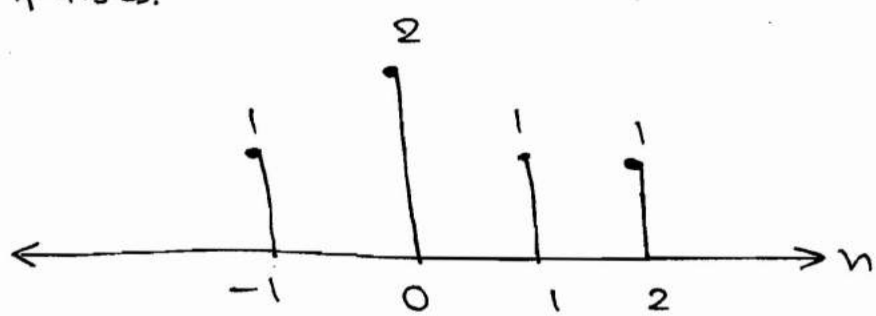
☆ LTI Systems :-

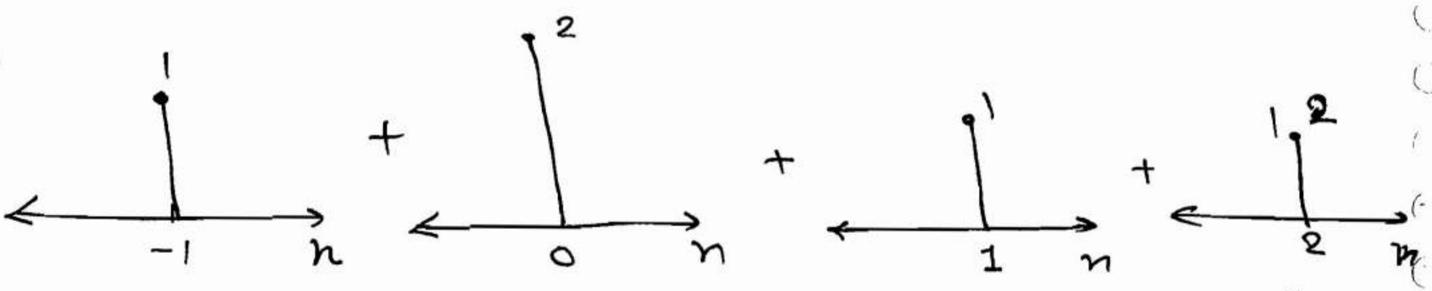
⇒ A LTI Systems is represented with respect to impulse response. (if input is impulse, output is impulse response).

⇒ Sifting property states that any signal can be produced as combination of impulses.

e.g.



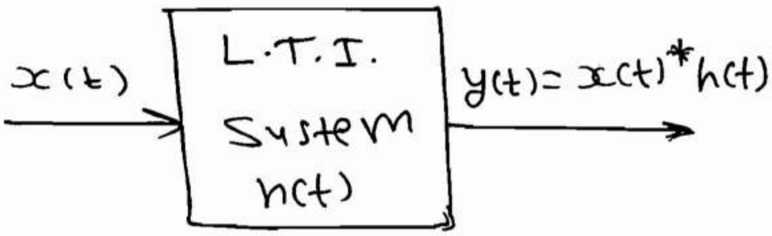
⇒

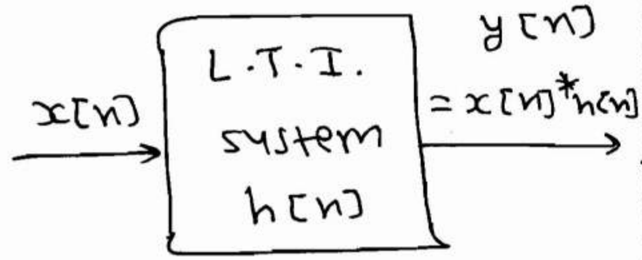


$x[-1]\,\delta[n+1]$ $\quad$ $x[0]\,\delta[n]$ $\quad$ $x[1]\,\delta[n-1]$ $\quad$ $x[2] \cdot \delta[n-2]$

⇒ Convolution is formal mathmatical operation, just as multiplication, addition while convolution takes two signals and produces a third signal.

* **Continuous Convolution** **Discrete Convolution.**





$$y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau.$$

$$= \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau)\, d\tau.$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k].$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n-k] \cdot h[k].$$

Steps:

1. $x(t) \longrightarrow x(\tau)$,
   $h(t) \longrightarrow h(\tau)$.

2. Folding: $x(-\tau)$, $h(-\tau)$.

3. Shifting: $x(t-\tau)$, $h(t-\tau)$

4. multiplication: $x(t-\tau) \cdot h(\tau)$, $h(t-\tau) \cdot x(\tau)$.

Steps:

1. $x[n] \longrightarrow x[k]$,
   $h[n] \longrightarrow h[k]$.

2. Folding: $x[-k]$, $h[-k]$

3. Shifting: $x[n-k]$. $h[n-k]$.

4. Multiplation: $x[k] \cdot h[n-k]$, $x[n-k] \cdot h[k]$

→ By using Convolution we are finding the zero state response for a given input and system. (zero initial condition).

→ Sliding one signal over the folded and shifted version of the other signal is a concept of Convolution.

$$P_1(s) = s^2 + 2s + 1$$

$$P_2(s) = s^2 + 3s + 4.$$

$s^2 + 2s + 1$ (fixed)

$4 + 3s + s^2$ (folding) ~~3s+~~

$4 + 3s \mid s^2 \longrightarrow = s^4$

$4\ \cancel{3s} \mid 3s + s^2 \longrightarrow = 3s^3 + 2s^3 = 5s^3$

$4 + 3s + s^2 \longrightarrow = 4s^2 + 6s^2 + s^2 = 11s^2$

$4 + 3s \longrightarrow = 8s + 3s = 11s$

$4 \longrightarrow = 4$

$$s^4 + 5s^3 + 11s^2 + 11s + 4.$$

*

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot \overbrace{h(t-\tau)}^{\text{Shifting}}\, d\tau$$

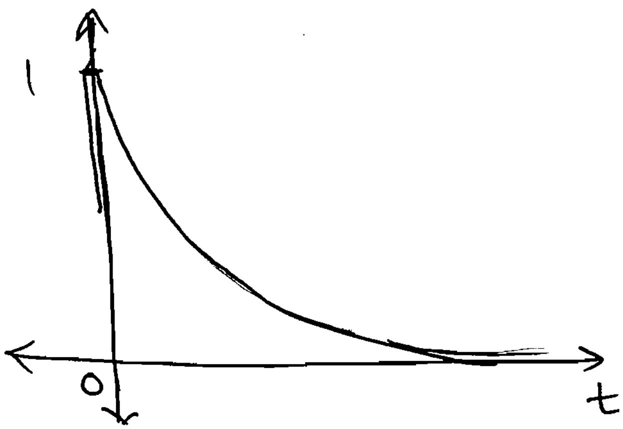
① folding

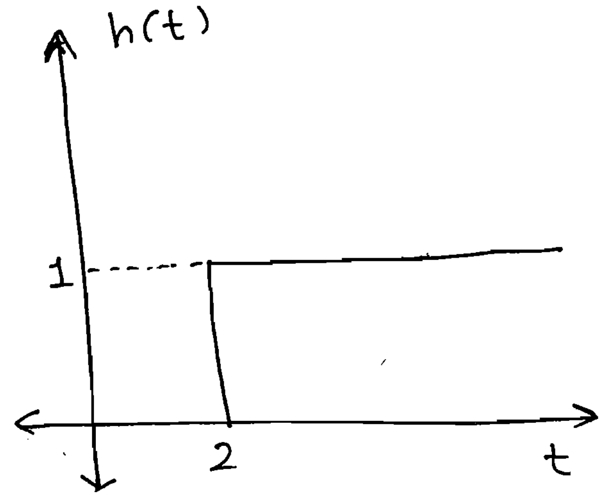
multiplication

P-1 Find the convalution of the two signal $e^{-3t} \cdot u(t)$ & $h(t) = u(t-2)$.

Soln:

$x(t) = e^{-3t} \cdot u(t)$.





Steps:

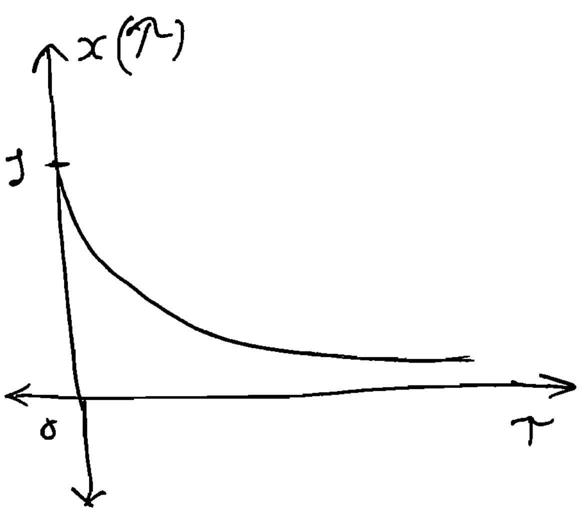
① **Obtaining the limits of $y(t)$:-**

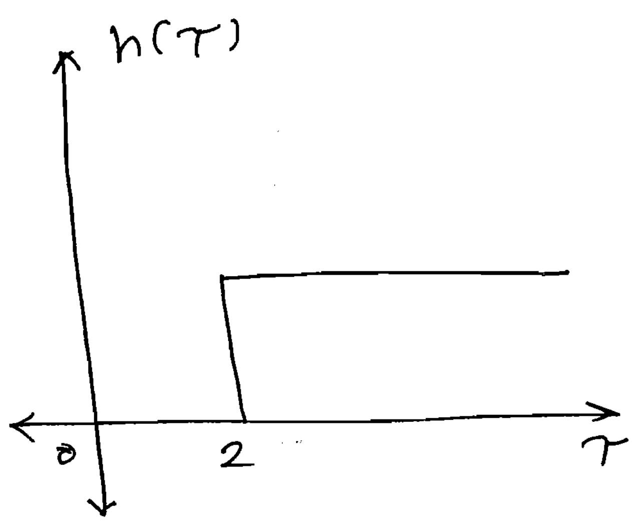
Sum of lower limits < t < sum of upper limits.

$\therefore \quad 0 + 2 < t < \infty + \infty$

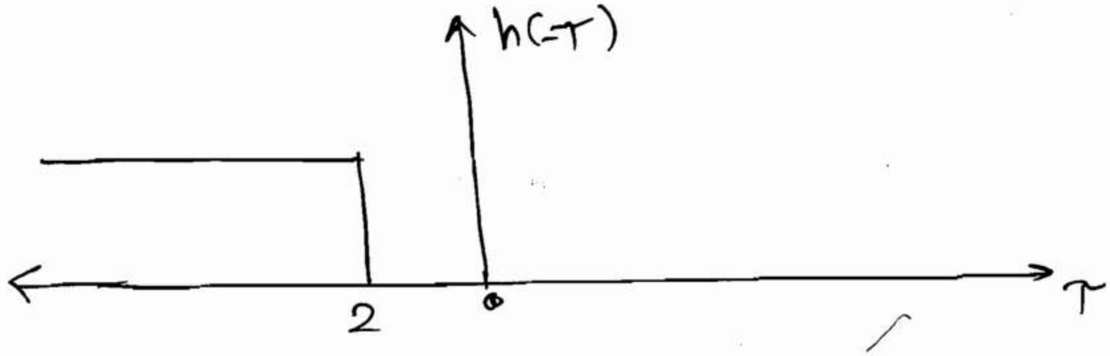
$\Rightarrow \quad \boxed{2 < t < \infty}$

② **Change axis from "t" to "τ".**

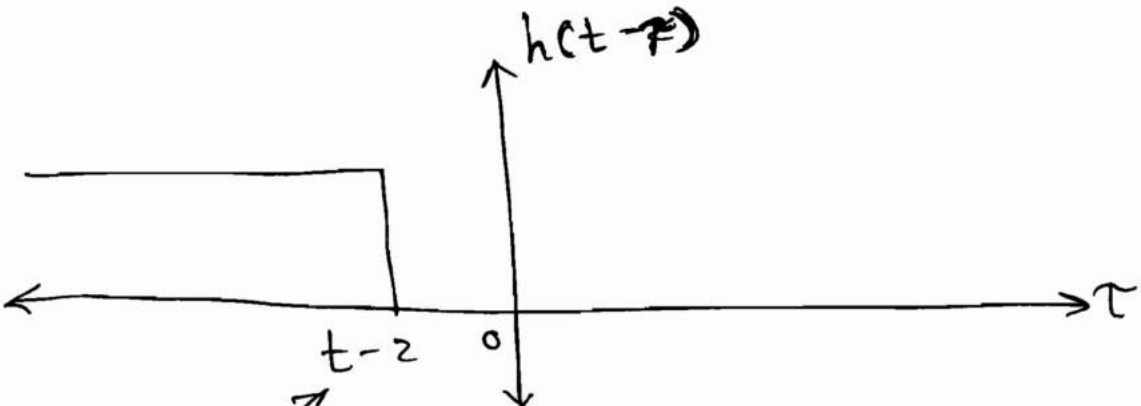




③ Folding / Flipping:



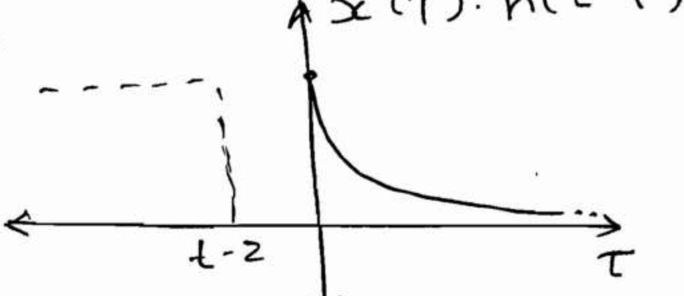
④ Shifting



Add variable i.e here (-2) to the t.

