CHAPTER 6

Term-I

APPLICATIONS OF DERIVATIVES

Syllabus

➢ ***Applications of derivatives: increasing/decreasing functions, tangents & normals, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).***

STAND ALONE MCQs (1 Mark each)

Q. 1. A ladder, 5 metre long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metre from the wall is :

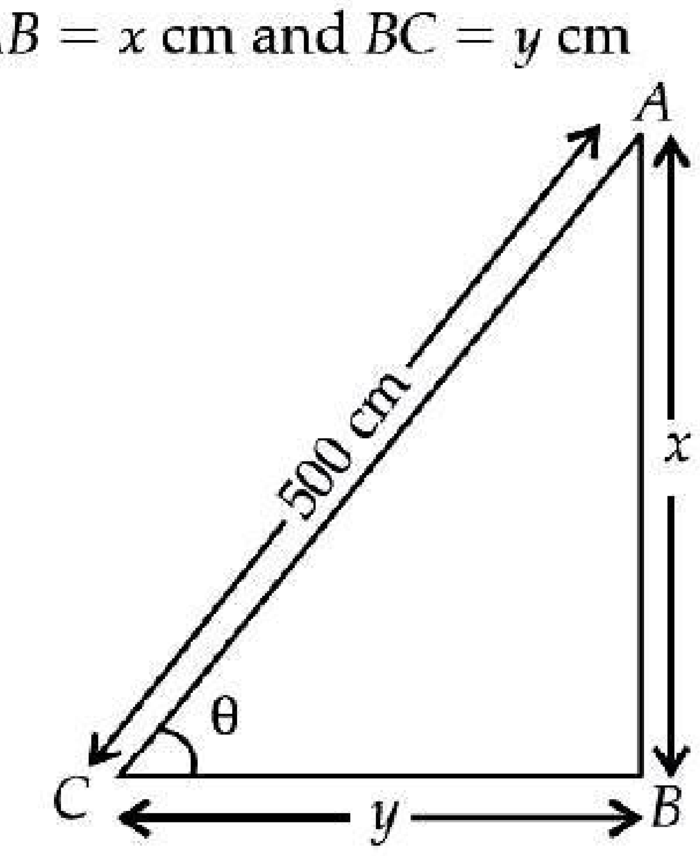
(A) $\frac{1}{10}$ radian/sec **(B)** $\frac{1}{20}$ radian/sec

(C) 20 radian/sec **(D)** 10 radian/sec

Ans. Option (B) is correct.

Explanation : Let the angle between floor and the ladder be θ.

Let $AB = x$ cm and $BC = y$ cm



$\therefore \quad \sin\theta = \frac{x}{500}$ and $\cos\theta = \frac{y}{500}$

$\Rightarrow \quad x = 500\sin\theta$ and $y = 500\cos\theta$

Also, $\frac{dx}{dt} = 10$ cm/s

$\Rightarrow \quad 500.\cos\theta.\frac{d\theta}{dt} = 10$ cm/s

$\Rightarrow \quad \frac{d\theta}{dt} = \frac{10}{500\cos\theta} = \frac{1}{50\cos\theta}$

For $y = 2$ m $= 200$ cm,

$$\frac{d\theta}{dt} = \frac{1}{50.\frac{y}{500}} = \frac{10}{y} = \frac{10}{200} = \frac{1}{20} \text{ rad/s}$$

Q. 2. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then at $x = 3$ the slope of curve is changing at ________ units/sec.

(A) –72 **(B)** –36

(C) 24 **(D)** 48

Ans. Option (A) is correct.

Explanation: Given

curve is $y = 5x - 2x^3$

or $\frac{dy}{dx} = 5 - 6x^2$

or $m = 5 - 6x^2$ $\left[\text{slope } m = \frac{dy}{dx}\right]$

$\frac{dm}{dt} = -12x\frac{dx}{dt}$

$= -24x$

$\left.\frac{dm}{dt}\right|_{x=3} = -72$

Q. 3. The contentment obtained after eating x units of a new dish at a trial function is given by the function $f(x) = x^3 + 6x^2 + 5x + 3$. The marginal contentment when 3 units of dish are consumed is ____________.

(A) 60 **(B)** 68

(C) 24 **(D)** 48

Ans. Option (B) is correct.

Explanation:

$f(x) = x^3 + 6x^2 + 5x + 3$

$\frac{df(x)}{dx} = 3x^2 + 12x + 5$

At $x = 3$,

Marginal contentment

$= 3 \times (3)^2 + 12 \times 3 + 5$

$= 27 + 36 + 5$

$= 68$ units.

Q. 4. A particle moves along the curve $x^2 = 2y$. The point at which, ordinate increases at the same rate as the abscissa is ________

(A) $(1, 2)$ **(B)** $\left(\frac{1}{2}, 1\right)$

(C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ **(D)** $\left(1, \frac{1}{2}\right)$

Ans. Option (D) is correct.

Explanation:

$x^2 = 2y$ ...(1)

$\Rightarrow 2x\frac{dx}{dt} = 2\frac{dy}{dt}$ $\left(\text{given } \frac{dy}{dt} = \frac{dx}{dt}\right)$

$\Rightarrow 2x\frac{dx}{dt} = 2\frac{dx}{dt}$

$\Rightarrow x = 1$

from (1), $y = \frac{1}{2}$

so point is $\left(1, \frac{1}{2}\right)$

Q. 5. The curve $y = x^{1/5}$ has at (0, 0)

(A) a vertical tangent (parallel to y-axis)

(B) a horizontal tangent (parallel to x-axis)

(C) an oblique tangent

(D) no tangent

Ans. Option (A) is correct.

Explanation : Given that, $y = x^{1/5}$

On differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$

$\therefore \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5} \times (0)^{-4/5} = \infty$

So, the curve $y = x^{1/5}$ has a vertical tangent at (0, 0), which is parallel to y-axis.

Q. 6. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(A) $3x - y = 8$ **(B)** $3x + y + 8 = 0$

(C) $x + 3y \pm 8 = 0$ **(D)** $x + 3y = 0$

Ans. Option (C) is correct.

Explanation : We have, the equation of the curve is $3x^2 - y^2 = 8$ ....(i)

Also, the given equation of the line is $x + 3y = 8$.

$\Rightarrow 3y = 8 - x$

$\Rightarrow y = -\frac{x}{3} + \frac{8}{3}$

Thus, slope of the line is $-\frac{1}{3}$ which should be equal to slope of the equation of normal to the curve.

On differentiating equation (i) with respect to x, we get

$6x - 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$ = Slope of the curve

Now, slope of normal to the curve

$= -\frac{1}{\left(\frac{dy}{dx}\right)}$

$= -\frac{1}{\left(\frac{3x}{y}\right)}$

$= -\frac{y}{3x}$

$\therefore -\left(\frac{y}{3x}\right) = -\frac{1}{3}$

$\Rightarrow -3y = -3x$

$\Rightarrow y = x$

On substituting the value of the given equation of the curve, we get

$3x^2 - x^2 = 8$

$\Rightarrow 2x^2 = 8$

$\Rightarrow x^2 = 4$

$\Rightarrow \quad x = \pm 2$

For $\quad x = 2$

$3(2)^2 - y^2 = 8$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

and for $\quad x = -2,$

$3(-2)^2 - y^2 = 8$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

So, the points at which normal is parallel to the given line are (±2, ±2).
Hence, the equation of normal at (±2, ±2) is

$$\Rightarrow \quad y - (\pm 2) = -\frac{1}{3}[x - (\pm 2)]$$

$$\Rightarrow 3[y - (\pm 2)] = -[x - (\pm 2)]$$

$$\therefore \quad x + 3y \pm 8 = 0$$

Q. 7. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is :

(A) 1 (B) 0

(C) –6 (D) 6

Ans. Option (D) is correct.

Explanation : Given that, $ay + x^2 = 7$ and $x^3 = y$
On differentiating both equations with respect to x, we get

$$a.\frac{dy}{dx} + 2x = 0 \quad \text{and} \quad 3x^2 = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a} \quad \text{and} \quad \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} = m_1$$

$$\text{and} \quad \left(\frac{dy}{dx}\right)_{(1,1)} = 3.1$$

$$= 3 = m_2$$

Since, the curve cuts orthogonally at (1, 1).

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad \left(\frac{-2}{a}\right).3 = -1$$

$$\therefore \quad a = 6$$

Q. 8. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is :

(A) $x + 5y = 2$ (B) $x - 5y = 2$

(C) $5x - y = 2$ (D) $5x + y = 2$

Ans. Option (A) is correct.

