-3)}$, we get

$\Rightarrow 1, \frac{n-4}{5}, \frac{(n-4)(n-5)}{6.5}$ are in A.P.

$\Rightarrow 1, \frac{n-4}{5}, \frac{n^2-9n+20}{30}$ are in A.P.

$\therefore \frac{n-4}{5} - 1 = \frac{n^2-9n+20}{30} - \frac{n-4}{5}$

$\Rightarrow \frac{n-9}{5} = \frac{n^2-15n+44}{30}$

$\Rightarrow 6(n-9) = n^2 - 15n + 44$

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow (n-1)(n-14) = 0$

$\therefore n = 7, 14$

Question 16.

Prove that $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{(n!)^2}$.

Solution:

We know $C_0 + C_1 + C_2 + \ldots\ldots\ldots C_n = 2^n$

and $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots\ldots\ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$

Taking $r = 0$ we get

$C_0C_0 + C_1C_1 + C_2C_2 + \ldots\ldots\ldots + C_nC_n = {}^{2n}C_n$

(*i.e.*) $C_0{}^2 + C_1{}^2 + C_2{}^2 + \ldots\ldots\ldots\ldots + C_n{}^2 = {}^{2n}C_n = \frac{2n!}{n!(2n-n)!} = \frac{2n!}{n!n!} = \frac{2n!}{(n!)^2}$

Ex 5.2

Question 1.
Write the first 6 terms of the sequences whose n^{th} terms are given below and classify them as arithmetic progression, geometric progression, arithmetico-geometric progression, harmonic progression and none of them.

(i) $\frac{1}{2^{n+1}}$ (ii) $\frac{(n+1)(n+2)}{n+3(n+4)}$ (iii) $4\left(\frac{1}{2}\right)^n$

(iv) $\frac{(-1)^n}{n}$ (v) $\frac{2n+3}{3n+4}$ (vi) $\frac{3n-2}{3^{n-1}}$

Solution:

(i) $t_n = \frac{1}{2^{n+1}}$

$$t_1 = \frac{1}{2^{1+1}} = \frac{1}{2^2}\ ;\ t_2 = \frac{1}{2^3}\ ;\ t_3 = \frac{1}{2^4}\ ;\ t_4 = \frac{1}{2^5}\ ;\ t_5 = \frac{1}{2^6}\ ;\ t_6 = \frac{1}{2^7}$$

So, the first six terms are $\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}$ and $\frac{1}{2^7}$ which is a G.P. with $a = \frac{1}{2^2}$ and $r = \frac{1}{2}$.

(ii) $t_n = \frac{(n+1)(n+2)}{n+3(n+4)}$

$$t_1 = \frac{(1+1)(1+2)}{1+3(1+4)} = \frac{2 \times 3}{1+15} = \frac{6}{16} = \frac{3}{8}$$

$$t_2 = \frac{(3)(4)}{2+3(6)} = \frac{12}{2+18} = \frac{12}{20} = \frac{3}{5}$$

$$t_3 = \frac{(4)(5)}{3+3(7)} = \frac{20}{24} = \frac{5}{6}$$

$$t_4 = \frac{(5)(6)}{4+3(8)} = \frac{30}{4+24} = \frac{30}{28} = \frac{15}{14}$$

$$t_5 = \frac{(6)(7)}{5+3(9)} = \frac{42}{32} = \frac{21}{16}$$

$$t_6 = \frac{(7)(8)}{6+3(10)} = \frac{56}{36} = \frac{14}{9}$$

So the first six terms are $\frac{3}{8}, \frac{3}{5}, \frac{5}{6}, \frac{15}{14}, \frac{21}{16}, \frac{14}{9}$.

It is not a G.P. or A.P. or H.P. or A.G.P.

(iii) $t_n = 4\left(\frac{1}{2}\right)^n$

$t_1 = 4\left(\frac{1}{2}\right)^1 = 2$

$t_2 = 4\left(\frac{1}{2}\right)^2 = 4 \times \frac{1}{4} = 1$

$t_3 = 4\left(\frac{1}{2}\right)^3 = \frac{4}{8} = \frac{1}{2}$; $t_4 = 4\left(\frac{1}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}$

$t_5 = 4\left(\frac{1}{2}\right)^5 = \frac{4}{32} = \frac{1}{8}$; $t_6 = 4\left(\frac{1}{2}\right)^6 = \frac{4}{64} = \frac{1}{16}$

So the first six terms are 2, 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$ which is a G.P. with $a = 2$ and $r = \frac{1}{2}$.

(iv) $t_n = \frac{(-1)^n}{n}$

$t_1 = \frac{(-1)^1}{1} = -1$; $t_2 = \frac{(-1)^2}{2} = \frac{1}{2}$

$t_3 = \frac{(-1)^3}{3} = -\frac{1}{3}$; $t_4 = \frac{(-1)^4}{4} = \frac{1}{4}$

$t_5 = \frac{(-1)^5}{5} = -\frac{1}{5}$; $t_6 = \frac{(-1)^6}{6} = \frac{1}{6}$

So the first 6 terms are $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}$.

It is not an A.P. or G.P. or H.P. or A.G.P

(v) $t_n = \frac{2n+3}{3n+4}$

$t_1 = \frac{2+3}{3+4} = \frac{5}{7}$; $t_2 = \frac{4+3}{6+4} = \frac{7}{10}$

$t_3 = \frac{6+3}{9+4} = \frac{9}{13}$; $t_4 = \frac{8+3}{12+4} = \frac{11}{16}$

$t_5 = \frac{10+3}{15+4} = \frac{13}{19}$; $t_6 = \frac{12+3}{18+4} = \frac{15}{22}$

So the first 6 terms are $\frac{5}{7}, \frac{7}{10}, \frac{9}{13}, \frac{11}{16}, \frac{13}{19}$ and $\frac{15}{22}$.

It is not an A.P. or G.P. or H.P. or A.G.P.

(vi) $t_n = \frac{3n-2}{3^{n-1}}$

$t_1 = \frac{3-2}{3^0} = 1$; $t_2 = \frac{6-2}{3^1} = \frac{4}{3}$

$t_3 = \frac{9-2}{3^2} = \frac{7}{9}$; $t_4 = \frac{12-2}{3^3} = \frac{10}{27}$

$t_5 = \frac{15-2}{3^4} = \frac{13}{81}$; $t_6 = \frac{18-2}{3^5} = \frac{16}{243}$

So the first 6 terms are $1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \frac{13}{81}$ and $\frac{16}{243}$

i.e., $\frac{1}{3^0}, \frac{4}{3^1}, \frac{7}{3^2}, \frac{10}{3^3}, \frac{13}{3^4}, \frac{16}{3^5}$.

It is a A.G.P.

Question 2.

Write the first 6 terms of the sequences whose n^{th} term a_n is given below.

(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

Solution:

$a_1 = 1 + 1 = 2$; $a_2 = 2$

$a_3 = 3 + 1 = 4$; $a_4 = 4$

$a_5 = 5 + 1 = 6$; $a_6 = 6$

So, the first 6 terms are 2, 2, 4, 4, 6, 6

(ii) $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

Solution:

$a_1 = 1$; $a_2 = 2$, $a_3 = 3$

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 \Rightarrow a_4 = 6$

$a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11 \Rightarrow a_5 = 11$

$a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20 \Rightarrow a_6 = 20$

So the first 6 terms are 1, 2, 3, 5, 8, 13.

$$(iii)\ a_n = \begin{cases} n & \text{if } n \text{ is } 1, 2 \text{ or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3 \end{cases}$$

Solution:

$a_1 = 1 \quad ; \quad a_2 = 2 \quad ; \quad a_3 = 3$

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 \quad \Rightarrow a_4 = 6$

$a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11 \quad \Rightarrow a_5 = 11$

$a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20 \Rightarrow a_6 = 20$

So the first 6 terms are 1, 2, 3, 6, 11, 20.

