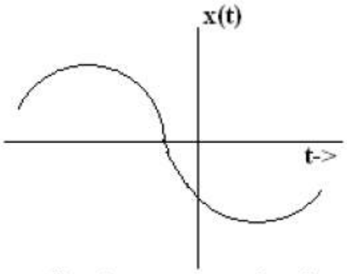
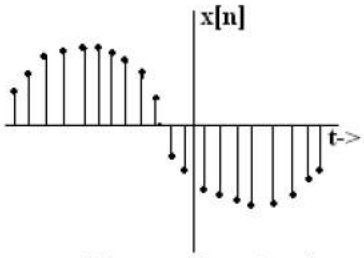
Signals and systems

Continuous time Signal: Independent variables are continuous with respect to time represented as x(t), mathematically as a function of t

Discrete time Signal: Independent variables are defined only at discrete time of same intervals of n represented as x[n] mathematically as a function of n



Continuous time signal

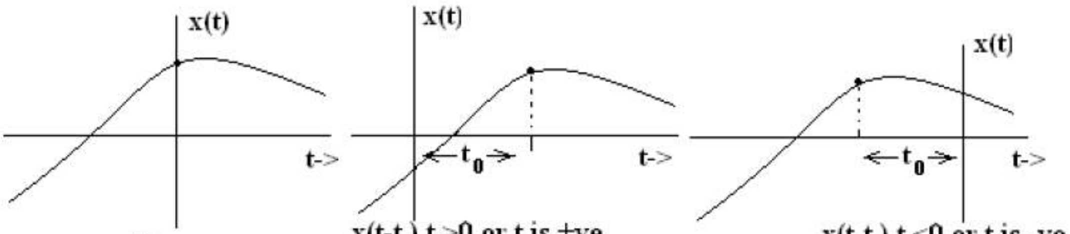


Discrete time signal

Sampling: Creation of discrete time signal from continuous time signal by defining successive samples of continuous time signal for a constant interval of time.

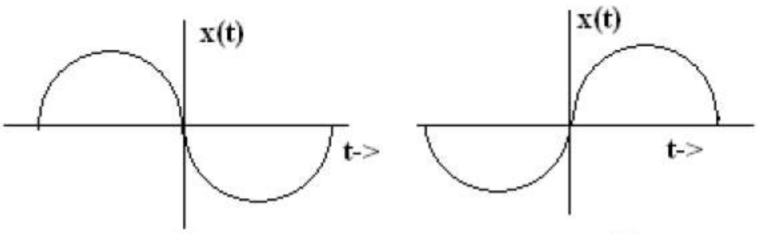
Transformation of independent variable:-

1. **Time shift's:** For both continuous and discrete time signal if x(t) represents a signal then $x(t-t_0)$ represents: -
   a. Delayed version of x(t) if $t_0>0$ or to is positive.
   b. Advanced version of x(t) if $t_0<0$ or t0 is negative.



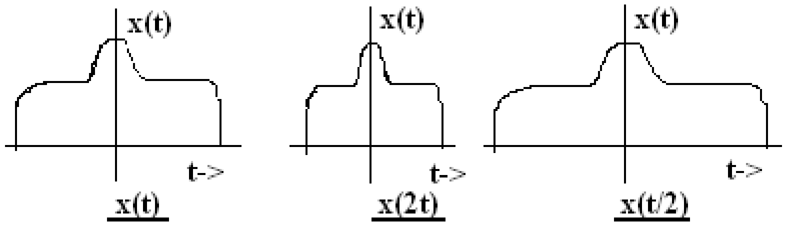
x(t) $x(t-t_0)$ $t_0>0$ or t_0 is +ve $x(t-t_0)$ $t_0<0$ or t_0 is -ve

2. **Time reversal:** To obtain the mirror image of a function of both continuous and discrete time signal. That is if x(t) is a function then x(-t) is its mirror image.



x(t) x(-t)

3. **Time scaling:** if x(t) is a signal (continuous/discrete) then x(αt) is the time scaled version of x(t) where α is a constant.

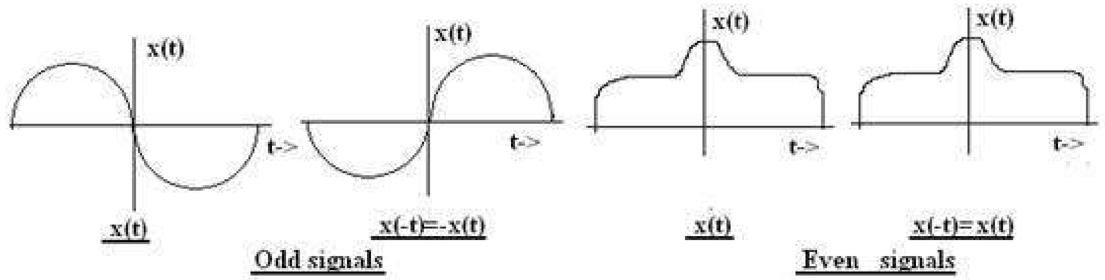


4. **Periodic signal:** A periodic signal has a property that there is a positive value of T such that x(t) = x(t+nT) for n = 0, 1, 2 … and T is a constant known as period.

5. **Even or odd signal:** From time reversal if the mirror image of the signal is same as that of the image. That is if x(-t) = x(t) then the signal is even else if the signal x(-t) = -x(t) then the signal is odd. Note that any signal can be broken to sum of signal one which is even and other is odd as given below: -

   Even {x(t)}=1/2[x(t)+x(-t)]

   Odd {x(t)}=1/2[x(t)-x(-t)]



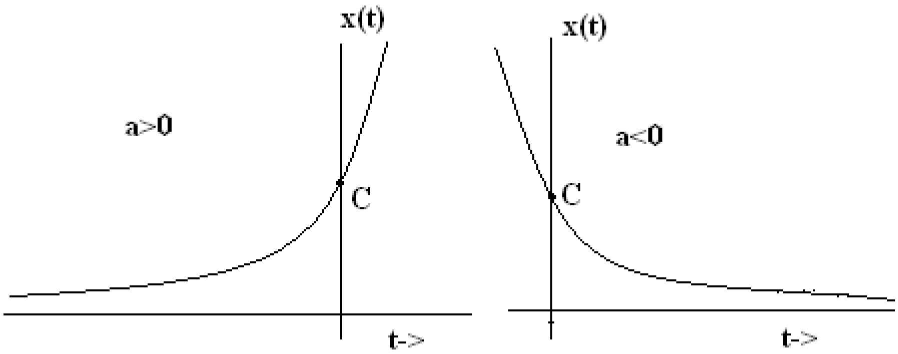
<u>Exponential and sinusoidal signals:-</u>

Continuous time complex exponential signal is of the form

$x(t) = Ce^{at}$

where C and a are generally complex number.