Explanation : Given that the equation of curve is

$$y(1 + x^2) = 2 - x \quad \text{...(i)}$$

On differentiating with respect to x, we get

$$\therefore \quad y.(0 + 2x) + (1 + x^2).\frac{dy}{dx} = 0 - 1$$

$$\Rightarrow \quad 2xy + (1 + x^2)\frac{dy}{dx} = -1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1 - 2xy}{1 + x^2} \quad \text{...(ii)}$$

Since, the given curve passes through x-axis,

i.e., $\quad y = 0$

$$\therefore \quad 0(1 + x^2) = 2 - x$$

[By using Eq. (i)]

$$\Rightarrow \quad x = 2$$

So the curve passes through the point (2, 0).

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,\,0)} = \frac{-1 - 2 \times 0}{1 + 2^2} = -\frac{1}{5}$$

= Slope of the curve

$\therefore$ Slope of tangent to the curve $= -\frac{1}{5}$

$\therefore$ Equation of tangent to the curve passsing through (2, 0) is

$$y - 0 = -\frac{1}{5}(x - 2)$$

$$\Rightarrow \quad y + \frac{x}{5} = +\frac{2}{5}$$

$$\Rightarrow \quad 5y + x = 2$$

Q. 9. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are :

(A) (2, –2), (–2, –34) (C) (2, 34), (–2, 0)

(B) (0, 34), (–2, 0) (D) (2, 2), (–2, 34)

Ans. Option (D) is correct.

Explanation : The equation of the curve is given by

$$y = x^3 - 12x + 18$$

On differentiating with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 12$$

So, the slope of line parallel to the x-axis,

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 - 12 = 0$$

$$\Rightarrow \quad x^2 = \frac{12}{3}$$

$$= 4$$

$$\therefore \quad x = \pm 2$$

For $\quad x = 2,$

$$y = 2^3 - 12 \times 2 + 18$$

$$= 2$$

and for $\quad x = -2,$

$$y = (-2)^3 - 12 \times (-2) + 18$$

$$= 34$$

So, the points are (2, 2) and (−2, 34).

Q. 10. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at :

(A) (0, 1) (B) $\left(-\frac{1}{2}, 0\right)$

(C) (2, 0) (D) (0, 2)

Ans. Option (B) is correct.

Explanation : The equation of the curve is given by $y = e^{2x}$

Since, it passes through the point (0, 1).

$$\therefore \quad \frac{dy}{dx} = e^{2x}.2$$
$$= 2.e^{2x}$$
$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)} = 2.e^{2.0}$$
$$= 2$$

= Slope of tangent to the curve.

$\therefore$ Equation of tangent is

$$y - 1 = 2(x - 0)$$
$$\Rightarrow \quad y = 2x + 1$$

Since, tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis, *i.e.*, $y = 0$.

$$\therefore \quad 0 = 2x + 1$$
$$\Rightarrow \quad x = -\frac{1}{2}$$

So, the required point is $\left(-\frac{1}{2}, 0\right)$.

Q. 11. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is :

(A) $[-1, \infty)$ (B) $[-2, -1]$

(C) $(-\infty, -2]$ (D) $[-1, 1]$

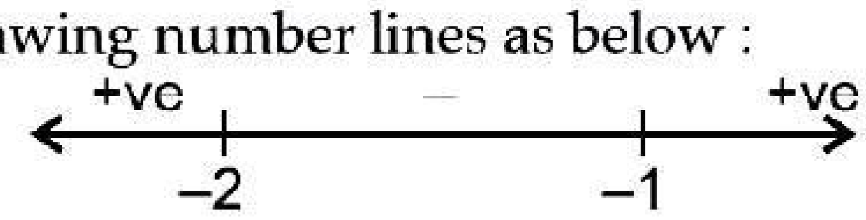
Ans. Option (B) is correct.

Explanation : Given that,

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$
$$f'(x) = 6x^2 + 18x + 12$$
$$= 6(x^2 + 3x + 2)$$
$$= 6(x + 2)(x + 1)$$

So, $f'(x) \leq 0$, for decreasing.

On drawing number lines as below :



We see that $f'(x)$ is decreasing in $[-2, -1]$.

Q. 12. $y = x(x-3)^2$ decreases for the values of x given by :

(A) $1 < x < 3$ (B) $x < 0$

(C) $x > 0$ (D) $0 < x < \frac{3}{2}$

Ans. Option (A) is correct.

Explanation : Given that,

$$y = x(x-3)^2$$

$$\therefore \quad \frac{dy}{dx} = x.2(x-3).1 + (x-3)^2.1$$
$$= 2x^2 - 6x + x^2 + 9 - 6x$$
$$= 3x^2 - 12x + 9$$
$$= 3(x^2 - 3x - x + 3)$$
$$= 3(x-3)(x-1)$$

So, $y = x(x-3)^2$ decreases for (1, 3).

[Since, $y' < 0$ for all $x \in (1, 3)$, hence y is decreasing on (1, 3)].

Q. 13. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(A) increasing in $\left(\pi, \frac{3\pi}{2}\right)$

(B) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(C) decreasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(D) decreasing in $\left(0, \frac{\pi}{2}\right)$

Ans. Option (B) is correct.

Explanation : Given that,

$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$

On differentiating with respect to x, we get

$$f'(x) = 12\sin^2 x.\cos x - 12\sin x.\cos x + 12\cos x$$
$$= 12[\sin^2 x.\cos x - \sin x.\cos x + \cos x]$$
$$= 12\cos x[\sin^2 x - \sin x + 1]$$
$$\Rightarrow f'(x) = 12\cos x[\sin^2 x + 1(1 - \sin x)]$$
$$\Rightarrow 1 - \sin x \geq 0 \text{ and } \sin^2 x \geq 0$$
$$\Rightarrow \sin^2 x + 1 - \sin x \geq 0$$

Hence, $f'(x) > 0$, when $\cos x > 0$, *i.e.*, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

So, $f(x)$ is increasing when $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

and $f'(x) < 0$, when $\cos x < 0$, *i.e.*, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Hence, $f'(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Since $\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

Q. 14. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$.

(A) $\sin 2x$ (B) $\tan x$
(C) $\cos x$ (D) $\cos 3x$

Ans. Option (C) is correct.

Explanation : In the given interval $\left(0, \frac{\pi}{2}\right)$

$$f(x) = \cos x$$

On differentiating with respect to x, we get

$$f'(x) = -\sin x$$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Q. 15. The function $f(x) = \tan x - x$

(A) always increases
(B) always decreases
(C) never increases
(D) sometimes increases and sometimes decreases

Ans. Option (A) is correct.

Explanation : We have,

$$f(x) = \tan x - x$$

On differentiating with respect to x, we get

$$f'(x) = \sec x - 1$$

$$\Rightarrow \quad f'(x) > 0, \forall x \in R$$

So, $f(x)$ always increases.

Q. 16. Let the $f: R \to R$ be defined by $f(x) = 2x + \cos x$, then f:

(A) has a minimum at $x = \pi$
(B) has a maximum, at $x = 0$
(C) is a decreasing function
(D) is an increasing function

Ans. Option (D) is correct.

Explanation : Given that,

$$f(x) = 2x + \cos x$$

Differentiating with respect to x, we get

$$f'(x) = 2 + (-\sin x)$$
$$= 2 - \sin x$$

Since, $f'(x) > 0, \forall x \in R$

Hence, $f(x)$ is an increasing function.

Q. 17. If x is real, the minimum value of $x^2 - 8x + 17$ is

(A) –1 (B) 0
(C) 1 (D) 2

Ans. Option (C) is correct.

Explanation : Let,

$$f(x) = x^2 - 8x + 17$$

On differentiating with respect to x, we get

$$f'(x) = 2x - 8$$

So, $f'(x) = 0$

$\Rightarrow \quad 2x - 8 = 0$

So, $f'(x) = 0$

$\Rightarrow \quad 2x - 8 = 0$

$\Rightarrow \quad 2x = 8$

$\therefore \quad x = 4$

Now, Again on differentiating with respect to x, we get

$$f''(x) = 2 > 0, \forall x$$

So, $x = 4$ is the point of local minima.

Minimum value of $f(x)$ at $x = 4$

$$f(4) = 4 \cdot 4 - 8 \cdot 4 + 17 = 1$$

Q. 18. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is

(A) 126 (B) 0
(C) 135 (D) 160

Ans. Option (B) is correct.

