Chapter – 7

Matrices and Determinants

Ex 7.1

Question 1.
Construct an m × n matrix A = [a_{ij}], where a_{ij} is given by

(i) $a_{ij} = \frac{(i-2j)^2}{2}$ **with** $m = 2, n = 3$ **(ii)** $a_{ij} = \frac{|3i-4j|}{4}$ **with** $m = 3, n = 4$

Solution:

(i) $a_{ij} = \frac{(i-2j)^2}{2}$

Here m = 2, n = 3
So we have to construct a matrix of order 2 × 3

Now A matrix of order 2 × 3 will be of the form $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$$\text{Here} \quad a_{ij} = \frac{(i-2j)^2}{2}$$

$$\therefore \quad a_{11} = \frac{[1-2(1)]^2}{2} = \frac{(1-2)^2}{2} = \frac{1}{2} = \frac{1}{2}$$

$$a_{12} = \frac{[1-2(2)]^2}{2} = \frac{(1-4)^2}{2} = \frac{9}{2}$$

$$a_{13} = \frac{[1-2(3)]^2}{2} = \frac{(1-6)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{[2-2(1)]^2}{2} = \frac{(2-2)^2}{2} = 0$$

$$a_{22} = \frac{[2-2(2)]^2}{2} = \frac{(2-4)^2}{2} = \frac{4}{2} = 2$$

$$a_{23} = \frac{[2-2(3)]^2}{2} = \frac{(2-6)^2}{2} = \frac{16}{2} = 8$$

$$\therefore \quad A = \begin{pmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 8 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 9 & 25 \\ 0 & 4 & 16 \end{pmatrix}$$

(ii) Here m = 3 and n = 4
So we have to construct a matrix order 3 × 4
The general form of a matrix of order 3 × 4 will be

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

Here $$a_{ij} = \frac{|3i-4j|}{4}$$

$$\therefore \quad a_{11} = \frac{|3-4|}{4} = \frac{1}{4}$$

$$a_{12} = \frac{|3-4(2)|}{4} = \frac{|3-8|}{4} = \frac{5}{4}$$

$$a_{13} = \frac{|3-4(3)|}{4} = \frac{|3-12|}{4} = \frac{9}{4}$$

$$a_{14} = \frac{|3-4(4)|}{4} = \frac{|3-16|}{4} = \frac{13}{4}$$

$$a_{21} = \frac{|3(2)-4(1)|}{4} = \frac{|6-4|}{4} = \frac{2}{4} = \frac{1}{2}$$

$$a_{22} = \frac{|3(2)-4(2)|}{4} = \frac{|6-8|}{4} = \frac{2}{4} = \frac{1}{2}$$

$$a_{23} = \frac{|3(2)-4(3)|}{4} = \frac{|6-12|}{4} = \frac{6}{4} = \frac{3}{2}$$

$$a_{24} = \frac{|3(2)-4(4)|}{4} = \frac{|6-16|}{4} = \frac{10}{4} = \frac{5}{2}$$

$$a_{31} = \frac{|3(3)-4(1)|}{4} = \frac{|9-4|}{4} = \frac{5}{4}$$

$$a_{32} = \frac{|3(3)-4(2)|}{4} = \frac{|9-8|}{4} = \frac{1}{4}$$

$$a_{33} = \frac{|3(3)-4(3)|}{4} = \frac{|9-12|}{4} = \frac{3}{4}$$

$$a_{34} = \frac{|3(3)-4(4)|}{4} = \frac{|9-16|}{4} = \frac{7}{4}$$

$$\therefore \quad A = \begin{pmatrix} \frac{1}{4} & \frac{5}{4} & \frac{9}{4} & \frac{13}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{5}{4} & \frac{1}{4} & \frac{3}{4} & \frac{7}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{pmatrix}$$

Question 2.
Find the values of p, q, r and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$$

Solution:
When two matrices (of the same order) are equal then their corresponding entries are equal.

$$\text{Here} \begin{pmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{pmatrix}$$

$p^2 - 1 = 1$ ………… (1)
$- 31 - q^3 = - 4$ ……….. (2)
$r + 1 = 3/2$ ……… (3)
$s - 1 = - \pi$ ……….. (4)
(1) $\Rightarrow p^2 - 1 \Rightarrow p^2 = 1 + 1 = 2$
$p = \pm\sqrt{2}$

(2) ⇒ – 31 – q^3 – 4
31 + q^3 = 4
q^3 = 4 – 31 = – 27
$q^3 = (-3)^3$
q = – 3
(3) ⇒ r + 1 = 3/2
r = 32 – 1 = 1/2
(4) ⇒ s – 1 = – π
s = 1 – π
∴ The required values are
p = ±√2,
q = 3,
r = 1/2,
s = 1 – π

Question 3.

Determine the value of x + y if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 2x & + & y & 4x \\ 5x & - & 7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y & - & 13 \\ y & x & + & 6 \end{bmatrix}$$

Equating the corresponding entries
2x + y = 7 .... (1)
4x = 7y – 13
4x – 7y = – 13 .... (2)
5x – 7 = y
5x – y = 7 .... (3)
4x = x + 6
4x – x = 6
3x = 6
x = 6/3 = 2 .... (4)
Substituting for x in equation (1)
(1) ⇒ 2 × 2 + y = 7
y = 7 – 4 = 3

The required values are x = 2 and y = 3
x + y = 2 + 3 = 5
x + y = 5

Question 4.
Determine the matrices A and B if they satisfy

$$2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0 \text{ and } A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

Solution:

$$2A - B + \begin{pmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad 2A - B = -\begin{pmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{pmatrix} \quad (1)$$

$$A - 2B = \begin{pmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{pmatrix} \quad (2)$$

Solving (1) and (2)

$$(1) \Rightarrow \quad 2A - B = \begin{pmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{pmatrix} \quad (1)$$

$$(2) \times 2 \Rightarrow \quad 2A - 4B = 2\begin{pmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 16 \\ -4 & 2 & -14 \end{pmatrix} \quad (3)$$

$$(1)-(3) \Rightarrow \quad 3B = \begin{pmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 4 & 16 \\ -4 & 2 & -14 \end{pmatrix}$$

$$\text{(i.e.)} \quad 3B = \begin{pmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{pmatrix} + \begin{pmatrix} -6 & -4 & -16 \\ 4 & -2 & 14 \end{pmatrix}$$

$$3B = \begin{pmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{pmatrix}$$

$$B = \frac{1}{3}\begin{pmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{pmatrix}$$

Substituting B value in (2) we get

$$A - 2\left(\frac{1}{3}\right)\begin{pmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -24 & 4 & -32 \\ 16 & -8 & 26 \end{pmatrix}$$

$$= \frac{1}{3} \times 3\begin{pmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -24 & 4 & -32 \\ 16 & -8 & 26 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 9 & 6 & 24 \\ -6 & 3 & -21 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -24 & 4 & -32 \\ 16 & -8 & 26 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{pmatrix}$$

Question 5.

If A = $\begin{bmatrix} \mathbf{1} & \boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$, then compute A^4

Solution:

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$$A^2 = A \times A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a + a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

Now

$$A^4 = A^2 \times A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4a \\ 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 4a \\ 0 & 1 \end{pmatrix}$$

Question 6.

Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(i) Show that $A_\alpha A_\beta = A_{\alpha+\beta}$.

(ii) Find all possible real values of satisfying the condition $A_\alpha + A^T_\alpha = 1$.

Solution:

$$A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\therefore \quad A_\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$A_{\alpha+\beta} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

(*i*) To Prove: $A_\alpha \cdot A_\beta = A_{\alpha+\beta}$

$$\text{LHS} = A_\alpha \cdot A_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = A_{\alpha+\beta} = \text{RHS}$$

(*ii*) $A_\alpha + A^T_\alpha = I$

$$\Rightarrow \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} + \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = I$$

$$\begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad 2\cos\alpha = 1$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$$

General solution is $\alpha = 2n\pi + \pi/3,\ n \in Z$

Question 7.

If A = $\begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ such that (A – 2I) (A – 3I) = 0, find the value of x.

Solution:

Given $$A = \begin{pmatrix} 4 & 2 \\ -1 & x \end{pmatrix}$$

$\therefore$ $$A - 2I = \begin{pmatrix} 4 & 2 \\ -1 & x \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ -1 & x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & x-2 \end{pmatrix}$$

$$A - 3I = \begin{pmatrix} 4 & 2 \\ -1 & x \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & x \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & x-3 \end{pmatrix}$$

Given $$(A - 2I)(A - 3I) = 0$$

$$\Rightarrow \quad \begin{pmatrix} 2 & 2 \\ -1 & x-2 \end{pmatrix} \downarrow \begin{pmatrix} 1 & 2 \\ -1 & x-3 \end{pmatrix} = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 0 & 4 + 2(x-3) \\ -1 - (x-2) & -2 + (x-2)(x-3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad -1 - x + 2 = 0 \ (i.e.) - x + 1 = 0$$

$$\Rightarrow \quad -x = -1$$

$$\Rightarrow \quad x = 1$$

Question 8.