Question 3.

Write the n_{th} term of the following sequences.

Solution:

(i) 2, 2, 4, 4, 6, 6……

Solution:

$$a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

$(ii)\ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

Solution:

Nr: 1, 2, 3, ……$t_n = n$

Dr: 2, 3, 4, …..$t_n = n + 1$

So the n^{th} term is $t_n = \frac{n}{n+1}\ \forall n \in \mathrm{N}$

$(iii)\ \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \ldots$

Solution:

Nr: 1, 3, 5, 7, . . .which is an A.P. $a = 1$, $d = 3 – 1 = 2$

$t_n = a + (n – 1)d$

$t_n = 1 + (n – 1)2 = 1 + 2n – 2 = 2n – 1$.

Dr : 2, 4, 6, 8, . . .

So the n^{th} term is $2 + (n – 1)2 = 2 + 2n – 2 = 2n$.

$\therefore t_n = \frac{2n-1}{2n}, \forall n \in N$

(iv) 6, 10, 4, 12, 2, 14, 0, 16, -2,....

Solution:

$t_1 = 6$; $t_2 = 10$

$t_3 = 4$; $t_4 = 12$

$t_5 = 2$; $t_6 = 14$

$t_7 = 0$; $t_8 = 16$

When n is odd, the sequence is 6, 4, 2, 0,…

(i.e.) a = 6 and d = 4 – 6 = -2.

So, $t_n = 6 + (n - 1)(-2) = 6 - 2n + 2 = 8 - 2n$

When n is even, the sequence is 10, 12, 14, 16,…

Here a = 10 and d = 12 – 10 = 2

$t_n = 10 + (n - 1)2 = 10 + 2n - 2 = 2n + 8$ (i.e.) $8 + 2n$

$$\therefore t_n = \begin{cases} 8 - 2n & \text{when } n \text{ is odd} \\ 8 + 2n & \text{when } n \text{ is even} \end{cases} \quad \text{or} \quad t_n = \begin{cases} 7 - n & \text{when } n \text{ is odd} \\ 8 + n & \text{when } n \text{ is even} \end{cases}$$

Question 4.

The product of three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an AP. Find the numbers in GP.

Solution:

The 3 numbers in a G.P. is taken as ar, a, ar

Their product is 5832.

$$\Rightarrow \frac{a}{r} \times a \times ar = 5832 \ (i.e.)\ a^3 = 5832 = 18^3 \Rightarrow a = 18$$

$\therefore$ The 3 numbers are $\frac{18}{r}$, 18 , $18r$

When 6 added to t_2 and 9 added to t_3 we get $\frac{18}{r}$, 24 , $18r + 9$ which is an A.P.

$$\frac{18}{r}, 24, 18r + 9 \text{ are in A.P.}$$

$$\Rightarrow \quad 24 - \frac{18}{r} = 18r + 9 - 24$$

$$(i.e.) \quad 24 - 9 + 24 = 18r + \frac{18}{r}$$

$$18r + \frac{18}{r} = 39$$

$$(\div \text{ by } 3) \ \ 6r + \frac{6}{r} = 13$$

$6r^2 + 6 = 13$
$6r^2 - 13r + 6 = 0$
$(3r - 2)(2r - 3) = 0$
$r = 2/3$ or $3/2$

When $a = 18$ and $r = 2/3$, the G.P. is $18, 18 \times \frac{2}{3}, 18 \times \left(\frac{2}{3}\right)^2, \ldots\ldots$

$(i.e.)$ 18, 12, 8, . . .

When $a = 18$ and $r = 3/2$, the G.P. is $18, 18 \times \frac{3}{2}, 18 \times \frac{3}{2} \times \frac{3}{2}, \ldots\ldots$

$(i.e.)$ $18, 27, \frac{81}{2}, \ldots$

Question 5.

Write the n^{th} term of the sequence $\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2}, \ldots$ as a difference of two terms.

Solution:

$$t_1 = \frac{3}{1^2 2^2}, t_2 = \frac{5}{2^2 3^2}, t_3 = \frac{7}{3^2 4^2}$$

Nr : 3, 5, 7, . . . {A.P. $a = 3, d = 5 - 3 = 2$}

$$t_n = 3 + (n - 1)2 = 3 + 2n - 2 = 2n + 1$$

Dr : $1^2 2^2, 2^2 3^2, \ldots$

So $t_n = n^2 (n + 1)^2$

$$\therefore n^{th} \text{ term} = \frac{2n + 1}{n^2 (n + 1)^2} = \frac{(n + 1)^2 - n^2}{n^2 (n + 1)^2} = \frac{1}{n^2} - \frac{1}{(n + 1)^2}$$

Question 6.

If t_k is the k^{th} term of a G.P., then show that t_{n-k}, t_n, t_{n+k} also form a GP for any positive integer k.

Solution:

Let a be the first term and r be the common ratio.
We are given $t_k = ar^{k-1}$
We have to Prove : t_{n-k}, t_n, t_{n+k} form a G.P.

$$t_{n-k} = ar^{n-k-1}$$
$$t_n = ar^{n-1}$$
$$t_{n+k} = ar^{n+k-1}$$

Now $\frac{t_n}{t_{n-k}} = \frac{\cancel{a}r^{n-1}}{\cancel{a}r^{n-k-1}} = r^{n-1-n+k+1} = r^k$

Also $\frac{t_{n+k}}{t_n} = \frac{\cancel{a}r^{n+k-1}}{\cancel{a}r^{n-1}} = r^{n+k-1-n+1} = r^k$

Now $\frac{t_n}{t_{n-k}} = \frac{t_{n+k}}{t_n}$

$\Rightarrow$ t_{n-k}, t_n, t_{n+k} form a G.P.

Question 7.

If a, b, c are in geometric progression, and if $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, then prove that x, y, z are in arithmetic progression.

Solution:

Given a, b, c are in G.P.

$\Rightarrow b^2 = ac$

$\Rightarrow \log b^2 = \log ac$

(i.e.) $2\log b = \log a + \log c$ ...(1)

We are given $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$ (say)

$\Rightarrow$ $\log a^k = \frac{1}{x}$; $\log b^k = \frac{1}{y}$; $\log c^k = \frac{1}{z}$

$\Rightarrow$ $\log a^k = \frac{1}{x}$ $\Rightarrow$ $x = \log k^a$

Similarly $y = \log k^b$

$z = \log k^c$

Substituting these values in equation (1) we get $2y = x + z \Rightarrow x, y\ z$ are in A.P.

Question 8.

The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.

Solution:

Let the two numbers be a and b.