Explanation : Given that, the smallest value of polynomial is $f(x) = x^3 - 18x^2 + 96x$

On differentiating with respect to x, we get

$$f'(x) = 3x^2 - 36x + 96$$

So,

$$f'(x) = 0$$

$\Rightarrow \quad 3x^2 - 36x + 96 = 0$

$\Rightarrow 3(x^2 - 12x + 32) = 0$

$\Rightarrow \quad (x - 8)(x - 4) = 0$

$\Rightarrow \quad x = 8, 4 \in [0, 9]$

We shall now calculate the value of $f(x)$ at these points and at the end points of the interval [0, 9], *i.e.*, at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$f(4) = 4^3 - 18 \times 4^2 + 96 \times 4$$
$$= 64 - 288 + 384$$
$$= 160$$
$$f(8) = 8^3 - 18 \times 8^2 + 96 \times 8$$
$$= 128$$
$$f(9) = 9^3 - 18 \times 9^2 + 96 \times 9$$
$$= 729 - 1458 + 864$$
$$= 135$$
$$\text{and } f(0) = 0^3 - 18 \times 0^2 + 96 \times 0$$
$$= 0$$

Thus, we conclude that absolute minimum value of $f(x)$ in [0, 9] is 0 occurring at $x = 0$.

Q. 19. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(A) two points of local maximum
(B) two points of local minimum
(C) one maxima and one minima
(D) no maxima or minima

Ans. Option (C) is correct.

Explanation : We have,

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$
$$f'(x) = 6x^2 - 6x - 12$$

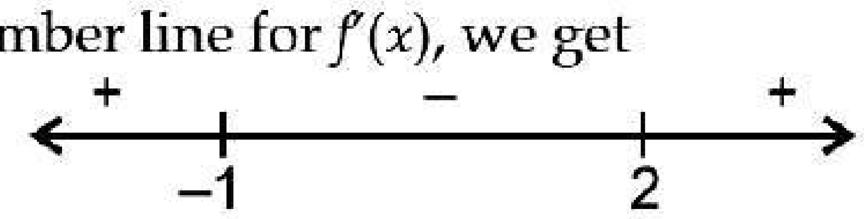
Now, $f'(x) = 0$

$\Rightarrow$ $6(x^2 - x - 2) = 0$

$\Rightarrow 6(x+1)(x-2) = 0$

$\Rightarrow$ $x = -1$ and $x = +2$

On number line for $f'(x)$, we get



Hence, $x = -1$ is point of local maxima and $x = 2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

Q. 20. The maximum value of $\sin x . \cos x$ is

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$

(C) $\sqrt{2}$ **(D)** $2\sqrt{2}$

Ans. Option (B) is correct.

Explanation : Let us assume that,

$$f(x) = \sin x . \cos x$$

Now, we know that

$$\sin x . \cos x = \frac{1}{2} \sin 2x$$

$\therefore$ $f'(x) = \frac{1}{2} . \cos 2x . 2 = \cos 2x$

Now, $f'(x) = 0$

$\Rightarrow$ $\cos 2x = 0$

$\Rightarrow$ $\cos 2x = \cos \frac{\pi}{2}$

$\Rightarrow$ $x = \frac{\pi}{4}$

Also, $f''(x) = \frac{d}{dx} . \cos 2x = -2 . \sin 2x$

$\therefore [f''(x)]_{\text{at } x = \frac{\pi}{4}} = -2 \sin 2 . \frac{\pi}{4}$

$= -2 \sin \frac{\pi}{2}$

$= -2 < 0$

$\therefore$ $x \ \frac{\pi}{4}$ is point of maxima.

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2} . \sin . 2 . \frac{\pi}{4}$$
$$= \frac{1}{2}$$

Q. 21. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is :

(A) 0 **(B)** 12

(C) 16 **(D)** 32

Ans. Option (B) is correct.

Explanation : Given that,

$$y = -x^3 + 3x^2 + 9x - 27$$

$\therefore$ $\frac{dy}{dx} = -3x^2 + 6x + 9$

$=$ Slope of the curve

and $\frac{d^2y}{dx^2} = -6x + 6 = -6(x-1)$

$\therefore$ $\frac{d^2y}{dx^2} = 0$

$\Rightarrow$ $-6(x-1) = 0$

$\Rightarrow$ $x = 1 > 0$

Now, $\frac{d^3y}{dx^3} = -6 < 0$

So, the maximum slope of given curve is at $x = 1$.

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x=1)} = -3 \times 1^2 + 6 \times 1 + 9$

$= 12$

Q. 22. The maximum value of $\left(\frac{1}{x}\right)^x$ is :

(A) e **(B)** e^e

(C) $e^{1/e}$ **(D)** $\left(\frac{1}{e}\right)^{1/e}$

Ans. Option (C) is correct.

Explanation:

Let $y = \left(\frac{1}{x}\right)^x$

$\Rightarrow$ $\log y = x . \log \frac{1}{x}$

$\therefore$ $\frac{1}{y} . \frac{dy}{dx} = x . \frac{1}{\frac{1}{x}} . \left(-\frac{1}{x^2}\right) + \log \frac{1}{x} . 1$

$= -1 + \log \frac{1}{x}$

$\therefore$ $\frac{dy}{dx} = \left(\log \frac{1}{x} - 1\right) . \left(\frac{1}{x}\right)^x$

Now, $\frac{dy}{dx} = 0$

$\Rightarrow$ $\log \frac{1}{x} = 1 = \log e$

$\Rightarrow$ $\frac{1}{x} = e$

$\Rightarrow$ $x = \frac{1}{e}$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

ASSERTION AND REASON BASED MCQs (1 Mark each)

Directions : In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false and R is True

Q. 1. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$ in rupees.

Assertion (A): The marginal revenue when $x = 5$ is 66.

Reason (R): Marginal revenue is the rate of change of total revenue with respect to the number of items sold at an instance.

Ans. Option (A) is correct.

Marginal revenue is the rate of change of total revenue with respect to the number of items sold at an instance. Therefore R is true.

$$R'(x) = 6x + 36$$
$$R'(5) = 66$$

$\therefore$ A is true.

R is the correct explanation of A.

Q. 2. The radius r of a right circular cylinder is increasing at the rate of 5 cm/min and its height h, is decreasing at the rate of 4 cm/min.

Assertion (A): When $r = 8$ cm and $h = 6$ cm, the rate of change of volume of the cylinder is 224π cm^3/min

Reason (R): The volume of a cylinder is $V = \frac{1}{3}\pi r^2 h$

Ans. Option (C) is correct.

Explanation: The volume of a cylinder is $V=\pi r^2 h$. So R is false.

$$\frac{dr}{dt} = 5 \text{ cm / min}, \frac{dh}{dt} = -4 \text{ cm / min}$$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = \pi[64 \times (-4) + 2 \times 6 \times 8 \times 5]$$

$$\left.\frac{dV}{dt}\right)_{r=8,\, h=6} = 224\pi \text{ cm}^3 / \text{min}$$

$\therefore$ Volume is increasing at the rate of 224π cm^3/min.

$\therefore$ A is true.

Q. 3. Assertion (A): For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then at $x = 3$ the slope of curve is decreasing at 36 units/sec.

Reason (R): The slope of the curve is $\frac{dy}{dx}$.

Ans. Option (D) is correct.

Explanation: The slope of the curve $y = f(x)$ is $\frac{dy}{dx}$. R is true.

Given curve is $y = 5x - 2x^3$

or $\frac{dy}{dx} = 5 - 6x^2$

or $m = 5 - 6x^2$ $\left[\text{slope } m = \frac{dy}{dx}\right]$

$$\frac{dm}{dt} = -12x\frac{dx}{dt} = -24x$$
$$\left[\because \frac{dx}{dt} = 2 \text{ units / sec}\right]$$

$$\left.\frac{dm}{dt}\right|_{x=3} = -72$$

Rate of Change of the slope is decreasing by 72 units/s.

A is false.

Q. 4. A particle moves along the curve $6y = x^3 + 2$.

Assertion (A): The curve meets the Y axis at three points.

Reason (R): At the points $\left(2, \frac{5}{3}\right)$ and $(-2, -1)$ the ordinate changes two times as fast as the abscissa.

Ans. Option (D) is correct.

Explanation:

On Y axis, $x = 0$. The curve meets the Y axis at only one point, *i.e.*, $\left(0, \frac{1}{3}\right)$.

Hence A is false.

$$6y = x^3 + 2$$

or $6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$

Given, $\frac{dy}{dt} = 2\frac{dx}{dt}$

or $12 = 3x^2$

or $x = \pm 2$

Put $x = 2$ and -2 in the given equation to get y

$\therefore$ The points are $\left(2, \frac{5}{3}\right), (-2, -1)$

R is true.

Q. 5. Assertion (A): At $x = \frac{\pi}{6}$, the curve $y = 2\cos^2(3x)$ has a vertical tangent.

Reason (R): The slope of tangent to the curve $y = 2\cos^2(3x)$ at $x = \frac{\pi}{6}$ is zero.

Ans. Option (D) is correct.