⑤ Multiplication:

Case-(i) $t-2<0$.



$y(t)=0$ for $t<2$

as. No overlapping

Case-(ii) $t-2>0$

$$\Rightarrow \quad y(t) = \int_0^{t-2} e^{-3\tau} \cdot u(\tau)\, d\tau.$$

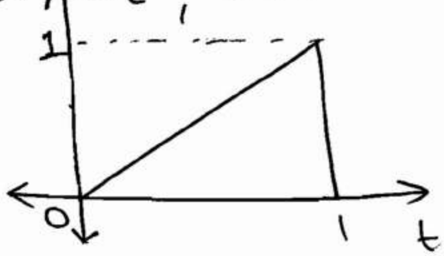
$$\therefore \boxed{y(t) = \frac{1 - e^{-3(t-2)}}{3} \; ; \quad t > 2.}$$

<u>Note:</u>

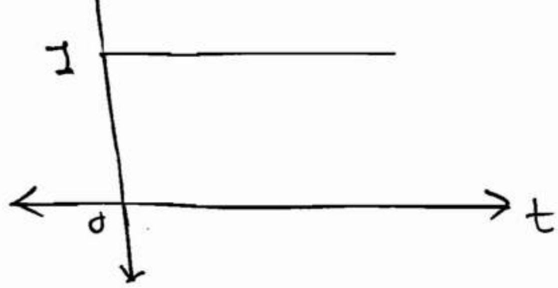
→ Convolution of two causal system is causal.

$\boxed{Q-2}$ Find the convolution of the signals shown in figure?

$x(t) = t, \quad 0 < t < 1$



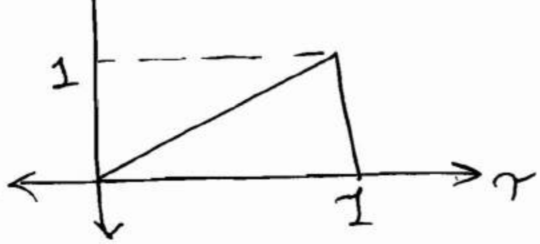
$h(t) = 1, \quad t > 0.$



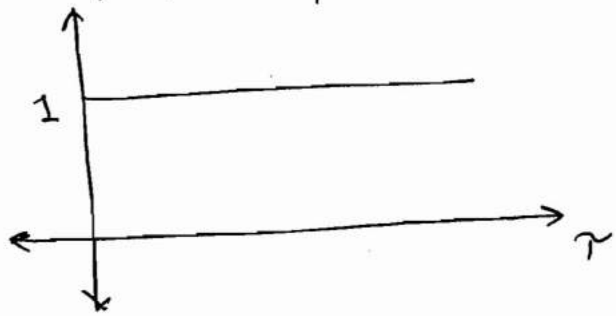
<u>Sol^n:</u> ① $0 + 0 < t < 1 + \infty.$

$$\Rightarrow \boxed{0 < t < \infty}.$$

② $x(\tau) = \tau, \quad 0 < \tau < 1.$



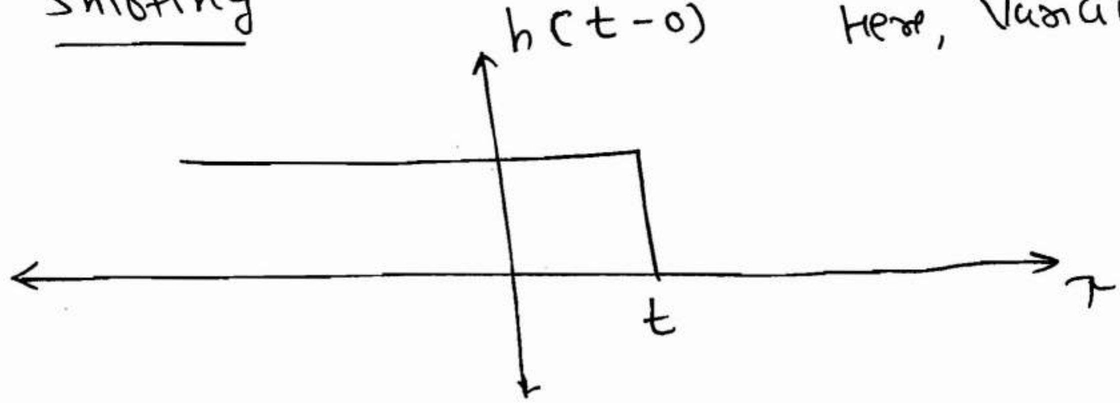
$h(\tau) = 1, \quad \tau > 0$



③ $h(-\tau)$

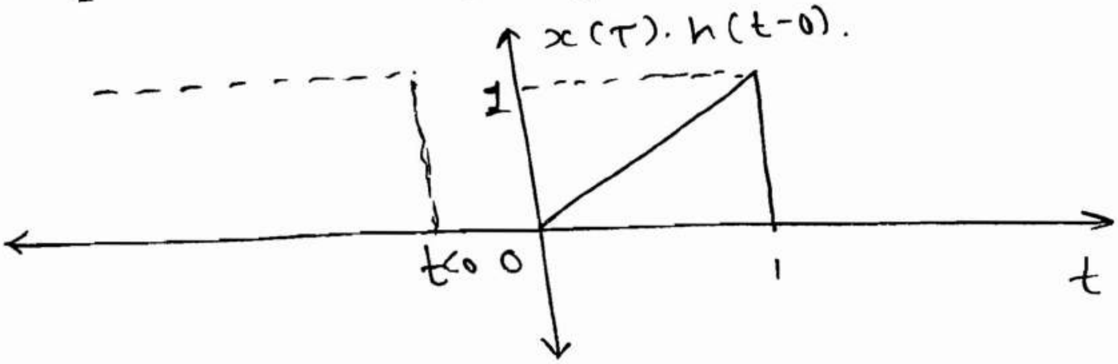
(4) <u>Shifting</u>

Here, Variable is 0.

