Unit 3

Matrices and Determinants

Teaching-Learning Points

● A matrix is an ordered rectangular array (arrangement) of numbers and enclosed by capital bracket [ ]. These numbers are called elements of the matrix. Matrix is denoted by capital letters of the English alphabet and its elements are denoted by small letters.

● In a matrix horizontal lines of numbers are called rows of the matrix and vertical lines are called columns of the matrix.

● A matrix having m rows and n columns is called a matrix of order m by n, written as m × n. In general a matrix of order m × n is written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i_1} & a_{i_2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{m_1} & a_{m_2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

(OR) $A = [a_{ij}]_{m \times n}$ where $1 \le i \le m$, $1 \le J \le n$, $m, n \in N$

● Types of matrices : A matrix $A = [a_{ij}]_{m \times n}$ is said to be a:

(i) Row matix if m = 1

(ii) Column matrix if n = 1

(iii) Zero/Null matrix if each of its elements is zero.

(iv) Square matrix if m = n

(v) Diagonal matrix, if m = n and $a_{ij} = 0$ when $i \neq J$

(vi) Scalar matrix, if m = n, $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = k$ when i = J

(vii) Unit/identity matrix if m = n, $a_{ij} = 0$ when i ¹ j

● Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]$ p × q are said to be equal (i.e. A = B) if m = p, n = q and $a_{ij} = b_{ij}$ for all i, j.

● Scalar multiplication of a matrix.

Let $A = [a_{ij}]_{m \times n}$ and k be any scalar,

then $KA = K[a_{ij}]_{m \times n} = [Ka_{ij}]_{m \times n}$.

i.e. to multiply a matrix by a scalar multiply each element of the matrix by the scalar.

● Addition of matrices:

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then

$A + B = [a_{ij} + b_{ij}]_{m \times n}$

● Properties of matrix addition.

(i) A + B = B + A (commutative law)

(ii) (A + B) + C = A + (B + C) [Associative law]

(iii) A + O = A = O + A, where O is Nuel matrix (existence of additive identity)

(iv) A + (–A) = O = (–A) + A [existence of additive inverse]

● A – B = A + (–B).

● Multiplication of matrices.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ be two matrices, then the product of A and B (i.e. AB) is defined if n = p and it is a matrix of order m × q.

● Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, then

A B = C (say) = $[c_{ij}]_{m \times b}$, where c_{ij} is obtained by taking ith row of A and jth column of B, multiplying their corresponding elements and taking sum of these produce.

(i) In general, AB ≠ BA.

(ii) (AB)C = A(BC)

(iii) A.I = IA where I is identity matrix.

(iv) A(B + C) = AB + AC (or) (A + B) C = AC + BC. (Distributive Law)

● Transpose of a matrix A is obtained by interchanging its rows and columns. It is denoted by A^1.

● Properties of transpose:

(i) $(KA)^1 = KA^1$ (ii) $(A + B)^1 = A^1 + B^1$

(iii) $AB)^1 = B^1A^1$ (iv) $(A^1)^1 = A$.

● A square matrix is called, a symmetric matrix if $A^1 = A$ and a skew symmetric matrix if $A^1 = -A$.

● For any square matrix A, $A + A^1$ is always symmetric matrix and $A - A^1$ is always skew symmetric matrix.

● Every square matrix can be expressed as a sum of symmetric and skew symmetric matrix.

i.e. $A = \left(\frac{A + A^1}{2}\right) + \left(\frac{A - A^1}{2}\right)$

● Elementary operations (transformations) of a matrix.

(i) Interchange of any two rows (or two columns)

i.e. $R_i \leftrightarrow R_j$ (or) $C_i \leftrightarrow C_j$

(ii) Multiplication of the elements of any row (or any column) by a non zero number i.e.

i.e. $R_i \to KR_i$ (or) $C_i \to KC_i$

(iii) Addition of the elements of any row (or any column) to the corresponding elements of any other row (or column) multiplied by any non zero number i.e.

i.e. $R_i \to R_i + KR_j$ (or) $C_i = C_i + KC_i$

● A square matrix A is said to be invertible if there exists another square matrix B of the same order such that AB = I = BA, then B is called the inverse of A and is denoted by A^{-1}.

● Properties of inverse of matrix.

(i) $AB = I \Rightarrow B = A^{-1}$ and $A = B^{-1}$ (ii) $(A^{-1})^{-1} = A$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$ (iv) $(A^{-1})^1 = (A^1)^{-1}$

● For finding the inverse of a square matrix by using.

(i) elementary row operations (transformations) we write A = IA.

(ii) elementary column operations (transformations) we write A = AI.

● A number which is associated to a square matrix $A = [a_{ij}]_{n \times n}$ is called a determinant of the matrix A and it is denoted as |A|.

● Determinant ca be expanded by using any row or column.

● Properties of determinants:

(i) $|A^1| = |A|$.

(ii) $R_i \leftrightarrow R_j$ (or) $C_i \leftrightarrow C_j \Rightarrow |A| = -|A|$

(iii) If R_i is identical to R_j or C_i is identical to $C_j \Rightarrow |A| = 0$.

(iv) $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k|A|$

In general $\left|kA_{n \times n}\right| = k^n \left|A_{n \times n}\right|$

(v) $R_i \to R_i + kR_j$ (or) $C_i = C_i + kC_j$

$\Rightarrow |A|$ remains unchanged

(vi) If some or all elements of a row (or column) of a determinant are expressed as the sum two or more terms, then the determinant can be expressed as a sum of two or more determinants.