If A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{b} & \mathbf{-1} \end{bmatrix}$, show that A^2 is a unit matrix.

Solution:

Given $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{pmatrix}$$

Now $$A^2 = A \times A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{pmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 9.

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + KI = 0$, find the value of k.

Solution:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$$

$$A^2 = A \times A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{pmatrix}$$

Given $A^3 - 6A^2 + 7A + kI = 0$

$$\Rightarrow \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{pmatrix} - 6\begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} + k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{pmatrix} + \begin{pmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{pmatrix} + \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} k-2 & 0 & 0 \\ 0 & k-2 & 0 \\ 0 & 0 & k-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow k - 2 = 0 \Rightarrow k = 2$$

Question 10.

Give your own examples of matrices satisfying the following conditions in each case:

(i) A and B such that $AB \neq BA$.

(ii) A and B such that $AB = 0 = BA$, $A \neq 0$ and $B \neq 0$.

(iii) A and B such that $AB = 0$ and $BA \neq 0$.

Solution:

(*i*) Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$

$$\text{Now } AB = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 12 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 14 & -1 \\ 19 & -2 \end{pmatrix}$$

Here $AB \neq BA$

(*ii*) Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Now $$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $$BA = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $AB = BA = 0$

(*iii*) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}$

Now $$AB = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $$BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $AB = 0$ But $BA \neq 0$

Question 11.

Show that f(x) f(y) = f(x + y), where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution:

Given $$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $$f(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$f(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To Prove: $f(x)\, f(y) = f(x+y)$

LHS $$f(x) \,.\, f(y) = \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x+y) = \text{RHS}$$

Question 12.
If A is a square matrix such that $A^2 = A$, find the value of $7A - (I + A)^3$.

Solution:
Given A is a square matrix such that $A^2 = A$.
$(I + A)^3 = (I + A)\,(I + A)\,(I + A)$
$= (I \,.\, I + I \,.\, A + A \,.\, I + A \,.\, A)\,(I + A)$
$= (I + A + A + A^2)\,(I + A)$
$= (I + 2A + A)\,(I + A)$
[Given $A^2 = A$]
$= (I + 3A)\,(I + A)$
$= I \,.\, I + I \,.\, A + 3A \,.\, I + 3A \,.\, A$

$= I + A + 3A + 3A^2$
$= I + 4A + 3A$
[Given $A^2 = A$]
$(I + A)^3 = I + 7A$
$\therefore 7A - (I + A)^2 = 7A - (I + 7A)$
$= 7A - I - 7A$
$7A - (I + A)^2 = - I$

Question 13.

Verify the property A (B + C) = AB + AC, when the matrices A, B, and C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Solution:

Given $A = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{pmatrix}$; $B = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{pmatrix}$

and $C = \begin{pmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$

Now $B + C = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{pmatrix}$

$$\text{LHS} = A(B + C) = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 14+0-15 & 16+0-3 \\ 7+4+25 & 8+4+5 \end{pmatrix} = \begin{pmatrix} -1 & 13 \\ 36 & 17 \end{pmatrix} \qquad (1)$$

$$AB = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 19 & 11 \end{pmatrix}$$

$$AC = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 17 \\ 17 & 6 \end{pmatrix}$$

$$RHS = AB + AC = \begin{pmatrix} -6 & -4 \\ 19 & 11 \end{pmatrix} + \begin{pmatrix} 5 & 17 \\ 17 & 6 \end{pmatrix} = \begin{pmatrix} -1 & 13 \\ 36 & 17 \end{pmatrix} \qquad (2)$$

$(1) = (2) \Rightarrow A(B+C) = AB + AC$

Question 14.
Find the matrix A which satisfies the matrix relation

$$A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}.$$

Solution:

Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

Here B is of order 2×3 and C is of order 2×3

Given AB = C

$\Rightarrow$ A should be a matrix of order 2×2

$$\left(\because \begin{array}{l} (2 \times \cancel{2}) \\ \qquad (\cancel{2} \times 3) \Rightarrow 2 \times 3 \end{array} \right)$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

Equating the corresponding elements

$a + 4b = -7 \quad (1)$

$2a + 5b = -8 \quad (2)$

$3a + 6b = -9 \quad (3)$

and

$c + 4d = 2 \quad (4)$

$2c + 5d = 4 \quad (5)$

$3c + 6d = 6 \quad (6)$

Solving (1) and (3)

$(1) \Rightarrow \quad a + 4b = -7$

$(3) \div 3 \Rightarrow \quad a + 2b = -3$

$(1) - (3) \Rightarrow \quad 2b = -4 \Rightarrow b = -2$

Substituting $b = -2$ in (1)

$a - 8 = -7$

$a = -7 + 8 = 1$

$\therefore \quad A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$

Solving (4) and (6)

$(4) \Rightarrow c + 4d = 2$

$(6) \div 3 \Rightarrow c + 2d = 2 \quad (6)$

$(4) - (6) \Rightarrow \quad 2d = 0 \Rightarrow d = 0$

Substituting $d = 0$ in (4)

$c + 0 = 2$

$c = 2$

Question 15.

If $A^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$, verify the following

(i) $(A + B)^T = A^T + B^T = B^T + A^T$

(ii) $(A - B)^T = A^T - B^T$

(iii) $(B^T)^T = B$.

Solution:

Given $A^T = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{pmatrix}$

$B = \begin{pmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix}$

(*i*) To verify $(A+B)^T = A^T + B^T = B^T + A^T$

$$A+B = \begin{pmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -2 & 3 \\ 12 & 5 & 1 \end{pmatrix}$$

$$\therefore\ (A+B)^T = \begin{pmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{pmatrix} \qquad (1)$$

$$A^T = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix}$$

$$\therefore\ A^T + B^T = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{pmatrix} \qquad (2)$$

$$\text{Also } B^T + A^T = \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{pmatrix} \qquad (3)$$

Here $(1) = (2) = (3)$

$$\Rightarrow (A+B)^T = A^T + B^T = B^T + A^T$$

(*ii*) To verify $(A-B)^T = A^T - B^T$

$$A-B = \begin{pmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -2 & 1 & -1 \\ -7 & -5 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & -5 & 5 \end{pmatrix}$$

$$\therefore \qquad (A-B)^T = \begin{pmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{pmatrix} \qquad (1)$$

$$\text{Also } A^T = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix}$$

$$\therefore \quad A^T - B^T = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} -2 & -7 \\ 1 & -5 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{pmatrix} \qquad (2)$$

Here $(1) = (2) \Rightarrow (A - B)^T = A^T - B^T$

(*iii*) To verify $(B^T)^T = B$

Here $B = \begin{pmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{pmatrix} \qquad (1)$

$$\therefore \quad B^T = \begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix} \text{ and } (B^T)^T = \left\{\begin{pmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{pmatrix}\right\}^T$$

$$= \begin{pmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{pmatrix} \qquad (2)$$

$(1) = (2) \Rightarrow (B^T)^T = B$

Question 16.
If A is a 3 × 4 matrix and B is a matrix such that both ATB and BAT are defined, what is the order of the matrix B?

Solution:
Given Order of A = 3 × 4
∴ Order of $A^T = 4 \times 3$
Given that A^TB is defined.

∴ Number of columns of A^T = Number of rows of B
Number of rows of B = 3
Also given BA^T is defined.
∴ Number of columns of B = Number of rows of A^T

Number of columns of B = 4
∴ Order of B = 3 × 4

Question 17.
Express the following matrices is the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ **and** **(ii)** $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Solution:

(i) Let $A = \begin{pmatrix} 4 & -2 \\ 3 & -5 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 4 & 3 \\ -2 & -5 \end{pmatrix}$

Let $P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{pmatrix} 4 & -2 \\ 3 & -5 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ -2 & -5 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 8 & 1 \\ 1 & -10 \end{pmatrix}$

Now $P^T = \frac{1}{2}\begin{pmatrix} 8 & 1 \\ 1 & -10 \end{pmatrix} = P \Rightarrow P$ is a symmetric matrix

Let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\left[\begin{pmatrix} 4 & -2 \\ 3 & -5 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ -2 & -5 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}$

Here $Q^T = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix} = -Q \Rightarrow Q$ is a skew symmetric matrix

$A = P + Q = \frac{1}{2}\left[\begin{pmatrix} 8 & 1 \\ 1 & -10 \end{pmatrix} + \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}\right]$

(ii) Let $A = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$

Let $P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} + \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{pmatrix}$

Now $P^T = \begin{pmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{pmatrix} = P \Rightarrow P$ is a symmetric matrix.