1. **Real and exponential signal** that is if 'C' and 'a' are real then the signal is as follows:



2. **Periodic complex exponential and sinusoidal signal**: That is ‘a’ is imaginary

$x(t) = Ce^{jwot}$

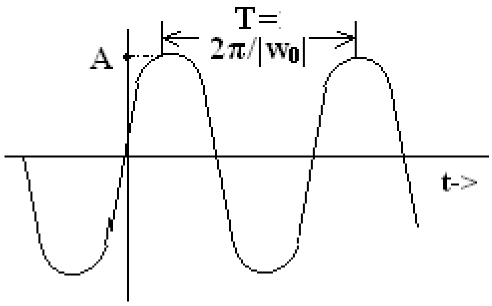
$x(t) = Ce^{jwo(t+T)} = C.\ e^{jwot}.e^{jwoT}$ but $e^{jwoT}=1$

i.e. if $w_0=0$ then $x(t)=1$ and if $w_0\neq 0$ then $T=2\pi/|w_0|$

Signal closely related is $x(t)= a\cos(w_0t+\varphi)$

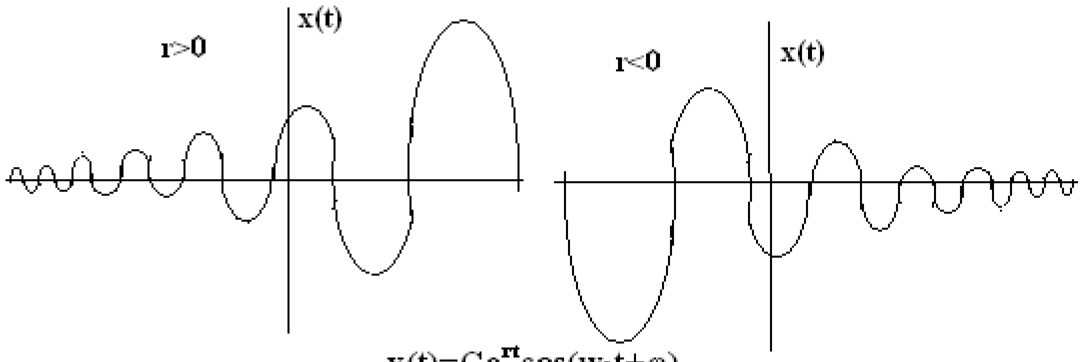
Euler’s relation: $e^{jwot}=\cos w_0t+j\sin w_0t$

$A\cos(w_0t+\varphi)=A.\mathrm{Re}\{e^{j(wot+\varphi)}\}$ and $A\sin(w_0t+\varphi)=A.\mathrm{Im}\{e^{j(wot+\varphi)}\}$



3. **Growing and decaying sinusoidal signal:**

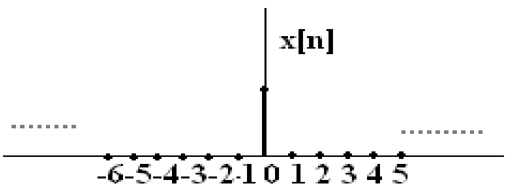
$x(t)=Ce^{rt}\cos(w_0t+\varphi)$ if $r>0$ then growing signal and if $r<0$ then decaying signal

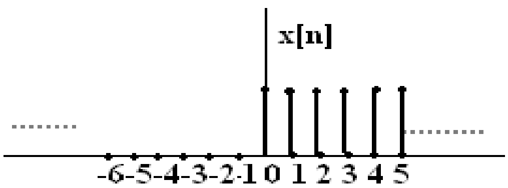


$x(t)=Ce^{rt}\cos(w_0t+\varphi)$

Sinusoidal signal multiplied by decaying exponential is referred as damped exponential. Similarly for the discrete time characteristic where t becomes n.

Unit impulse and unit step function:





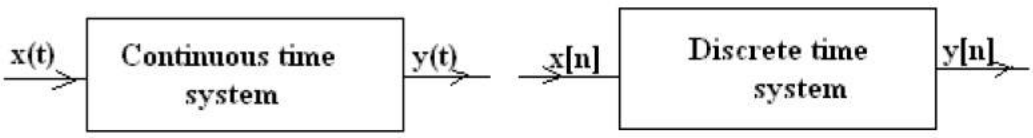
$\Delta_n = \begin{cases} 0, & n\neq 0 \\ 1, & n=0 \end{cases}$ $\qquad$ $\Delta_n = \begin{cases} 0, & n<0 \\ 1, & n\geq 0 \end{cases}$

Name	*Continuous*	*Discrete*
Unit Step function	$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
Ramp signal	$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$r[n]=nu(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
Impulse function	$\delta(t) = 0, t \neq 0$	$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$
Rectangular pulse function	$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1, & \lvert t \rvert \leq \tau/2 \\ 0, & \lvert t \rvert > \tau/2 \end{cases}$	$rect[\frac{n}{2N}] = \begin{cases} 1, & \lvert n \rvert \leq N \\ 0, & \lvert n \rvert > N \end{cases}$
Triangular pulse	$tri\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \lvert\frac{t}{\tau}\rvert, & t \leq \lvert\tau\rvert \\ 0, & t > \lvert\tau\rvert \end{cases}$	$tri[\frac{n}{N}] = \begin{cases} 1 - \frac{\lvert n \rvert}{N}, & \lvert n \rvert \leq N \\ 0, & elsewhere \end{cases}$
Signum signal	$Sgn(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$	$Sgn[n] = \begin{cases} 1, & n > 0 \\ -1, & n < 0 \end{cases}$
Sinusoidal signal	$x(t) = \sin(2\pi f_0 t + \theta)$	$X[n] = \sin(2\pi f_0 n + \theta)$
Sinc function	$\sin(\omega_0 t) = \frac{\sin(\pi\omega_0 t)}{\pi\omega_0 t}$	$\sin[\omega_0 n] = \frac{\sin(\pi\omega_0 n)}{\pi\omega_0 n}$

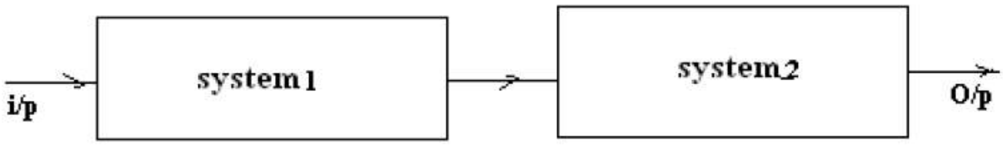
Important Properties of Signals:

Signals in term of unit step and vice versa	$r(t) = t\,u(t)$ $u(t) = \frac{d}{dt} r(t)$ $\delta(t) = \frac{d}{dt} u(t)$ $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$ $sgn = u(t) - u(-t)$ $sgn = 2u(t) - 1$ $\Pi\left(\frac{t}{\tau}\right) = u\left(t + \frac{t}{\tau}\right) - u(t - \frac{t}{\tau})$	*Impulse properties*	$\int_{-\infty}^{\infty} \delta(t) dt = 1$ $\delta(at) = \frac{1}{\lvert a \rvert}\delta(t)$ $\delta(at + b) = \frac{1}{\lvert a \rvert}\delta(t + \frac{b}{a})$ $\int_{-\infty}^{\infty} \emptyset(t)\delta(t-\lambda)dt = \emptyset(\lambda)$ $\emptyset(t)\delta(t-\lambda) = \emptyset(\lambda)\delta(t-\lambda)$
Time period of linear combination of two signals	Sum of signals is periodic if $\frac{T_1}{T_2} = \frac{m}{n}$ = rational number The fundamental period of g(t) is given by nT1 = mT2 provided that the values of m and n are chosen such that the greatest common divisor (gcd) between m and n is 1	Odd and even & symmetry	$x_e(t) = x_e(-t)$ $x_o(t) = -x_o(-t)$ $x(t) = x_e(t) + x_o(t)$ $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$
Combined operation	*x(t)⇒ Kx(t) +C* Scale by *K* then shift by *C* .... x(t)⇒ x(αt −β) Shift by β : [x(t -β)] Then Compress by a:[x(t- β) ⇒ x (αt-β)] OR Compress by α: [x(t)⇒ x(αt)] then Shift by $\frac{\beta}{\alpha}$:[x(αt) ⇒x{α (t- $\frac{\beta}{\alpha}$)} = x (αt −β)}]	Derivative of impulse (doublet)	$\frac{d}{dt}\delta(t) = \delta'(t) = \begin{cases} undefined, & t = 0 \\ 0, & otherwise \end{cases}$ $\delta'(at) = \frac{1}{a\lvert a \rvert}\delta'(t)$ $\int_{-\infty}^{\infty} x(t)\delta'(t-\lambda)dt = -x'(\lambda)$ $x(t)\delta'^{(t)} = x(0)\delta'^{(t)} - x'(0)\delta(t)$
Energy and power	Periodic signals have infinite energy hence power type signals.		

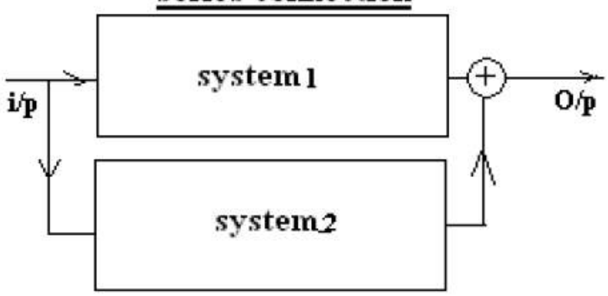
Systems: -



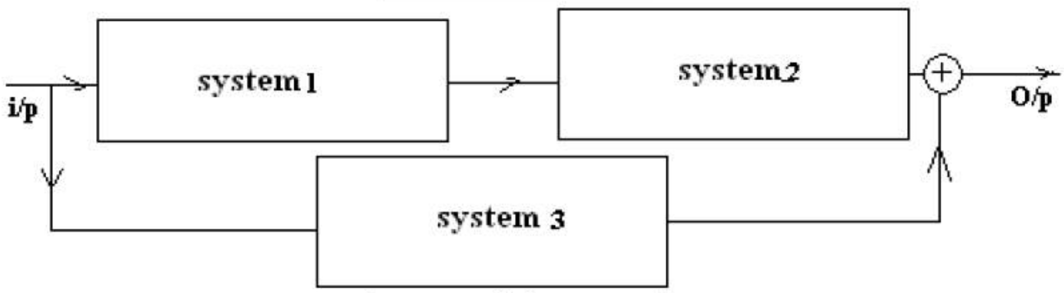
System is an interconnection of components, devices or sub-systems. Various method of interconnection of systems areas follows: -



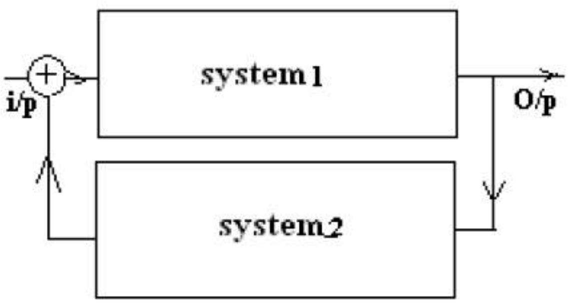
series connection



parallel connection



series-parallel connection

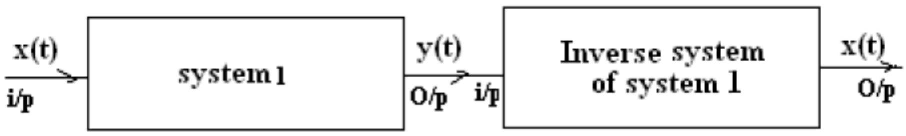


Feedback connection

System properties:-

1. **System with and without memory:** If there is a time delay then the system is said to be having a memory and if there is no time delay then the system is memory-less. E.g.
   - $y[n] = (2x[n] – x2[n])2$ //memory-less
   - $y[n] = x[n – 1]$ //with memory
   - $y(t)=1/C\int x(\tau)d\tau$ (c = capacitance) //with memory

2. **Invertibility and inverse system:** If distinct input leads to distinct output then the system is said to be invertible. If output of a system given to a second system and if we obtain the input of first system as the output of second system then the second system is said to be the inverse of the first one.



3. **Causal system:** A system is said to be causal if output of that system dependents only on the present and the past value of the input and not on the future value of the input. E.g.
   - $y(t) = x(t- 4)$, $y(t) = x(t)$ //causal
   - $y(t) = x(t+t_0)$, $y(t) = x(-t)$ //non-causal

4. **Stability:** If the Region of convergence (ROC) of the Laplace transform X(s) include the entire jω axis. ROC is the region where the Laplace transform is valid. It depends on position.

5. **Time invariance:** if the behavior of the signal and characteristics is fixed over a given time i.e. the system response don’t change with time then the system is known as time invariant. E.g.
   - $y(t) = cow\ \omega t$, $y(t) = x(-t)$ //time-invariant
   - $y(t) = tcow\ \omega t$ //time-dependent

6. **Linearity:** To satisfy the condition of linearity it should follow the following condition: -
   - Superposition theorem: it states that if $y(t) = x(t)$ such that
     If $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$ then if $y_3(t) = x_1(t) + x_2(t)$
     Then $y_3(t) = y_1(t) +y_2(t)$
     And if $y_3(t) = a.x_1(t) + b.x_2(t)$
     Then $y_3(t) = a.y_1(t) + b.y_2(t)$
   - If $y(t) = x(t) = 0$ at $t = 0$.
   - $y(t) = x(t)$ as the degree one.

Linear time-invariant system and convolution integral:

LTI Systems: They are the systems that are linear and time invariant in nature.