Their A.M. $= \dfrac{a+b}{2}$

G.M. $= \sqrt{ab}$ and H.M. $= \dfrac{2ab}{a+b}$

We are given A.M. = G.M.+ 10 = H.M.+ 16

$$(i.e.) \quad \frac{a+b}{2} = \sqrt{ab} + 10 \qquad \ldots(1)$$

$$\text{and} \quad \frac{a+b}{2} = \frac{2ab}{a+b} + 16 \qquad \ldots(2)$$

$$\text{from (2)} \quad \frac{a+b}{2} - \frac{2ab}{a+b} = 16$$

$$\Rightarrow \quad (a+b)^2 - 4ab = 16\,(2)\,(a+b)$$

$$(i.e.) \quad (a-b)^2 = 32\,(a+b) \qquad \ldots(3)$$

$$(1) \Rightarrow \frac{a+b}{2} = \sqrt{ab} + 10$$

$$\Rightarrow \quad a+b = 2\sqrt{ab} + 20$$

$$\Rightarrow \quad a+b-20 = 2\sqrt{ab}$$

So, $(a + b - 20)^2 = 4ab$

(i.e.) $(a + b)^2 + 400 - 40(a + b) = 4ab$

$(a + b)^2 - 4ab = 40(a + b) - 400$

from(3) $(a + b)^2 - 4ab = 32(a + b)$

$\Rightarrow 32(a + b) = 40(a + b) - 400$

($\div$ by 8) $4(a + b) = 5(a + b) - 50$

$4a + 4b = 5a + 5b - 50$

$a + b = 50$

$a = 50 - b$

Substituting $a = 50 - b$ in (3) we get

$(50 - b - b)^2 = 32(50)$

$(50 - 2b)^2 = 32 \times 50$

$$[2(25-b)]^2 = 32 \times 50$$

(*i.e.*) $$4(25-b)^2 = 32 \times 50$$

$$\Rightarrow \quad (25-b)^2 = \frac{32 \times 50}{4} = 8 \times 50 = 400 = 20^2$$

$$\Rightarrow \quad 25 - b = \pm 20$$

$25 - b = 20$	$25 - b = -20$
$\Rightarrow b = 25 - 20 = 5$	$b = 25 + 20 = 45$

When b = 5, a = 50 – 5 = 45
When b = 45, a = 50 – 45 = 5
So the two numbers are 5 and 45.

Question 9.
If the roots of the equation $(q - r)x^2 + (r - p)x + p - q = 0$ are equal, then show that p, q and r are in AP.

Solution:
The given quadratic equation is $(q - r)x^2 + (r - p)x + (p - q) = 0$
Given that the roots of the equation are equal. ∴ The discriminant is equal to zero.
$(r - p)^2 - 4(q - r)(p - q) = 0$
$r^2 - 2rp + p^2 - 4(pq - q^2 - rp + qr) = 0$
$r^2 - 2rp + p^2 - 4pq + 4q^2 + 4rp - 4qr = 0$
$r^2 + p^2 + 4q^2 + rp - 4pq - 4rq = 0$
$r^2 + p^2 + (-2q)^2 + 2r.p + 2p(-q) + 2(-2q)r = 0$
$(r + p - 2q)^2 = 0$
$r + p - 2q = 0 \Rightarrow 2q = p + r$
∴ p, q , r are in A. P.

Question 10.
If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a G.P., show that $(q - r) \log a + (r - p) \log b + (p - q) \log c = 0$.

Solution:
Let the G.P. be $l, lk, lk^2, \ldots$
We are given $t_p = a$, $t_q = b$, $t_r = c$

$\Rightarrow a = l k^{p-1}$; $b = l k^{q-1}$; $c = l k^{r-1}$

$$a = l k^{p-1} \Rightarrow \log a = \log l + \log k^{p-1} = \log l + (p-1) \log k$$
$$b = l k^{q-1} \Rightarrow \log b = \log l + \log k^{q-1} = \log l + (q-1) \log k$$
$$c = l k^{r-1} \Rightarrow \log c = \log l + \log k^{r-1} = \log l + (r-1) \log k$$

LHS = (q – r) log a + (r – p) log b + (p – q) log c

$= (q-r)[\log l + (p-1)\log k] + (r-p)[\log l + (q-1)\log k] +$
$(p-q)[\log l + (r-1)\log k]$

$= \log l\,[p-q+q-r+r-p] + \log k\,[(q-r)(p-1) + (r-p)(q-1) +$
$(p-q)(r-1)]$

$= \log l\,(0) + \log k\,[p(q-r) + q(r-p) + r(p-q) - (q-r+r-p+p-q)]$

$= 0 =$ RHS.

Ex 5.3

Question 1.
Find the sum of the first 20-terms of the arithmetic progression having the sum of first 10 terms as 52 and the sum of the first 15 terms as 77.

Solution:
Let 'a' be the first term and d be the common difference of A.P.

Let the A.P. be $a, a+d, a+2d, \ldots$ we are given $S_{10} = 52$ and $S_{15} = 77$.

(i.e.) $\frac{10}{2}\left[2a+(10-1)d\right] = 52$ and $\frac{15}{2}\left[2a+(15-1)d\right] = 77$

(i.e) $2a + 9d = 52 \times \frac{2}{10} = \frac{104}{10} = \frac{52}{5}$ and $2a + 14d = \frac{77 \times 2}{15} = \frac{154}{15}$

$$2a + 9d = \frac{52}{5} \qquad \ldots(1)$$

$2a + 14d = \frac{154}{15}$ ...(2)

$(1) - (2) \Rightarrow -5d = \frac{52}{5} - \frac{154}{15} = \frac{156-154}{15} = \frac{2}{15}$

$d = \frac{-2}{15 \times 5} = {}^{-2}\!/_{75}$

Substituting $d = \frac{-2}{75}$ in (1) we get $2a + 9\left(\frac{-2}{75}\right) = \frac{52}{5}$

$2a = \frac{52}{5} + \frac{18}{75} = \frac{52}{5} + \frac{6}{25} = \frac{260+6}{25} = \frac{266}{25}$

$\Rightarrow a = \frac{266}{25 \times 2} = \frac{133}{25}$

Now $a = \frac{133}{25}$ and $d = \frac{-2}{75}$

$\therefore S_{20} = \frac{20}{2}\left[2a + (20-1)d\right] = 10\left[2a + 19d\right] = 10\left[2\left(\frac{133}{25}\right) + 19\left(\frac{-2}{75}\right)\right]$

$= 10\left[\frac{266}{25} - \frac{38}{75}\right] = \frac{10}{75}\left[798 - 38\right] = \frac{10}{75}(760) = \frac{7600}{75} = \frac{304}{3}$

Question 2.

Find the sum up to the 17th term of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots$

Solution:

$t_1 = \frac{1^3}{1}; t_2 = \frac{1^3+2^3}{1+3}; t_3 = \frac{1^3+2^3+3^3}{1+3+5} \ldots$

$\therefore t_n = \frac{1^3+2^3+3^3+\ldots n^3}{1+3+5+\ldots . n \text{ terms}} = \frac{\Sigma n^3}{n^2} \quad \left(\frac{2n-1+1}{2}\right)^2$

$$=\frac{n^2(n+1)^2}{4(n^2)}=\frac{(n+1)^2}{4}=\frac{n^2+2n+1}{4}$$

$$\therefore S_n=\Sigma t_n=\frac{1}{4}\sum n^2+2n+1=\frac{1}{4}\Sigma n^2+\Sigma 2n+\Sigma 1$$

$$=\frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}+\frac{2(n)(n+1)}{2}+n\right]$$

To find S_{17} put $n = 17$

$$S_{17}=\frac{1}{4}\left[\frac{17\times 18\times 35}{6}+\frac{2(17)(18)}{2}+17\right]$$

$$=\frac{1}{4}\ [17\times 105+17\times 18+17]$$

$$=\frac{17(105+18+1)}{4}=17\times 31=527$$

Question 3.
Compute the sum of first n terms of the following series:
(i) 8 + 88 + 888 + 8888 + ……
(i) 6 + 66 + 666 + 6666 + …..