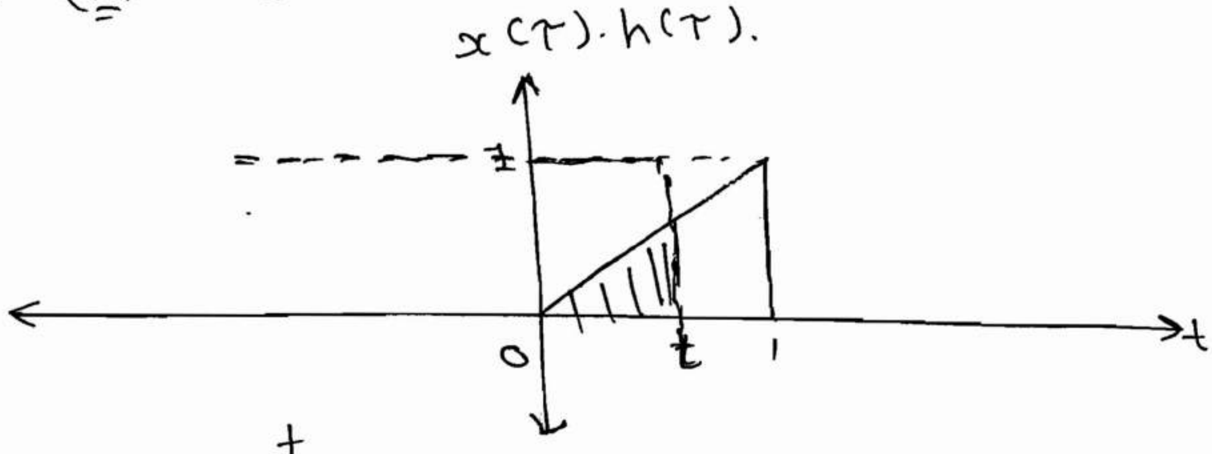


(5) <u>Multiplication</u> & <u>Integration.</u>

Case-(i) $t<0 \Rightarrow y(t)=0$



Case-(ii) $0<t<1$

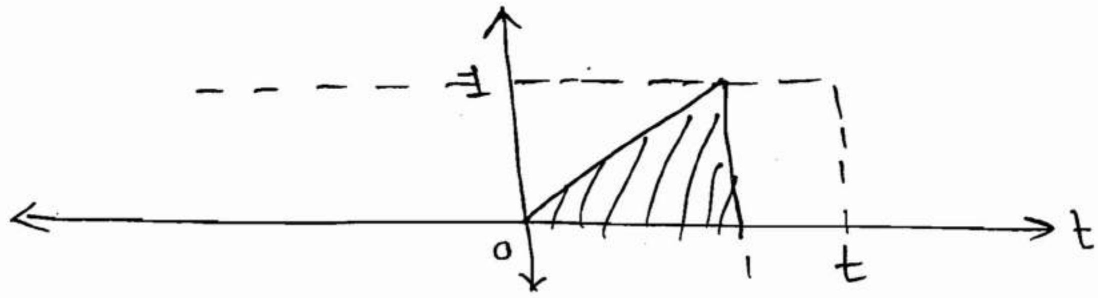


$$y(t) = \int_0^t (\tau)\cdot(1)\cdot d\tau.$$

$$y(t) = t^2/2 \quad ; \quad 0<t<1.$$

Case-(iii) $t>1$

$\Rightarrow$

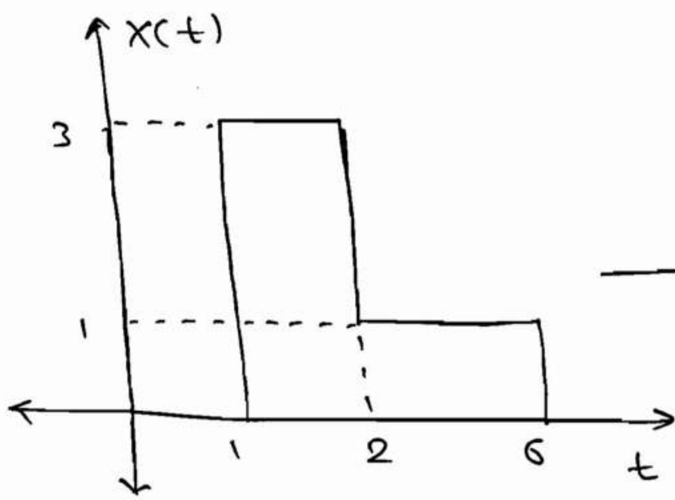


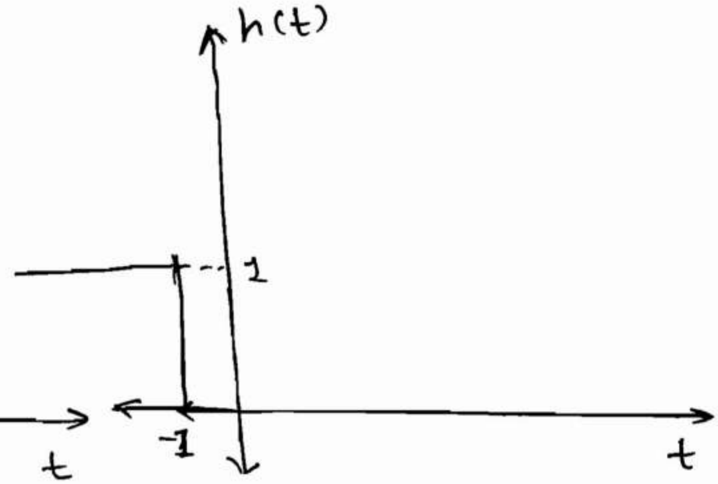
So, $y(t) = 0 \;;\; t<0$

$= t^2/2 \;;\; 0<t<1$

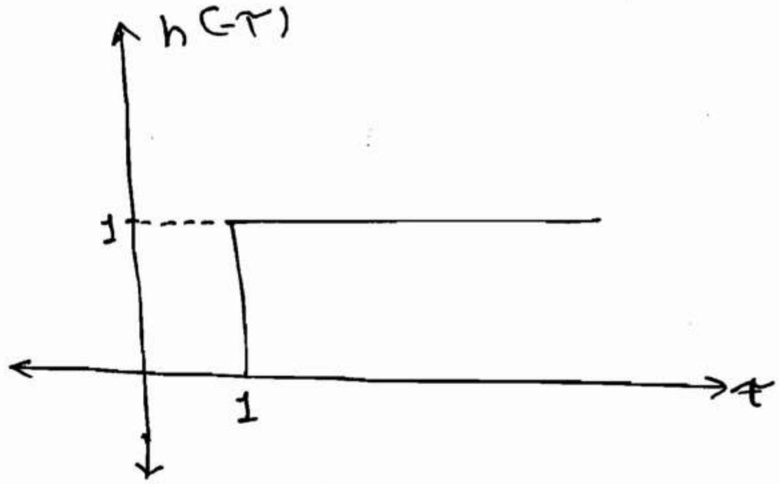
$= 1/2 \;;\; t>1.$

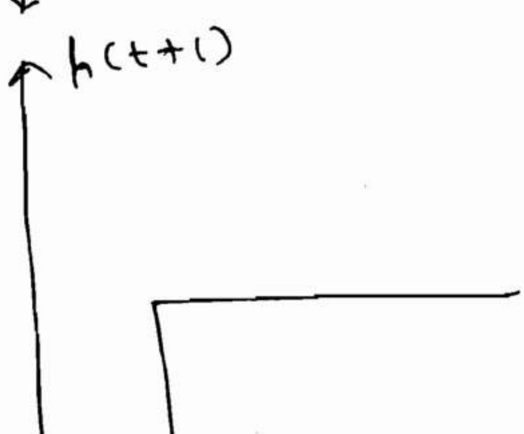
$\boxed{P-3}$



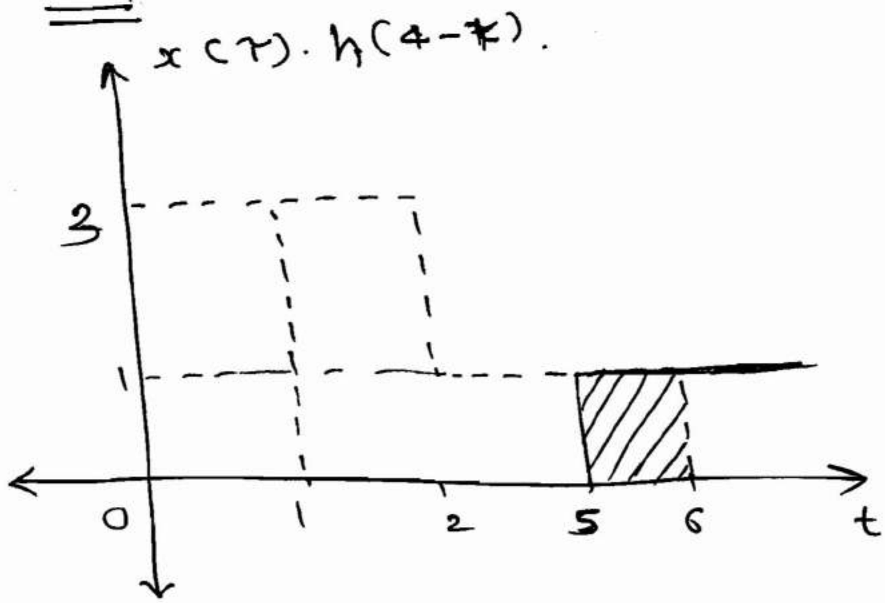


Soln:



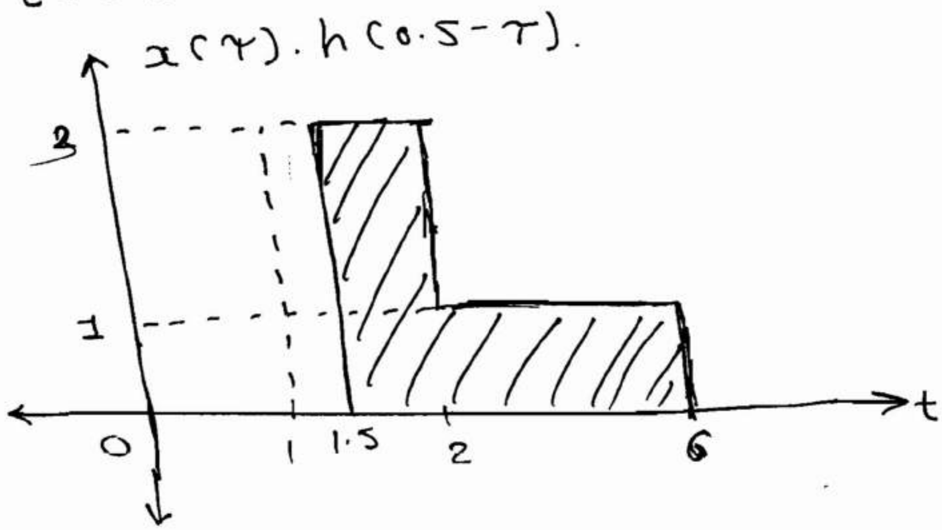


$\Rightarrow$ at $\underline{\underline{t=4}}$.



$$y(t) = \int_{5}^{6} (1)(1).\,d\tau \;=\; 1.$$

$\Rightarrow$ at $t = 0.5$



$$y(t) = \int_{1.5}^{6} x(\tau).\, t\,(0.5-\tau).\,d\tau.$$

$$= \int_{1.5}^{2} (3).\,d\tau + \int_{2}^{6} (1).\,d\tau.$$

$$= 3\,(2-1.5) + (6-\cancel{1.5}\,2)$$

$$1.5 + 4$$

P 2.1.4. Suppose $z(t) = \int_{-\infty}^{+\infty} x(-\tau + a)\, h(t+\tau)\, d\tau$.

Express $z(t)$ in terms of $y(t) = x(t) * h(t)$.

Soln:

take $t + \tau = \lambda$.

$\Rightarrow t = \lambda - \tau. \quad dt = d\lambda$

$-\tau = t - \lambda$.

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau.$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau.$$

$$z(t) = \int_{-\infty}^{+\infty} x(t - \lambda + a) \cdot h(\lambda)\, d\lambda.$$

$$= \int_{-\infty}^{\infty} x(\underline{(t+a)} - \lambda) \cdot h(\lambda)\, d\lambda.$$

$$\therefore \boxed{z(t) = x(t+a) * h(t+a).}$$

P 2.1.5.

(a) $x(t+5) * \delta(t-7) =$ ______.

(b) $x(t) * \delta(at+b) =$ ______.

Soln: (a) $x(t+5) * \delta(t-7)$

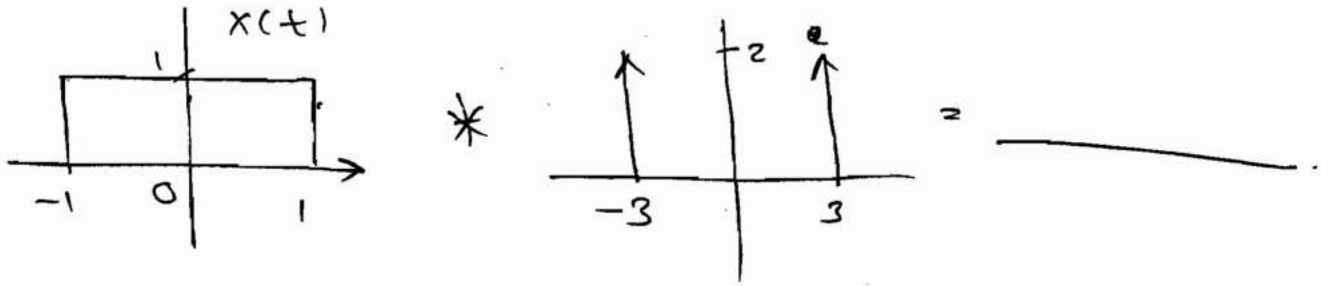
$= x(t+5-7) \quad (\because x(t) * \delta(t - t_0)$

(b) $x(t) * \delta(at+b)$

$$= x(t) * \left[\frac{1}{a}\, \delta(t + b/a)\right].$$

$$= \frac{1}{a}\, x(t + b/a).$$

(c)



Sol^{n}. $x(t) = u(t+1) - u(t-1).$

$h(t) = \delta(t+3) + \delta(t-3).$

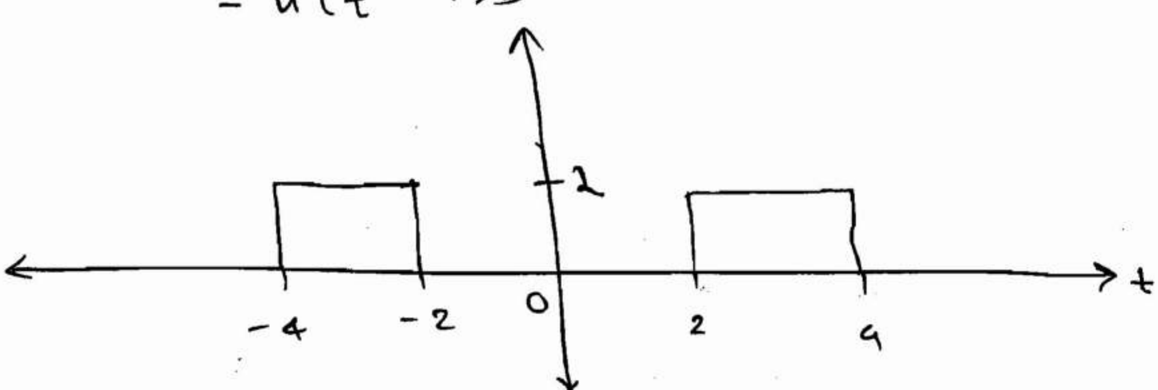
$$\therefore\ y(t) = x(t) * h(t)$$

$$= [u(t+1) \ - u(t-1)] * [2\delta(t+3) + 2\delta(t-3)].$$

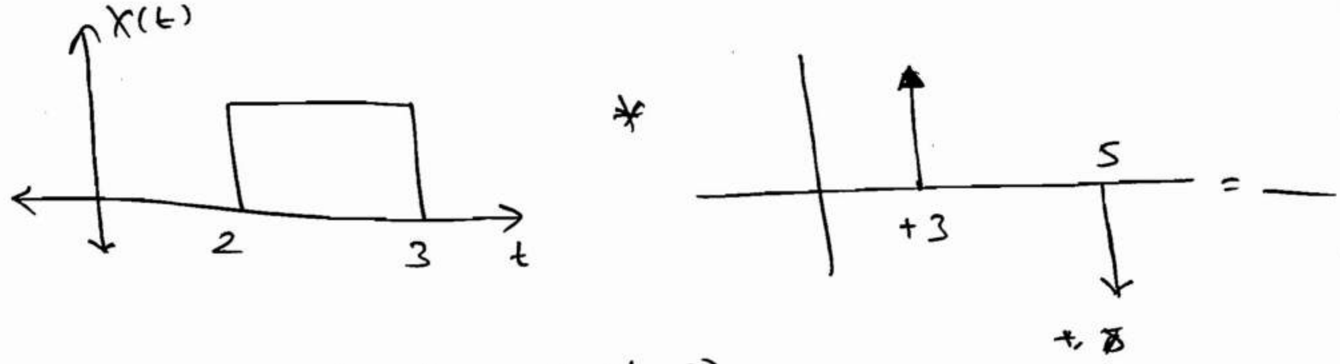
$$= 2[u(t+1) * \delta(t+3) \ + u(t+1) * \delta(t-3).$$
$$- u(t-1) * \delta(t+3) \ - u(t-1) * \delta(t-3)]$$

$$= 2[u(t+1+3) \ + u(t+1-3) - u(t-1+3)$$
$$- u(t-1-3)]$$

$$= 2[u(t+4) + u(t-2) - u(t+2)$$
$$- u(t-4)]$$



$\boxed{a}$



Sol^n:

$$x(t) = u(t-2) - u(t-3).$$

$$h(t) = \delta(t-3) - \delta(t-5).$$

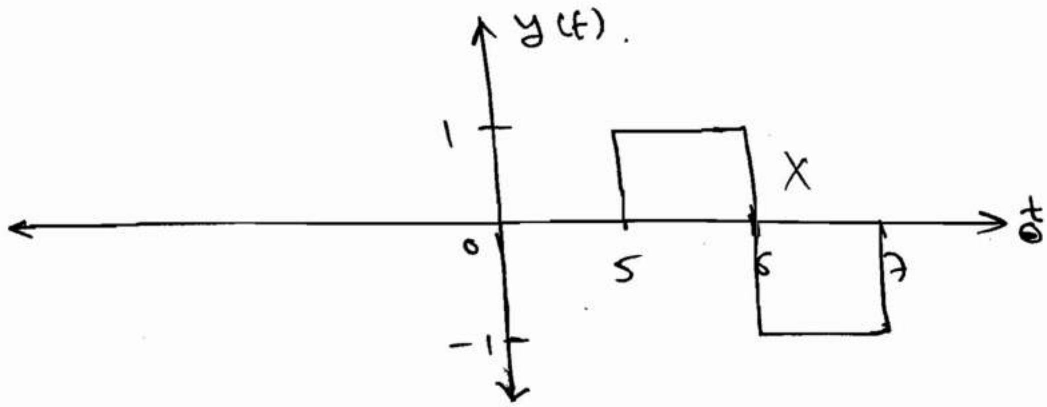
$$\therefore \; y(t) = x(t) * h(t).$$

$$= u(t-2) * \delta(t-3) - u(t-3) * \delta(t-3)$$

$$+ u(t-2) * \delta(t-5) - u(t-3) * \delta(t-5).$$

$$= u(t-5) - u(t-6) + u(t-7) - u(t-8).$$

$$\therefore \; y(t) = u(t-5) - 2u(t-6) + u(t-7) + u(t-8)$$



* Convolution property:

$$\boxed{x(t) * \delta(t-t_0) = x(t-t_0).}$$

e.g. ① $x(t) * \delta(t-13)$

$= x(t-13).$

② $x(t+5) * \delta(t-13).$

③ $x(t) * \delta(2t+3)$.

$= x(t) * \left[\frac{1}{2}\,\delta(t+3/2)\right]$ $\left(\because \delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)\right)$.

$= \frac{1}{2}\cdot\left[x(t) * \delta(t+3/2)\right]$

$= \frac{1}{2} \times x(t+3/2)$.

$\boxed{\text{P 2.1.3. (b)}}$

The impulse response of the continuous time system is given by $h(t) = \delta(t-1) + \delta(t-3)$. The value of the step response at $t=2$ is.

(a) 0 (B) 1 (C) 2 (D) 3.

Solⁿ:

$x(t) = u(t)$.