Convolution: if h[n] is the impulse response of the system then if x[n] is the input and y[n] is the output then,

y[n] = x[n] * h[n] **// convolution**

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k].h[n-k]$$ **//convolution sum or superposition sum**

Properties of LTI system:

1. **Commutative:** x1[n] * x2[n] = x2[n] * x1[n]
2. **Distributive:** x[n] * (h1[n] + h2[n]) = x[n] * h1[n] + x[n] * h2[n]
3. **Associative:** x[n] * (h1[n] * h2[n]) = (x[n] * h1[n]) * h2[n]
4. **LTI system with and without memory**
5. **Invertibility and inverse system**
6. **Causality of LTI system**
7. **Stability of LTI system**
8. **Unit step response of LTI system:** By convolution δ[n] = u[n] * h[n] there for h[n] in discrete time LTI system is h[n] = δ[n]. δ[n-1]

Causal LTI system described by:

1. Linear constant coefficient differential equation: dy(t)/dt + 2y(t) = x(t)
2. Linear constant coefficient difference equation

$$\sum_{k=0}^{n} a_k . y[n-k] = \sum_{k=0}^{m} b_k . x[n-k]$$

$$y[n] = 1/a_0 \left\{ \sum_{k=0}^{n} a_k . y[n-k] - \sum_{k=0}^{m} b_k . x[n-k] \right\}$$

Discrete-time LTI signals and systems:

Area under impulse	$\sum_n \delta(n) = 1$
Multiplication by impulse	$f(n)\,\delta(n) = f(0)\,\delta(n)$
Shifted impulse	$f(n)\,\delta(n-n_o) = f(n_o)\,\delta(n-n_o)$
Convolution	$f(n) * g(n) = \sum_k f(k)\,g(n-k)$
Convolution with an impulse	$f(n) * \delta(n) = f(n)$
Convolution with a shifted impulse	$f(n) * \delta(n-n_o) = f(n-n_o)$
Transfer function	$H(z) = \sum_n h(n)\,z^{-n}$
Frequency response	$H^f(\omega) = \sum h(n)\,e^{-j\omega n}$
Frequency response their connection	$H^f(\omega) = H(e^{j\omega})$ provided unit circle $\subset$ ROC

Continuous-time LTI signals and systems:

Area under impulse	$\int \delta(t)\,dt = 1$
Multiplication by impulse	$f(t)\,\delta(t) = f(0)\,\delta(t)$
Shifted impulse	$f(t)\,\delta(t-t_o) = f(t_o)\,\delta(t-t_o)$
Convolution	$f(t) * g(t) = \int f(\tau)\,g(t-\tau)\,d\tau$
Convolution with an impulse	$f(t) * \delta(t) = f(t)$
Convolution with a shifted impulse	$f(t) * \delta(t-t_o) = f(t-t_o)$
Transfer function	$H(s) = \int h(t)\,e^{-st}\,dt$
Frequency response	$H^f(\omega) = \int h(t)\,e^{-j\omega t}\,dt$
Frequency response their connection	$H^f(\omega) = H(j\omega)$ provided $j\omega$-axis $\subset$ ROC

Fourier series representation of periodic signal

Fourier series representation of continuous time periodic signals can be given as a linear combination of harmonically related complex exponentials:-

A signal is periodic if x(t) = x(t + nT) for all t where T is a constant known as period and n = 0, 1, 2, … and T = $\omega_0/2\pi$.

E.g. x(t) = $\cos\omega_0 t$, x(t) = $e^{j\omega_0 t}$

Complex exponential

$\Phi_k(t) = e^{ik\omega 0t} = e^{jk(2\pi/T)t}$ k= 0, ±1, ±2…

i.e. $x(t) = \sum_{-\infty}^{\infty} a_k.\, e^{jk\omega 0t} = \sum_{-\infty}^{\infty} a_k.e^{jk(2\pi/T)t}$

if k = 0, x (t) is constant

k = ±1 fundamental frequency ω_0 is the fundamental components known as first harmonic component.

k = ±2 second harmonic component.

k ± n is n^{th} harmonic component.

This representation is known as Fourier series representation of periodic signal.

To determine a_k multiply both with $e^{-jk\omega 0t}$

i.e. $x(t)\, e^{-jk\omega 0t} = \sum_{-\infty}^{\infty} a_k.\, e^{jk\omega 0t}\, e^{-jk\omega 0t} = \sum_{-\infty}^{\infty} a_k.e^{jk(2\pi/T)t}\, e^{-jk\omega 0t}$

Integrating from 0 to T

i.e. $\int_0^T x(t)\, e^{-jk\omega 0t} = \int_0^T \sum_{-\infty}^{\infty} a_k.\, e^{jk\omega 0t}\, e^{-jk\omega 0t} = \sum_{-\infty}^{\infty} \int_0^T a_k[e^{jk(k-n)\omega 0t}dt]$

Euler's formula:

$\int_0^T a_k[e^{jk(k-n)\omega 0t}dt] = \int_0^T [\cos(k-n)\omega_0 t + j\sin(k-n)\omega_0 t]dt$

For k ≠ n cos(k-n) and sin(k-n) are periodic and for k=n cos(k-n)=1 and ans is T.
There for

i.e. $\int_0^T x(t)\, e^{-j(k-n)\omega 0t}$ = T, k=n
0, k≠n

Then for $\mathbf{a_n = 1/T \int_0^T x(t)\, e^{-jn\omega 0t}\, dt}$ **and for** $\mathbf{a_0 = 1/T \int_0^T x(t)\, dt}$

Dirichlets condition:-

- The signal must be absolutely integrable over any period, i.e $\int_T^{\propto} |x(t)| dt < \infty$
- In any finite interval of time, there are not more than a finite maxima and minima in a single period.
- In any finite interval of time, there are only a finite number of discontinuities.

Properties of continuous time Fourier series: -

1. **Linearity:** Let x(t) and y(t) has the same period T and let their fourier series coefficient be a_k and b_k respectively then since x(t) and y(t) are of same period T it is followed that the combination of both the signal will be also periodic with period T. i.e.
   $x(t) \leftrightarrow a_k$ (with period T), $y(t) \leftrightarrow b_k$ (with period T) then
   $z(t) = Ax(t) + By(t) \leftrightarrow C_k = Aa_k + Bb_k$ with period T

2. **Time shifting:**
   $x(t) \leftrightarrow a_k$ then $x(t\text{-}t0) \leftrightarrow e^{jk\omega oto}.a_k$
   i.e. magnitude of the Fourier series coefficient remains unaltered.

3. **Time reversal:**
   $x(t) \leftrightarrow a_k$ (with period T), then $x(-t) \leftrightarrow a_{-k}$ (with period T) if x(t) is even then Fourier series coefficient is also even and if x(t) is odd then the Fourier series coefficient is also odd.

4. **Time scaling:** It changes the period. If x(t) has a period T then x(αt) will have a period P = T/α and the Fourier series coefficient will not change.

5. **Multiplication:** $x(t) \leftrightarrow a_k$ (with period T), $y(t) \leftrightarrow b_k$ (with period T) then

   $$x(t).y(t) \leftrightarrow h_k = \sum_{k=-\infty}^{\infty} a_L.b_{k-L} \text{ with period T}$$

6. **Conjugate and conjugate symmetry:** $x(t) \leftrightarrow a_k$ (with period T), then $x^*(t) \leftrightarrow a^*_{-k}$ (with period T) where '*' represents complex conjugate.

7. **Frequency shifting:** i.e. multiplication with $e^{-jm\omega 0t} = x(t).\ e^{-jm\omega 0t}$ the Fourier series coefficient will be a_{k-m}

8. **Periodic convolution:** $x(t) \leftrightarrow a_k$, $y(t) \leftrightarrow b_k$ (with same period T) then $\int x(\tau).y(t-\tau)d\tau$ will have the coefficient as $T.\ a_k\ b_k$

9. **Differentiation:** dx(t)/dt has the Fourier coefficient $jkw_0.a_k$

10. **Integration:** $\int x(t)dt$ (finite value) (its periodic only if $a_0 = 0$) has the Fourier series coefficient a_k /jkw_0.