Solution:

(*i*) $S_n = 8 + 88 + 888 + \ldots\ n$ terms $= 8\ [1 + 11 + 111 + \ldots\ldots\ n$ terms$]$

$$=\frac{8}{9}\ [9+99+999+\ldots\ldots\ n \text{ terms}]$$

$$=\frac{8}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\ldots n \text{ terms}\right]$$

$$=\frac{8}{9}\left[10+10^2+\ldots m \text{ terms}-(1+1+1\ldots\ldots n \text{ terms})\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right] = \frac{8}{9}\left[\frac{10(10^n - 1)}{9} - n\right]$$

$$= \frac{80}{81}(10^n - 1) - \frac{8n}{9}$$

(ii) $S_n = 6 + 66 + 666 + \ldots\ldots n \text{ terms} = 6\,[1 + 11 + 111 + \ldots\ldots n \text{ terms}]$

$$= \frac{6}{9}\,[9 + 99 + 999 + \ldots\ldots n \text{ terms}]$$

$$= \frac{6}{9}\left[(10-1) + (10^2 - 1) + (10^3 - 1) + \ldots\ldots n \text{ terms}\right]$$

$$= \frac{6}{9}\left[10 + 10^2 + 10^3 + \ldots\ldots m \text{ terms} - (1 + 1 + 1 \ldots\ldots n \text{ terms})\right]$$

$$= \frac{6}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right] = \frac{6}{9}\left(\frac{10^n - 1}{9} - n\right)$$

$$= \frac{6}{9}\left[\frac{10(10^n - 1) - 9n}{9}\right] = \frac{6}{81}\left[10(10^n - 1) - 9n\right]$$

Question 4.

Compute the sum of first n terms of $1 + (1 + 4) + (1 + 4 + 4^2) + (1 + 4 + 4^2 + 4^3) + \ldots\ldots$

Solution:

$t_1 = 1, t_2 = 1 + 4, t_3 = 1 + 4 + 4^2, t_n = 1 + 4 + 4^2 + \ldots\ldots n \text{ terms}$ which is a G.P

$$\therefore t_n = \frac{1(4^n - 1)}{4-1} = \frac{4^n - 1}{3}$$

So $S_n = \Sigma t_n = \dfrac{\Sigma 4^n - 1}{\Sigma 3} = \dfrac{\Sigma 4^n - \Sigma 1}{\Sigma 3}$

$$\Sigma 4^n = 4^1 + 4^2 + \ldots\ldots n \text{ terms} = \frac{4(4^n - 1)}{4-1} = \frac{4(4^n - 1)}{3}$$

So $\dfrac{\Sigma 4^n - \Sigma 1}{\Sigma 3} = \dfrac{\dfrac{4(4^n-1)}{3} - n}{3} = \dfrac{4(4^n-1)-3n}{9} = \dfrac{4}{9}(4^n-1) - \dfrac{n}{3}$

Question 5.
Find the general term and sum to n terms of the sequence
$1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \ldots\ldots\ldots$

Solution:
$t_1 = 1, t_2 = \dfrac{4}{3}, t_3 = \dfrac{7}{9}, t_4 = \dfrac{10}{27}$
Numerator 1, 4, 7, 10, ... (A.P $a = 1$, $d = 4 - 1 = 3$)
So $t_n = a + (n-1)d = 1 + (n-1)3 = 1 + 3n - 3 = 3n - 2$
Denominator 1, 3, 9, 27, which is a G.P. a = 1, r = 3
So $t_n = ar^{n-1} = 1(3^{n-1}) = 3^{n-1}$
$\therefore t_n = \dfrac{3n-2}{3^{n-1}}$
It is an arithmetic Geometric series. Here the n^{th} term is $t_n = [a + (n-1)d]r^{n-1}$ where a = 1, d = 3 and r = 1/3
Now the sum to n terms is

$$S_n = \frac{a - [a+(n-1)d]r^n}{1-r} + dr\left(\frac{1-r^{n-1}}{(1-r)^2}\right)$$

$$= \frac{1-[1+(n-1)3]\frac{1}{3^n}}{1-\frac{1}{3}} + 3\times\frac{1}{3}\left(\frac{1-\frac{1}{3^{n-1}}}{\left(1-\frac{1}{3}\right)^2}\right)$$

$$= \frac{1-(3n-2)\frac{1}{3^n}}{\frac{2}{3}} + \frac{3^{n-1}-1}{3^{n-1}\left(\frac{2}{3}\right)^2} = \frac{3^n-(3n-2)}{2(3^{n-1})} + \frac{3^{n-1}-1}{4(3^{n-3})}$$

Question 6.
Find the value of *n*, if the sum to *n* terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\ldots\ldots$ **is** $435\sqrt{3}$.

Solution:

$$t_1 = \sqrt{3}, t_2 = \sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3}, t_3 = \sqrt{243} = \sqrt{81 \times 3} = 9\sqrt{3}$$

Here $t_1 = \sqrt{3}, t_2 = 5\sqrt{3}, t_3 = 9\sqrt{3}$

(i.e) $a = \sqrt{3}, d = 5\sqrt{3} - \sqrt{3} = 4\sqrt{3}$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = 435\sqrt{3} \text{ (given)}$$

$$\Rightarrow \frac{n}{2}\left[2\sqrt{3} + (n-1)4\sqrt{3}\right] = 435\sqrt{3}$$

$$\Rightarrow \frac{n\cancel{\sqrt{3}}}{2}[2 + 4n - 4] = 435\cancel{\sqrt{3}}$$

$$\Rightarrow n[4n - 2] = 870$$

$$4n^2 - 2n - 870 = 0$$

$$(\div by\ 2)\ 2n^2 - n - 435 = 0$$

$$2n^2 - 30n + 29n - 435 = 0 \Rightarrow 2n(n - 15) + 29(n - 15) = 0$$

$$(2n + 29)(n - 15) = 0 \Rightarrow n = \frac{-29}{2} \text{ or } 15$$

$n = \frac{-29}{2}$ not possible, So $n = 15$

Question 7.

Show that the sum of $(m + n)^{th}$ and $(m - n)^{th}$ term of an A.P. is equal to twice the m^{th} term.

Solution:

The n^{th} term of an A.P is $T_n = a + (n - 1) d$

$T_{m+n} = a + (m + n - 1) d$

$T_{m-n} = a + (m - n - 1) d$

$T_{m+n} + T_{m-n} = a + (m + n - 1) d + a + (m - n - 1)d$

$= 2a + [m + n - 1 + m - n - 1) d$

$= 2a + [2m - 2] d$

$= 2a + 2 (m - 1) d$

$= 2 [a + (m - 1] d$

$T_{m+n} + T_{m-n} = 2T_m$

Hence the sum of the (m + n)th term and (m – n)th term of A.P is equal to twice the mth term.

Question 8.
A man repays an amount of ₹ 3250 by paying ₹ 20 in the first month and then increases the payment by ₹ 15 per month. How long will it take him to clear the amount?

Solution:
$a = 20, d = 15, S_n = 3250$ to find n.

$$\text{Now } S_n = \frac{n}{2}\left[2a+(n-1)d\right] = 3250$$

$$\Rightarrow \frac{n}{2}\left[40+(n-1)15\right] = 3250$$

$$n\,[40 + 15n - 15] = 3250 \times 2$$

$$n\,[25 + 15n] = 6500$$

$$5n\,[5 + 3n] = 6500$$

$$\Rightarrow n(5+3n) = \frac{6500}{5} = 1300$$

$3n^2 + 5n - 1300 = 0$
$3n^2 - 60n + 65n - 1300 = 0$
$3n (n - 20) + 65 (n - 20) = 0$
$(3n + 65) (n - 20) = 0$
$n = -65/3$ or 20
$n = -65/3$ is not possible
$\therefore n = 20$
So he will take 20 months to clear the amount.

Question 9.
In a race, 20 balls are placed in a line at intervals of 4 meters with the first ball 24 meters away from the starting point. A contestant is required to bring the balls back to the starting place one at a time. How far would the contestant run to bring back all balls?

Solution:
$t_1 = 24 \times 2 = 48$, $t_2 = 48 + 8 = 56$ or $(24 + 4)2$, $t_3 = (28 + 4)2 = 64$ which is

an A.P.
Here a = 48,
d = 56 – 48 = 8

$\therefore S_{20} = \frac{20}{2}[2a + (20-1)d]$ = 10 [2(48) + 19 (8)] = 10 (96 + 152) = 10 (248) = 2480 m

The contestant has to run 2480 m to bring all the balls.