Explanation:

Given $y = 2\cos^2(3x)$

$$\frac{dy}{dx} = 2\times 2\times \cos(3x)\times(-\sin 3x)\times 3$$

$$\frac{dy}{dx} = -6\sin 6x$$

$$\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{6}} = -6\sin\pi = -6\times 0 = 0$$

∴ R is true.

Since the slope of tangent is zero, the tangent is parallel to the X-axis. That is the curve has a horizontal tangent at $x = \frac{\pi}{6}$. Hence A is false.

Q. 6. Assertion (A): The equation of tangent to the curve $y = \sin x$ at the point (0, 0) is $y = x$.

Reason (R): If $y = \sin x$, then $\frac{dy}{dx}$ at $x = 0$ is 1.

Ans. Option (A) is correct.

Explanation: Given $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

Slope of tangent at (0, 0) $= \left[\frac{dy}{dx}\right]_{(0,0)}$

$$= \cos 0° = 1$$

∴ R is true.

Equation of tangent at (0, 0) is

$$y - 0 = 1(x - 0)$$

$\Rightarrow$ $y = x.$

Hence A is true.

R is the correct explanation of A.

Q. 7. Assertion (A): The slope of normal to the curve $x^2 + 2y + y^2 = 0$ at (–1, 2) is –3.

Reason (R): The slope of tangent to the curve $x^2 + 2y + y^2 = 0$ at (–1, 2) is $\frac{1}{3}$.

Ans. Option (A) is correct.

Explanation:

Given $x^2 + 2y + y^2 = 0$

$$2x + 2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2+2y) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2(1+y)} = -\frac{x}{1+y}$$

Slope of tangent at (–1, 2)

$$\left[\frac{dy}{dx}\right]_{(-1,2)} = \frac{-(-1)}{1+2} = \frac{1}{3}$$

Hence R is true.

Slope of normal at (–1, 2)

$$= \frac{-1}{\text{Slope of tangent}} = -3.$$

Hence A is true.

R is the correct explanation for A.

Q. 8. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$.

Assertion (A): The value of a is ±2

Reason (R): The value of b is ±7

Ans. Option (C) is correct.

Explanation:

$$y^2 = ax^3 + b$$

Differentiate with respect to x,

$$2y\frac{dy}{dx} = 3ax^2$$

or $\frac{dy}{dx} = \frac{3ax^2}{2y}$

or $\frac{dy}{dx} = \frac{3ax^2}{\pm 2\sqrt{ax^3+b}}$ $[\because y^2 = ax^3 + b]$

or $\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3a(2)^2}{\pm 2\sqrt{a(2)^3+b}} = \frac{12a}{\pm 2\sqrt{8a+b}} = \frac{6a}{\pm\sqrt{8a+b}}$

Since (2, 3) lies on the curve

$$y^2 = ax^3 + b$$

or $9 = 8a + b$ ...(i)

Also from equation of tangent

$$y = 4x - 5$$

slope of the tangent = 4

∴ $\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{6a}{\pm\sqrt{8a+b}}$ becomes

$$4 = \frac{6a}{\pm\sqrt{9}} \qquad \{\text{from (i)}\}$$

$$\therefore \quad 4 = \frac{6a}{\pm 3}$$

$$\therefore \quad 4 = \frac{6a}{3} \text{ or } 4 = \frac{6a}{-3}$$

either, $a = 2$ or $a = -2$

For $a = 2$,

$9 = 8(2) + b$

or $b = -7$

$\therefore$ $a = 2$ and $b = -7$

and for $a = -2$,

$9 = 8(-2) + b$

or $b = 25$

or $a = -2$ and $b = 25$

Hence A is true and R is false.

Q. 9. Assertion (A): The function $f(x) = x^3 - 3x^2 + 6x - 100$ is strictly increasing on the set of real numbers.

Reason (R): A strictly increasing function is an injective function.

Ans. Option (B) is correct.

Explanation:

$$f(x) = x^3 - 3x^2 + 6x - 100$$

$$f'(x) = 3x^2 - 6x + 6$$

$$= 3[x^2 - 2x + 2]$$

$$= 3[(x-1)^2 + 1]$$

since $f'(x) > 0$; $x \in R$

$f(x)$ is strictly increasing on R.

Hence A is true.

For a strictly increasing function,

$x_1 > x_2$

$\Rightarrow$ $f(x_1) > f(x_2)$

i.e.; $x_1 = x_2$

$\Rightarrow$ $f(x_1) = f(x_2)$

Hence, a strictly increasing function is always an injective function.

So R is true.

But R is not the correct explanation of A.

Q. 10. Consider the function $f(x) = \sin^4 x + \cos^4 x$.

Assertion (A): $f(x)$ is increasing in $\left[0, \frac{\pi}{4}\right]$

Reason (R): $f(x)$ is decreasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Ans. Option (B) is correct.

Explanation:

$$f(x) = \sin^4 x + \cos^4 x$$

or $f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$$= -4\sin x\cos x[-\sin^2 x + \cos^2 x]$$

$$= -2\sin 2x \cos 2x$$

$$= -\sin 4x$$

On equating,

$f'(x) = 0$

or $-\sin 4x = 0$

or $4x = 0, \pi, 2\pi, \ldots\ldots\ldots$

or $x = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

Sub-intervals are $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

or $f'(x) < 0$ in $\left[0, \frac{\pi}{4}\right]$

or $f(x)$ is decreasing in $\left[0, \frac{\pi}{4}\right]$

and, $f'(x) > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\therefore$ $f'(x)$ is increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Both A and R are true. But R is not the correct explanation of A.

Q. 11. Assertion (A): The function $y = [x(x-2)]^2$ is increasing in $(0, 1) \cup (2, \infty)$

Reason (R): $\frac{dy}{dx} = 0$, when $x = 0, 1, 2$.

Ans. Option (B) is correct.

Explanation:

$$y = [x(x-2)]^2$$

$$= [x^2 - 2x]^2$$

$$\therefore \quad \frac{dy}{dx} = 2(x^2 - 2x)(2x - 2)$$

$$\text{or} \quad \frac{dy}{dx} = 4x(x-1)(x-2)$$

On equating $\frac{dy}{dx} = 0$,

$4x(x-1)(x-2) = 0 \Rightarrow x = 0, x = 1, x = 2$

$\therefore$ Intervals are $(-\infty, 0), (0,1), (1,2), (2, \infty)$

Since, $\frac{dy}{dx} > 0$ in $(0,1)$ or $(2, \infty)$

$\therefore$ $f(x)$ is increasing in $(0,1) \cup (2, \infty)$

Both A and R are true. But R is not the correct explanation of A.

Q. 12. Assertion (A): The function $y = \log(1 + x) - \frac{2x}{2+x}$ is a decreasing function of x throughout its domain.

Reason (R): The domain of the function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is $(-1, \infty)$

Ans. Option (D) is correct.

Explanation:

$\log(1 + x)$ is defined only when $x + 1 > 0$ or $x > -1$.

Hence R is true.

$y = \log(1+x) - \frac{2x}{2+x}$

Diff. w.r.t. 'x',

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{[(2+x)(2) - 2x]}{(2+x)^2}$$
$$= \frac{1}{1+x} - \frac{[4 - 2x - 2x]}{(2+x)^2}$$
$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$
$$= \frac{(2+x)^2 - 4(1+x)}{(2+x)^2(1+x)}$$
$$= \frac{4 + x^2 + 4x - 4 - 4x}{(2+x)^2(1+x)}$$
$$= \frac{x^2}{(2+x)^2(1+x)}$$

For increasing function,

$$\frac{dy}{dx} \geq 0$$

or $\frac{x^2}{(2+x)^2(x+1)} \geq 0$

or $\frac{(2+x)^2(x+1)x^2}{(2+x)^4(x+1)^2} \geq 0$

or $(2+x)^2(x+1)x^2 \geq 0$

When $x > -1$,

$\frac{dy}{dx}$ is always greater than zero.

$\therefore$ $y = \log(1+x) - \frac{2x}{2+x}$

is always increasing throughout its domain.

Hence A is false.

Q. 13. The sum of surface areas (S) of a sphere of radius 'r' and a cuboid with sides $\frac{x}{3}$, x and $2x$ is a constant.

Assertion (A): The sum of their volumes (V) is minimum when x equals three times the radius of the sphere.

Reason (R): V is minimum when $r = \sqrt{\frac{S}{54+4\pi}}$

Ans. Option (A) is correct.