11. **Conjugate symmetry for real signals:** i.e. if x(t) is real then $a_k = {}^*a_k$, re{ a_k }=re{ a_{-k} },im{ a_k }= -im{ a_{-k} }, | a_k |=| a_{-k} | and $\measuredangle a_k = -\measuredangle a_{-k}$

12. **Real and even:** If x(t) is real and even then their coefficient is also real and even.

13. **Real and odd:** If x(t) is real and odd then the coefficient is purely imaginary and odd.

14. **Decomposition of real signal:** $x_e(t) = Ev\{x(t)\}$ [x(t) is real] then coefficient Re{ a_k }and $x_o(t) = Od\{x(t)\}$ [x(t) is real] then coefficient is Im{ a_k }

Fourier series representation of discrete time periodic signals

Linear combination of harmonically related complex exponentials:-
A signal is periodic if x[n] = x(n + mN) for all n where N is a constant known as period and m = 0, 1, 2, … and $N = \omega_0/2\pi$. E.g. $x[n] = \cos\omega_0 t$, $x[n] = e^{j\omega_0 n}$

Complex exponential

$\Phi_k[n] = e^{ik\omega 0n} = e^{jk(2\pi/N)n}$ k= 0, ±1, ±2…

$\Phi_k[n] = \Phi_{k+N}[n]$

i.e. $\Phi_0[n] = \Phi_N[n]$, $\Phi_1[n] = \Phi_{N+1}[n]$

$x[0] = \sum_{k=<N>} a_k.$, $x[1] = \sum_{k=<N>} a_k.\ e^{jk(2\pi/N)}$

i.e. $x[n] = \sum_{k=<N>} a_k.\ e^{jk\omega 0n} = \sum_{k=<N>} a_k.e^{jk(2\pi/N)n}$

if k = 0, 1, …. N -1 or K=3, 4, …, N+2 etc…

K= N is N^{th} harmonic component or N successive integers.

This representation is known as Fourier series representation of periodic signal of discrete type.

To determine a_k multiply both with $e^{-jr\omega 0n}$

i.e. $x[n]\ e^{-jr\omega 0n} = \sum_{k<n>} a_k.\ e^{jk\omega 0n}\ e^{-jr\omega 0n} = \sum_{k=<n>} a_k.e^{jk(k-r)(2\pi/N)n}$

There for inner most sum if k = r is N and if K ≠ r is 0.

$a_r = 1/N \sum_{k=<N>} x[n]\ e^{-jr\omega 0n}$ where $\omega_0 = 2\pi/N$

$a_k = 1/N \sum_{k=<N>} x[n]\ e^{-jk\omega 0n}$ where $\omega_0 = 2\pi/N$

Properties of discrete time Fourier series: -

1. **Linearity:** Let x[n] and y[n] has the same period N and let their Fourier series coefficient be a_k and b_k respectively then since x[n] and y[n] are of same period N it is followed that the combination of both the signal will be also periodic with period N. i.e.
   $x[n] \leftrightarrow a_k$ (with period N), $y[n] \leftrightarrow b_k$ (with period N) then
   $z[n] = Ax[n] + By[n] \leftrightarrow C_k = Aa_k + Bb_k$ with period N

2. **Time shifting:**
   $x[n] \leftrightarrow a_k$ then $x[n-n0] \leftrightarrow e^{-jk\omega ono}.a_k$
   i.e. magnitude of the Fourier series coefficient remains unaltered.

3. **Time reversal:**
   $x[n] \leftrightarrow a_k$ (with period N), then $x[-n] \leftrightarrow a_{-k}$ (with period N) if x[n] is even then Fourier series coefficient is also even and if x[n] is odd then the Fourier series coefficient is also odd.

4. **Time scaling:** It changes the period. If x[n] has a period N then x[αn] will have a period P = N/α and the Fourier series coefficient will not change.

5. **Multiplication:** $x[n] \leftrightarrow a_k$ (with period N), $y[n] \leftrightarrow b_k$ (with period N) then
   $x[n].y[n] \leftrightarrow h_k = \sum_{k=<N>} a_L.b_{k-L}$ with period N

6. **Conjugate and conjugate symmetry:** $x[n] \leftrightarrow a_k$ (with period N), then
   $x^*[n] \leftrightarrow a^*_{-k}$ (with period N) where '*' represents complex conjugate.

7. **Frequency shifting:** i.e. multiplication with $e^{-jm\omega 0n}$ => $x[n].\ e^{-jm\omega 0n}$ the Fourier series coefficient will be a_{k-m}

8. **Periodic convolution:** $x[n] \leftrightarrow a_k$, $y[n] \leftrightarrow b_k$ (with same period N) then $\sum_{k=<n>} x(r).y[n-r]d\tau$ will have the coefficient as $N.\ a_k\ b_k$

9. **First difference:** x[n]-x[n-1] has the Fourier coefficient $(1-e^{-jk\omega 0}).a_k$

10. **Running sum:** Finite value and periodic only if a_0=0.
    $\sum_{K=-\infty}^{N} x[k]$ has the Fourier series coefficient as $(1/(1-e^{-jk\omega 0})).a_k$

11. **Conjugate symmetry for real signals:** i.e. if x[n] is real then $a_k = {}^*a_k$, re{ a_k }=re{ a_{-k} },im{ a_k }= -im{ a_{-k} }, | a_k |=| a_{-k} | and $\measuredangle a_k = -\measuredangle a_{-k}$

12. **Real and even:** If x[n] is real and even then their coefficient is also real and even.

13. **Real and odd:** If x[n] is real and odd then the coefficient is purely imaginary and odd.

14. **Decomposition of real signal:** $x_e[n]$ = Ev{x[n]} [x[n] is real] then coefficient Re{ a_k }and $x_o[n]$ = Od{x[n]} [x[n] is real] then coefficient is Im{ a_k }

<u>Paseval's relation:</u>

1. <u>Continuous time periodic signal:</u>

   $$1/T \int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$ where a_k is Fourier series coefficient and T is the time period of the signal.
   I.e. average power or energy per unit time in one period.
   Total paseval's relation = sum of the average power in all harmonic components.

2. <u>Discrete time periodic signal:</u>

   $$1/N \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$ where a_k is Fourier series coefficient and N is the time period of the signal.
   I.e. | a_k | = average power or energy per unit time in one period.
   Total paseval's relation = sum of the average power in all harmonic components.