Question 10.
The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of the 2^{nd} hour, 4^{th} hour, and n^{th} hour?

Solution:
Number of bacteria present initially = 30
Number of bacteria at the end of 1 hour = 2 × 30 = 60
Number of bacteria at the end of two hours = 2 × 60 = 120
Number of bacteria at the end of three hours = 2 × 120 = 240
Number of bacteria at the end of four hours = 2 × 240 = 480
- -
∴ The sequence of a number of bacterias at the end of every hour is
30, 60, 120, 240, 480, …………………………..
30, 30 × 2, 30 × 4, 30 × 8, 30 × 16, …………………
30, 30 × 2, 30 × 2^2, 30 × 2^3, 30 × 2^4, …………………
This sequence is a Geometric sequence with first term a = 30, common ratio r = 2

Number of bacteria at the end of n^{th} hour $t_{n+1} = a.\ r^n$
$t_{n+1} = 30(2^n)$
Number of bacteria at the end of 2^{nd} hour = 120
Number of bacteria at the end of 4^{th} hours = 480
Number of bacteria at the end of n^{th} hour = $30(2^n)$

Question 11.
What will ₹ 500 amount to in 10 years after its deposit in a bank which pays an annual interest rate of 10% compounded annually?

Solution:

$P = ₹\ 500$, C.I = 10%, Amount after 1 year $= P\left(1+\frac{r}{100}\right) = 500 + \left(1+\frac{10}{100}\right) = 500\left(\frac{11}{10}\right)$

Amount after 2 years $= 500\left(\frac{11}{10}\right)^2$

Now $a = 500\left(\frac{11}{10}\right)$, $r = \dfrac{500\left(\frac{11}{10}\right)^2}{500\left(\frac{11}{10}\right)} = \frac{11}{10}$

$t_n = ar^{n-1}$, $t_{10} = 500\left(\frac{11}{10}\right)\left(\frac{11}{10}\right)^9 = 500\left(\frac{11}{10}\right)^{10} = 500\,(1.1)^{10} = 500\,(2.5937) = 1296.87$

Aliter:

Amount after 1 year $= 500\left(1+\frac{10}{100}\right) = 500\left(\frac{11}{10}\right)$

Amount after 2 years $= 500\left(\frac{11}{10}\right)\left(\frac{11}{10}\right) = 500\left(\frac{11}{10}\right)^2$

So amount after 10 years $= 500\left(\frac{11}{10}\right)^{10} = 1296.87$

Question 12.

In a certain town, a viral disease caused severe health hazards upon its people disturbing their normal life. It was found that on each day, the virus which caused the disease spread in Geometric Progression. The amount of infectious virus particle gets doubled each day, being 5 particles on the first day. Find the day when the infectious virus particles just grow over 1,50,000 units?

Solution:

$a = 5$, $r = 2$, $t_n > 150000$, To find 'n'

$t_n = ar^{n-1} = 150000$ (*i.e*) $5(2)^{n-1} = 150000$

$\Rightarrow 2^{n-1} = \frac{150000}{5} = 30000$

$2^{15} = 32768$ and $2^{14} = 16384$

We are given $2^{n-1} > 30000 \Rightarrow 2^{15} = 32768 > 30000 \Rightarrow n - 1 = 14 \Rightarrow n = 15$

On the 15th day it will grow over 150000 units.

Ex 5.4

Question 1.
Expand the following in ascending powers of x and find the condition on x for which the binomial expansion is valid.

Solution:

$$(i)\ \frac{1}{5+x}=\frac{1}{5\left(1+\frac{x}{5}\right)}=\frac{1}{5}\left(1+\frac{x}{5}\right)^{-1}=\frac{1}{5}\left\{1-\frac{x}{5}+\left(\frac{x}{5}\right)^2-\left(\frac{x}{5}\right)^3\ldots\right\}$$

Hence $\left|\frac{x}{5}\right|<1 \Rightarrow \therefore |x|<5$

$$=\frac{1}{5}-\frac{x}{5^2}+\frac{x^2}{5^3}-\frac{x^3}{5^4}\ldots$$

$$(ii)\ \frac{2}{(3+4x)^2}=\frac{2}{\left[3\left(1+\frac{4}{3}x\right)\right]^2}=\frac{2}{9\left(1+\frac{4}{3}x\right)^2}$$

$$=\frac{2}{9}\left(1+\frac{4}{3}x\right)^{-2}=\frac{2}{9}\left[1-2\left(\frac{4}{3}x\right)+3\left(\frac{4}{3}x\right)^2\ldots\right]$$

$$=\frac{2}{9}\left[1-\frac{8}{3}x+\frac{16}{9}x^2\ldots\right]$$

Hence $\left|\frac{4x}{3}\right|<1 \Rightarrow \therefore |x|<3/4$

$$(iii)\ (5+x^2)^{\frac{2}{3}}=\left\{5\left(1+\frac{x^2}{5}\right)\right\}^{2/3}$$

$$= 5^{\frac{2}{3}}\left[\left(1+\frac{x^2}{5}\right)^{2/3}\right] = 5^{2/3}\left\{1+\frac{2}{3}\left(\frac{x^2}{5}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2.1}\left(\frac{x^2}{5}\right)^2 \dots\right\}$$

$$= 5^{2/3}\left\{1+\frac{2x^2}{15}-\frac{\cancel{2}}{9\times\cancel{2}}\left(\frac{x^4}{25}\right)\dots\right\}$$

$$= 5^{\frac{2}{3}}\left\{1+\frac{2x^2}{15}-\frac{x^4}{225}\dots\right\}$$

Hence $\left|\frac{x^2}{5}\right| < 1 \Rightarrow |x^2| < 5$

(iv) $$(x+2)^{-\frac{2}{3}} = \frac{1}{(x+2)^{2/3}} = \frac{1}{2^{2/3}\left(1+\frac{x}{2}\right)^{2/3}} = 2^{-\frac{2}{3}}\left(1+\frac{x}{2}\right)^{-2/3}$$

$$= 2^{-2/3}\left(1-\frac{2}{3}\left(\frac{x}{2}\right)+\frac{\left(\frac{-2}{3}\right)\left(\frac{-2-1}{3}\right)\left(\frac{x}{2}\right)^2}{2!}-\dots\right)$$

$$= 2^{-2/3}\left(1-\frac{x}{3}+\frac{10}{18}\frac{x^2}{4}-\dots\right)$$

$$= 2^{-2/3}\left\{1-\frac{x}{3}+\frac{5x^2}{36}-\frac{5}{81}x^3\dots\right\}$$

Hence $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

Question 2.

Find $\sqrt[3]{1001}$ approximately (two decimal places).

Solution:

$$\sqrt[3]{1001}=(1001)^{1/3}=(1000+1)^{1/3}=\left\{1000\left(1+\frac{1}{1000}\right)\right\}^{1/3}=(1000)^{1/3}\left[1+\frac{1}{10^3}\right]^{1/3}$$

$$=10\left\{1+\frac{1}{3}\left(\frac{1}{10^3}\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{1}{10^3}\right)^2\dots\right\}$$

$$=10\left\{1+\frac{1}{3000}-\frac{2}{18000000}\dots\right\}=10[1+0.000333..]=10(1.000333)$$

$$=10.0033$$

Question 3.

Prove that $\sqrt[3]{x^3+6}-\sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficiently large.