Explanation:

Given $S = 4\pi r^2 + 2\left[\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right]$

$S = 4\pi r^2 + 6x^2$

or $x^2 = \frac{S - 4\pi r^2}{6}$

and $V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3}$

$\therefore$ $V = \frac{4}{3}\pi r^3 + \frac{2}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{3/2}$

$$\frac{dV}{dr} = 4\pi r^2 + \left(\frac{S - 4\pi r^2}{6}\right)^{1/2}\left(\frac{-8\pi r}{6}\right)$$

$$\frac{dV}{dr} = 0$$

or $r = \sqrt{\frac{S}{54+4\pi}}$

Now $\frac{d^2V}{dr^2} = 8\pi r + \left(\frac{-8\pi}{6}\right)\left(\frac{S - 4\pi r^2}{6}\right)^{1/2} + \frac{1}{2}\left(\frac{S - 4\pi r^2}{6}\right)^{-1/2} \cdot \left(\frac{-8\pi r}{6}\right)$

at $r = \sqrt{\frac{S}{54+4\pi}}$; $\frac{d^2V}{dr^2} > 0$

$\therefore$ for $r = \sqrt{\frac{S}{54+4\pi}}$ volume is minimum

i.e., $r^2(54 + 4\pi) = S$

or $r^2(54 + 4\pi) = 4\pi r^2 + 6x^2$

or $6x^2 = 54r^2$

or $x^2 = 9r^2$

or $x = 3r$

Hence both A and R are true.

R is the correct explanation of A.

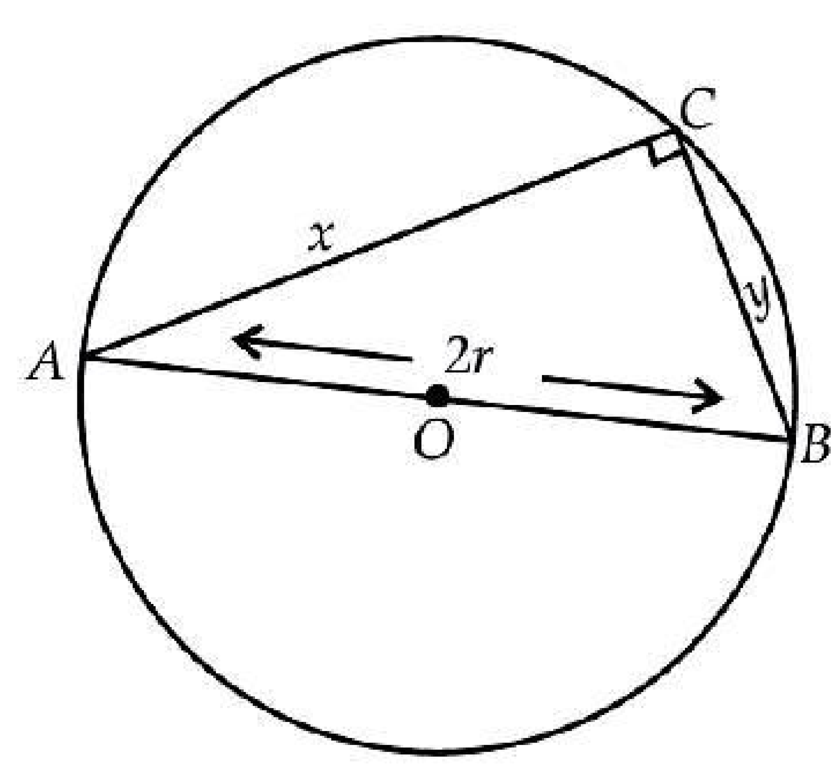
Q. 14. AB is the diameter of a circle and C is any point on the circle.

Assertion (A): The area of ΔABC is maximum when it is isosceles.

Reason (R): ΔABC is a right-angled triangle.

Ans. Option (A) is correct.

Explanation:



Let the sides of rt. ΔABC be x and y.

$\therefore$ $x^2 + y^2 = 4r^2$

and A = Area of $\Delta = \frac{1}{2}xy$

Let, $S - A^2$

$= \frac{1}{4}x^2y^2$

$$= \frac{1}{4} x^2(4r^2 - x^2)$$

$$= \frac{1}{4} (4r^2x^2 - x^4)$$

$\therefore$ $$\frac{dS}{dx} = \frac{1}{4} [8r^2x - 4x^3]$$

or $$\frac{dS}{dx} = 0$$

or $$x^2 = 2r^2 \text{ or } x = \sqrt{2}r$$

and $$y^2 = 4r^2 - 2r^2 = 2r^2$$

or $$y = \sqrt{2}r$$

i.e., $$x = y \text{ and } \frac{d^2S}{dx^2} = (2r^2 - 3x^2)$$

$$= 2r^2 - 6r^2 < 0$$

or Area is maximum, when Δ is isosceles.

Hence A is true.

Angle in a semicircle is a right angle.

$\therefore \angle C = 90°$

⇒ ΔABC is a right-angled triangle.

∴ R is true.

R is the correct explanation of A.

Q. 15. A cylinder is inscribed in a sphere of radius R.

Assertion (A): Height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$ units.

Reason (R): The maximum volume of the cylinder is $\frac{4\pi R^3}{\sqrt{3}}$ cubic units.

Ans. Option (C) is correct.

Explanation: Let the radius and height of cylinder be r and h respectively

$\therefore$ $$V = \pi r^2 h \quad ...(i)$$

But $$r^2 = R^2 - \frac{h^2}{4}$$

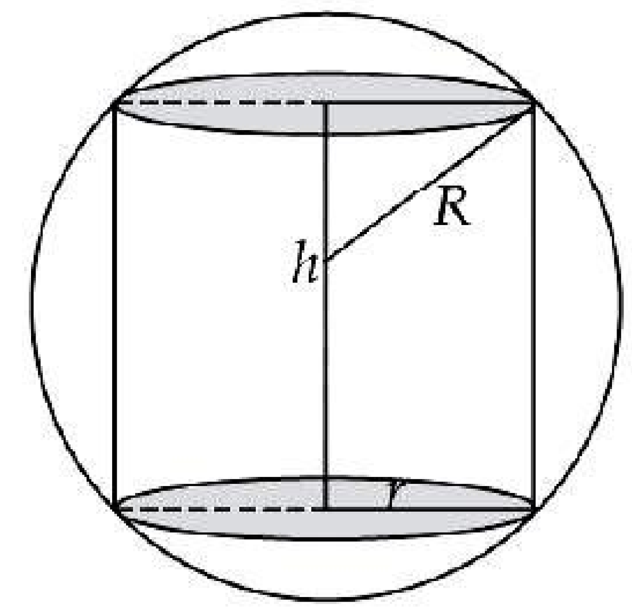
$\therefore$ $$\pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(R^2h - \frac{h^3}{4}\right)$$

or $$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$$

For maximum or minimum

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h^2 = \frac{4R^2}{3}$$

or $$h = \frac{2R}{\sqrt{3}}$$



and $$\frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right) < 0$$

$$\text{Maximum volume} = \pi \cdot \left[R^2 \cdot \frac{2R}{\sqrt{3}} - \frac{1}{4}\left(\frac{2R}{\sqrt{3}}\right)^3\right]$$

$$= \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units}$$

Hence A is true and R is false.

Q. 16. Assertion (A): The altitude of the cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Reason (R): The maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

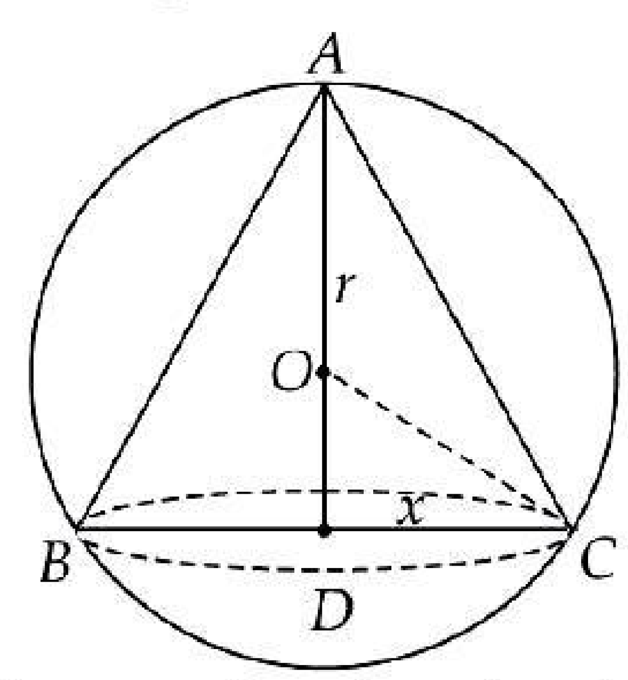
Ans. Option (B) is correct.

Explanation: Let radius of cone be x and its height be h.