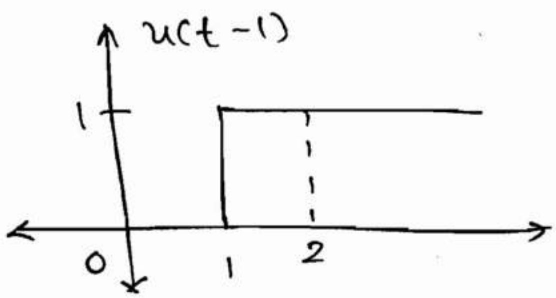
$y(t) = h(t)$.

$y(t) = x(t) * h(t)$.

$\therefore y(t) = u(t) * [\delta(t-1) + \delta(t-3)]$.

$y(t) = u(t-1) + u(t-3)$.

$\therefore y(2) = u(1) + u(-1) = 1+0 = 1$.



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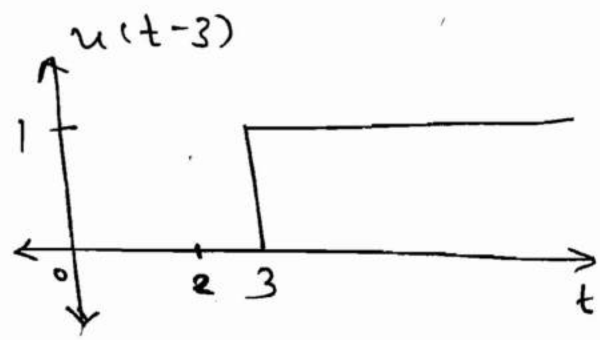


So, Ans ⇒ (B) 1.

$\boxed{P\ 2.1.6}$ Explain the difference between the following operations?

(a) $\left[e^{-t} u(t)\right] \delta(t-1)$.

$\Rightarrow$ it is <u>product property</u> of impulse

i.e. $\boxed{\begin{aligned} & x(t) \cdot \delta(t-t_0) \\ &= x(t-t_0) \cdot \delta(t-t_0). \end{aligned}}$

So, $\left[e^{-t} \cdot u(t)\right] \delta(t-1]$

$= e^{-1} \cdot u(1) \cdot \delta[t-1]$ $(\because t_0 = 1]$.

$= e^{-1} \cdot \delta[t-1]$.

(b) $\displaystyle\int_{-\infty}^{+\infty} e^{-t} \cdot u(t) \cdot \delta(t-1)\, dt$.

$\Rightarrow$ It is a <u>sifting property</u> of impulse.

i.e. $\boxed{\displaystyle\int_{t_1}^{t_2} x(t) \cdot \delta(t-t_0)\, dt = x(t_0) \quad ; \ t_1 \le t_0 \le t_2}$

So, $\boxed{\displaystyle\int_{-\infty}^{+\infty} e^{-t} \cdot u(t) \cdot \delta(t-1)\, dt}$

$= e^{-1} \cdot u(1)$ $(\because t_0 = 1$ & $x(t) = e^{-t} \cdot u(t)$.
& $-\infty \le 1 \le +\infty)$.

$= e^{-1}$

$\Rightarrow$ It is the ~~Pro~~ Convolution property of Impulse.

i.e. $\boxed{x(t) * \delta(t-t_0) = x(t-t_0).}$

So, $e^{-t} \cdot u(t) * \delta(t-1)$.

$= e^{-(t-1)} \cdot u(t-1)$.

$\boxed{P\ 2.1.7}$ Let $x(t) = u(t-3) - u(t-5)$ &

$h(t) = e^{-3t} \cdot u(t)$. Find $\frac{d}{dt} x(t) * h(t)$.

Soln:

$x(t) = u(t-3) - u(t-5)$.

$\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$.

~~Now,~~ $\cancel{\frac{d}{dt} x(t) * h(t)} = \cancel{[(\delta(t-3) - \delta(t-5)] *}$
$\cancel{[u(t-3) - u(t-5)]}$.

$\cancel{= u(t-3) * \delta(t-3) - u(t-5) * \delta(t-5)}$
$\cancel{- u(t-5) * \delta(t-3) + u(t-5) * \delta(t-5)}$.
$\cancel{= u(t-6) - u(t-10) - u(t-8) + u(t-10)}$.

Now, $\frac{dx(t)}{dt} * h(t)$.

$= [\delta(t-3) - \delta(t-5)] * e^{-3t} \cdot u(t)$.

$= e^{-3t} \cdot u(t) * \delta(t-3) - e^{-3t} \cdot u(t) * \delta(t-5)$

Note: In LTI system, whatever happens in i/p same thing happens in o/p.

i.e. $\frac{d}{dt} x(t) * h(t) = \frac{d}{dt} . y(t).$

⇒ In general,

$$\boxed{\begin{array}{l} x^{m}(t) * h^{n}(t) = y^{m+n}(t). \\ m, n \rightarrow \text{order of differentiation} \end{array}}$$

⇒ $\boxed{x(t-\alpha) * h(t-\beta) = y(t-\alpha-\beta).}$

i.e. whatever be the delay in input same delay will occur in o/p as it is time Invariant system.

⇒ Area:

		Area
$x(t)$	$\longrightarrow$	A_x
$h(t)$	$\longrightarrow$	A_h.
$y(t)$	$\longrightarrow$	$A_x . A_h$.

Proof:

$$y(t) = \int x(\tau) . h(t-\tau) \, d\tau.$$

$$\text{Area } A_h = \int y(t) . dt = \iint x(\tau) . h(t-\tau) . d\tau . dt$$

$\therefore \boxed{A_h = A_x \cdot A_h.}$

⇒ Scaling:

$$\boxed{x(\alpha t) * h(\alpha t) = \frac{1}{|\alpha|} \cdot y(\alpha t).}$$

Note:
⇒ In above case, property is valid only when the scaling factor in x & h should be same.

i.e. if $x(\alpha t) * h(\beta t) = ?$

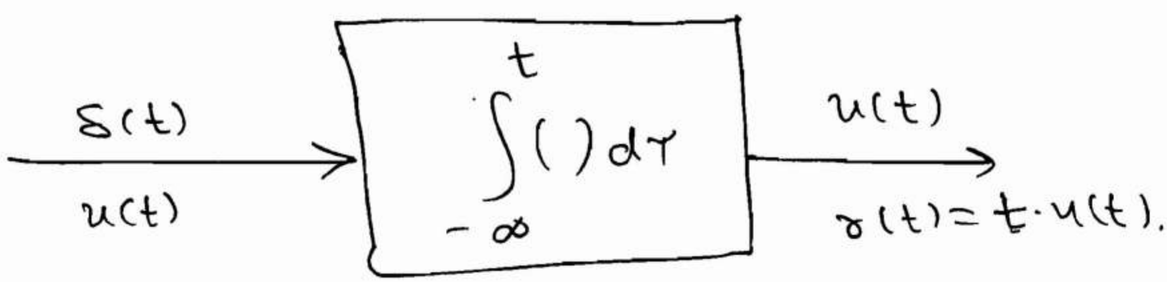
& $\alpha \neq \beta$

then No Comment.

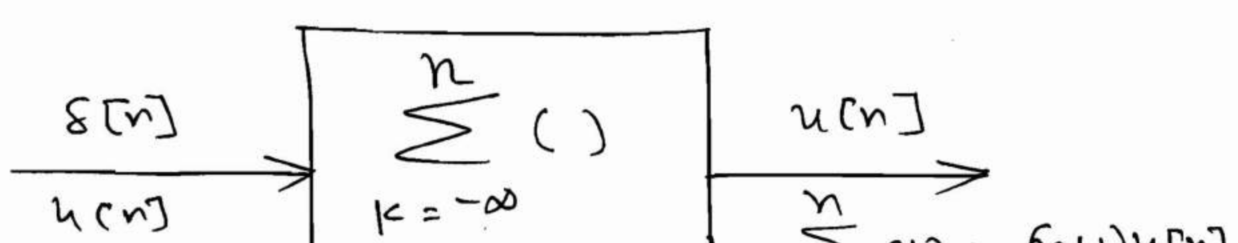
*

$$\boxed{\begin{aligned} u(t) * u(t) &= t \cdot u(t) \\ u[n] * u[n] &= (n+1)\,u[n]. \end{aligned}}$$

⇒



⇒



$\boxed{Q}$ Convolution of $u[n]$ with $u[n-4]$ at $n=5$ is ___ ?

$\underline{\underline{Sol^n}}$:

$$y[n] = u[n] * u[n-4].$$

$$= (n+1-4)\,u[n]. \quad = \underline{(n+1-4)\,u[n]}.$$

$$\therefore y[5] = (5+1-4].$$

$$\therefore \boxed{y[5] = 2.}$$

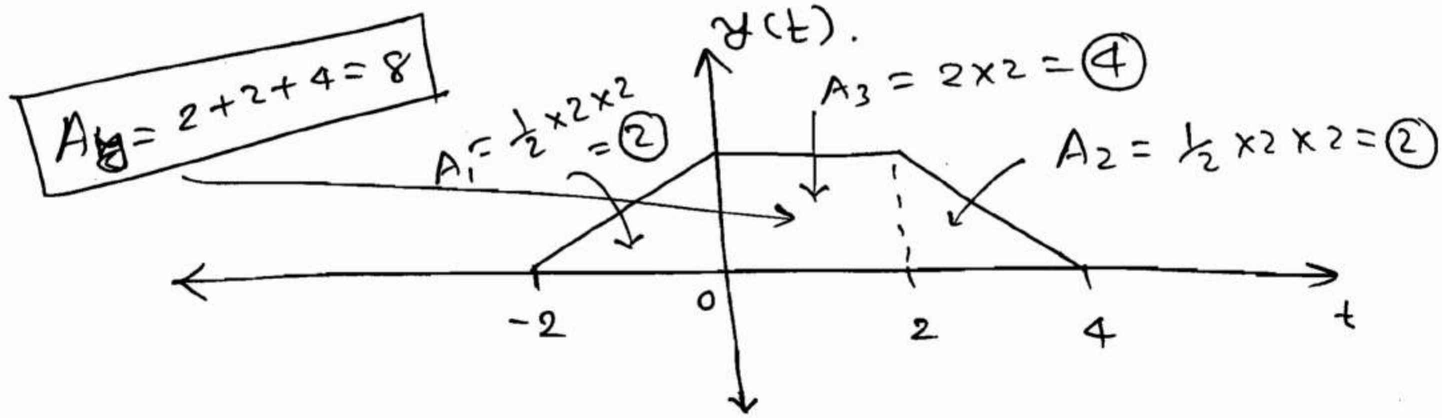
$\boxed{Q}$ $[u(t+1) - u(t-3)] * [u(t+1) - u(t-1)] =$ ___ .

$\underline{\underline{Sol^n}}$:

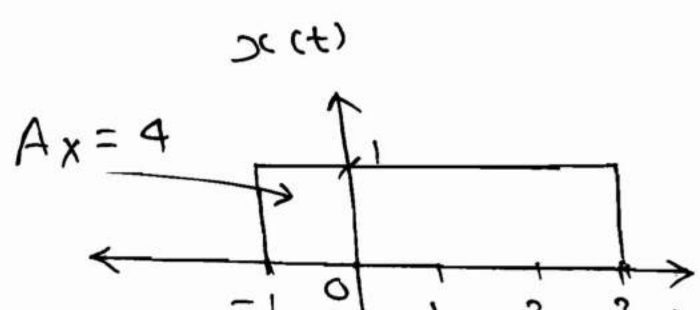
$$y(t) = u(t+1) * u(t+1) - u(t+1) * u(t-1)$$
$$- u(t-3) * u(t+1) + u(t-3) * u(t-1).$$

$$= r(t+1+1) - r(t+1-1) - r(t-3+1) + r(t-3-1).$$

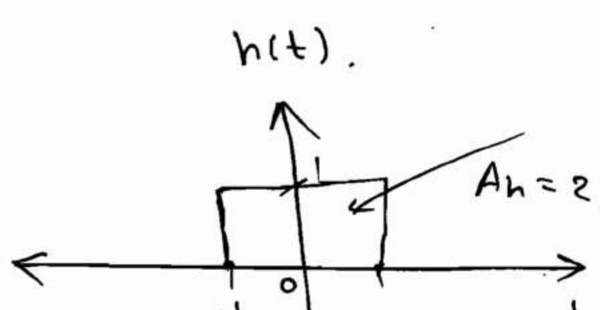
$$= r(t+2) - r(t) - r(t-2) + r(t-4).$$



$\Uparrow$



$*$



Note: → Convolution of two unequal length rectangular functions is a Trapezium.

→ If length is same then it is a triangle.