Solution:

When x is large then $\frac{1}{x}$ will be small (*i.e.*) $\left|\frac{1}{x}\right|<1$

$$\text{So } \sqrt[3]{x^3+6}-\sqrt[3]{x^3+3}=\left(x^3+6\right)^{1/3}-\left(x^3+3\right)^{1/3}$$

$$=\left\{x^3\left(1+\frac{6}{x^3}\right)\right\}^{1/3}-\left\{x^3\left(1+\frac{3}{x^3}\right)\right\}^{1/3}$$

$$=x\left(1+\frac{6}{x^3}\right)^{1/3}-x\left(1+\frac{3}{x^3}\right)^{1/3}=x\left[1+\frac{1}{3}.\frac{6}{x^3}\dots\right]-x\left[1+\frac{1}{3}\cdot\frac{3}{x^3}\dots\right]$$

$$=\left(x+\frac{2}{x^2}\dots\right)-\left(x+\frac{1}{x^2}\dots\right)=x+\frac{2}{x^2}\dots\dots-x-\frac{1}{x^2}\dots.$$

$$=\frac{1}{x^2}\text{ (approximately)}$$

Question 4.

Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1-x+\frac{x^2}{2}$ when x is very small.

Solution:

$$\text{LHS} = \sqrt{\frac{1-x}{1+x}} = \frac{(1-x)^{1/2}}{(1+x)^{1/2}} = (1-x)^{1/2}(1+x)^{-1/2}$$

$$= \left(1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2\cdot 1}x^2 \ldots\ldots\right)\left(1 - \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2\cdot 1}x^2 \ldots\right) = \left(1 - \frac{x}{2} - \frac{x^2}{8}\ldots\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8}\ldots\right)$$

$$= 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} \ldots = \frac{1-2x}{2} + \frac{4x^2}{8} + \ldots$$

$$= 1 - x + \frac{x^2}{2} = \text{RHS}$$

Question 5.
Write the first 6 terms of the exponential series

(i) e^{5x}

(ii) e^{-2x}

(iii) $e^{\frac{x}{2}}$

Solution:

$$(i)\ e^x = 1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} \ldots$$

$$\text{So } e^{5x} = 1 + \frac{5x}{\angle 1} + \frac{(5x)^2}{\angle 2} + \frac{(5x)^3}{\angle 3} + \frac{(5x)^4}{\angle 4} + \ldots$$

$$= 1 + 5x + \frac{25x^2}{2} + \frac{125x^3}{6} + \frac{625x^4}{24} + \frac{3125}{120}x^5 + \frac{15625}{720}x^6 \ldots$$

$$= 1 + 5x + \frac{25x^2}{2} + \frac{125x^3}{6} + \frac{625x^4}{24} + \frac{625x^5}{24} + \frac{3125x^6}{144} \ldots$$

$$(ii)\quad e^{-2x} = 1 + \frac{(-2x)}{\angle 1} + \frac{(-2x)^2}{\angle 2} + \frac{(-2x)^3}{\angle 3} - \ldots$$

$$= 1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6} + \frac{16x^4}{24} - \frac{32x^5}{120} + \frac{64x^6}{720} - \ldots$$

$$= 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} + \frac{4x^6}{45} - \ldots$$

$$(iii)\quad e^{\frac{1}{2}x} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}x\right)^2}{\angle 2} + \frac{\left(\frac{1}{2}x\right)^3}{\angle 3} \ldots$$

$$= 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \ldots$$

Question 6.
Write the first 4 terms of the logarithmic series
(i) $\log(1 + 4x)$,
(ii) $\log(1 - 2x)$,
(iii) $\log\left(\frac{1+3x}{1-3x}\right)$
(iv) $\log\left(\frac{1-2x}{1+2x}\right)$.
Find the intervals on which the expansions are valid.

Solution:

$$(i)\ \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots,\quad \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots$$

$$\log(1 + 4x) = 4x - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} \ldots$$

Hence $|4x| < 1 \Rightarrow |x| < 1/4$

$$= 4x - \frac{16x^2}{2} + \frac{64x^3}{3} - \frac{256x^4}{4} \ldots$$

$$= 4x - 8x^2 + \frac{64}{3}x^3 - 64x^4 \ldots$$

(*ii*) $\log(1-2x) = -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} \ldots = -2x - \frac{4x^2}{2} - \frac{8x^3}{3} - \frac{16x^4}{4} \ldots$

$$= -2x - 2x^2 - \frac{8x^3}{3} - 4x^4$$

Hence $|2x| < 1 \Rightarrow |x| < 1/2$

(*iii*) $\log\left(\frac{1+3x}{1-3x}\right) = \log(1+3x) - \log(1-3x)$

$$= \left[3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} \ldots\right] - \left[-3x - \frac{(3x)^2}{2} - \frac{(3x)^3}{3} - \frac{(3x)^4}{4} \ldots\right]$$

$$= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} \ldots + 3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{3} + \ldots$$

$$= 2\left(3x + \frac{(3x)^3}{3} + \frac{(3x)^5}{5} + \frac{(3x)^7}{7} \ldots\right)$$

Hence $|3x| < 1 \Rightarrow |x| < 1/3$

(*iv*) $\log\left(\frac{1-2x}{1+2x}\right) = \log(1-2x) - \log(1+2x)$

$$= \left[-2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} \ldots\right] - \left[2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} \ldots\right]$$

$$= -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} \ldots \quad -2x + \frac{(2x)^2}{2} - \frac{(2x)^3}{3} + \frac{(2x)^4}{4} \ldots$$

$$= -2\left(2x + \frac{(2x)^3}{3} + \frac{(2x)^5}{5} + \frac{(2x)^7}{7} \ldots\right)$$

Hence $|2x| < 1 \Rightarrow |x| < 1/2$

Question 7.

If $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$ **then show that** $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$

Solution:

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

(i.e) $y = -\left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\right] = -\log(1 - x)$

(i.e) $y = -\log(1-x) = \log\frac{1}{1-x}$

So $\log \frac{1}{1-x} = y$

$$\Rightarrow \frac{1}{1-x} = e^y \Rightarrow 1 - x = \frac{1}{e^y} = e^{-y} \Rightarrow 1 - x = e^{-y} \Rightarrow 1 - e^{-y} = x$$

(i.e) $x = 1 - \left[1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} \ldots\right] = 1 - 1 + y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} \ldots$

(i.e) $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} \ldots$

Question 8.

If $p - q$ **is small compared to either** p **or** q**, then show that** $\sqrt[n]{\frac{p}{q}} \simeq \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q}$.
Hence find $\sqrt[8]{\frac{15}{16}}$.

Solution:

$$\text{RHS} = \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \frac{n(p+q)+(p-q)}{n(p+q)-(p-q)} = \frac{1+\frac{1}{n}\left(\frac{p-q}{p+q}\right)}{1-\frac{1}{n}\left(\frac{p-q}{p+q}\right)} = \frac{\left(1+\frac{p-q}{p+q}\right)^{\frac{1}{n}}}{\left(1-\frac{p-q}{p+q}\right)^{1/n}}$$

$$= \left(\frac{p}{q}\right)^{1/n} = \sqrt[n]{\frac{p}{q}} = \text{LHS}$$

To find $\sqrt[8]{\frac{15}{16}}$ we take $n = 8, p = 15, q = 16$.

So $\sqrt[8]{\frac{15}{16}} = \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \frac{9\times15+7\times16}{7\times15+9\times16} = \frac{135+112}{105+144} = \frac{247}{249} = 0.99196$

Question 9.

Find the coefficient of x^4 in the expansion of $\frac{3-4x+x^2}{e^{2x}}$.