● $|AB| = |A|\,|B|$

● $|A^{-1}| = \dfrac{1}{|A|}$

● A square matrix is said to be singular if $|A| = 0$ and non singular if $|A| \neq 0$

● Using determinants Area if a ΔABC with vertices $A(x_1\ y_1)$, $B(x_2\ y_2)$, $C(x_3\ y_3)$ is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 3 \end{vmatrix}$$

● Minor of an element a_{ij} of a determ inants is the determinant obtained by deleting the ith row and yth column in which elements a_{ij} lies. It is denoted by M_{ij}.

● Cofactor of an element a_{ij} of a determinant is denoted as A_{ij} and is defined as $A_{ij} = (-1)^{i + y} M_{ij}$

● Adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$ where A_{ij} is the cofactor of the element a_{ij} and it is denoted as adj.A

● A(adjA) = (adjA) A = |A| I

● $|adjA| = |A|^{n-1}$, where A is a square matrix of order n.

● A square matrix A is invertible if and only |A| ≠ 0 if A is a non singular matrix

$A^{-1} = \frac{adj A}{|A|}$.

● If A and B are non singular matrices of the same order then AB and BA are also non singular matrices of the same order.

● A system of linear equations in three variables can be expressed in a matrix equation.

For example, $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

● By solving the matrix equation AX = B or $X = A^{-1}B$ we get solution of given system of equations.

● If |A| ≠ 0, then the system of equations has unique solution and hence it is consistent.

● If |A| = 0 and (adjA) B ≠ 0 (zero matrix), then the system of equations has no solution and hence it is inconsistent.

● If |A| = 0 and (adjA) B = 0 (zero matrix), then the system of equations may be either consistent or inconsistent according as the system has either infinitely many solutions or no solution.

Question for Practice

Very short answer questions carrying one mark each:

1. construct a 2 × 2 matrix A = [a_{ij}], where $a_{iJ} = 3\hat{i} - \hat{J}$

2. If $\begin{bmatrix} a+b & 2 \\ 5 & a \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, find the values of a and b.

3. If $a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find a and b.

4. If a matrix has 12 elements, then how many possible orders it can have.

5. If A = $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, $0 < x < \frac{\pi}{2}$ and $A + A^1 = I$, then find the value of x.

6. If $\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$ = P + θ where P is symmetric and θ is skew symmetric matrix, find θ.

7. Evaluate $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

8. Find the value of x if $\begin{vmatrix} x & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$

9. What is the value of the determinant $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 11 & 13 & 15 \end{vmatrix}$

10. Find cofactor of a_{23} in $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

11. Find the minor of a_{12} in the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -3 & 2 & -1 \end{vmatrix}$

12. Let A be a square matrix of order 3 and $|A| = 5$, find the value of $|2A|$.

13. Let A be a square matrix of order 3×3 and $|A| = 2$, then find $|\text{adj}A|$.

14. If A is invertible matrix of order 2 and $|A| = -11$, then find $|A^{-1}|$.

15. If a singular matrix A is given by $\begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}$, then find the value of x.

Short answer questions carrying 4 marks each:

16. Express the following matrix as a sum of a symmetric and a skew symmetric matrix.

$$\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$$

17. Given that A = $\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the real number k such that $A^2 - kA + 2I = 0$.

18. Given that A = $\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & 5 \end{bmatrix}$, verify that $(AB)^1 = B^1A^1$.

19. If A = $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ show that $A^2 - 6A + 17\,I = 0$. Hence find A^{-1}.

20. Show that x = 2 is one of the roots of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$ = 0. Find other roots also.

21. Using properties of determinants show that:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

22. $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

23. $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x + y + z)^3$

24. $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a + b + c)(a^2 + b^2 + c^2)$

25. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2$.

Long Answer type questions carrying 6 marks each:

26. Find the inverse of the matrix using elementary transformations: $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

27. Solve the following system of linear equations using inverse of matrix.

2x + 3y + 4z = 8, 3x + y – z = 2, 4x – y – 5z = –9.

28. If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Hence solve the following system of linear equations:

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

29. If A = $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & -1 \end{bmatrix}$, find A^{-1} and hence solve the following system of linear equations:

$x + 2y + 5z = 10$ [Hint : $(A^1)^{-1} = (A^{-1})^1$

$x - y - z = -2$

$2x + 3y - z = -11$

30. If A = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$. Find AB and hence solve the following system of equation.

$x - y + 2z = 1$ [Hint : $AB = I \Rightarrow A^{-1} = B$]

$2y - 3z = 1$

$3x - 2y + 4z = 2$

Answers

1. $\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ 2. a = 8, b = –2 3. a = 3, b = –4 4. 6

5. $\frac{\pi}{3}$ 6. $\frac{1}{2}\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ 7. 1 8. 1
9. 0 10. –1 11. 10 12. 40

13. 4 14. $\frac{-1}{11}$ 15. 2

16. $\begin{bmatrix} 1 & -3/2 & 1/2 \\ -3/2 & 8 & 9/2 \\ 1/2 & 9/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 9/2 & 9/2 \\ -9/2 & 0 & -3/2 \\ -9/2 & 3/2 & 0 \end{bmatrix}$ 18. 1 19. $A^{-1} = \dfrac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

20. 1, –3.

26. $\begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix}$ 27. $x = 1$, $y = -2$, $z = 3$. 28. $\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$ $x = 1$, $y = -2$, $z = 3$.

29. $A^{-1} = \dfrac{1}{27}\begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}$ $x = -1$, $y = -2$, $z = -3$. 30. $x = 0$, $y = 5$, $z = 3$.