Note:- → Convolution of the function to it self is equal to integration of that f^n. This statement is valid only for unit step f^n, $u(t)$.

i.e. $$\boxed{\begin{array}{c} x(t) * x(t) = \int_0^t x(\tau)\,d\tau. \\ \text{where, } x(\tau) = u(\tau). \end{array}}$$

⇒ i.e. $$\boxed{u(t) * u(t) = r(t) = t \cdot u(t) = \int_0^t u(\tau)\,d\tau.}$$

* Convolution using Differentiation:-

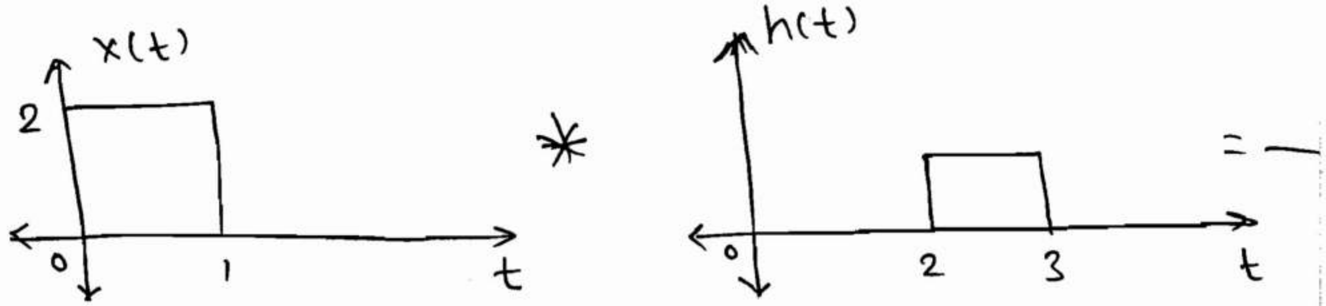
⇒ $$y(t) = x(t) * h(t).$$

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} * h(t)$$

$$\therefore \quad \frac{dy(t)}{dt} = x(t) * \frac{d}{dt} h(t).$$

$$\Rightarrow \quad \boxed{y(t) = \int_{-\infty}^{t} \left[x(\tau) \cdot \frac{d}{d\tau} h(\tau)\right] d\tau.}$$

(a)



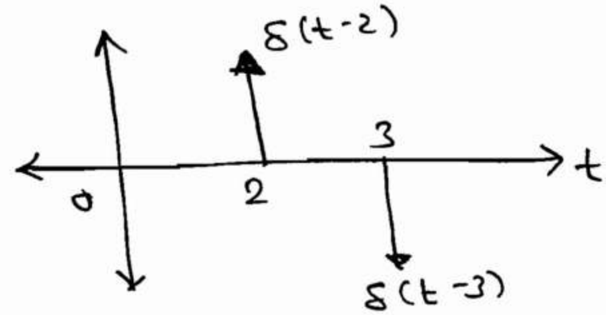
Sol^n:

$$x(t) = u(t) - u(t-1).$$

$$h(t) = u(t-2) - u(t-3).$$

$$\Rightarrow \quad \frac{d}{dt} \cdot h(t) = \delta(t-2) - \delta(t-3).$$

=



Now, $y(t) = x(t) * h(t)$.

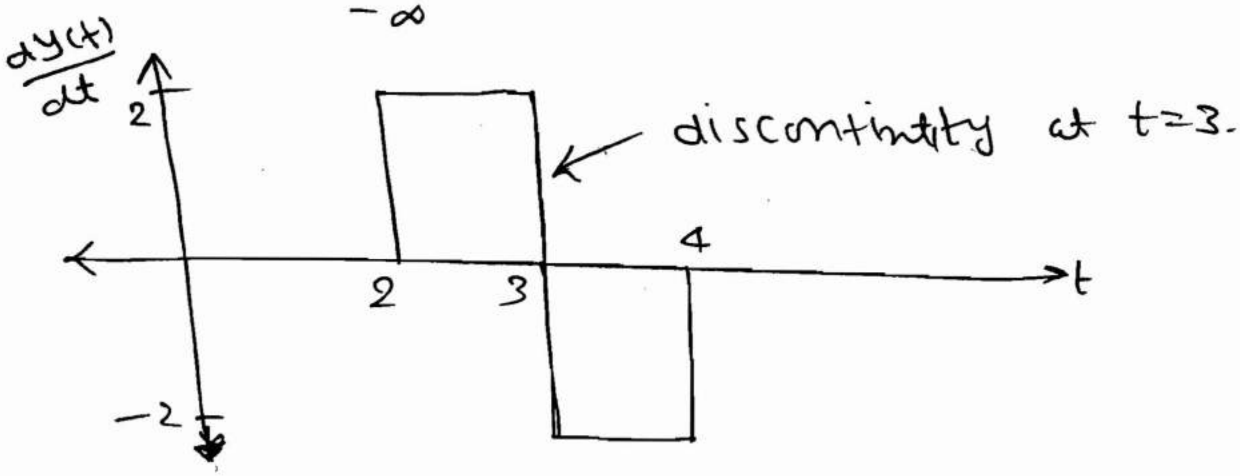
$$\Rightarrow \quad \frac{dy(t)}{dt} = x(t) * \frac{dh(t)}{dt}.$$

$$= x(t) * \frac{d}{dt} [u(t-2) - u(t-3)].$$

$$= x(t) * \delta(t-2) - x(t) * \delta(t-3).$$

$$\frac{dy(t)}{} = x(t-2) - x(t-3).$$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{t} [x(t-2) - x(t-3)]\,dt.$$



$$\Rightarrow \quad y(t) = \int_{-\infty}^{t} [x(t-2) - x(t-3)]\,dt.$$

Case-(i): $\quad 2 \le t < 3.$

$$y(t) = \int_{2}^{t} 2\,dt = 2(t-2).$$

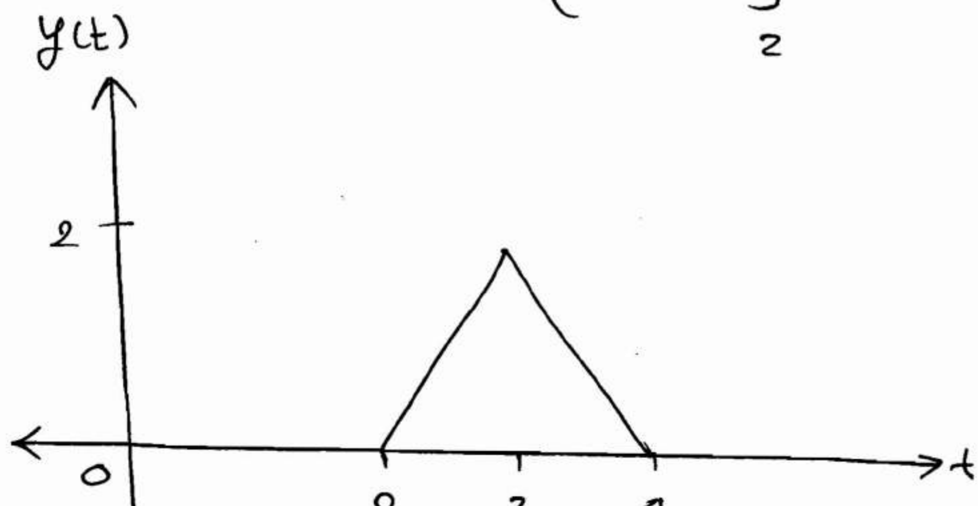
$t=3$

Case-(ii): $\quad 3 \le t < 4$

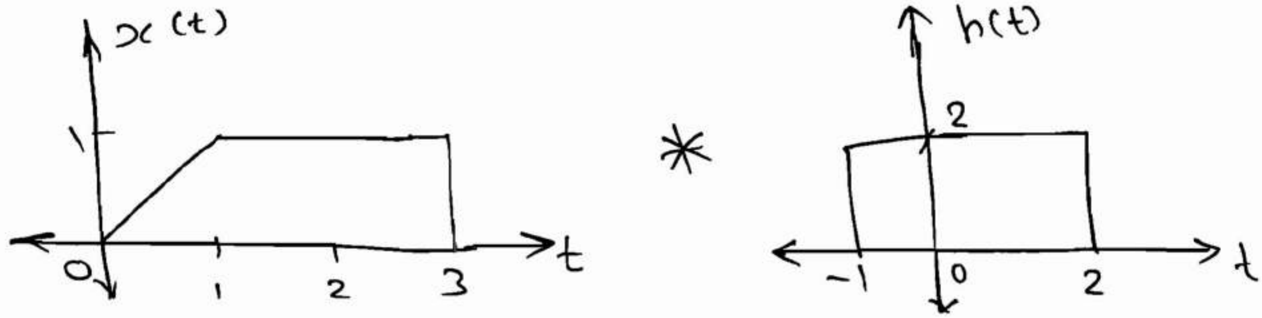
$$y(t) = 2 + \int_{3}^{t} (-2)\,dt = -2t + 8.$$

$$\left(\because \int_{2}^{3} 2\,dt + \int_{3}^{t} -2\,dt.\right)$$

So,



a



Solⁿ:

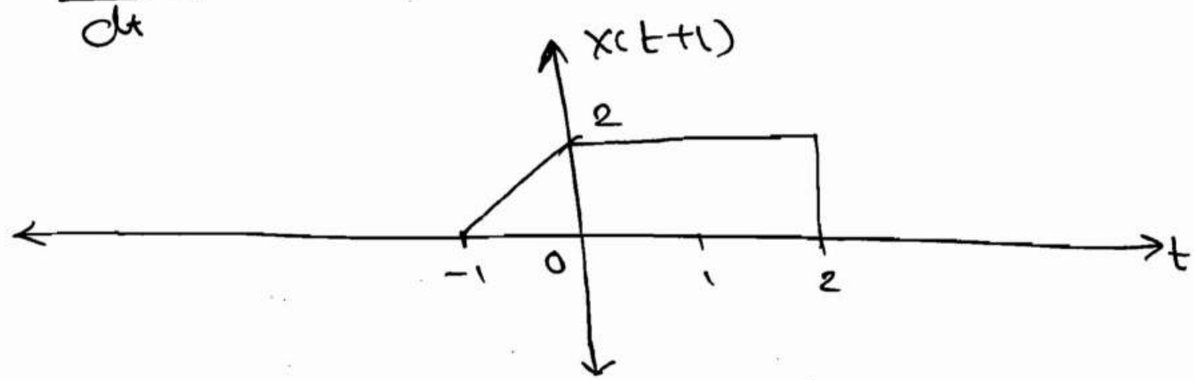
$$y(t) = x(t) * h(t).$$

$$\frac{dh(t)}{dt} = 2\,\delta(t+1) - 2\,\delta(t-2).$$

$$\frac{dy(t)}{dt} = x(t) * \frac{dh(t)}{dt}.$$

$$= x(t) * [2\,\delta(t+1) - 2\,\delta(t-2)].$$

$$\therefore \frac{dy(t)}{dt} = 2\,x(t+1) - 2\,x(t-2).$$



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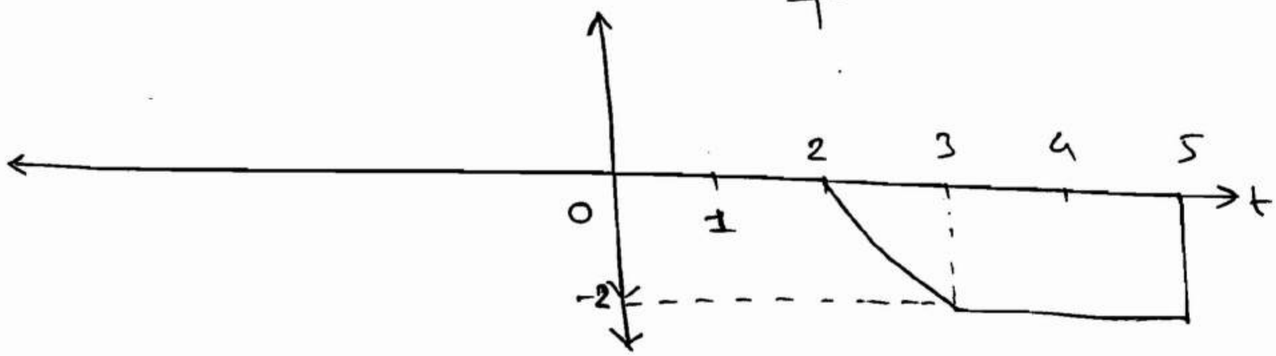

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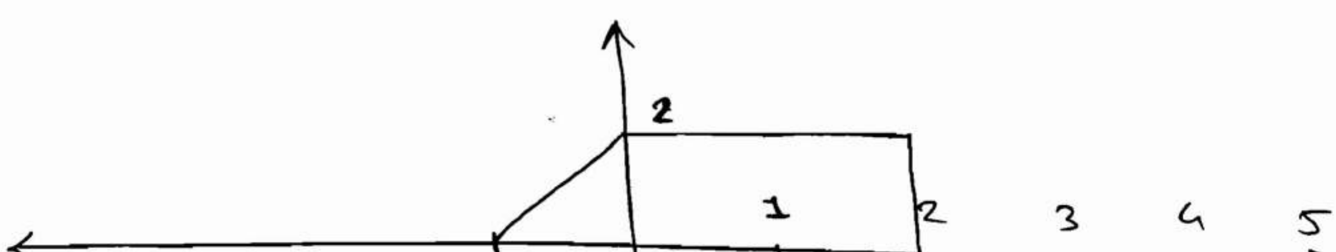

So, $$y(t) = \int_{-\infty}^{t} \left(\frac{dy(t)}{dt}\right) . dt .$$

Now, <u>Case - (i)</u> : $-1 \le t < 0.$

$$y(t) = \int_{-1}^{t} 2(t+1) . dt$$

$$= \left[t^2 + t\right]_{-1}^{t} = t^2 + t - 1 + 1$$

$$= t^2 + t \; ; \; -1 \le t < 0.$$

<u>case - (ii)</u> : $0 \le t < 2.$

$$y(t) = \int_{0}^{t} (2) . dt = 2t \; ; \; 0 \le t < 2.$$

$t = 2$
as discon^t at
$t = 2$.