Let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} - \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

Now $Q^T = \frac{1}{2}\begin{pmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{pmatrix} = -Q \Rightarrow$ Q is a skew symmetric matrics

$$A = P + Q = \frac{1}{2}\left\{\begin{pmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{pmatrix}\right\}$$

Question 18.

Find the matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$

Solution:

Let $B = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$

B is of order 3×2 and C is of order 3×3

So $\left\{\begin{matrix} \text{Now } (3 \times 2) \\ \times \\ (2 \times 3) \Rightarrow (3 \times 3) \end{matrix}\right\}$

$\Rightarrow$ A^T should be of order 2×3

$\Rightarrow$ A should be of order 3×2

Let $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ so $A^T = \begin{pmatrix} a & c & c \\ b & d & f \end{pmatrix}$

Here $BA^T = C$

$$\Rightarrow \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a-b & 2c-d & 2e-f \\ a & c & e \\ -3a+4b & -3c+4d & -3e+4f \end{pmatrix} = \begin{pmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$$

equating the corresponding elements we get

$$\begin{array}{r|r|r} 2a-b=-1 & 2c-d=-8 & 2e-f=-10 \\ a=1 & c=2 & e=-5 \\ \Rightarrow 2-b=-1 & \Rightarrow 4-d=-8 & \Rightarrow \quad -10-f=-10 \\ -b=-3 & -d=-8-4=-12 & -f=-10+10=0 \\ b=3 & \Rightarrow \quad d=12 & \Rightarrow f=0 \end{array}$$

$$\therefore \quad A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 12 \\ -5 & 0 \end{pmatrix}$$

Question 19.

If $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \boldsymbol{x} & \mathbf{2} & \boldsymbol{y} \end{bmatrix}$ is a matrix such that $AA^T = 9I$, find the values of x and y.

Solution:

$$\text{Given } A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix} \begin{pmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & x+4+2y \\ 0 & 9 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{pmatrix}$$

$$= 9I \text{ (given)}$$

$$\Rightarrow \begin{pmatrix} 9 & 0 & x+4+2y \\ 0 & 9 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{pmatrix} = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$\Rightarrow \quad x + 4 + 2y = 0 \quad (1)$

and $\quad 2x + 2 - 2y = 0 \quad (2)$

Solving (1) and (2)

$(1) \Rightarrow \quad x + 2y = -4$

$(2) \Rightarrow \quad 2x - 2y = -2$

$(1) + (2) \Rightarrow \quad 3x = -6$

$x = -2$

Substituting $x = -2$ in (1)

$-2 + 2y = -4$

$2y = -4 + 2 = -2$

$y = -1$

So $\quad x = -2;\ y = -1$

Question 20.

(i) For what value of x, the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is skew-symmetric.

(ii) If $\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of p, q and r.

Solution:

(*i*)
$$A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{pmatrix}$$

Given A is skew symmetric

$$\Rightarrow \quad A^T = -A$$

$$\Rightarrow \quad \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & x^3 & 0 \end{pmatrix} = -A = -\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -x^3 \\ -2 & 3 & 0 \end{pmatrix}$$

$$\Rightarrow \quad -x^3 = -3 \Rightarrow x^3 = 3 \Rightarrow x = 3^{1/3}$$

(*ii*) Let
$$A = \begin{pmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{pmatrix}$$

Given A is a skew symmetric matrix

$$\Rightarrow \quad A^T = -A$$

$$\Rightarrow \quad \begin{pmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -p & -3 \\ -2 & -q^2 & 1 \\ -r & -1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{array}{r|l} & r = -3; \quad q^2 = -q^2 \\ -p = 2 & \Rightarrow 2q^2 = 0 \\ p = -2 & q^2 = 0 \\ & q = 0 \end{array}$$

$$\therefore \quad p = -2; q = 0; r = -3$$

Question 21.
Construct the matrix $A = [a_{ij}]_{3\times3}$, where $a_{ij} = i- j$. State whether A is symmetric or skew- symmetric.

Solution:
Given A is a matrix of order 3×3

$$\therefore\ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We are given $a_{ij} = i - j$

$$\therefore \quad \begin{array}{l|l|l} a_{11} = 1-1=0 & a_{21} = 2-1=1 & a_{31} = 3-1=2 \\ a_{12} = 1-2=-1 & a_{22} = 2-2=0 & a_{32} = 3-2=1 \\ a_{13} = 1-3=-2 & a_{23} = 2-3=-1 & a_{33} = 3-3=0 \end{array}$$

$$\therefore\ A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } A^T = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

Here $A^T = -A$
⇒ A is skew-symmetric

Question 22.
Let A and B be two symmetric matrices. Prove that AB = BA if and only if AB is a symmetric matrix.

Solution:
Let A and B be two symmetric matrices
⇒ $A^T = A$ and $B^T = B$ …………….. (1)
Given that AB = BA (2)
To prove AB is symmetric:

Now $(AB)^T = B^TA^T = BA$
(from(1)) But $(AB)^T = AB$ by ………….. (2)
⇒ AB is symmetric.

Conversely, let AB be a symmetric matrix.
⇒ $(AB)^T = AB$
i.e. $B^TA^T = AB$

i.e. BA = AB (from (1))
⇒ AB is symmetric

Question 23.
If A and B are symmetric matrices of the same order, prove that
(i) AB + BA is a symmetric matrix.
(li) AB – BA is a skew-symmetric matrix.

Solution:
Given A and B are symmetric matrices of the same order.
∴ $A^T = A$, $B^T = B$
(i) AB + BA is a symmetric matrix
$(AB + BA)^T = (AB)^T + (BA)^T$
$= B^TA^T + A^TB^T$
= BA + AB
= AB + BA
∴ AB + BA is a symmetric matrix.

(ii) AB – BA is a skew – symmetric matrix
$(AB – BA)^T = (AB)^T (BA)^T$
$= B^TA^T – A^TB^T$
= BA – AB
$(AB – BA)^T = – (AB – BA)$
∴ AB – BA is a skew symmetric matrix.

Question 24.
A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins, and almonds.

The pack-I contains 100 gm of cashew nuts, 100 gm of raisins, and 50 gm of almonds. Pack-II contains 200 gm of cashew nuts, 100 gm of raisins, and 100 gm of almonds. Pack-III contains 250 gm of cashew nuts, 250 gm of raisins, and 150 gm of almonds. The cost of 50 gm of cashew nuts is ₹ 50, 50 gm of raisins is ₹ 10, and 50 gm of almonds is₹ 60. What is the cost of each gift pack?

Solution:

	Cashew nuts	Raisins	Almonds
Pack I	100 gm	100 gm	50 gm
Pack II	200 gm	100 gm	100 gm
Pack III	250 gm	250 gm	150 gm

Cashew	50 gm	Rs. 50
Raisins	50 gm	Rs. 10
Almonds	50 gm	Rs. 60

$\therefore$ Cost per gram.

Cashew	1 gm	Rs. 1
Raisins	1 gm	Rs. $\frac{1}{5}$
Almonds	1 gm	Rs. $\frac{6}{5}$

Cost of each pack:

$$\begin{pmatrix} 100 & 100 & 50 \\ 200 & 100 & 100 \\ 250 & 250 & 150 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{5} \\ \frac{6}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 100+20+60 \\ 200+20+120 \\ 250+50+180 \end{pmatrix} = \begin{pmatrix} 180 \\ 340 \\ 480 \end{pmatrix}$$

$\therefore$ Cost of pack I = ₹ 180

Cost of pack II = ₹ 340

Cost of pack III = ₹ 480

Ex 7.2

Question 1.

Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$.

Solution:

$$\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = \begin{vmatrix} s & a^2 & a^2+b^2+c^2 \\ s & b^2 & a^2+b^2+c^2 \\ s & c^2 & a^2+b^2+c^2 \end{vmatrix} (C_3 \longrightarrow C_3 + C_2)$$

Taking s from C_1 and $a^2 + b^2 + c^2$ from C_3 as common factors are get

$$(s)(a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix} = 0 \qquad (\because C_1 = C_3)$$

Question 2.

Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0.$

Solution:

$$\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b, R_3 by c and dividing by abc we get

$$\frac{1}{abc}\begin{vmatrix} a(b+c) & abc & ab^2c^2 \\ b(c+a) & abc & a^2bc^2 \\ c(a+b) & abc & a^2b^2c \end{vmatrix}$$

$$= \frac{(abc)^2}{abc}\begin{vmatrix} ab+ac & 1 & bc \\ bc+ab & 1 & ca \\ ac+bc & 1 & ab \end{vmatrix} = (abc)\begin{vmatrix} ab+bc+ca & 1 & bc \\ ab+bc+ca & 1 & ca \\ ab+bc+ca & 1 & ab \end{vmatrix} (C_1 \to C_1 + C_3)$$

$$= (ab+bc+ca)(abc)\begin{vmatrix} 1 & 1 & bc \\ 1 & 1 & ca \\ 1 & 1 & ab \end{vmatrix} = 0 \qquad (\because C_1 = C_2)$$

Question 3.

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

Solution:

LHS

Taking a from C_1, b from C_2 and c from C_3 we get

$$(abc)\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= (abc)\begin{vmatrix} -b & c-b & c \\ a & -c & a-c \\ b & b+c & c \end{vmatrix} \begin{matrix} (R_1 \to R_1 - R_2) \\ (R_1 \to R_2 - R_3) \end{matrix}$$

$$= (abc)\begin{vmatrix} 0 & 2c & 2c \\ a & -c & a-c \\ b & b+c & c \end{vmatrix} (R_1 \to R_1 + R_3)$$

Taking $2c$ as a common factor from R_1 we get

$$= (abc)(2c)\begin{vmatrix} 0 & 1 & 1 \\ a & -c & a-c \\ b & b+c & c \end{vmatrix}$$

$$= (2c)(abc)\begin{vmatrix} 0 & 0 & 1 \\ a & -a & a-c \\ b & b & c \end{vmatrix} \quad (C_2 \to C_2 - C_3)$$

Expanding along R_1 we get

(2c) (abc) (1) [ab + ab] = abc (2c) (2ab)

1 = (abc) (4abc) = $4a^2b^2c^2$

= RHS

Question 4.

Prove that $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Solution:

$$\text{LHS} = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ 1 & 1 & 1+c \end{vmatrix} \begin{pmatrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{pmatrix}$$

Expanding along R_1

$$a\begin{vmatrix} b & -c \\ 1 & 1+c \end{vmatrix} + b\begin{vmatrix} 0 & -c \\ 1 & 1+c \end{vmatrix} + 0$$

$$= a\,[b\,(1+c)+c] + b\,[0+c]$$

$$= a\,(b+bc+c) + bc = ab + abc + ac + bc$$

$$= abc + ab + bc + ac = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= \text{RHS}$$

Question 5.

Prove that $$\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0 \cdot$$

Solution:

$$\text{LHS}=\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix}$$

$$=\begin{vmatrix} \sec^2\theta-\tan^2\theta & \tan^2\theta & 1 \\ \tan^2\theta-\sec^2\theta & \sec^2\theta & -1 \\ 38-36 & 36 & 2 \end{vmatrix} C_1 \to C_1 - C_2 = \begin{vmatrix} 1 & \tan^2\theta & 1 \\ -1 & \sec^2\theta & -1 \\ 2 & 36 & 2 \end{vmatrix}$$

$$= 0 \ (\because C_1 = C_3)$$
$$= \text{RHS}$$

Question 6.

Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0.$

Solution:

$$\text{LHS} = \begin{vmatrix} x+2a & y+2b & 2+2c \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} x & y & z \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$= \underset{(\because R_1 = R_2)}{0} + 2\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$= 0 + 2(0) \ (\because R_1 = R_3)$$
$$= 0 = \text{RHS}$$

Question 7.

Write the general form of a 3×3 skew-symmetric matrix and prove that its determinant is 0.

Solution:

$$\text{Let} \quad A = \begin{pmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{pmatrix}$$

$$\text{Here} \quad A^T = \begin{pmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} = -A$$

$\Rightarrow$ A is a skew symmetric matrix

$$\text{Now} \quad |A| = \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix} = 0\,(0 + c^2) - a\,[0 - bc] + b\,[-ac + 0]$$
$$= abc - abc = 0$$

Question 8.

$$\textbf{If} \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0,$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = \begin{vmatrix} a & b & a\alpha \\ b & c & b\alpha \\ a\alpha + b & b\alpha + c & -(b\alpha + c) \end{vmatrix} (C_3 \to C_3 - C_2)$$

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(b\alpha + c) \\ & & -(a\alpha^2 + b\alpha) \end{vmatrix} (C_3 \to C_3 - \alpha C_1)$$

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix}$$

expanding along C_3

we get – $(a\alpha^2 + 2b\alpha + c)\ [ac - b^2]$
So $\Delta = 0 \Rightarrow (a\alpha^2 + 2b\alpha + c)\ (ac - b^2) = -0 = 0$
$\Rightarrow a\alpha^2 + 2b\alpha + c = 0$ or $ac - b^2 = 0$
(i.e.) a is a root of $ax^2 + 2bx + c = 0$
or $ac = b^2$
$\Rightarrow$ a, b, c are in G.P.

Question 9.

Prove that $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$.

Solution:

$$\text{LHS} = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ac \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a - b & a^2 - bc - b^2 + ac \\ 0 & b - c & b^2 - ac - c^2 + ab \\ 1 & c & c^2 - ab \end{vmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & a - b & (a^2 - b^2) + (ac - bc) \\ 0 & b - c & (b^2 - c^2) + (ab - ac) \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a - b & (a + b)(a - b) + c(a - b) \\ 0 & b - c & (b + c)(b - c) + a(b - c) \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a - b & (a - b)(a + b + c) \\ 0 & b - c & (b - c)(a + b + c) \\ 1 & c & c^2 - ab \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ from R_1 and R_2 respectively

we get $(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & c & c^2-ab \end{vmatrix}$

expanding along C_1

$(a-b)(b-c)\{0-0+1[(a+b+c)-(a+b+c]\}=0=\text{RHS}$

Question 10.

If a, b, c are p^{th}, q^{th} and r^{th} terms of an A.P., find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$.

Solution:

We are given $a = t_p, b = t_q$ and $c = t_r$

Let a be the first term and d be the common difference

$$\begin{aligned} \text{So} \quad t_p &= a+(p-1)d = a \\ t_q &= a+(q-1)d = b \\ \text{and} \quad t_r &= a+(r-1)d = c \end{aligned}$$

To find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$:

$$\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} t_p & t_q & t_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} t_p-t_q & t_q-t_r & t_r \\ p-q & q-r & r \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_1 \\ C_2 \to C_2 - C_3 \end{matrix}$$

expanding along R_3

$1\{(t_p-t_q)(q-r)-(p-q)(t_q-t_r)\}$

$= qt_p - rt_p - qt_q + rt_q - pt_q + pt_r - qt_q - qt_r$

$$\text{So}\quad t_p = a+(p-1)\,d = a$$
$$t_q = a+(q-1)\,d = b$$
$$\text{and}\quad t_r = a+(r-1)\,d = c$$

To find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$:

$$\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} t_p & t_q & t_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} t_p - t_q & t_q - t_r & t_r \\ p-q & q-r & r \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_1 \\ C_2 \to C_2 - C_3 \end{matrix}$$

expanding along R_3

$1\{(t_p - t_q)(q-r) - (p-q)(t_q - t_r)\}$

$= qt_p - rt_p - qt_q + rt_q - pt_q + pt_r - qt_q - qt_r$

$= (p-q)tr + (q-r)\,tp + (r-p)\,tq$

$= (p-q)\,[a+(r-1)d + (q-r)\,[a+(p-1)d] + (r-p)\,[(a+(q-1)d]$

$= a[p-q+q-r+r-p] + d\,[(r-1)(p-q) + (p-1)(q-r) + (q-1)(r-p)]$

$= a(0) + d(0) = 0$

Question 11.

Show that $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4

Solution:

$$\text{LHS} = \begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$$

Multiplying R_1 by a, R_2 by b and R_3 by c and
taking out a from C_1 b from C_2 and c from C_3 we get

=

$$\frac{abc}{abc}\begin{vmatrix} a^2+x^2 & a^2 & a^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2+x^2 & a^2+b^2+c^2+x^2 & a^2+b^2+c^2+x^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix} R_1 \to R_1 + R_2 + R_3$$

$$= (a^2+b^2+c^2+x^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

$$= (a^2+b^2+c^2+x^2)\begin{vmatrix} 0 & 0 & 1 \\ -x^2 & x^2 & b^2 \\ 0 & -x^2 & c^2+x^2 \end{vmatrix} \begin{array}{l} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{array}$$

expanding along R_1 we get

$$= (a^2+b^2+c^2+x^2)\,[1(-x^2)(-x^2) - 0]$$

$$= x^4\,(a^2+b^2+c^2+x^2)$$

$\Rightarrow$ The given determinant is divisible by x^4.