Solution:

$$\frac{3-4x+x^2}{e^{2x}} = (3-4x+x^2)\ e^{-2x}$$

$$= (3-4x+x^2)\left[1+\frac{(-2x)}{1!}+\frac{(-2x)^2}{\angle 2}+\frac{(-2x)^3}{\angle 3}\dots\right]$$

Coefficient of x^4: $3\left[\frac{(-2)^4}{4!}\right]-4\left[\frac{(-2)^3}{3!}\right]+1\left(\frac{(-2)^2}{2!}\right)$

$$= 3\left[\frac{16}{24}\right]+(-4)\frac{(-8)}{6}+\frac{4}{2} = \frac{48}{24}+\frac{32}{6}+2$$

$$= 2+\frac{16}{3}+2 = \frac{6+16+6}{3} = \frac{28}{3}$$

Question 10.

Find the value of $\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right)$.

Solution:

$$S_{\infty}=\frac{1}{1}\left(1+\frac{1}{9}\right)+\frac{1}{3}\left(\frac{1}{9}+\frac{1}{9^2}\right)+\frac{1}{5}\left(\frac{1}{9^2}+\frac{1}{9^5}\right)+\frac{1}{7}\left(\frac{1}{9^3}+\frac{1}{9^7}\right)+\ldots$$

$$=\left(1+\frac{1}{3}.\frac{1}{9}+\frac{1}{5}.\frac{1}{9^2}+\ldots\right)+\left(\frac{1}{9}+\frac{1}{3}.\frac{1}{9^3}+\frac{1}{5}.\frac{1}{9^5}\right)\ldots$$

$$=\left(1+\frac{1}{3}.\frac{1}{3^2}+\frac{1}{5}.\frac{1}{3^4}+\frac{1}{7}.\frac{1}{3^6}+\ldots\right)+\left[\frac{1}{9}+\frac{1}{3}\left(\frac{1}{9}\right)^3+\frac{1}{5}\left(\frac{1}{9}\right)^5+\ldots\right]$$

Multiplying and dividing the I summation by 3 we get

$$3\left(\frac{1}{3}+\frac{1}{3}.\frac{1}{3^3}+\frac{1}{5}.\frac{1}{3^5}+\ldots\right)+\frac{1}{9}+\frac{1}{3}\left(\frac{1}{9}\right)^3+\ldots$$

$$=\left[3\left(\frac{1}{3}\right)+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\ldots\right]+\left[\frac{1}{9}+\frac{\left(1/9\right)^3}{3}+\frac{\left(1/9\right)^5}{5}+\ldots\right]$$

$$\left\{\text{We know } \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right\}$$

$$=3\times\frac{1}{2}\log\left(\frac{1+1/3}{1-1/3}\right)+\frac{1}{2}\log\left(\frac{1+1/9}{1-1/9}\right)$$

$$=\frac{1}{2}\left\{3\log\frac{4/3}{2/3}\right\}+\frac{1}{2}\log\frac{10/9}{8/9}=\frac{1}{2}\left[3\log 2+\log\frac{10}{8}\right]=\frac{1}{2}\left[\log 2^3+\log\frac{10}{8}\right]$$

$$=\frac{1}{2}\left[\log 8\cdot\frac{10}{8}\right]=\frac{1}{2}\log e^{10}\quad[\log a+\log b=\log ab]$$

Ex 5.5

Choose the correct or the most suitable answer:

Question 1.
The value of 2 + 4 + 6 + … + 2n is …..
(a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+1)}{2}$ (c) $\frac{2n(2n+1)}{2}$ (d) $n(n+1)$

Solution:
(d) n(n + 1)

Hint . $2+4+6+\ldots\ldots+2n=2(1+2+3+\ldots\, n)=2\left[\frac{n(n+1)}{2}\right]=n(n+1)$

Question 2.
The coefficient of x^6 in $(2 + 2x)^{10}$ is ……….
(a) ${}^{10}C_6$
(b) 2^6
(c) ${}^{10}C_6 2^6$
(d) ${}^{10}C_6 2^{10}$

Solution:
(d) ${}^{10}C_6 2^{10}$
Hint.
$t_{r+1} = 2^{10}({}^nC_r)$
To find coefficient of x_6 put r = 6
∴ coefficient of x_6 = 210 $[{}^{10}C_6]$

Question 3.
The coefficient of x^8y^{12} in the expansion of $(2x + 3y)^{20}$ is …….
(a) 0
(b) $2^8 3^{12}$
(c) $2^8 3^{12} + 2^{12} 3^8$
(d) ${}^{20}C_8 2^8 3^{12}$

Solution:

(d) ${}^{20}C_8 2^8 3^{12}$

Hint. $t_{r+1} = {}^{20}C_r (2x)^{20-r} (3y)^r = {}^{20}C_r 2^{20-r} (x)^{20-r} 3^r y^r \quad [{}^nC_r = {}^nC_{n-r}]$

To find coefficient of x^8 put $20 - r = 8 \Rightarrow r = 12$

To find coefficient of y^{12} put $r = 12 \therefore r = 12$

Coefficient $= {}^{20}C_{12} (2)^{20-12} 3^{12} = {}^{20}C_{12} 2^8 3^{12} = {}^{20}C_8 2^8 3^{12}$

Question 4.
If ${}^nC_{10} > {}^nC_r$ for all possible r, then a value of n is ……..
(a) 10
(b) 21
(c) 19
(d) 20

Solution:
(d) 20
Hint.
$20C_{10} > 20C_r$ for all possible value of r.
n = 20

Question 5.
If a is the arithmetic mean and g is the geometric mean of two numbers, then ……..
(a) $a \leq g$
(b) $a \geq g$
(c) $a = g$
(d) $a > g$

Solution:
(b) $a \geq g$
Hint. AM ≥ GM
$\therefore a \geq g$

Question 6.
If $(1 + x^2)^2 (1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots. + x^{n+4}$ and if a_0, a_1, a_2 are in AP, then n is ……
(a) 1

(b) 2
(c) 3
(d) 4

Solution:
(b or c)n = 2 or 3

Hint. $\left(1+x^2\right)^2(1+x)^n=\left(1+2x^2+x^4\right)\left(1+nx+\frac{n(n-1)}{2}x^2\ldots\right)$

$$=1+nx+\frac{n(n-1)}{2}x^2+2x^2\ldots.=a_0+a_1x+a_2x^2+\ldots \text{ (given)}$$

$$\Rightarrow a_0=1, a_1=n, a_2=\frac{n(n-1)}{2}+2 \,(i.e)\, a_2=\frac{n^2-n+4}{2}$$

Given a_0 ,a_1, a_2 are in A.P.

$$2a_1=a_0+a_2\Rightarrow 2n=1+\frac{n^2-n+4}{2}$$

$$\Rightarrow 2n=\frac{n^2-n+6}{2}\Rightarrow n^2-n+6=4n$$

(i.e) $n^2-5n+6=0\Rightarrow n=2$ or 3

Question 7.
If a, 8, b are in A.P, a, 4, b are in G.P, if a, x, b are in HP then x is ……
(a) 2
(b) 1
(c) 4
(d) 16

Solution:
(a) 2

Hint: a, 8, b are in AP $\Rightarrow \frac{a+b}{2}=8 \Rightarrow a+b=16$

a, 4, b are in GP $\Rightarrow ab=4^2=16$

Now a, x, b are in HP $\Rightarrow x=\frac{2ab}{a+b}=\frac{2(16)}{16}=2$

Question 8.

The sequence $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}+\sqrt{2}}, \frac{1}{\sqrt{3}+2\sqrt{2}}$... form an ....

(a) AP
(b) GP
(c) HP
(d) AGP
Solution:
(c) HP

Question 9.

The HM of two positive numbers whose AM and GM are 16, 8 respectively is ....

(a) 10
(b) 6
(c) 5
(d) 4

Solution:

(d) 4
Hint.
Let the two numbers be a and b

$$\text{AM} = \frac{a+b}{2} = 16 \Rightarrow a+b = 32$$

$$\text{GM} = \sqrt{ab} = 8 \Rightarrow ab = 64$$

$$\text{HM} = \frac{2ab}{a+b} = \frac{2\times 64}{32} = 4$$

Question 10.