<u>Case - (iii)</u> : $2 \le t < 3.$

$$y(t) = \int_{2}^{t} \left(-2(t-2)\right) . dt + 4.$$

$$= \int_{2}^{t} (-2t + 4)\, dt + 4.$$

$$= 4 + \left[4t - t^2\right]_{2}^{t}$$

$$= \cancel{4} + 4t - t^2 - \cancel{8} + \cancel{4}.$$

$$\therefore \; y(t) = 4t - t^2 \; ; \; 2 \le t < 3.$$

Case - (iv) :

$$y(t) = \int_3^t (-2)\,dt = -2\,(t-3) = 6-t \quad ; 3 \le t < 5$$

$$\therefore \; y(t) = t^2 + t \quad ; -1 \le t < 0.$$

$$= 2t \quad ; 0 \le t < 2.$$

$$= 4t - t^2 \quad ; 2 \le t < 3.$$

$$y(t) = 6 - t \quad ; 3 \le t < 5.$$

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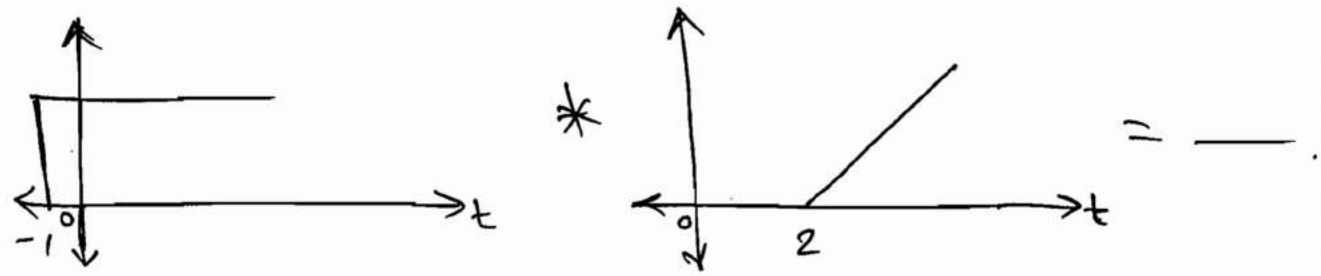
$\boxed{Q}$



<u>Sol^n:</u> Assume, $g(t) = x(t) * h(t)$.

Let, $x(t) = u(t)$.

$h(t) = r(t)$.

$$g(t) = u(t) * r(t).$$

$$\therefore \; g'(t) = \delta(t) * r(t).$$

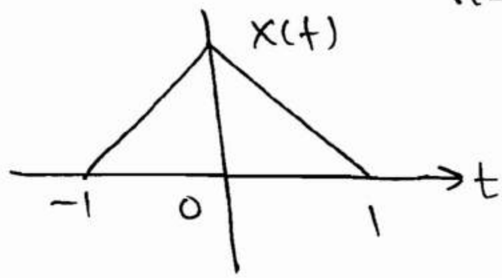
$$g'(t) = r(t) \quad (\because t_0 = 0).$$

$$\therefore \; g(t) = \int_0^t t \cdot dt = t^2/2 \cdot u(t).$$

$$\therefore x(t+1) * r(t-2) = g(t+1-2).$$

$$= g(t-1).$$

P 2.1.8 An Input signal $x(t)$ shown in figure is applied to the system with impulse response $h(t) = \sum_{n=-\infty}^{\infty} \delta(t-3n)$ Find the output.



Soln:

$$h(t) = \sum_{n=-\infty}^{+\infty} \delta(t-3n).$$

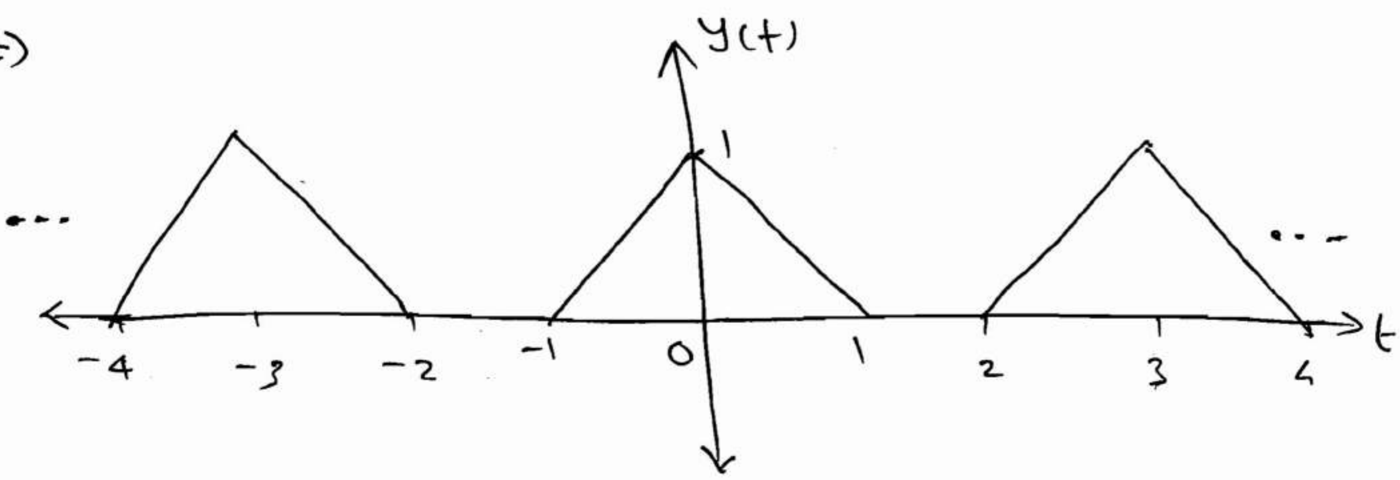
$$= \cdots + \delta(t+6) + \delta(t+3) + \delta(t) + \delta(t-3) + \delta(t-6) + \cdots$$

$$y(t) = x(t) * h(t).$$

$$= x(t) * (\cdots + \delta(t+6) + \delta(t+3) + \delta(t) + \delta(t-3) + \delta(t-6) + \cdots).$$

$$\therefore\ y(t) = \cdots + x(t+6) + x(t+3) + x(t) + x(t-3) + x(t-6) + \cdots$$

$\Rightarrow$



Note: Convolution of a general signal with periodic train of impulses is periodic repetation of general signal.

$\Rightarrow$ System length should be ~~the~~ more than the input signal length so that we can retain the information without distroying.

* Discrete Convolution:

P 2.1.11 Consider the signal $h(n) = \left[\frac{1}{2}\right]^{n-1} \cdot \{u(n+3) - u[n-10]\}$.

Such that $h(n-k) = \begin{cases} \left(\frac{1}{2}\right)^{n-k-1} ; & A \le k \le B \\ 0 ; & \text{elsewhere} \end{cases}$ Find A & B?

Soln:

$$h(n) = \left[\frac{1}{2}\right]^{n-1} \cdot (1) ; \quad -3 \le n \le 9.$$

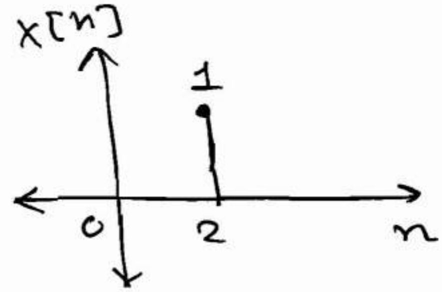
$$\therefore h(n-k) = \left[\frac{1}{2}\right]^{n-k-1} ; \quad -3 \le n-k \le 9$$

$$-3-n \le -k \le 9-n.$$

$$\therefore \boxed{n-9 \le k \le n+3.}$$

P 2.1.12 A linear system with Input $x(n)$ & output $y(n)$ related as

$g(n) = u(n) - u(n-4)$. Find $y(n)$ when $x(n) = \delta(n-2)$.



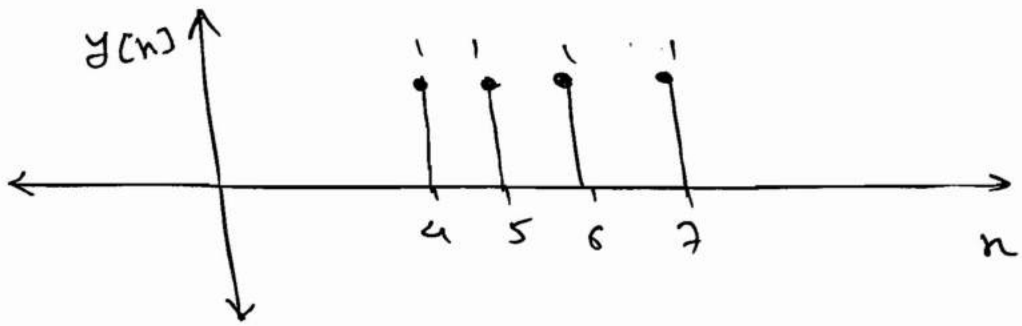
Sol^n: Here, $x[n] = \delta[n-2]$

$\Rightarrow x[2] = 1$

$\therefore y(n) = x(2) \cdot g(n-4)$.

$y[n] = g(n-4)$

$\therefore y[n] = u[n-4] - u[n-8]$.



$\rightarrow$ above system is time variant.

* Convolution Property of Discrete Impulse.

$\Rightarrow$

$$\boxed{x(n) * \delta(n-n_0) = x(n-n_0).}$$

$\boxed{P\ 2.1.14}$ Find the convolution of

$x(n) = \{1, 2, 3, 4\}$ & $h(n) = \{1, 2, 1, -1\}$.

Sol^n: There are total ~~three~~ four methods for convolution:

① Sliding strip. ③ Sum by column.

① Sliding Strip method:

$$\begin{array}{lcccccccc} \rightarrow h[k] & & & & 1 & 2 & 1 & -1 \\ x[-k] & 4 & 3 & 2 & 1 & & & \\ \hline \end{array}$$

$y(0) = 1$

$$\begin{array}{lcccccc} \rightarrow h[k] & & & 1 & 2 & 1 & -1 \\ & & & \times & \times & & \\ & 4 & 3 & 2 & 1 & & \\ \hline \end{array}$$

$y(1) = 2 + 2 = 4.$

$$\begin{array}{lccccc} \rightarrow h[k] & & 1 & 2 & 1 & -1 \\ & & \times & \times & \times & \\ & 4 & 3 & 2 & 1 & \\ \hline \end{array}$$

$y(2) = 3 + 4 + 1 = 8$

$$\begin{array}{lcccc} \rightarrow h[k] & 1 & 2 & 1 & -1 \\ & \times & \times & \times & \times \\ & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

$y(3) = 4 + 6 + 3 - 1$
$= 12.$

$$\begin{array}{lccccc} \rightarrow h[k] & 1 & 2 & 1 & -1 & \\ & & \times & & & \\ & & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

$y(4) = 8 + 3 - 2 = 9$

$$\begin{array}{lcccccc} \rightarrow h[k] & 1 & 2 & 1 & -1 & & \\ & & & & 4 & 3 \; 2 & 1 \\ \hline \end{array}$$

$y(5) = 4 - 3 = 1.$

$$\begin{array}{lccccccc} \rightarrow h(k) & 1 & 2 & 1 & -1 & & & \\ & & & & 4 & 3 & 2 & 1 \\ \hline \end{array}$$

$y(6) = -4 = -4.$

$\therefore \; y(n) = \{1, 4, 8, 12, 9, 1, -4\}.$

② Array Method:

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
1	1	2	3	4

$y(n) = \{1, 4, 8, 11, 9, 1, -4\}.$

③ <u>Sum by Column</u>:

$$\Rightarrow \quad \begin{array}{lcccccccc} x[n] = & 1 & 2 & 3 & 4 & & & \\ h[n] = & 1 & 2 & 1 & -1 & & & \\ \hline & 1 & 2 & 3 & 4 & & & \\ & + & 2 & 4 & 6 & 8 & & \\ & & + & 1 & 2 & 3 & 4 & \\ & & \dots & & -1 & -2 & -3 & -4 \\ \hline \end{array}$$

$$y[n] = \{1, 4, 8, 19, 9, 1, -4\}.$$

④ ~~Polynomial Division method~~:

$\Rightarrow$ $\cancel{x(n) = \{1, 2, 3, 4\}}$

$\cancel{\omega^0 \ \omega^1 \ \omega^2 \ \omega^3}$

④ <u>Recursive Division</u>:-

$$y[n] = \sum x(n-k) \cdot h(k).$$

$\underline{n=0}$ $\quad y[0] = x(0) \cdot h(0).$

$\underline{n=1}$ $\quad y[1] = x[0] \cdot h[1] + x[1] \cdot h[0]$

$\underline{n=2}$ $\quad y[2] = x[0] \cdot h[0] + x[1] \cdot h[1]$

$\qquad\qquad + x[2] \cdot h[0].$

* Deconvolution (System Identification):

Q i/p $x[n] = \{1, 2, 3, 4\}$

o/p $y[n] = \{1, 4, 8, 11, 9, 1, -4\}$.

$h[n] = ?$

Sol^n: Method-1: Sum by Column:

$\Rightarrow$

$$\begin{array}{lccccccc}
x[n] \rightarrow & 1 & 2 & 3 & 4 & & & \\
h[n] \rightarrow & a & b & c & d & & & \\
\hline
 & a & 2a & 3a & 4a & & & \\
 & & b & 2b & 3b & 4b & & \\
 & & & c & 2c & 3c & 4c & \\
 & & & & d & 2d & 3d & 4d \\
\hline
y[n] \rightarrow & 1 & 4 & 8 & 11 & 9 & 1 & -4
\end{array}$$

$\Rightarrow \boxed{a=1}, \boxed{d=-1}$,

$2a+b=4$

$2+b=4$

$\Rightarrow \boxed{b=2}$

$3a+2b+c = 8$

$3+4+c = 8$

$\therefore \boxed{c=1}$

$$\boxed{h[n] = \{1, 2, 1, -1\}.}$$

Method-2: Recursive division:

$\Rightarrow y[n] = \sum x[n-k] \cdot h[k]$.