Question 12.

If a, b, c are all positive, and are pth, qth and rth terms of a G.P., show that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$$

Solution:

Let the G.P. be $k, kx, kx^2, \ldots\ldots$.

$$a = t_p = k\,x^{p-1}$$
$$b = t_q = k\,x^{q-1}$$
$$\text{and} \quad c = t_r = k\,x^{r-1}$$

$$\text{Now} \quad \text{L.H.S.} = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$$

$$\begin{vmatrix} \log a - \log b & p-q & 0 \\ \log b - \log c & q-r & 0 \\ \log c & r & 1 \end{vmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_1 \to R_2 - R_3 \\ \\ \end{matrix}$$

expanding along C_3

$[(\log a - \log b)(q-r) - (\log b - \log c)(p-q)]$

$$= \left(\log \frac{a}{b}\right)(q-r) - \left(\log \frac{b}{c}\right)(p-q)$$

$$\left\{\begin{matrix} \text{Here } \dfrac{a}{b} = \dfrac{kx^{p-1}}{kx^{q-1}} = x^{p-1-q+1} = x^{p-q} \\ \text{and } \dfrac{b}{c} = \dfrac{kx^{q-1}}{kx^{r-1}} = x^{q-1-r+1} = x^{q-r} \end{matrix}\right\}$$

$$= (q-r)\left[\log x^{p-q}\right] - (p-q)\left[\log x^{q-r}\right]$$
$$= (q-r)(p-q)\log x - (p-q)(q-r)\log x$$
$$= 0 = \text{RHS}$$

Question 13.

Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ if x, y, z ≠ 1.

Solution:

Expanding the determinant along R_1

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

$= 1[1 - \log_z y \log_y z] - \log_x y[\log_y x - \log_z x \log_y z] + \log_x z$

$= 1[1 - 1] - \log_x y [\log_y x - \log_y x] + \log_x z [\log_z x - \log_z x] = 0$

Question 14.

If $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$, prove that $\sum_{k=1}^{n} \det(A^k) = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$.

Solution:

$$A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix} \quad |A| = \begin{vmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - 0 = \frac{1}{4}$$

$$A^2 = A \times A = \begin{pmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \alpha \\ 0 & \frac{1}{4} \end{pmatrix}$$

$$|A^2| = \begin{vmatrix} \frac{1}{4} & \alpha \\ 0 & \frac{1}{4} \end{vmatrix} = \frac{1}{4} \times \frac{1}{4} - 0 = \left(\frac{1}{4}\right)^2 = \frac{1}{4^2}$$

$$|A^k| = \frac{1}{4^k}$$

$$\text{So, } \sum_{k=1}^{n} \det(A^k) = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots.. + \frac{1}{4^n}$$

Which is a G.P with $a = \frac{1}{4}$ and $r = \frac{1}{4}$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{4}\left[1-\left(\frac{1}{4}\right)^n\right]}{1-\frac{1}{4}}$$

$$= \frac{\frac{1}{4}\left[1-\frac{1}{4^n}\right]}{\frac{3}{4}} = \frac{1}{4}\times\frac{4}{3}\left[1-\frac{1}{4^n}\right]$$

$$= \frac{1}{3}\left[1-\frac{1}{4^n}\right].$$

Question 15.
Without expanding, evaluate the following determinants:

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ (ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Solution:

(i) $$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix} = x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6 & 9 & 12 \end{vmatrix}$$

$$= (x)(3)\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 3x(0) = 0 \qquad (\therefore R_1 = R_3)$$

(ii) $$\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} R_1 = R_1 + R_2$$

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (\because R_1 = R_3)$$

Question 16.
If A is a square matrix and $|A| = 2$, find the value of $|AA^T|$.

Solution:
Given $|A| = 2$
[Property 1: The determinant of a matrix remains unaltered if its rows are changed into columns and columns. That is, $|A| = |A^T|$]
$|A^T| = |A| = 2$
$\therefore |A A^T| = |A|\,|A^T|$
$= 2 \times 2 = 4$

Question 17.
If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$, find the value of $|3AB|$.

Solution:
Given A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$
[It A is a square matrix of order n then $\det(kA) = |kA| = k^n |A|$.]
A and B are square matrices of order 3. Therefore,
AB is also a square matrix of order 3.
$|3\,AB| = 3^3\,|AB|$
$= 27\,|A|\,|B|$
$= 27 \times -1 \times 3$
$|3\,AB| = -81$

Question 18.
If $\lambda = -2$, determine the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$.

Solution:
Given $\lambda = -2$
$\therefore 2\lambda = -4;\ \lambda^2 = (-2)^2;\ 3\lambda^2 + 1 = 3(4) + 1 = 13$
$6\lambda - 1 = 6(-2) - 1 = -13$

So $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 1 \\ 4 & 0 & 13 \\ -1 & -13 & 0 \end{vmatrix}$

expanding along R_1

0(0) + 4 (0 + 13) + 1 (-52 + 0) = 52 – 52 = 0
Aliter: The determinant value of a skew-symmetric matrix is zero

Question 19.

Determine the roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

Solution:

Now $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = (5)\begin{vmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 2x & x^2 \end{vmatrix}$

(Taking 5 as a common factor from C_3)

$$= (5)(2)\begin{vmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & x & x^2 \end{vmatrix}$$

(Taking 2 as a common factor from C_2)

$$= 10\begin{vmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & x & x^2 \end{vmatrix}$$

$$= 10\begin{vmatrix} 0 & 3 & 3 \\ 0 & -1-x & 1-x^2 \\ 1 & x & x^2 \end{vmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \\ \end{matrix}$$

expanding along C_1

$10\ \{1[3(1-x^2) - (-1-x)(3)]\}$

$$= 10[3(1-x^2) + 3(1+x)]$$
$$= 10[3(1+x)(1-x) + 3(1+x)]$$
$$= 10[3(1+x)(1-x+1)]$$
$$= 30(1+x)(2-x)$$

Given the determinant value is 0
⇒ 30(1 + x) (2 – x) = 0
⇒ 1 + x = 0 or 2 – x = 0

$\Rightarrow x = -1$ or $x = 2$
So, $x = -1$ or 2.

Question 20.
Verify that det (AB) = (det A) (det B)
for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$.

Solution:

$$A = \begin{pmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{pmatrix}$$

$$\text{Now} \quad AB = \begin{pmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{pmatrix}$$

$$= \begin{bmatrix} -20 & 10 & 2 \\ 64 & 52 & 38 \\ -49 & -17 & -19 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{vmatrix}$$

(expanding along R_2)

$$= (-1)[-15 + 6] - 7[12 - 6]$$
$$= (-1)(-9) - 7(6)$$
$$= 9 - 42 = -33 = \boxed{|A| = -33} \quad \ldots(i)$$

$$|B| = \begin{vmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{vmatrix}$$

(expanding along C_3)

$$= 3(-14-36) - 0(-14-36) + 5(4+6)$$
$$= 3(-50) + 5(10) = -150 + 50 = -100$$
$$\boxed{|B| = -100} \quad ...(ii)$$

$$|AB| = \begin{vmatrix} -20 & 10 & 2 \\ 64 & 52 & 38 \\ -49 & -17 & -19 \end{vmatrix}$$

{(-20)(52) (-19) + (10)(38)(—49) + (2)(64)(-17)} – {(-49)(52) (2) + (-17)(38)(-20) + (-19)(64)(10)}

= (19760 – 18620 – 2176) – (-5096 + 12920 – 12160)

= (19760 + 5096 + 12160) – (18620 + 2176 + 12920)

= 37016 – 33716 = 3300 ….(3)

Now (1) × (2) = (3)

(i.e.,) (-33) (-100) = 3300

⇒ det (AB) = (det A), (det B)

Question 21.

Using cofactors of elements of the second row, evaluate |A|,

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

where

Solution:

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

expanding along R_2

$$\text{Cofactor of } \quad 2 = -\begin{bmatrix} 3 & 8 \\ 2 & 3 \end{bmatrix} = (-9+16) = 7$$

$$\text{Cofactor of } \quad 0 = +\begin{bmatrix} 5 & 8 \\ 1 & 3 \end{bmatrix} = 15-8 = 7$$

$$\text{Cofactor of } \quad 1 = -\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix} = -(10-3) = -7$$

$$\therefore \quad |A| = 2(7) + 0(7) + (1)(-7)$$
$$= 14 - 7 = 7$$

Ex 7.3

Solve the following problems by using factor theorem

Question 1.