$\therefore$ $$OD = (h - r)$$

Volume of cone

$$(V) = \frac{1}{3}\pi x^2 h \quad ...(i)$$



In ΔOCD, $x^2 + (h - r)^2 = r^2$ or $x^2 = r^2 - (h - r)^2$

$\therefore$ $$V = \frac{1}{3}\pi h\{r^2 - (h - r)^2\}$$

$$= \frac{1}{3}\pi(-h^3 + 2h^2r)$$

or $$\frac{dV}{dh} = \frac{\pi}{3}(-3h^2 + 4hr)$$

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h = \frac{4r}{3}$$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h+4r)$$

$$= \frac{\pi}{3}\left(-6\left(\frac{4r}{3}\right)+4r\right)$$

$$= -\frac{4\pi r}{3} < 0$$

$\therefore$ at $h = \frac{4r}{3}$, Volume is maximum

Maximum volume

$$= \frac{1}{3}\pi\cdot\left\{-\left(\frac{4r}{3}\right)^3 + 2\left(\frac{4r}{3}\right)^2 r\right\}$$

$$= \frac{8}{27}\cdot\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{8}{27} \text{ (volume of sphere)}$$

Hence both A and R are true.
R is not the correct explanation of A.

CASE-BASED MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions, on the basis of the same:

The Relation between the height of the plant (y in cm) with respect to exposure to sunlight is governed by the following equation $y = 4x - \frac{1}{2}x^2$ where x is the number of days exposed to sunlight.

[CBSE QB 2021]

Q. 1. The rate of growth of the plant with respect to sunlight is ______ .

(A) $4x - \left(\frac{1}{2}\right)x^2$ (B) $4 - x$

(C) $x - 4$ (D) $x - \frac{1}{2}x^2$

Ans. Option (B) is correct.

Explanation:

$$y = 4x - \frac{1}{2}x^2$$

$\therefore$ rate of growth of the plant with respect to sunlight

$$= \frac{dy}{dx}$$

$$= \frac{d}{dx}\left[4x - \frac{1}{2}x^2\right]$$

$$= (4-x) \text{ cm / day}$$

Q. 2. What is the number of days it will take for the plant to grow to the maximum height?

(A) 4 (B) 6

(C) 7 (D) 10

Ans. Option (A) is correct.

Explanation:

$$\frac{dy}{dx} = 4 - x$$

The number of days it will take for the plant to grow to the maximum height,

$$\frac{dy}{dx} = 0$$

$$4 - x = 0$$

$$x = 4 \text{ Days.}$$

Q. 3. What is the maximum height of the plant?

(A) 12 cm (B) 10 cm

(C) 8 cm (D) 6 cm

Ans. Option (C) is correct.

Explanation: We have, number of days for maximum height of plant

= 4 Days

$\therefore$ Maximum height of plant

$$\Rightarrow \quad y_{(x=4)} = 4\times4 - \frac{1}{2}\times4\times4$$

$$= 16 - 8$$

$$= 8 \text{ cm}$$

Q. 4. What will be the height of the plant after 2 days?

(A) 4 cm (B) 6 cm

(C) 8 cm (D) 10 cm

Ans. Option (B) is correct.

Explanation: Height of plant after 2 days

$$= y_{(x=2)} = 4\times2 - \frac{1}{2}\times2\times2$$

$$= 8 - 2$$

$$= 6 \text{ cm}$$

Q. 5. If the height of the plant is 7/2 cm, the number of days it has been exposed to the sunlight is ______ .

(A) 2 (B) 3
(C) 4 (D) 1

Ans. Option (D) is correct.

Explanation:

Given, $y = \frac{7}{2}$

i.e., $4x - \frac{1}{2}x^2 = \frac{7}{2}$

$$8x - x^2 - 7$$
$$x^2 - 8x + 7 = 0$$
$$x^2 - 7x - x + 7 = 0$$
$$x(x-7) - (x-7) = 0$$
$$x = 1, 7$$

We will take $x = 1$, because it will take 4 days for the plant to grow to the maximum height i.e. 8 cm and $\frac{7}{2}$ cm is not maximum height so, it will take less than 4 days. *i.e.*, 1 Day.

II. Read the following text and answer the following questions on the basis of the same:

P(x) = $-5x^2 + 125x + 37500$ is the total profit function of a company, where x is the production of the company. **[CBSE QB 2021]**

Q. 1. What will be the production when the profit is maximum?

(A) 37,500 (B) 12.5
(C) – 12.5 (D) – 37,500

Ans. Option (B) is correct.

Explanation: We, have

$$P(x) = -5x^2 + 125x + 37500$$
$$P'(x) = -10x + 125$$

For maximum profit

$$P'(x) = 0$$
$$-10x + 125 = 0$$
$$-10x = -125$$
$$x = \frac{125}{10}$$
$$= 12.5$$

Q. 2. What will be the maximum profit?

(A) ₹ 38,28,125 (B) ₹ 38,281.25
(C) ₹ 39,000 (D) None of these

Ans. Option (B) is correct.

Explanation: Maximum profit

$$= P(12.5)$$
$$= -5(12.5)^2 + 125 \times 12.5 + 37500$$
$$= -781.25 + 1562.5 + 37500$$
$$= 38,281.25$$

Q. 3. Check in which interval the profit is strictly increasing .

(A) (12.5, ∞)
(B) for all real numbers
(C) for all positive real numbers
(D) (0, 12.5)

Ans. Option (D) is correct.

Q. 4. When the production is 2 units what will be the profit of the company?

(A) 37,500 (B) 37,730
(C) 37,770 (D) None of these

Ans. Option (B) is correct.

Explanation: When production is 2 units, then profit of company = $P(2)$

$$= -5 \times 2^2 + 125 \times 2 + 37500$$
$$= -20 + 250 + 37500$$
$$= 37,730$$

Q. 5. What will be production of the company when the profit is ₹ 38,250?

(A) 15 (B) 30
(C) 10
(D) data is not sufficient to find

Ans. Option (C) is correct.

Explanation:

$$\text{Profit} = 38,250$$

i.e., $-5x^2 + 125x + 37,500 = 38,250$

$$5x^2 - 125x + 750 = 0$$
$$x^2 - 25x + 150 = 0$$
$$x(x-15) - 10(x-15) = 0$$
$$(x-10)(x-15) = 0$$
$$x = 10, 15$$
$$P(x) = -5x^2 + 125x + 37500$$
$$P(10) = -5 \times 10^2 + 125 \times 10 + 37500$$
$$= -500 + 1250 + 37500$$
$$= ₹\ 38,250$$

Hence, production of company is 10 units when the profit is ₹38250.

III. Read the following text and answer the following questions on the basis of the same:

The shape of a toy is given as $f(x) = 6(2x^4 - x^2)$. To make the toy beautiful 2 sticks which are perpendicular to each other were placed at a point (2, 3), above the toy. **[CBSE QB-2021]**

Q. 1. Which value from the following may be abscissa of critical point?

(A) $\pm 1/4$

(B) ± 12

(C) ± 1

(D) None of these

Ans. Option (B) is correct.

Q. 2. Find the slope of the normal based on the position of the stick.

(A) 360

(B) –360

(C) $\frac{1}{360}$

(D) $\frac{-1}{360}$

Ans. Option (D) is correct.

Explanation: Slope of the normal based on the position of the stick

$$= \frac{-1}{f'(x)}$$

$$f'(x) = 6[8x^3 - 2x]$$
$$f'(2) = 6[8 \times 8 - 2 \times 2]$$
$$= 6[64 - 4]$$
$$= 360$$
$$\therefore \quad \text{Slope} = \frac{-1}{360}$$

Q. 3. What will be the equation of the tangent at the critical point if it passes through (2, 3)?

(A) $x + 360y = 1082$

(B) $y = 360x - 717$

(C) $x = 717y + 360$

(D) None of these

Ans. Option (B) is correct.

Explanation: We have

$$\left.\frac{dy}{dx}\right]_{(2,\,3)} = 360$$

$$\therefore \quad (y - y') = \frac{dy}{dx}(x - x')$$
$$(y - 3) = 360(x - 2)$$
$$y - 3 = 360x - 720$$
$$y = 360x - 717$$

Q. 4. Find the second order derivative of the function at $x = 5$.

(A) 598 **(B)** 1,176

(C) 3,588 **(D)** 3,312

Ans. Option (C) is correct.

Explanation:

$$f(x) = 6(2x^4 - x^2)$$
$$f'(x) = 6[8x^3 - 2x]$$
$$f''(x) = 6[24x^2 - 2]$$
$$f''(5) = 6[24 \times 25 - 2]$$
$$= 6[600 - 2]$$
$$= 3588$$

Q. 5. At which of the following intervals will $f(x)$ be increasing?

(A) $\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

(B) $\left(\frac{-1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(C) $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

(D) $\left(-\infty, \frac{-1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Ans. Option (B) is correct.