<u>*Fourier transforms representation of periodic signal*</u>

Representation of a-periodic signal:

$$x(t) = \begin{cases} 1 & |t| < T \\ 0, & T_1 < |t| < T/2 \end{cases} \quad \text{period } T$$

<u>Formula for Fourier transform</u>

if x(t) is a signal:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t).\, e^{-j\omega t}\, dt$$

<u>Formula for inverse Fourier transform</u>

if X(jω) is the transformed signal:

$$x(t) = 1/2\pi \int_{-\infty}^{\infty} X(j\omega).\, e^{-j\omega t}\, d\omega$$

Properties of continuous time Fourier transforms: -

1. **Linearity:** Let x(t) and y(t) has the same period T and let their transform be
   $x(t) \leftrightarrow X(j\omega)$ and $y(t) \leftrightarrow Y(j\omega)$ then
   $z(t) = Ax(t) + By(t) \leftrightarrow A\,X(j\omega) + B\,Y(j\omega)$ with period T

2. **Time shifting:** $x(t) \leftrightarrow X(j\omega)$ then $x(t\text{-}t0) \leftrightarrow e^{j\omega oto}\,X(j\omega)$

3. **Time scaling :** $x(t) \leftrightarrow X(j\omega)$ then $x(\alpha t) \leftrightarrow 1/|\alpha|\,X(j\omega/\alpha)$

4. **Multiplication:** $x(t) \leftrightarrow a_k$ (with period T), $y(t) \leftrightarrow b_k$ (with period T) then

   $$x(t).y(t) \leftrightarrow h_k = \sum_{k=-\infty}^{\infty} a_L.b_{k-L} \text{ with period T}$$

5. **Conjugate and conjugate symmetry:** $x(t) \leftrightarrow X(j\omega)$ (with period T), then
   $x^*(t) \leftrightarrow X^*(-j\omega)$ (with period T) where '*' represents complex conjugate.
   $X(-j\omega) = X^*(j\omega)$ if x(t) is real.

6. **Frequency shifting:** i.e. multiplication with $e^{-jm\omega 0t} = x(t).\ e^{-j\omega 0t}$ the Fourier transforms will be $X(j(\omega\text{-}\ \omega_0))$

7. **Convolution:** If y(t) = x(t) * h(t) (where * represents convolution) then $Y(j\omega)= X(j\omega) \times H(j\omega)$.

8. **Differentiation:** $dx(t)/dt \leftrightarrow j\omega\,X(j\omega)$

9. **Integration:** $\int x(t)dt \leftrightarrow 1/j\omega\,X(j\omega) + \pi X(0)\delta(\omega)$

10. **Multiplication by t:** $tx(t) \leftrightarrow jd/d\omega\,X(j\omega)$

11. **Multiplication:** $x(t).\ y(t) \leftrightarrow 1/2\pi\,[X(j\omega) * Y(j\omega)]$ where * referred as amplitude modulation or convolution.

12. **Conjugate symmetry for real signals:** i.e. if x(t) is real then $X(j\omega) = X^*(j\omega)$, $re\{X(j\omega)\}=re\{X(-j\omega)\}, im\{X(j\omega)\}= -im\{X(-j\omega)\}$, $|X(j\omega)|=|X(-j\omega)|$ and $\measuredangle X(j\omega) = -\measuredangle X(-j\omega)$;

13. **Real and even:** If x(t) is real and even then $X(j\omega)$ is also real and even.

14. **Real and odd:** If x(t) is real and odd then $X(j\omega)$ is purely imaginary and odd.

15. **Decomposition of real signal:** $x_e(t) = Ev\{x(t)\}$ [x(t) is real] then $F_s=Re\{X(j\omega)\}$ and $x_o(t) = Od\{x(t)\}$ [x(t) is real] then $F_s= Im\{X(j\omega)\}$

16. **Time reversal:** $x(-t) \leftrightarrow X(-j\omega)$

Fourier transforms representation of discrete time periodic signals

Representation of Fourier Transform: -

$$X[e^{j\omega}] = \sum_{N=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Representation of inverse Fourier transform: -

$$x[n] = 1/2\pi \int_{2\pi} X(e^{j\omega}).\, e^{j\omega n}\, d\omega$$

Properties of continuous time Fourier transforms: -

1. **Linearity:** Let x[n] and y[n] has the same period N and let their transform be $x[n] \leftrightarrow X[e^{j\omega}]$ and $y[n] \leftrightarrow Y[e^{j\omega}]$ then
   $z[n] = Ax[n] + By[n] \leftrightarrow A\,Y[e^{j\omega}] + B\,Y[e^{j\omega}]$ with period N
2. **Time shifting:** $x[n] \leftrightarrow X[e^{j\omega}]$ then $x[t\text{-}t0] \leftrightarrow e^{j\omega no}\, X[e^{j\omega}]$
3. **Time scaling :** $x[n] \leftrightarrow X[e^{j\omega}]$ then $x[\alpha t] \leftrightarrow 1/|\alpha|\, X[e^{j\omega}/\alpha]$
4. **Multiplication:** $x[n] \leftrightarrow X[e^{j\omega}]$ (with period N), $y[n] \leftrightarrow Y[e^{j\omega}]$ (with period N) then
   $$x[n].y[n] \leftrightarrow 1/2\pi \int_{2\pi} X[e^{j\theta}].Y[e^{j(\omega-\theta)}]$$
5. **Conjugate and conjugate symmetry:** $x[n] \leftrightarrow X[e^{j\omega}]$ (with period N), then $x^*[n] \leftrightarrow X^*[e^{-j\omega}]$ (with period T) where '*' represents complex conjugate.
6. **Frequency shifting:** i.e. multiplication with $e^{-j\omega 0n}$ => $x[n].\ e^{-j\omega 0n}$ the Fourier transforms will be $X[e^{j(\omega-\omega 0)}]$
7. **Convolution:** If $y[n] = x[n] * h[t]$ (where * represents convolution) then
   $$Y[e^{j\omega}] = X[e^{j\omega}] \times H[e^{j\omega}].$$
8. **Difference in time:** $x[n]-x[n-1] \leftrightarrow (1- e^{j\omega})X(e^{j\omega})$
9. **Difference in frequency:** $nx[n] \leftrightarrow jdX[e^{j\omega}]/d\omega$
10. **Accumulation:** $\sum_{k=-\infty}^{N} x[k] \leftrightarrow 1/(1- e^{-j\omega})X[e^{j\omega}] + \pi X[e^{j\theta}] \sum_{k=-\infty}^{\infty} \delta[\omega-2\pi k]$
11. **Conjugate symmetry for real signals:**
    if x[n] is real then
    $Y[e^{j\omega}] = X^*[e^{-j\omega}]$, $re\{Y[e^{j\omega}]\}=re\{X[e^{-j\omega}]\}$, $im\{Y[e^{j\omega}]\}= -im\{X[e^{-j\omega}]\}$,
    $|Y[e^{j\omega}]| = |X[e^{-j\omega}]|$ and $\measuredangle X[e^{jw}] = -\measuredangle X[e^{-jw}]$
12. **Real and even:** If x[n] is real and even then $X[e^{j\omega}]$ is also real and even.
13. **Real and odd:** If x[n] is real and odd then $X[e^{j\omega}]$ is purely imaginary and odd.
14. **Decomposition of real signal:** $x_e[t] = Ev\{x[n]\}$ (x[n] is real) then $F_s=Re\{X[e^{j\omega}]\}$ and $x_o[t] = Od\{x[n]\}$ (x[n] is real) then $F_s= Im\{Y[e^{j\omega}]\}$
15. **Time reversal:** $x[-t] \leftrightarrow X[e^{-j\omega}]$