Show that < $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$ = (x – a)² (x + 2a)

Solution:

$$\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$$

put $x = a$ we get $\begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix} = 0$

Since all the three rows are identical we get $(x-a)^2$ as a factor

Now, put $x = -2a$.

We get
$$\Delta = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix}$$
$$= \begin{vmatrix} 0 & a & a \\ 0 & -2a & a \\ 0 & a & -2a \end{vmatrix} \quad C_1 = C_1 + C_2 + C_3 = 0$$

⇒ (x + 2d) is a factor of A.

Now degree of Δ is 3 ($x \times x \times x = x^3$) and we have 3 factors for A
∴ There can be a constant as a factor for A.
(i.e.,) $\Delta = k(x - a)^2 (x + 2d)$
equating coefficient of x^3 on either sides we get k = 1
∴ $\Delta = (x - a)^2 (x + 2a)$

Question 2.

Show that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}$$

Put $a = 0$ in Δ

$$\text{then } \Delta = \begin{vmatrix} b+c & -c & -b \\ b-c & c & b \\ c-b & c & b \end{vmatrix} = 0 \quad (\because C_2 \text{ and } C_3 \text{ are proportional})$$

∴ $a - 0$ (i.e.,) a is a factor.

Similarly b and c are factors of Δ.

The product of the leading diagonal elements is (b + c) (c + a) (a + b)
The degree is 3. And we got 3 factors for Δ ∴ m = 3 – 3 = 0

∴ there can be a constant k as a factor for Δ.

$$(i.e.)\quad \Delta = \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = k\,abc$$

To find k:

put $a = 1, b = 1, c = 1$

$$\therefore\quad \Delta = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = k(1)(1)(1)$$

$$\Rightarrow\quad 2(4-0)\,0-0 = k$$

$$8 = k$$

So, $\Delta = 8\,abc$ $(i.e.,)$ $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8\,abc$

Question 3.

Solve $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0.$

Solution:

Put $x = -(a+b+c)$

$$\text{Then}\quad \Delta = \begin{vmatrix} -(a+b+c)+a & b & c \\ a & -(a+b+c)+b & c \\ a & b & -(a+b+c)+c \end{vmatrix}$$

$$= \begin{vmatrix} -b-c & b & c \\ a & -a-c & c \\ a & b & -a-b \end{vmatrix} C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 0 & b & c \\ 0 & -a-c & c \\ 0 & b & -a-b \end{vmatrix} = 0 \text{ (expanding along } C_1)$$

$$\Rightarrow \quad x = -(a+b+c) \text{ is a root.}$$

Put $x = 0$, then

$$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0 \quad (\because R_1 = R_2 = R_3)$$

⇒ x = 0, 0 are roots.
Now the degree of the leading diagonal elements is 3.
∴ the equation is of degree 3, so the roots are 0, 0, – (a + b + c)

Question 4.

Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$ = (a + b + c) (a – b) (b – c) (c – a)

Solution:

$$\text{Let} \quad \Delta = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

$$\text{Put } a = b\text{, then} \quad \Delta = \begin{vmatrix} b+c & b & b^2 \\ b+c & b & b^2 \\ 2b & c & c^2 \end{vmatrix} = 0 \ (\because R_1 = R_2)$$

⇒ (a – b) is a factor of Δ.
Similarly, (b – c) and (c – a) are factors of Δ.

The degree of the product of elements along the leading diagonal is 1 + 1 + 2 = 4 and we got 3 factors for Δ.
m = 4 – 3 = 1

∴ There can be one more factor symmetric with a, b, c which is of the form k (a

+ b + c).

$$\text{Now } \Delta = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = k(a+b+c)(a-b)(b-c)(c-a)$$

To find k : Put $a = 0, b = 1, c = 2$

$$\text{We get } \Delta = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = k(0+1+2)(0-1)(1-2)(2-0)$$

(*i.e.,*) $3(4-2) = k(3)(-1)(-1)(2)$ (*i.e.,*) $6 = 6k \Rightarrow k = 1$

$$\therefore \Delta = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$$

Question 5.

Solve $\begin{vmatrix} \mathbf{4-x} & \mathbf{4+x} & \mathbf{4+x} \\ \mathbf{4+x} & \mathbf{4-x} & \mathbf{4+x} \\ \mathbf{4+x} & \mathbf{4+x} & \mathbf{4-x} \end{vmatrix} = \mathbf{0}.$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

Put $x = -12$. We get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ -8 & 16 & -8 \\ -8 & -8 & 16 \end{vmatrix} = 0$$

$\Rightarrow \quad x = -12$ is a root.

Put $x = 0$ we get $$\Delta = \begin{vmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0$$

all the 3 rows are identical

$$\Rightarrow \quad (x-0)^2 = x^2 \text{ is a factor}$$

So, $$\Delta = 0 \Rightarrow x^2 (x+12) = 0$$

$$\Rightarrow \quad x = 0, 0, -12.$$

Question 6.

Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$

Solution:

Let $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Put $x = y$

We get $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ y & y & z \\ y^2 & y^2 & z^2 \end{vmatrix} = 0 \ (\because C_1 = C_2)$$

$\Rightarrow (x - y)$ is a factor of Δ.

Similarly $(y - z)$ and $(z - x)$ are factors of Δ.
Now degree of $\Delta = 0 + 1 + 2 = 3$ and we have 3 factors of Δ.
and so there can be a constant k as a factor of Δ.

$$\text{So,} \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = k(x-y)(y-z)(z-x)$$

put $x = 0, y = 1$ and $z = 2$ we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{vmatrix} = k(0-1)(1-2)(2-0)$$

$\Rightarrow 2k = 2 \Rightarrow k = 1$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

Ex 7.4

Question 1.
Find the area of the triangle whose vertices are (0, 0), (1, 2) and (4, 3)

Solution:
Area of triangle with vertices

$0(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

∴ Area of A with vertices (0, 0), (1, 2) and (4, 3) is

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\{1(3-8)\}$$

$$= \frac{1}{2}(-5) = 5/2 \text{ sq. unit} = 2.5 \text{ sq. unit}$$

(as the area cannot be negative).

Question 2.
If (k, 2), (2, 4) and (3, 2) are vertices of the triangle of area 4 square units then

determine the value of k.

Solution:

Area of Δ with vertices $(k, 2)$ $(2, 4)$ and $(3, 2)$ $= \frac{1}{2}\begin{vmatrix} k & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 4$(given)

$$\Rightarrow \quad \begin{vmatrix} k & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 2(4) = 8$$

$(i.e.,)$ $k(4 - 2) - 2(2 - 3) + 1(4 - 12) = \pm 8$

$(i.e.,)$ $2k - 2(-1) + 1(-8) = \pm 8$

$(i.e.,)$ $2k + 2 - 8 = 8$	$2k + 2 - 8 = -8$
$(i.e.,)$ $2k = 8 + 8 - 2 = 14$	$\Rightarrow 2k = -8 + 8 - 2$
$k = 14/2 = 7$	$2k = -2$
$\therefore k = 7.$	$k = -1$
	So $k = 7$ (or) $k = -1$

Question 3.
Identify the singular and non-singular matrices:

$$(i) \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad (ii) \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} \qquad (iii) \begin{vmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{vmatrix}$$

Solution:
(i) For a given square matrix A,
1. If $|A| = 0$ then it is a singular matrix.
2. If $|A| \neq 0$ then it is a non singular matrix.

$$(i) \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48) - 2(36-42) + 3(32-35)$$

$$= 1(-3) - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9 = 0$$

⇒ A is a singular matrix.

(*ii*) Let $A = \begin{pmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{pmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} \text{ (expanding along } R_2\text{)}$$

$$= -6(21-25) - 4(10+3)$$

$$= -6(-4) - 4(13)$$

$$= 24 - 52 = -28 \neq 0 \quad \therefore \text{ A is a non singular matrix}$$

(*iii*) Let $A \begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

Which is a skew symmetric matrix

∴ $|A| = 0$ ⇒ A is a singular matrix.

Question 4.

Determine the value of a and b so that the following matrices are singular:

(*i*) $A = \begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$ (*ii*) $B = \begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

Solution:

(i) $$A = \begin{pmatrix} 7 & 3 \\ -2 & a \end{pmatrix}$$

Given A is a singular matrix

$$\Rightarrow \quad |A| = 0 \Rightarrow \begin{vmatrix} 7 & 3 \\ -2 & a \end{vmatrix} = 0$$

(i.e.) $7a + 6 = 0$

$$7a = -6 \Rightarrow a = -6/7.$$

$$B = \begin{pmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{pmatrix}$$

Given B is a singular matrix $\Rightarrow |B| = 0$

$$\text{Now} \quad |B| = \begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} b & 0 & 7 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} (R_1 \Rightarrow R_1 + R_3)$$

expanding along R_1

$b(4 + 4) + 7(-6 - 1) = 0$ (given)

$8b + 7(-7) = 0$

(i.e.,) $8b - 49 = 0 \Rightarrow 8b = 49 \Rightarrow b = 49/8$

Question 5.