If S_n denotes the sum of n terms of an AP whose common difference is d, the value of

$S_n - 2S_{n-1} + S_{n-2}$ is ......

(a) d
(b) 2d
(c) 4d
(d) d^2

Solution:
(a) d

Hint. $S_n - 2S_{n-1} + S_{n-2} = S_n - S_{n-1} - S_{n-1} + S_{n-2}$

$$= (S_n - S_{n-1}) - (S_{n-1} - S_{n-2})$$

Now $S_n = t_1 + t_2 + ...t_n$

$S_{n-1} = t_1 + t_2 + ...t_{n-1}$ and $S_{n-2} = t_1 + t_2 + ...t_{n-2}$

So $S_n - S_{n-1} = t_n$ and $S_{n-1} - S_{n-2} = t_{n-1}$

Now $(S_n - S_{n-1}) - (S_{n-1} - S_{n-2}) = t_n - t_{n-1}$

$= [a + (n-1)\,d] - [a + (n-1-1)\,d]$

$= a + nd - d - a - nd + 2d = d$

Question 11.
The remainder when 38^{15} is divided by 13 is …….
(a) 12
(b) 1
(c) 11
(d) 5

Solution:
(a) 12
Hint.
$(38)^{15} = (39 - 1)^{15}$
$= (39)^{15} - 15C_1(39)^{14} + 15C_2(39)^{13} + \ldots\ldots\ldots + 15C_{14}(39) - 1$

In the Binomial expansion, all the terms except the last term (- 1) are divisible by 13
∴ The remainder = 13 – 1 = 12

Question 12.
The n^{th} term of the sequence 1, 2, 4, 7, 11, …… is
(a) $n^3 + 3n^2 + 2n$
(b) $n^3 - 3n^2 + 3n$
(c) $\frac{n(n+1)(n+2)}{3}$
(d) $\frac{n^2 - n + 2}{2}$

Solution:

(d) $\frac{n^2-n+2}{2}$

Question 13.

The sum up to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\ldots$ is ....

Solution:

(a) $\sqrt{2n+1}$ (b) $\frac{\sqrt{2n+1}}{2}$ (c) $\sqrt{2n+1}-1$ (d) $\frac{\sqrt{2n+1}-1}{2}$

Solution:

(d) $\frac{\sqrt{2n+1}-1}{2}$

Hint. $\frac{1}{\sqrt{3}+1}=\frac{1}{\sqrt{3}+\sqrt{1}}\times\frac{\sqrt{3}-\sqrt{1}}{\sqrt{3}-1}=\frac{\sqrt{3}-1}{3-1}=\frac{\sqrt{3}-\sqrt{1}}{2}$

$$\frac{1}{\sqrt{5}+\sqrt{3}}=\frac{1}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{5}-\sqrt{3}}{5-3}=\frac{\sqrt{5}-\sqrt{3}}{2}$$

$$\text{So } \frac{1}{\sqrt{3}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\ldots=\frac{\sqrt{3}-\sqrt{1}}{2}+\frac{\sqrt{5}-\sqrt{3}}{2}+\ldots\left(\frac{\sqrt{2n+1}-\sqrt{2n+1}}{2}\right)$$

$$=\frac{\sqrt{2n+1}-1}{2}$$

Question 14.

The n^{th} term of the sequence $\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16}$ is ....

(a) 2^n-n-1 (b) $1-2^{-n}$ (c) $2^{-n}+n-1$ (d) 2^{n-1}

Solution:

Hint. $t_1=\frac{1}{2}=1-\frac{1}{2}; t_2=\frac{3}{4}=1-\frac{1}{4}=1-\frac{1}{2^2},\ t_3=\frac{7}{8}=1-\frac{1}{8}=1-\frac{1}{2^3}$

$$\therefore t_n=1-\frac{1}{2^n}=1-2^{-n}$$

Question 15.

The sum up to n terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+$ is ....

(a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$ (c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1

Solution:

(c) $\frac{n(n+1)}{\sqrt{2}}$

Hint. $\sqrt{2}+2\sqrt{2}+3\sqrt{2}+\ldots$ n terms

$$=\sqrt{2}(1+2+3+\ldots n)=\sqrt{2}\ \frac{(n)(n+1)}{2}=\frac{n(n+1)}{\sqrt{2}}$$

Question 16.

The value of the series $\frac{1}{2}+\frac{7}{4}+\frac{13}{8}+\frac{19}{16}+\ldots$ is ....

(a) 14
(b) 7
(c) 4
(d) 6

Solution:
(a) 14

Hint. $\frac{1}{2}+\frac{7}{4}+\frac{13}{8}+\frac{19}{16}+$

It is an arithmetico geometric series

Here $a = 1$, $d = 7 - 1 = 6$, and $r = \frac{1}{2}$

$$S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}=\frac{1}{1-\frac{1}{2}}+\frac{(6)\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^2}=\frac{1}{\frac{1}{2}}+\frac{3}{\left(\frac{1}{2}\right)^2}=2+(3\times 4)=2+12=14$$

Question 17.
The sum of an infinite GP is 18. If the first term is 6, the common ratio is ....
(a) $\frac{1}{3}$
(b) $\frac{2}{3}$
(c) $\frac{1}{6}$
(d) $\frac{3}{4}$

Hint:
$\frac{6}{1-r}=18 \Rightarrow 6=18-18r$
18r = 18 – 6 = 12
r = 12/18 = 2/3

Question 18.
The coefficient of x^5 in the series e^{-2x} is ....
(a) $\frac{2}{3}$
(b) $\frac{3}{2}$
(c) $\frac{-4}{15}$
(d) $\frac{4}{15}$

Answer:

(c) $\frac{-4}{15}$

Hint. $e^{-2x}=1-\frac{2x}{\angle 1}+\frac{(2x)^2}{\angle 2}-\frac{(2x)^3}{\angle 3}+\frac{(2x)^4}{\angle 4}-\frac{(2x)^5}{\angle 5}...$

Coefficient of $x^5=\frac{-2^5}{\angle 5}=\frac{-32}{120}=\frac{-4}{15}$

Question 19.
The value of $\frac{1}{2!},+\frac{1}{4!}+\frac{1}{6!}+$ .... is ....

Answer:

(c) $\frac{(e-1)^2}{2e}$

Hint. $e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots, \; e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \ldots$

$$\frac{e^1 + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots.$$

$$\frac{1}{2!} + \frac{1}{4!} + \ldots. = \frac{e^1 + e^{-1}}{2} - 1$$

$$= \frac{e^1 - 2 + e^{-1}}{2} = \frac{(e-1)^2}{2}$$

Question 20.

The value of $1 - \frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right)^2 - \frac{1}{4}\left(\frac{2}{3}\right)^3 + \ldots$ is ....

(a) $\log\left(\frac{5}{3}\right)$ (b) $\frac{3}{2}\log\left(\frac{5}{3}\right)$ (c) $\frac{5}{3}\log\left(\frac{5}{3}\right)$ (d) $\frac{2}{3}\log\left(\frac{2}{3}\right)$

Solution:

(b) $\frac{3}{2}\log\left(\frac{5}{3}\right)$

Hint. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

$$\log\frac{(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots.$$

put $x = 2/3$

$$= 1 - \frac{(2/3)}{2} + \frac{(2/3)^2}{3} - \frac{(2/3)^3}{4} + \ldots.$$

$$= \log (1 + 2/3) / 2/3$$

$$= \frac{3}{2}\log\left(1 + 2/3\right) = \frac{3}{2}\log\frac{5}{3}$$