$n=0 \quad y[0] = x[0] \cdot h[0]. \longrightarrow$ find $h(0)$

$\underline{\underline{n=2}}$ $\quad y[2] = x[0]\cdot h[2] + x[1]\,h[1] + x[2]\cdot h[0].$

__Method-3:__ __Polynomial division.__

$\rightarrow \quad x[n] = \{\underset{\omega^0}{1}, \underset{\omega^1}{2}, \underset{\omega^2}{3}, \underset{\omega^3}{4}\}.$

$y[n] = \{\underset{\omega^0}{1}, \underset{\omega^1}{4}, \underset{\omega^2}{8}, \underset{\omega^3}{11}, \underset{\omega^4}{9}, \underset{\omega^5}{1}, \underset{\omega^6}{-4}\}.$

$$1+2\omega+\omega^2-\omega^3$$

$$1+2\omega+3\omega^2+4\omega^3 \;\big)\; 1+4\omega+8\omega^2+11\omega^3+9\omega^4+\omega^5 \mp 4\omega^6$$

$$\underline{1+2\omega+3\omega^2+4\omega^3}$$

$$2\omega+5\omega^2+7\omega^3+9\omega^4+\omega^5 \mp 4\omega^6$$

$$\underline{2\omega+4\omega^2+6\omega^3+8\omega^4}$$

$$0+\omega^2+\omega^3+\omega^4+\omega^5 \mp 4\omega^6$$

$$\underline{\omega^2+2\omega^3+3\omega^4+4\omega^5}$$

$$-\omega^3+2\omega^4+6\omega^5-4\omega^6$$

$$\underline{\omega^3-2\omega^4-6\omega^5+4\omega^6}$$

$$0 \quad 0 \quad 0 \quad 0$$

So, $\boxed{h[n] = \{1, 2, 1, -1\}.}$

* ordinary convolution | Periodic (or) circular convolution.

Ordinary convolution	Periodic (or) circular convolution
① $x_1[n] \rightarrow N_1$ Samples.	① $x_p[n] \rightarrow N_1$
② $x_2[n] \rightarrow N_2$ Samples.	② $h_p[n] \rightarrow N_2$
$y[n] \rightarrow (N_1+N_2-1)$.	$y_p[n] \rightarrow \max.(N_1, N_2)$

$\boxed{a}$ Find periodic (or) circular convolution of $x_p[n] = \{1\ 2\ 3\ 4\}$ & $h_p[n] = \{1\ 2\ 1\ -1\}$.

Sol^n:

$$
\begin{array}{lccccccc}
x_p[n] = & 1 & 2 & 3 & 4 & & & \\
h_p[n] = & 1 & 2 & 1 & -1 & & & \\
\hline
\text{ordinary conv.} = & 1 & 4 & 8 & 11 & 9 & 1 & -4 \\
\text{wrap around last half of } y[n] \quad + & 9 & 1 & -4 & 0 & & & \\
\hline
& 10 & 5 & 4 & 11 & & &
\end{array}
$$

Periodic conv. o/p $y[n]$.

Note: Linear conv. + Aliasing = periodic conv.

⇒ Matrix method:

{1 2 3 4 1 2 3 4 1 2 3 4 1 2 3 4}

$$
\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{bmatrix}
\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = [10\ 5\ 4\ 11]
$$

$\boxed{P\ 2.1.17}$ Given $x = [a, b, c, d]$ as the input to an LTI system produces an output $y = [x, x, x, x, \ldots$ repeated N times$]$. The impulse response of the system is ——.

Soln:

$$y = [\underbrace{a\ b\ c\ d}_{\delta(n)}\ \underbrace{a\ b\ c\ d}_{\delta(n-4)}\ \underbrace{a\ b\ c\ d}_{\delta(n-8)} \ldots N \text{ times}].$$

$\therefore$

$$\boxed{\delta = \sum_{n=0}^{N-1} \delta(n-4i).}$$

$\boxed{Q}$ Given $y(t) = e^{-t}.u(t) * \sum_{k=-\infty}^{\infty} \delta(t-3k)$ such that $y(t) = Ae^{-t}$ for $0 \le t < 3$ find A?

Soln:

$$y(t) = e^{-t}.u(t) * [\cdots + \delta(t+6) + \delta(t+3) + \delta(t) + \delta(t-3) + \delta(t-6) + \cdots].$$

So, $y(t) = + \cdots + e^{-(t+6)} u(t+6) + e^{-(t+3)}. u(t+3)$

$+ e^{-t}. u(t) + e^{-(t-3)} \cdot u(t-3) + e^{-(t-6)} \cdot u(t-6) + \cdots$

as $y(t) = Ae^{-t}$ for $0 \le t < 3$

$$y(t) = Ae^{-t} = \cdots + e^{-(t+6)} + e^{-(t+3)} + e^{-t}.$$

$$A = 1 + e^{-3} + e^{-6} + e^{-9} + \cdots$$

$$\therefore \boxed{A = \frac{1}{1 - e^{-3}}} \qquad \left(\because A_n = \frac{a}{1-r}\right).$$

$\boxed{P\ 2.1.18}$ Suppose for an L.T.I. system if the input applied is $\delta(n) - \delta(n-1)$, the output is observed to be $\delta(n) - \delta(n-1) + 2\delta(n-3)$. Find the output due to the input $7\delta(n) - 7\delta(n-2)$?

Soln:

i/p	o/p
$\delta(n) - \delta(n-1)$	$\delta(n) - \delta(n-1) + 2\delta(n-3)$.
$7\delta(n) - 7\delta(n-2)$	

Let, $x_1(n) = \delta(n) - \delta(n-1)$.

$x_1(n-1) = \delta(n-1) - \delta(n-2)$.

$\therefore x_1(n) + x_1(n-1) = \delta(n) - \delta(n-2)$.

so, $x(n) \longrightarrow \delta(n) - \delta(n-1) + 2\delta(n-3)$.

$y_2(n) = 7y_1(n) + 7y_1(n-1)$.

$= 7\delta(n) - 7\cancel{\delta(n-1)} + 14\delta(n-3)$
$+ 7(\cancel{\delta(n-1)}) - 7\delta(n-2) + 14\delta(n-4)$.

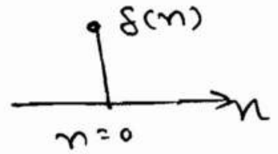
$$\therefore \boxed{y_2(n) = 7\delta(n) - 7\delta(n-2) + 14\delta(n-3) + 14\delta(n-4)}$$

* **Properties of LTI Systems:**

① **Causal:**

⇒ Before the application of input the ~~output~~ as impulse at $n=0$. we can not get the output as impulse response before $n=0$. i.e. $\boxed{h(n)=0 \text{ for } n<0.}$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t).$$



$\Rightarrow$ o/p $h(n)$ = I. Response
$= 0 \quad ; n<0.$

② **Stable:**

$$\Rightarrow \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty = \text{finite}.$$

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty.$$

Proof:

$$|y(t)| = \int_{-\infty}^{\infty} \left| x(t-\tau).\, h(\tau) \right| d\tau$$

B.I.B.O.

$$= \int_{-\infty}^{\infty} \underbrace{|x(t-\tau)|}_{\text{finite}} \cdot |h(\tau)| \cdot d\tau$$

$$= \text{finite} \int^{+\infty} |h(\tau)|\, d\tau$$

$$\therefore \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty = \text{finite}.$$

$\therefore$ Area under absolute value of impulse response is finite.

③ Static (or) Memory less:

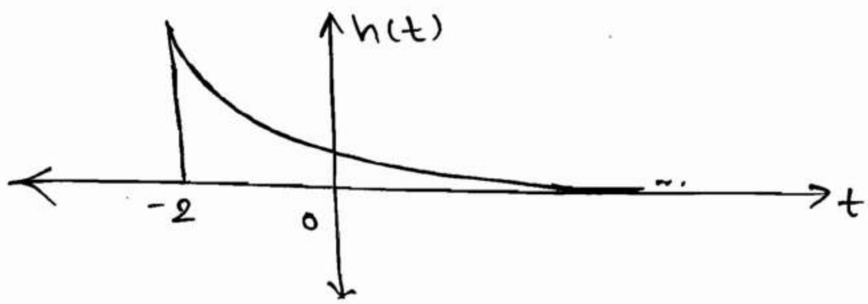
$$h(t) = K\,\delta(t).$$

(or) $h(t) = 0$, for $t \neq 0$.

Note: All Property indicate internal behaviour of the system.

$\boxed{Q}$ $h(t) = e^{-t} . u(t+2).$

Soln: $h(t) = e^{-t} . u(t+2).$

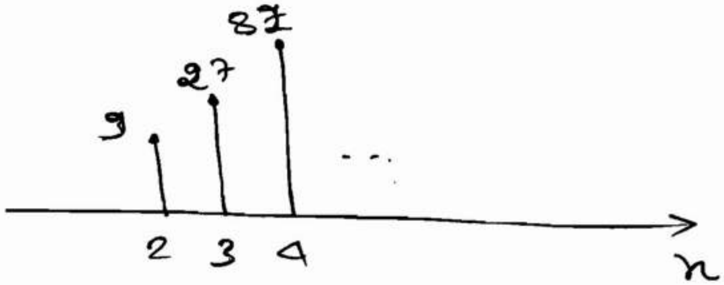


$h(t) \neq 0$ for $t < 0$. So, Non- Causal.

Stability: $$\int_{-2}^{\infty} e^{-t} . dt = \text{finite} = \text{stable}.$$

$\boxed{Q}$ $h[n] = 3^{n} . u[n-2].$

$\Rightarrow$



$\Rightarrow$

$$\sum_{n=2}^{\infty} 3^n = 9 + 27 + 81 + \cdots = \infty.$$

$$= \text{Unstable.}$$

$\rightarrow$ h(n) is define for positive index, n

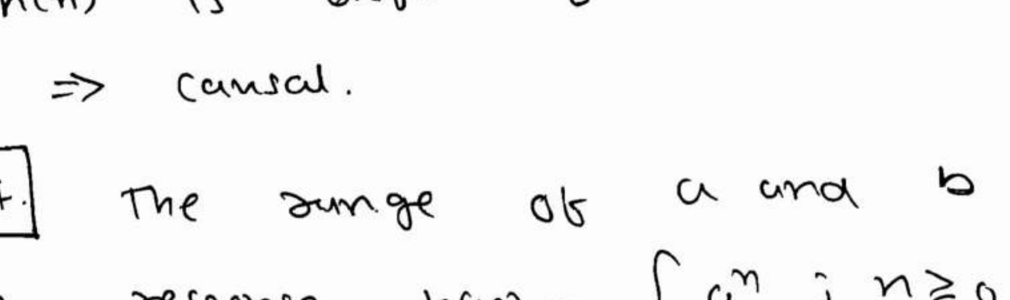
$\Rightarrow$ causal.

$\boxed{\text{P2.2.4.}}$ The range of a and b for the impulse response $h(n) = \begin{cases} a^n & ; n \geq 0 \\ b^n & ; n < 0 \end{cases}$ to be stable is ____.

Sol^n: $\boxed{\text{Q - 2.2.5}}$ Given $h(t) = e^{\alpha t} \cdot u(t) + e^{\beta t} \cdot u(-t)$. Find α & β for stable.

$$h(t) = e^{\alpha t} \cdot u(t) + e^{\beta t} \cdot u(-t).$$

i.e.



$$\int_{-\infty}^{\infty} |h(t)| \cdot dt.$$

$$= \int_{-\infty}^{0} e^{4t} \cdot dt + \int_{0}^{\infty} e^{-3t} \cdot dt.$$

$$= \frac{1}{4} + \frac{1}{3}$$

$$= 7/12 \cdot = \text{finite.}$$

Sol^n of $\boxed{P\ 2.2.4.}$:

for stable: $\displaystyle\sum_{n=-\infty}^{\infty} h(n) = \text{finite} < \infty.$

$$= \sum_{n=-\infty}^{-1} b^n + \sum_{n=0}^{\infty} a^n.$$

① $\displaystyle\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad \boxed{|a| < 1.}$ ✓

② $\displaystyle\sum_{n=-\infty}^{-1} b^n$, let $-n = m$

$$\therefore \Rightarrow \sum_{m=1}^{\infty} b^{-m}. = 1 + b^{-1} + b^{-2} + b^{-3} + \cdots - 1.$$

$$= \frac{1}{1-b^{-1}} - 1, \quad |b^{-1}| < 1.$$

$$\Rightarrow \boxed{|b| > 1}$$

✓

* ~~Series~~ 1.