If cos 2θ = 0, determine $\begin{bmatrix} \theta & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}^2$

Solution:

Given $\cos 2\theta = 0$

$$\Rightarrow \quad 2\theta = \pi/2 \Rightarrow \theta = \pi/4$$

$$\therefore \quad \cos\theta = \cos\pi/4 = 1/\sqrt{2}$$

and

$$\sin\theta = \sin\pi/4 = 1/\sqrt{2}$$

$$\text{Let } \Delta = \begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= 0\,() - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - 0\right) + \frac{1}{\sqrt{2}}\left(0 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{-1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{-2}{2\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}$$

$$\therefore \quad \Delta^2 = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Question 6.

Find the value of the product; $\begin{bmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{bmatrix} \times \begin{bmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{bmatrix}$

Solution:

$$\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_3 64 \times \log_2 3 + \log_4 3 \times \log_3 4 & \log_3 64 \times \log_8 3 + \log_3 64 + \log_3 4 \\ \log_3 8 \times \log_2 3 + \log_4 9 \times \log_3 4 & \log_3 8 \times \log_8 3 + \log_4 9 \times \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_2 64 + 1 & \log_8 64 + 1 \\ \log_2 8 + \log_3 9 & 1 + \log_3 9 \end{vmatrix}$$

$$\begin{bmatrix} \log_2 64 = \log_2 2^6 = 6 \log_2 2 = 6 \\ \log_8 64 = \log_8 8^2 = 2 \log_8 8 = 2 \\ \log_2 8 = \log_2 2^3 = 3 \log_2 2 = 3 \\ \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2 \end{bmatrix}$$

$$= \begin{vmatrix} 6+1 & 2+1 \\ 3+2 & 1+2 \end{vmatrix} = \begin{vmatrix} 7 & 3 \\ 5 & 3 \end{vmatrix} = 21 - 15 = 6.$$

Ex 7.5

Question 1.

If $a_{ij} = \frac{1}{2}(3i - 2j)$ and $A = [a_{ij}]_{2\times2}$ is

Solution:

(*b*) Hint: $a_{ij} = \frac{1}{2}(3i - 2j)$

A is a matrix of order 2 × 2 $\therefore A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$a_{11} = \frac{1}{2}(3-2) = \frac{1}{2}\ ;\ a_{12} = \frac{1}{2}(3-4) = -\frac{1}{2};$$

$$a_{21} = \frac{1}{2}(6-2) = \frac{4}{2} = 2 : a_{22} = \frac{1}{2}(6-4) = \frac{2}{2} = 1$$

$$\therefore A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 2 & 1 \end{pmatrix}$$

Question 2.

What must be the matrix X, if $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -6 \\ 4 & -2 \end{bmatrix}$

Solution:

(a) Hint : $2X + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 7 & 2 \end{pmatrix}$

$$\Rightarrow \quad 2X = \begin{pmatrix} 3 & 8 \\ 7 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3-1 & 8-2 \\ 7-3 & 2-4 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & -2 \end{pmatrix}$$

$$\therefore \; X = \frac{1}{2}\begin{pmatrix} 2 & 6 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

Question 3.

Which one of the following is not true about the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?

(a) a scalar matrix
(b) a diagonal matrix
(c) an upper triangular matrix
(d) A lower triangular matrix

Solution:
(b) a diagonal matrix

Question 4.
If A and B are two matrices such that A + B and AB are both defined, then …………
(a) A and B are two matrices not necessarily of same order.
(b) A and B are square matrices of same order.
(c) Number of columns of a is equal to the number of rows of B.
(d) A = B.

Solution:
(b) A and B are square matrices of same order.

Question 5.

If $A = \begin{bmatrix} \lambda & 1 \\ -1 & -\lambda \end{bmatrix}$, then for what value of λ, $A^2 = 0$?

(a) 0
(b) ± 1
(c) -1
(d) 1

Solution:

(b) Hint: $A^2 = A \times A = \begin{pmatrix} \lambda & 1 \\ -1 & -\lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

$$= \begin{pmatrix} \lambda^2 - 1 & 0 \\ 0 & \lambda^2 - 1 \end{pmatrix} = 0 \text{ given}$$

$\Rightarrow \quad \lambda^2 - 1 = 0 \Rightarrow \lambda^2 = 1; \lambda = \pm 1$

Question 6.

If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$; and $(A + B)^2 = A^2 + B^2$, then the values of a and b are ....

(a) a = 4, b = 1
(b) a = 1, b = 4
(c) a = 0, b = 4
(d) a = 2, b = 4

Solution:

(b) Hint: $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}; B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$

$$A + B = \begin{pmatrix} 1+a & 0 \\ 2+b & -2 \end{pmatrix}$$

$$(A + B)^2 = \begin{pmatrix} 1+a & 0 \\ 2+b & -2 \end{pmatrix}\begin{pmatrix} 1+a & 0 \\ 2+b & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (1+a)^2 & 0 \\ (1+a)(2+b) - 2(2+b) & 4 \end{pmatrix} \qquad \text{....(1)}$$

$$A^2 = A \times A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B^2 = B \times B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}\begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix} = \begin{pmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{pmatrix}$$

$$\text{Now } A^2 + B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{pmatrix} \qquad \text{...(2)}$$

Given $\quad (A+B)^2 = A^2 + B^2$

$\Rightarrow \quad (1) = (2)$

$$\begin{pmatrix} (1+a)^2 & 0 \\ (1+a)(2+b) - 2(2+b) & 4 \end{pmatrix} = \begin{pmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{pmatrix}$$

$\Rightarrow \quad a - 1 = 0$

$\Rightarrow a = 1$

$\therefore a = 1, b = 4$

Question 7.

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to ………….

(a) (2, -1)
(b) (-2, 1)
(c) (2, 1)
(d) (-2, -1)

Solution:

(*d*) Hint: $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{pmatrix}$

So, $A^T = \begin{pmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{pmatrix}$

Given $AA^T = 9I$

$$= 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{pmatrix}\begin{pmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$\Rightarrow$ $a + 2b = -4$ ...(1)

and $2a - 2b = -2$ ...(2)

$(1) + (2) \Rightarrow$ $3a = -6 \Rightarrow a = -6/3 = -2$

Substituting $a = -2$ in (1) we get

$-2 + 2b = -4$

$2b = -4 + 2 = -2$

$b = -2/2 = -1$

$\therefore a = 2, 1$ $b = -1$

Question 8.

If A is a square matrix, then which of the following is not symmetric?

(a) $A + A^T$

(b) AA^T

(c) A^TA

(d) $A - A^T$

Solution:

(b) AA^T

Question 9.
If A and B are symmetric matrices of order n, where ($A \neq B$), then …………….
(a) A + B is skew-symmetric
(b) A + B is symmetric
(c) A + B is a diagonal matrix
(d) A + B is a zero matrix

Solution:
(b) A + B is symmetric

Question 10.
If $A = \begin{bmatrix} a & x \\ y & a \end{bmatrix}$ and if xy = 1, then det (AA^T) is equal to ………….
(a) $(a - 1)^2$
(b) $(a^2 + 1)^2$
(c) $a^2 - 1$
(d) $(a^2 - 1)^2$

Solution:
(*d*) Hint:
$$A = \begin{pmatrix} a & x \\ y & a \end{pmatrix} \therefore A^T = \begin{pmatrix} a & y \\ x & a \end{pmatrix}$$
$$|A| = \begin{vmatrix} a & x \\ y & a \end{vmatrix} = a^2 - xy = a^2 - 1$$
$$|A^T| = \begin{vmatrix} a & y \\ x & a \end{vmatrix} = a^2 - xy = a^2 - 1$$
$$\therefore \quad |AA^T| = |A||A^T| = (a^2 - 1)(a^2 - 1) = (a^2 - 1)^2$$

Question 11.
The value of x, for which the matrix $A = \begin{bmatrix} e^{x-2} & e^{7+x} \\ e^{2+x} & e^{2x+3} \end{bmatrix}$ is singular is …………
(a) 9
(b) 8
(c) 7
(d) 6

Solution:
(b) Hint: Given A is a singular matrix ⇒ $|A| = 0$

(i.e.) $\begin{vmatrix} e^{x-2} & e^{7+x} \\ e^{2+x} & e^{2x+3} \end{vmatrix} = 0$

⇒ $e^{x-2}.e^{2x+3} - e^{2+x}.e^{7+x} = 0$
⇒ $e^{3x+1} - e^{9+2x} = 0$
⇒ $e^{3x+1} = e^{9+2x}$
⇒ $3x + 1 = 9 + 2x$
⇒ $3x - 2x = 9 - 1$
⇒ $x = 8$

Question 12.
If the points (x, -2), (5, 2), (8, 8) are collinear, then x is equal to …………
(a) -3
(b) 1/3
(c) 1
(d) 3

Solution:
(d) Hint: Given that the points are collinear
So, area of the triangle formed by the points = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(2-8) + 2(5-8) + 1(40-16) = 0$$
$$-6x - 6 + 24 = 0$$
$$-6x = -18 \Rightarrow x = 3$$

Question 13.