Explanation: For increasing

$$f'(x) > 0$$
$$6(8x^3 - 2x) > 0$$

i.e., $x(4x^2 - 1) > 0$

$\Rightarrow$ $4x^2 - 1 > 0$

and $x > 0$

$$4x^2 > 1$$

$\Rightarrow$ $x^2 > \frac{1}{4}$

$\Rightarrow$ $x > \frac{1}{2}$

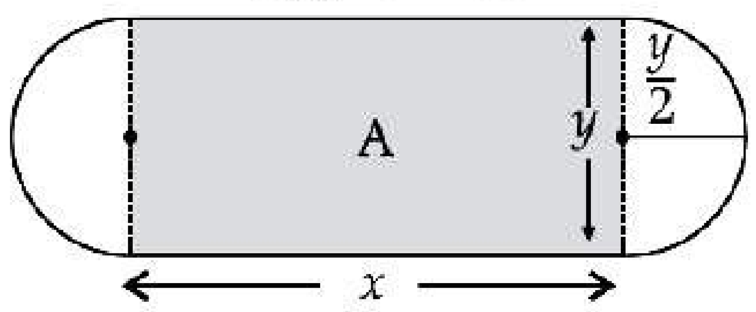
and $x > -\frac{1}{2}$

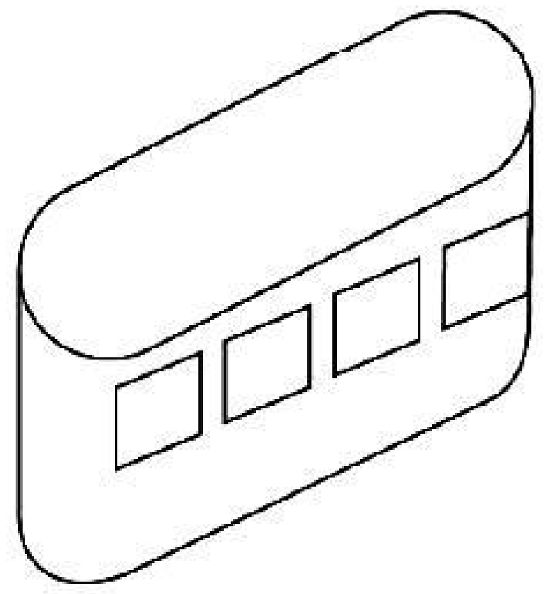
i.e., $x \in \left(\frac{-1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

VI. Read the following text and answer the following questions, on the basis of the same:

An architect designs a building for a multi-national company. The floor consists of a rectangular region with semicircular ends having a perimeter of 200 m as shown below:

Design of Floor





Building

Q. 1. If x and y represents the length and breadth of the rectangular region, then the relation between the variables is :

(A) $x + \pi y = 100$ (B) $2x + \pi y = 200$

(C) $\pi x + y = 50$ (D) $x + y = 100$

Ans. Option (B) is correct.

Explanation:

$$\text{Perimeter} = x + x + \frac{\pi y}{2} + \frac{\pi y}{2}$$

$$200 = 2x + \frac{2\pi y}{2}$$

$$200 = 2x + \pi y \qquad \text{...(i)}$$

Q. 2. The area of the rectangular region A expressed as a function of x is :

(A) $\frac{2}{\pi}(100x - x^2)$ (B) $\frac{1}{\pi}(100x - x^2)$

(C) $\frac{x}{\pi}(100 - x)$ (D) $\pi y^2 + \frac{2}{\pi}(100x - x^2)$

Ans. Option (A) is correct.

Explanation:

$$\text{Area (A)} = x \times y$$

$$= x \times \left(\frac{200 - 2x}{\pi}\right) \qquad \text{[from (i)]}$$

$$= \frac{2}{\pi}[100x - x^2] \qquad \text{...(ii)}$$

Q. 3. The maximum value of area A is :

(A) $\frac{\pi}{3200}\text{m}^2$ (B) $\frac{3200}{\pi}\text{m}^2$

(C) $\frac{5000}{\pi}\text{m}^2$ (D) $\frac{1000}{\pi}\text{m}^2$

Ans. Option (C) is correct.

Explanation:

$$\frac{dA}{dx} = \frac{2}{\pi}[100 - 2x]$$

$$\frac{dA}{dx} = \frac{4}{\pi}[500 - x]$$

For maxima,

$$\frac{dA}{dx} = 0$$

$$x = 50 \qquad \text{...(i)}$$

$$A = \frac{2}{\pi}[100 \times 50 - 50 \times 50]$$

[from (ii)]

$$= \frac{2}{\pi}[5000 - 2500]$$

$$= \frac{2}{\pi} \times 2500$$

$$= \frac{5000}{\pi}\ \text{m}^2$$

Q. 4. The CEO of the multi-national company is interested in maximizing the area of the whole floor including the semi-circular ends. For this to happen the value of x should be

(A) 0 m (B) 30 m

(C) 50 m (D) 80 m

Ans. Option (A) is correct.

Q. 5. The extra area generated if the area of the whole floor is maximized is :

(A) $\frac{3000}{\pi}\text{m}^2$

(B) $\frac{5000}{\pi}\text{m}^2$

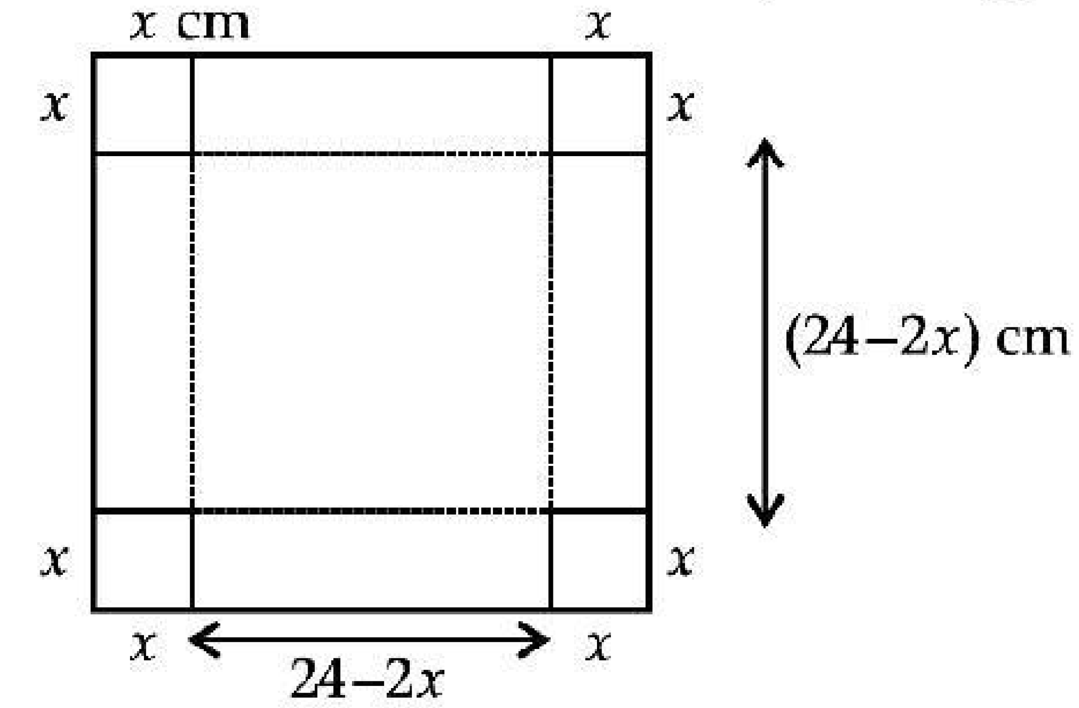
(C) $\frac{7000}{\pi}\text{m}^2$

(D) No change. Both areas are equal.

Ans. Option (D) is correct.

V. Read the following text and answer the following questions. On the basis of the same:

An open box is to be made out of a piece of cardboard measuring (24 cm × 24 cm) by cutting of equal squares from the corners and turning up the sides. **[CBSE SQP 2020-21]**



Q. 1. Find the volume of that open box ?

(A) $4x^3 - 96x^2 + 576x$ (B) $4x^3 + 96x^2 - 576x$

(C) $2x^3 - 48x^2 + 288x$ (D) $2x^3 + 48x^2 + 288x$

Ans. Option (A) is correct.

Explanation:

Volume of open box $= \text{length} \times \text{breadth} \times \text{height}$

$= (24-2x)\times(24-2x)\times x$

$= (4x^3 - 96x^2 + 576x)\ \text{cm}^3$

Q. 2. Find the value of $\frac{dV}{dx}$?

(A) $12(x^2 + 16x - 48)$ **(B)** $12(x^2 - 16x + 48)$

(C) $6(x^2 + 8x - 24)$ **(D)** $6(x^2 - 8x + 24)$

Ans. Option (B) is correct.