Parseval's relation:

If x[n] and $X[e^{j\omega}]$ are Fourier transform pair then ,

$$\sum_{n=-\infty}^{\infty} |x[n]|2 = 1/2\pi \int_{2\pi} |X[e^{j\omega}]|2\, d\omega$$

Duality:

For discrete time signal the Fourier transform and its inverse are more or just same:

$$x[n] = 1/2\pi \int_{2\pi} X[e^{j\omega}]\, e^{j\omega n}\, d\omega$$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Similarly for continuous time:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t).\, e^{-j\omega t}\, dt$$

$$x(t) = 1/2\pi \int_{-\infty}^{\infty} X(j\omega).\, e^{-j\omega t}\, d\omega$$

Sampling theorem:

It states that the sampling frequency should be more than or equal to twice the max frequency component of the message signal (base band signal)

$$f_s \geq 2\ f_m$$

System characterized by linear constant coefficient differential equation:

$$H[e^{j\omega}] = Y[e^{j\omega}]/\ X[e^{j\omega}]$$

where, $x(t)$ is the input ($X[e^{j\omega}]$ its Fourier transform)
$y(t)$ is the output ($Y[e^{j\omega}]$ its Fourier transform) and
$h(t)$ is the impulse response of the system ($H[e^{j\omega}]$ its Fourier transform).

Fourier transform pairs:

$x(t)$	$X^f(\omega)$
$x(t)$	$\int x(t)\, e^{-j\omega t}\, dt$ (def.)
$\frac{1}{2\pi} \int X^f(\omega)\, e^{j\omega t}\, d\omega$	$X^f(\omega)$
$\delta(t)$	1
1	$2\,\pi\,\delta(\omega)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
$e^{j\omega_o t}$	$2\,\pi\,\delta(\omega - \omega_o)$
$\cos(\omega_o\, t)$	$\pi\,\delta(\omega + \omega_o) + \pi\,\delta(\omega - \omega_o)$
$\sin(\omega_o\, t)$	$j\,\pi\,\delta(\omega + \omega_o) - j\,\pi\,\delta(\omega - \omega_o)$
$\frac{\omega_o}{\pi}\ \mathrm{sinc}\left(\frac{\omega_o}{\pi} t\right)$	ideal LPF cut-off frequency ω_o
symmetric pulse width T, height 1	$\frac{2}{\omega}\ \sin\left(\frac{T}{2}\,\omega\right)$
impulse train period T, height 1	impulse train period, height $\omega_o = \frac{2\pi}{T}$

Laplace transform

Laplace transforms definition:

Laplace transforms:

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t).\ e^{-st}\ dt = X(s)$$ where s is complex variable and $s = \sigma + j\omega$

Inverse Laplace transforms:

$$L^{-1}\{X(s)\} = 1/2\pi \int_{0}^{t} X(s).\ e^{st}\ ds = x(t)$$

Laplace transforms **exist only if $\int x(t).\ e^{-st}\ dt$ exist**. I.e. $|\int x(t).\ e^{-st}\ dt| < \infty$

Initial and final value theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulse or higher order singularities at $t = 0$ and let $X(s)$ be the Laplace transform of $x(t)$ then,

$x(0^+) = \mathrm{Lim}_{s\to\infty} SX(s)$ {Initial value theorem }

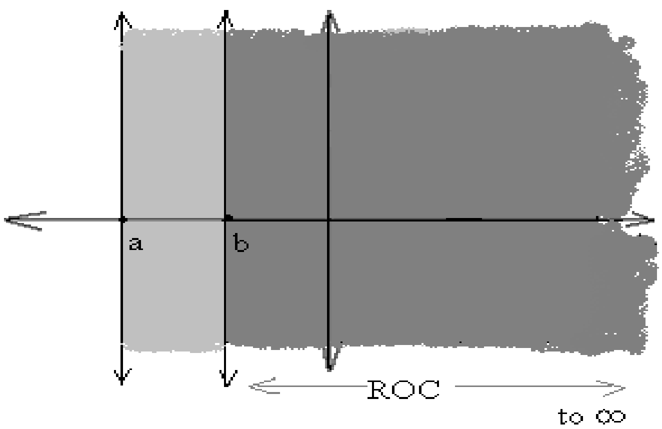
$\mathrm{Lim}_{t\to\infty} x(t) = \mathrm{Lim}_{s\to 0} SX(s)$ {Final value theorem }

ROC of Laplace transform:

ROC (region of convergence) of Laplace transform is the region in the x-y plane where Laplace transform is ROC.

For e.g. X(s) = 1/s+a is Valid if s > -a in the x- y plane. If there are more than one root the overlapping area is the ROC.

For e.g. X(s) = 1/((s+a)(s+b)) where |a| > |b| say then,

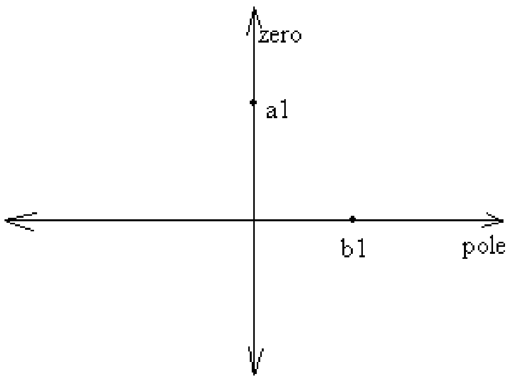


Poles and zeros:

Poles are the roots of the denominator of the fraction in the Laplace transform and zeroes are the roots of numerator of the fraction. For example as follows: -

$$X(s) = \frac{(s + a_1)(s + a_2) \dots\dots\dots\dots\dots.(s + a_n)}{(s + b_1)(s + b_2) \dots\dots\dots\dots\dots.(s + b_n)}$$

where a_1, a_2, …, an are known as zeroes and b_1, b_2, …. , bn are known as poles. They are represented in a pole zero diagram as



Properties of Laplace transform:

If x(t), $x_1(t)$, and $x_2(t)$ are three signals and X(s), $X_1(s)$, $X_2(s)$ are their Laplace transform respectively and a, b are some constant then,

1. **Linearity:** $ax_1(t) + bx_2(t) = aX_1(s) + bX_2(s)$
2. **Time shifting:** $x(t-t_0) = e^{-sto}.X(s)$
3. **Shifting in s domain:** $e^{sto}x(t) = X(s-s_0)$
4. **Time scaling:** x(at) = 1/|a| X(s/a)
5. **Conjugation:** x*(t) = X*(s*)
6. **Convolution:** $x_1(t)*x_2(t) = X_1(s).X_2(s)$
7. **Differential in time domain:** dx(t)/dt = sX(s)
8. **Differential in frequency domain:** -tx(t) = d/ds X(s)
9. **Integration in time domain:** Integration in time domain is division in frequency domain.

   i.e. $\int_{-\infty}^{t} x(\tau)d(\tau) = 1/s \,.\, X(s)$

Applications of Laplace transform:

- For a system with a rational system function causality of the system is equivalent to the ROC being to the right half plane to the right of the right most pole.
- An LTI system is stable if and only if the ROC of its system function H(s) include the entire jω axis. (i.e. Re(s) = 0)

Laplace transform pairs:

$x(t)$	$X(s)$	ROC	
$x(t)$	$\int x(t)\, e^{-st}\, dt$ (def.)		
$\delta(t)$	1	all s	
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$	
$e^{-a\,t}\, u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$	; a is real.
$\cos(\omega_o t)\, u(t)$	$\frac{s}{s^2+\omega_o^2}$	$\mathrm{Re}(s) > 0$	
$\sin(\omega_o t)\, u(t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\mathrm{Re}(s) > 0$	
$e^{-a\,t}\cos(\omega_o t)\, u(t)$	$\frac{s+a}{(s+a)^2+\omega_o^2}$	$\mathrm{Re}(s) > -a$	
$e^{-a\,t}\sin(\omega_o t)\, u(t)$	$\frac{\omega_o}{(s+a)^2+\omega_o^2}$	$\mathrm{Re}(s) > -a$	

Complex arithmetic operations:

Operation	***Formula***
Rectangular to Polar Conversion	$z = x + jy = re^{j\theta}$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$
Polar to Rectangular Conversion	$z = re^{j\theta} = r\,[\cos(\theta) + j\sin(\theta)] = x + jy$ where $r = \cos(\theta)$ and $y = r\sin(\theta)$
Add: $z_3 = z_1 + z_2$	$(x_1 + x_2) + j\,(y_1 + y_2)$
Subtract: $z_3 = z_1 - z_2$	$(x_1 - x_2) + j\,(y_1 - y_2)$
Multiply: $z_3 = z_1 z_2$ (polar form)	$(x_1 x_2 - y_1 y_2) + j\,(x_1 y_2 + y_1 x_2)$ $r_1 r_2\, e^{j(\theta_1+\theta_2)}$
Divide: $z_3 = z_1/z_2$ (polar form)	$\frac{(x_1 x_2 - y_1 y_2) - j\,(x_1 y_2 - y_1 x_2)}{x_2^2 + y_2^2}$ $\frac{r_1}{r_2}\, e^{j(\theta_1-\theta_2)}$

Z-transform and discrete Fourier transform

Z-transform definition:

If x(t) is the signal then its z transform X(z) is defined as follows: -

$$X(z) = \sum_{n=-\infty}^{\infty} x[n].\, z^{-n}$$

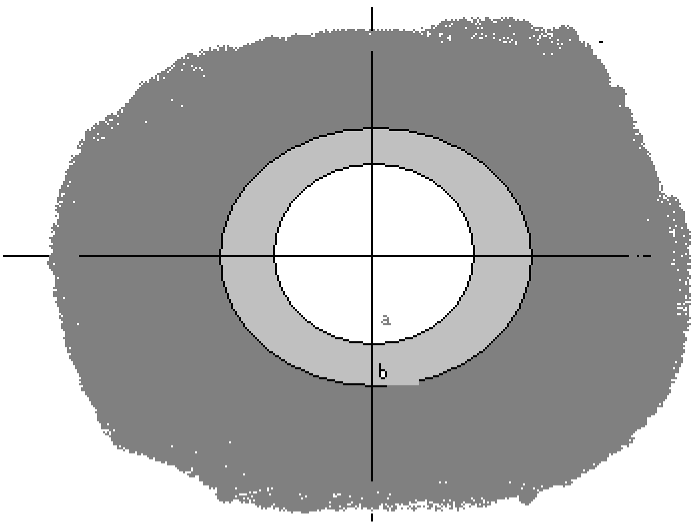
Note: Every transform is of the form $\int x(t).k(s,t)dt$.
For example of discrete Fourier transforms

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].(r.e^{j\omega})^{-n} \quad = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n}.r^{-n}$$

If r = 1 then, $X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\omega n}$

ROC of Z transform:

ROC (region of convergence) of Z transform is the region in the x-y plane where Z transform is valid. It dose not have any pole. It is a ring in z- plane centered about origin. If x[n] is of finite duration then ROC in the entire Z plane except possibly z = 0 and/or z = ∞.
For e.g. X(z) = 1/z+a is Valid if z > -a and we draw a circle with center origin and radius |a| in the z plane and if the transform is valid for values greater than a then the ROC is exterior of the circle and if less than a then interior of the circle.. If there are more than one root the overlapping area is the ROC.
For e.g. X(z) = 1/((z+a)(z+b)) where |a| < |b| say then,



Initial value theorem:

If x[n] = 0 for n < 0 then $x[0] = \text{Lim}_{z\to\infty} X(z)$

Properties of Z transform :

If x[n], $x_1[n]$, and $x_2[n]$ are three signals and X(z), $X_1(z)$, $X_2(z)$ are their Z transform respectively and a, b are some constant then,

1. **Linearity:** $ax_1[n] + bx_2[n] = aX_1(z) + bX_2(z)$
2. **Time shifting:** $x[n\text{-}n0] = z^{-no}.X(z)$
3. **Shifting in z domain:** $e^{j\omega on}x[n] = X(e^{-j\omega oz})$, $Z_0^n.x[n] = X(z/z_0)$, $a^n.x[n] = X(a^{-1}z)$
4. **Time reversal:** $x[-n] = x[z^{-1}]$
5. **Conjugation:** $x^*[n] = X^*(z^*)$
6. **Convolution:** $x_1[n]*x_2[n] = X_1(z).X_2(z)$
7. **First difference:** $x[n] – x[n-1] = (1-z^{-1})X(z)$
8. **Differential in z domain:** $nx[n] = -z.dX(z)/dz$
9. **Accumulation:** $\sum_{n=-\infty}^{\infty} x[n] = 1/|1\text{-}z\text{-}1| \,.\, X(z)$

Application of Z-transform:

1.A discrete time LTI system is causal if and only if the ROC of its system is the exterior of the circle including ∞.

2.A discrete time LTI system is stable if the function H(z) include the circle $|z| = 1$.

Z-transform transform pairs

$x(n)$	$X(z)$	ROC
$x(n)$	$\sum_n x(n)\, z^{-n}$ (def.)	
$\delta(n)$	1	all z
$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$a^n\, u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n\, u(-n-1)$	$\frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_o n)\, u(n)$	$\frac{z^2 - \cos(\omega_o)\, z}{z^2 - 2\cos(\omega_o)\, z + 1}$	$\lvert z\rvert > 1$
$\sin(\omega_o n)\, u(n)$	$\frac{\sin(\omega_o)\, z}{z^2 - 2\cos(\omega_o)\, z + 1}$	$\lvert z\rvert > 1$
$a^n \cos(\omega_o n)\, u(n)$	$\frac{z^2 - a\cos(\omega_o)\, z}{z^2 - 2a\cos(\omega_o)\, z + a^2}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(\omega_o n)\, u(n)$	$\frac{a\sin(\omega_o)\, z}{z^2 - 2a\cos(\omega_o)\, z + a^2}$	$\lvert z\rvert > \lvert a\rvert$