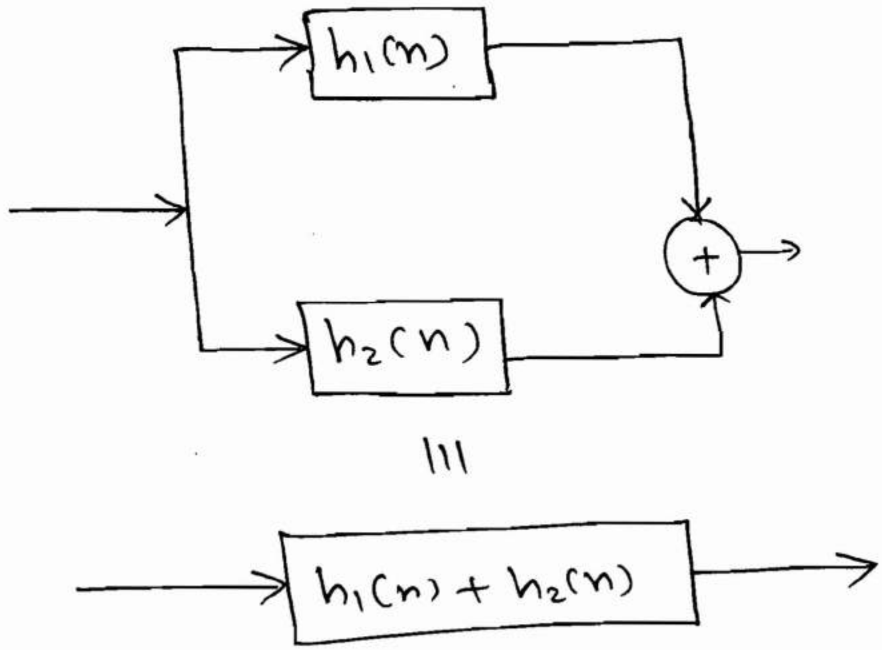
* **Blocks in series:**

$$\rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow \quad \equiv \quad \rightarrow \boxed{h_1(n) * h_2(n)} \rightarrow$$

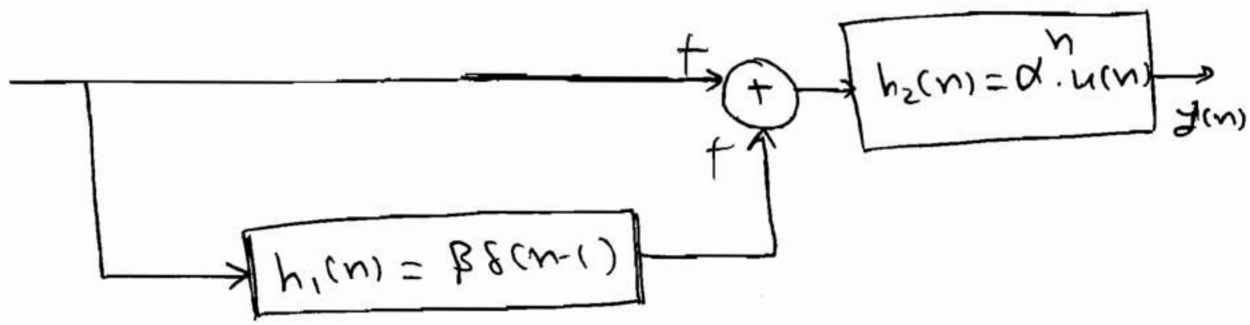
* Blocks in parallel:

$\Rightarrow$

$\Rightarrow$



$\boxed{P2.2.3}$ Consider the system in figure.



(a) Find. I.R. of overall system.

Soln:

$$h(n) = [\delta(n) + \beta\,\delta(n-1)] * [\alpha^n . u(n)].$$

$$= \delta(n) * \alpha^n . u(n) \quad + \quad \beta\,\delta(n-1) * \alpha^n . u(n).$$

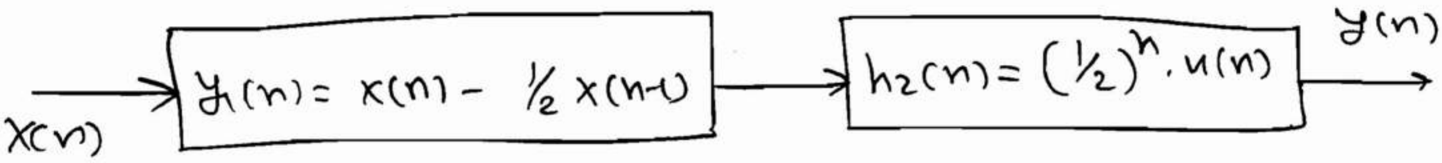
$$\boxed{h(n) = \alpha^n . u(n) + \beta . \alpha^{n-1} . u(n-1).}$$

(b) Is this system causal? Under what

$\Rightarrow$ yes, System is causal.

$\Rightarrow$ If $|\alpha| < 1$ then system is stable.

$\boxed{P\ 2.2.6}$ For the interconnected system shown in fig. find the overall impulse response.



Sol^n: for impulse response $x(n) = \delta(n)$.

$$\therefore h(n) = \left[\delta(n) - \frac{1}{2}\delta(n-1)\right] * \left[(\tfrac{1}{2})^n . u(n)\right].$$

$$= \delta(n) * \left(\frac{1}{2}\right)^n . u(n) - \frac{1}{2}\delta(n-1) * (\tfrac{1}{2})^n . u(n).$$

$$= \left(\frac{1}{2}\right)^n . u(n) - \frac{1}{2} . \left(\frac{1}{2}\right)^{n-1} . u(n-1).$$

$$= \left(\frac{1}{2}\right)^n . [u(n) - u(n-1)].$$

$$= \left(\frac{1}{2}\right)^n . \delta(n).$$

$$= \left(\frac{1}{2}\right)^0 . \delta(n).$$

$$= \delta(n).$$

(4) <u>Invertible</u> :

$$\therefore \boxed{h(n) * h_{inv}(n) = \delta(n).}$$

P 2.2.2 Consider the D.T. system S_1, with I.R. $h(n) = (1/5)^n \cdot u(n)$.

(a) Find 'A' such that

$$h(n) - A h(n-1) = \delta(n).$$

Soln:

$$h(n) - A h(n-1) = \delta(n).$$

$$\therefore \left(\frac{1}{5}\right)^n \cdot u(n) - A \cdot \left(\frac{1}{5}\right)^{n-1} \cdot h(n-1) = \delta(n).$$

$\therefore$ A should $1/5$ so that

$$\left(\frac{1}{5}\right)^n \left[u(n) - u(n-1)\right] = \delta(n).$$

$$\therefore \boxed{A = 1/5}$$

(b) using result from part-(a), determine the I.R. $g(n)$ of an LTI system S_2 which is inverse of S_1.

Soln:

$$h(n) * g(n) = \delta(n)$$

$$g(n) = h_{Inv}(n).$$

$\therefore$ here $h(n) - A h(n-1) = \delta(n).$

$$h(n) * \underbrace{\left[\delta(n) - A\,\delta(n-1)\right]}_{h_{inv} = g(n)} = \delta(n).$$

$$\therefore \boxed{g(n) = \delta(n) - A\,\delta(n-1).}$$

$\Rightarrow \quad y[n] = n\,x[n]$

T.V.   Non. Invertible.

$\Rightarrow \quad y[n] = x[n] \cdot x[n-3]$.

NL   N.I.

$\Rightarrow \quad y[n] = x[n] \cdot \sin\left[\frac{5\pi n}{6}\right]$.

T.V.   N.L.   N.I.

$\Rightarrow \quad y(n) = \sum_{k=-\infty}^{+\infty} x(k)$. $\Rightarrow$ Inverse is $y[n] - y[n-1]$.

L.T.I. System.

<u>Note</u>: Impulse response term is valid only for LTI system.

$\rightarrow \quad x(k) = \delta(k) \Rightarrow y(n) = h(n)$.

if $h(n) = u(n)$.

$h(n) * h_{inv}(n) = \delta[n]$.

$\therefore u[n] * \underbrace{[\delta[n] - \delta[n-1]]}_{h_{inv}(n)} = \delta[n]$.

$\boxed{\text{P2.2.7.}}$ Determine whether each of the following statements are TRUE (or) FALSE.

① The cascade of 4 non causal LTI

non Causal.

Soln:

$$\xrightarrow{} \boxed{u(n+2)} \xrightarrow{} \boxed{\delta(n-5)} \xrightarrow{} u(n+2-5)$$

$$= u(n-3).$$

Non-Causal $\qquad\qquad$ causal

there is chance of causality.

So, statement is false.

(2) If an LTI system is causal, it is stable.

Soln: let, $h(t) = u(t) \rightarrow$ LTI & causal.

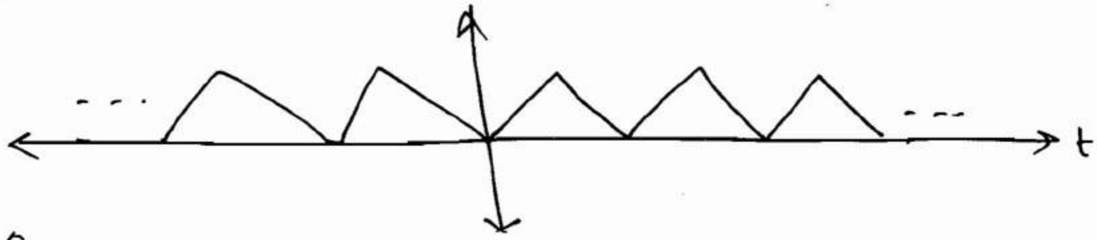
But $\displaystyle\int_{-\infty}^{\infty} |u(t)| = \int_{0}^{\infty} (1)\, dt = \infty$

= unstable.

So, False statement

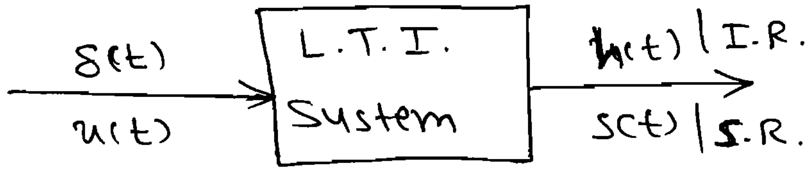
(3) If $h(t)$ is the I.R. of an LTI system which is periodic & non-zero, the system is unstable.

Soln: All periodic signals are ever lasting signal.



$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \infty \Rightarrow \text{Unstable.}$$

* Step Response :-



→ total internal behaviour of system is charu. by I.R.

→ Sudden change in system behaviour is char. by S.R.

$$\delta(t) = \frac{d}{dt} u(t).$$

$$\boxed{h(t) = \frac{d}{dt} s(t).}$$

&

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau.}$$

$\boxed{Q}$ $x(t) = u(t-2)$ & $h(t) = e^{-3t} u(t)$.

Sol^n:

O/P $y(t) = s(t-2)$.

$$s(t) = \int_{-\infty}^{t} e^{-3\tau} \cdot u(\tau)\, d\tau.$$

$$= \int_{0}^{t} e^{-3\tau} \cdot d\tau.$$

$$= \left[\frac{e^{-3\tau}}{-3} \right]_0^t$$

$\Rightarrow s(t-2) = y(t)$

$$= \frac{1 - e^{-3(t-2)}}{3}.$$

Note: whenever the one signal is step signal then we can find O/P by just integrating other signal.

$\boxed{Q\ 2.3.2.}$ Find the impulse response of the system if the step response is $s(t) = \cos\omega_0 t\, u(t)$.

Sol^n:

I.R. $h(t) = \frac{d}{dt}(s(t))$.

$\therefore h(t) = \frac{d}{dt}\left[\cos\omega_0 t \cdot u(t)\right]$.

$\therefore h(t) = -\omega_0 \sin\omega_0 t \cdot u(t) + \cos\omega_0 t \cdot \delta(t)$.

$= -\omega_0 \sin\omega_0 t \cdot u(t) + \left[\cos(\omega_0 \cdot 0)\right] \cdot \delta(t)$

$\therefore h(t) = -\omega_0 \sin\omega_0 t \cdot u(t) + \delta(t)$.

$\boxed{P\ 2.3.4}$ If the unit step response of a system is $(1-e^{-\alpha t})u(t)$, then its unit impulse response is ____.

Sol^n: $s(t) = \left[1 - e^{-\alpha t}\right]u(t)$.

$\therefore h(t) = \frac{d}{dt} s(t)$.

$= \frac{d}{dt}\left[u(t) - e^{-\alpha t} \cdot u(t)\right]$.

$$= \cancel{\delta(t)} - e^{-0}\cdot\cancel{\delta(t)} + \alpha\cdot e^{-\alpha t}\cdot u(t).$$

$$= \alpha\cdot e^{-\alpha t}\cdot u(t).$$

$\boxed{\text{P2.3.1.}}$ Find the step response of the System if the impulse response is

$$h(n) = (0.5)^n\cdot u(n).$$

Soln:

$$s(n) = \sum_{k=-\infty}^{\infty} h(k).$$

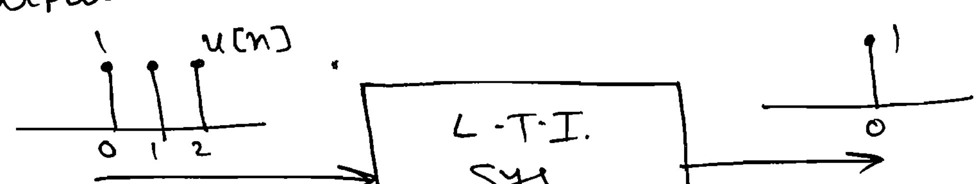
$$\therefore\ s(n) = \sum_{k=-\infty}^{+\infty} (0.5)^k\cdot u(k).$$

$$= \sum_{k=0}^{n} (0.5)^k.$$

$$= 1 + (0.5) + (0.5)^2 + (0.5)^3 + \dots + (0.5)^n.$$

$$s(n) = \frac{1-(0.5)^{n+1}}{1-0.5}\ ;\quad n \geq 0.$$

$\boxed{\text{P2.3.3.}}$ An LTI system with Input u(n) Produces the output as δ(n), then find the output due to the input n u(n)?



$\therefore \delta(n) = u(n) - u(n-1).$

$\therefore$

$$o/p = n u(n) - (n-1) u(n-1).$$

$$= n u(n) - n u(n-1) + u(n-1).$$

$$= n\,(u(n) - u(n-1)) + u(n-1).$$

$$= n\,\delta(n) + u(n-1).$$

$$= 0 + u(n-1).$$

$$\boxed{o/p = u(n-1).}$$

P2.3.5. Find the overall impulse response for the interconnected system shown in figure?

Soln:

$$h(t) = \Big[-\big[[h_2(t) + h_3(t)] * h_4(t)\big] + h_1(t)\Big] * h_5(t).$$