Explanation:

$$\frac{dV}{dx} = \frac{d}{dx}[4x^3 - 96x^2 + 576x]$$
$$= 12x^2 - 2\times 96x + 576$$
$$= 12[x^2 - 16x + 48]$$

Q. 3. Find the value of $\frac{d^2V}{dx^2}$?

(A) $24(x + 8)$ **(B)** $12(x - 4)$

(C) $24(x - 8)$ **(D)** $12(x + 4)$

Ans. Option (C) is correct.

Explanation:

$$\frac{d^2V}{dx} = \frac{d}{dx}\left[\frac{dV}{dx}\right]$$
$$= \frac{d}{dx}[12(x^2 - 16x + 48)]$$
$$= [12(2x-16)]$$
$$= 24(x-8)$$

Q. 4. Find the value of x other than 12 ?

(A) 3 **(B)** 9

(C) 1 **(D)** 4

Ans. Option (D) is correct.

Q. 5. Volume is maximum at what height of that open box ?

(A) 3 cm **(B)** 9 cm

(C) 1 cm **(D)** 4 cm

Ans. Option (D) is correct.

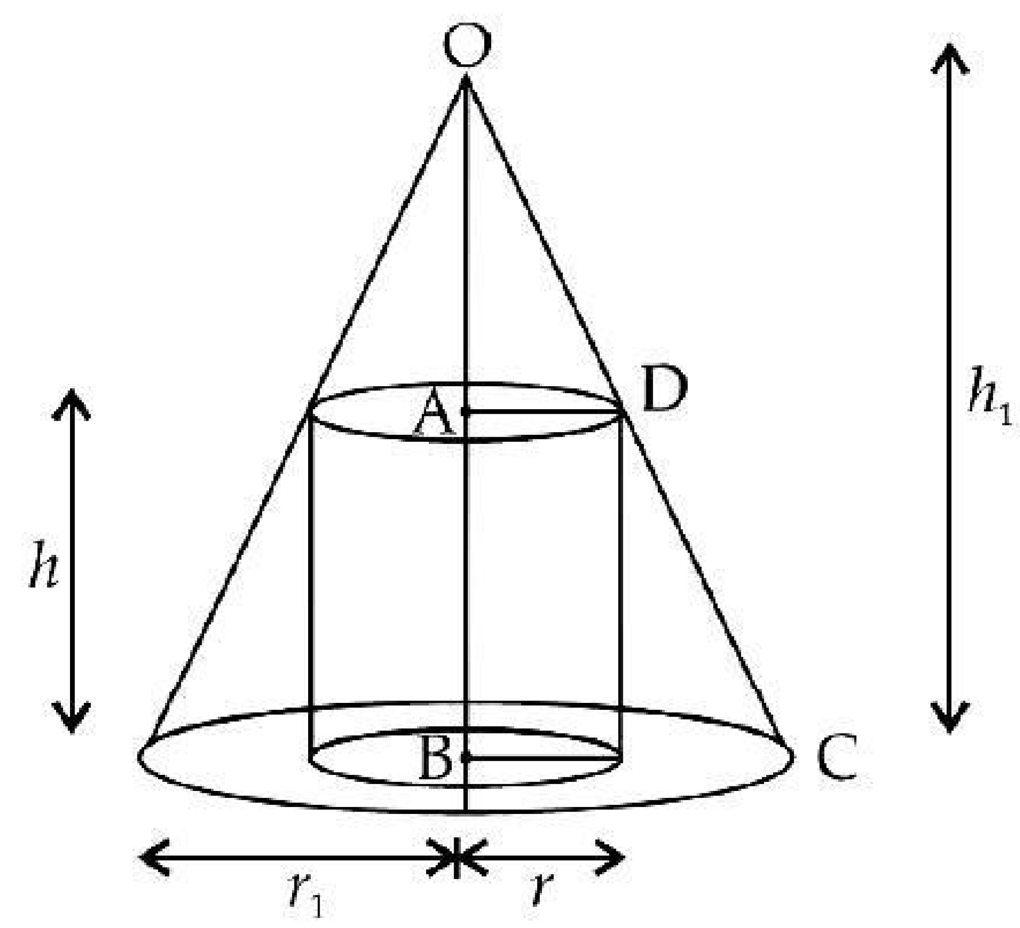
Explanation: For maximum value,

$$\frac{dV}{dx} = 0$$
$$i.e.,\ 12(x^2 - 16x + 48) = 0$$
$$x^2 - 16x + 48 = 0$$
$$x^2 - 4x - 12x + 48 = 0$$
$$x(x-4) - 12(x-4) = 0$$
$$(x - 4(x-12) = 0$$
$$x = 4, 12$$
$$V(x=4) = (24 - 2\times 4)(24 - 2\times 4)\times 4$$
$$= 16\times 16\times 4$$
$$= 1024\ \text{cm}^3$$
$$V(x=12) = (24 - 2\times 12)(24 - 2\times 12)\times 12$$
$$= 0$$

Hence, volume is maximum at height 4 cm of the open box.

VI. Read the following text and answer the following questions on the basis of the same:

A right circular cylinder is inscribed in a cone.



S = Curved Surface Area of Cylinder.

Q. 1. $\frac{r}{r_1} = ?$

(A) $\frac{h - h_1}{h_1}$ **(B)** $\frac{h_1 - h}{h_1}$

(C) $\frac{h - h_1}{h}$ **(D)** $\frac{h + h_1}{h_1}$

Ans. Option (B) is correct.

Explanation: In Δ DEC and Δ OBC

$$\frac{DE}{OB} = \frac{EC}{BC} \quad [\text{Since } \Delta DEC \sim \Delta OBC]$$
$$\frac{h}{h_1} = \frac{r_1 - r}{r_1}$$
$$r_1 h = r_1 h_1 - r h_1$$
$$r_1(h - h_1) = -r h_1$$
$$\text{or} \quad \frac{r}{r_1} = \frac{h_1 - h}{h_1}$$

Q. 2. Find the value of 'S' ?

(A) $\frac{2\pi r}{h}(h_1 - h)h$ **(B)** $\frac{2\pi r}{h_1}(h_1 - h)h$

(C) $\frac{2\pi r_1}{h_1}(h_1 - h)h$ **(D)** $\frac{2\pi r_1}{h_1}(h_1 + h)h$

Ans. Option (C) is correct.

Explanation: Curved surface area of cylinder,

$$S = \frac{2\pi r h_1 (r_1 - r)}{r_1}$$

$$= \frac{2\pi r}{r_1}(r_1 - r)h_1$$

$$= 2\pi r h_1 \times \frac{h}{h_1} \qquad \left[\because \frac{h}{h_1} = \frac{r_1 - r}{r_1}\right]$$

$$\frac{2\pi r_1 (h_1 - h).h}{h_1} \qquad \left[\because r = r_1 \frac{(h_1 - h)}{h_1}\right]$$

$$\therefore \quad S = \frac{2\pi r_1}{h_1}(h_1 - h).h$$

Q. 3. What is the value of $\frac{dS}{dh}$?

(A) $\frac{2\pi r_1}{h}(h_1 - 2h)$ (B) $\frac{2\pi r_1}{h_1}(h - 2h_1)$

(C) $\frac{2\pi r}{h}(h_1 - 2h)$ (D) $\frac{2\pi r_1}{h_1}(h_1 - 2h)$

Ans. Option (D) is correct.

Explanation:

$$\frac{dS}{dh} = \frac{2\pi r_1}{h_1}(h_1 - 2h)$$

Q. 4. Find the value of $\frac{d^2S}{dh^2}$?

(A) $-\frac{4\pi r_1}{h_1}$ (B) $-\frac{4\pi r}{h}$

(C) $-\frac{4\pi r_1}{h}$ (D) $\frac{4\pi r_1}{h}$

Ans. Option (A) is correct.

Explanation:

$$\frac{d^2S}{dh^2} = \frac{2\pi r_1}{h_1}(0 - 2)$$

$$= \frac{-4\pi r_1}{h_1}$$

Q. 5. What is the relation between r_1 and r ?

(A) $r_1 = \frac{r}{2}$ (B) $2r_1 = 3r$

(C) $r_1 = 2r$ (D) $\frac{r_1}{2} = \frac{r}{3}$

Ans. Option (C) is correct.

Explanation:

$$S = \frac{2\pi r}{r_1}(r_1 - r)h_1$$

$$S = \frac{2\pi h_1 (r r_1 - r^2)}{r_1}$$

$$\frac{dS}{dr} = \frac{2\pi h_1 (r_1 - 2r)}{r_1}$$

$$\frac{dS}{dr} = 0$$

$$\frac{2\pi h_1 (r_1 - 2r)}{r_1} = 0$$

$$\Rightarrow \quad r_1 - 2r = 0$$

$$r_1 = 2r$$