If $\begin{bmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{bmatrix} = \frac{abc}{2} \neq 0$, then the area of the triangle whose vertices are $\left(\frac{x_1}{a}, \frac{y_1}{a}\right), \left(\frac{x_2}{b}, \frac{y_2}{b}\right), \left(\frac{x_3}{c}, \frac{y_3}{c}\right)$ is ....

(a) $\frac{1}{4}$ (b) $\frac{1}{4}abc$ (c) $\frac{1}{8}$ (d) $\frac{1}{8}abc$

Solution:

(c) Hint: Given $\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2}$

$$\Rightarrow \quad 2\begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2} \Rightarrow \begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix} = \frac{abc}{4}$$

Now the area of the triangle with vertices $\left(\frac{x_1}{a}, \frac{y_1}{a}\right)\left(\frac{x_2}{b}, \frac{y_2}{b}\right)$ and $\left(\frac{x_3}{c}, \frac{y_3}{c}\right)$

$$\frac{1}{2}\begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{a} & 1 \\ \frac{x_2}{b} & \frac{y_2}{b} & 1 \\ \frac{x_3}{c} & \frac{y_3}{c} & 1 \end{vmatrix}$$

multiplying R_1 by a, R_2 by b and R_3 by c and dividing by abc we get

$$\frac{1}{2abc}\begin{vmatrix} x_1 & y_1 & a \\ x_2 & y_2 & b \\ x_3 & y_3 & c \end{vmatrix}$$

Interchanging C_1 and C_3, then C_2 and C_3

we get $$\frac{1}{2abc}(-)(-)\begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2abc} \times \frac{abc}{4} = \frac{1}{8}$$

Question 14.

If the square of the matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is the unit matrix of order 2, then α, β and γ should satisfy the relation.

(a) $1 + \alpha^2 + \beta\gamma = 0$
(b) $1 - \alpha^2 - \beta\gamma = 0$
(c) $1 - \alpha^2 + \beta\gamma = 0$
(d) $1 + \alpha^2 - \beta\gamma = 0$

Solution:

(b) Hint: Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$

Given A^2 is a unit matrix

$$\Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha^2 + \beta\gamma & \alpha\beta - \alpha\beta \\ \alpha\gamma - \alpha\gamma & \beta\gamma + \alpha^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \alpha^2 + \beta\gamma = 1 \Rightarrow 1 - \alpha^2 - \beta\gamma = 0$$

Question 15.

If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$ is ....

(a) Δ
(b) kΔ
(c) 3kΔ
(d) $k^3\Delta$

Solution:

(d) Hint: Given $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$

$$\therefore \begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix} = (k)(k)(k)\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$= k^3\Delta$$

Question 16.

A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is ……………

(a) 6
(b) 3
(c) 0
(d) -6

Solution:

(c) Hint

$$\text{Let}\Delta = \begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix}$$

$$= \begin{vmatrix} -x & -x & -x \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} R_1 \to R_1 + R_2 + R_3$$

Put $x = 0$ we get $\Delta = 0$

$\therefore$ $x = 0$ is a root

Question 17.

The value of the determinant of $A = \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix}$ is …………..

(a) -2abc

(b) abc
(c) 0
(d) $a^2 + b^2 + c^2$

Solution:

(*c*) Hint: $$|A| = \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$$

($\because$ the determinant of a skew symmetric matrix is 0)

Question 18.
If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in geometric progression with the same common ratio, then the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are
(a) vertices of an equilateral triangle
(b) vertices of a right-angled triangle
(c) vertices of a right-angled isosceles triangle
(d) collinear

Solution: (d) collinear

Question 19.

If $\lfloor . \rfloor$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0$, $0 \le y < 1$, $1 \le z \le 2$, then the value of the determinant $\begin{vmatrix} \lfloor x \rfloor + 1 & \lfloor y \rfloor & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor + 1 & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor & \lfloor z \rfloor + 1 \end{vmatrix}$ is ………….

(a) $\lfloor z \rfloor$

(b) $\lfloor y \rfloor$

(c) $\lfloor x \rfloor$

(d) $\lfloor x \rfloor + 1$

Solution:

(a) Hint: From the given values

$$\lfloor x \rfloor = -1 \; ; \lfloor y \rfloor = 0 \text{ and } \lfloor z \rfloor = 1$$

$$\therefore \text{ The given determinant is } \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1(0+1) = 1 = \lfloor z \rfloor$$

Question 20.

If a ≠ b, b, c satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0,$ then abc = ……………..

(a) a + b + c
(b) 0
(c) b^3
(d) ab + bc

Solution:

(c) Hint: Expanding along R_1,
$a(b^2 - ac) - 2b(3b - 4c) + 2c(3a - 4b) = 0$
$(b^2 - ac)(a - b) = 0$
$b^2 = ac$ (or) $a = b$
$\Rightarrow abc = b(b^2) = b^3$

Question 21.

If $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$, then B is given by ……………….

(a) B = 4A
(b) B = -4A
(c) B = -A
(d) B = 6A

Solution:

(b) Hint : $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$

Now $B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix} = (2)(2) \begin{vmatrix} -2 & 4 & 2 \\ 3 & 1 & 0 \\ -1 & 2 & 4 \end{vmatrix}$

$= 4 \begin{vmatrix} -2 & 4 & 2 \\ 3 & 1 & 0 \\ -1 & 2 & 4 \end{vmatrix} R_1 \leftrightarrow R_3$

$= (-4) \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix} = -4A \Rightarrow B = -4A$

Question 22.
If A is skew-symmetric of order n and C ¡s a column matrix of order n × 1, then C^T AC is ……………..
(a) an identity matrix of order n
(b) an identity matrix of order 1
(e) a zero matrix of order I
(d) an Identity matrix of order 2

Solution:
(c) a zero matrix of order I
Hint: Given A is of order n × n
C is of order n × 1
so, CT is of order 1 × n

Now C^TAC is of order $\begin{pmatrix} 1\times n & & \\ & n\times n & \\ & & n\times 1 \end{pmatrix} = 1\times 1$

Let it be equal to (x) say
Taking transpose on either side
$(C^T, AC)^T$ $(x)^T$.
(i.e.) $C^T(A^T)(C) = x$
$C^T(-A)(C) = x$

$\Rightarrow C^TAC = -x$
$\Rightarrow x = -x$
$\Rightarrow 2x = 0$
$\Rightarrow x = 0$

Question 23.

The matrix A satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is ....

(a) $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -4 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

Solution:

(c) Hint: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Given $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+3c & b+3d \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow c = 0$ and $d = -1$

$a + 3c = 1, \quad b + 3d = 1$

$\Rightarrow a + 0 = 1, \quad b + 3(-1) = 1$

$a = 1, \quad b - 3 = 1, b = 4$

$\therefore A = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}$

Question 24.

If $A + I = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, then (A + I) (A – I) is equal to ....

(a) $\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 4 \\ -8 & 9 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & -4 \\ -8 & -9 \end{bmatrix}$

Solution:

(a)

$$\text{Hint: } A + I = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$

$$\Rightarrow \quad A + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$

$$\therefore \quad A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 4 & 0 \end{pmatrix}$$

$$\text{Now} \quad A^2 = A \times A = \begin{pmatrix} 2 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -4 \\ 8 & -8 \end{pmatrix}$$

$$\therefore \quad (A + I)(A - I) = A^2 - I^2 = A^2 - I$$

$$= \begin{pmatrix} -4 & -4 \\ 8 & -8 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -4 \\ 8 & -9 \end{pmatrix}$$

Question 25.

Let A and B be two symmetric matrices of the same order. Then which one of the following statements is not true?

(a) A + B ¡s a symmetric matrix

(b) AB ¡s a symmetric matrix

(c) $AB = (BA)^T$

(d) $A^TB = AB^T$

Solution:

(b) AB ¡s a symmetric matrix