Three Dimensional Geometry

Short Answer Type Questions

1. **If the direction ratios of a line are $1,\ 1,\ 2,$ find the direction cosines of the line.**

Sol. The direction cosines are given by

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},m=\frac{b}{\sqrt{a^2+b^2+c^2}},n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Here $a,\ b,\ c$ are $1,\ 1,\ 2,$ respectively.

Therefore, $l=\frac{a}{\sqrt{1^2+1^2+2^2}},m=\frac{b}{\sqrt{1^2+1^2+2^2}},n=\frac{c}{\sqrt{1^2+1^2+2^2}}$

i.e., $l=\frac{1}{\sqrt{6}},m=\frac{1}{\sqrt{6}},n=\frac{2}{\sqrt{6}}$ i.e. $\pm\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$ are D.C's of the line.

2 **Find the direction cosines of the line passing through the points $P(2,\ 3,\ 5)$ and $Q(-1,\ 2,\ 4)$.**

Sol. The direction cosines of a line passing through the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are

$$\frac{x_2-x_1}{\mathrm{PQ}},\frac{y_2-y_1}{\mathrm{PQ}},\frac{z_2-z_1}{\mathrm{PQ}}.$$

Here PQ=$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$=\sqrt{(-1-2)^2+(2-3)^2+(4-5)^2}=\sqrt{9+1+1}=\sqrt{11}$

Here D.C.'s are

$$\pm\left(\frac{-3}{\sqrt{11}},\frac{-1}{\sqrt{11}},\frac{-1}{\sqrt{11}}\right)\text{ or }\pm\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{1}{\sqrt{11}}\right).$$

3. **If a line makes an angle of $30°,\ 60°,\ 90°$ with the positive direction of x, y, z-axes, respectively, then find its direction cosines.**

Sol. The direction cosines of a line which makes an angle of $\alpha,\ \beta,\ \gamma$ with the axes, are $cos\alpha,\ cos\beta,\ cos\gamma$

Therefore, D.C.'s of the line are $cos\,30°,\ cos\,60°,\ cos\,90°$ i.e., $\pm\left(\frac{\sqrt{3}}{2},\frac{1}{2},0\right)$

4. **The $x-coordinate$ of a point on the line joining the points Q(2, 2, 1) and R(5, 1, -2) is 4. Find its $z-coordinate$.**

Sol. Let the point P divide QR in the ratio $\lambda:1$, then the co-ordinate of P are

$$\left(\frac{5\lambda+2}{\lambda+1},\frac{\lambda+2}{\lambda+1},\frac{-2\lambda+1}{\lambda+1}\right)$$

But $x-coordinate$ of P is 4. Therefore,

$$\frac{5\lambda+2}{\lambda+1}=4\Rightarrow\lambda=2$$

Hence, the $z-coordinate$ of P is $\dfrac{-2\lambda+1}{\lambda+1}=-1$.

5. **Find the distance of the point whose position vector is $(2\hat{i}+\hat{j}-\hat{k})$ from the plane $\vec{r}\cdot(\hat{i}-2\hat{j}+4\hat{k})=9$**

Sol. Here $\vec{a}=2\hat{i}+\hat{j}-\hat{k}, \vec{n}=\hat{i}-2\hat{j}+4\hat{k}$ and d=9

So, the required distance is $\dfrac{\left|\left(2\hat{i}+\hat{j}-\hat{k}\right).\left(\hat{i}-2\hat{j}+4\hat{k}\right)-9\right|}{\sqrt{1+4+16}}$

$=\dfrac{|2-2-4-9|}{\sqrt{21}}=\dfrac{13}{\sqrt{21}}$.

6. **Find the distance of the point $(-2,4,-5)$ from the line $\dfrac{x+3}{3}=\dfrac{y-4}{5}=\dfrac{z+8}{6}$**

Sol. Here $P(-2,4,-5)$ is the given point.

Any point Q on the line is given by $(3\lambda-3,\ 5\lambda+4,\ (6\lambda-8)$,

$\overrightarrow{PQ}=3(\lambda-1)\hat{i}+5\lambda\hat{j}+(6\lambda-3)\hat{k}$.

Since $\overrightarrow{PQ}\perp(3\hat{i}+5\hat{j}+6\hat{k})$, we have

$3(3\lambda-1)+5(5\lambda)+6(6\lambda-3)=0$

$9\lambda+25\lambda+36\lambda=21,\ i.e.\ \lambda=\dfrac{3}{10}$

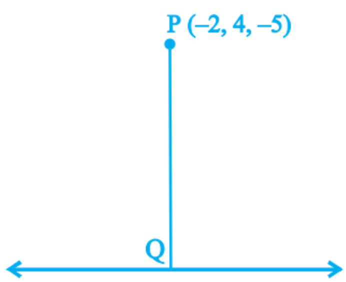


Fig. 11.1

Thus $\overrightarrow{PQ}=-\dfrac{1}{10}\hat{i}+\dfrac{15}{10}\hat{j}-\dfrac{12}{10}\hat{k}$

Hence $|\overrightarrow{PQ}|=\dfrac{1}{10}\sqrt{1+225+144}=\sqrt{\dfrac{37}{10}}$.

7. **Find the coordinates of the point where the line through $(3,-4,-5)$ *and* $(2,-3,1)$ crosses the plane passing through three points $(2,2,1)$, $(3,0,1)$ *and* $(4,-1,0)$.**

Sol. Equation of plane through three points $(2,2,1)$, $(3,0,1)$ *and* $(4,-1,0)$ is

$\left[(\vec{r}-(2\hat{i}+2\hat{j}+\hat{k})\right].\left[(\hat{i}-2j)\times(\hat{i}-\hat{j}-\hat{k})\right]=0$

i.e., $\vec{r}-(2\hat{i}+2\hat{j}+\hat{k})=7$ or $2x+y+z-7=0$ ...(1)

Equation of line through $(3,-4,-5)$ *and* $(2,-3,1)$ is

$$\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6} \quad ...(2)$$

Any point on line (2) is $(-\lambda+3,\ \lambda-4,\ 6\lambda-5)$. This point lies on plane (1). Therefore,

$2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-7=0,$ i.e., $\lambda=z$

Hence the required point is $(1,-2,7)$.

Long Answer Type Questions

8. Find the distance of the point (–1, –5, – 10) from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$.

Sol. We have $\vec{r}=2\hat{i}-j+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$

Solving these two equations, we get $\left[(2\hat{i}-\hat{j}+2\hat{k})+\lambda(3\hat{i}+4\hat{j}+2\hat{k})\right]\cdot(\hat{i}-\hat{j}+\hat{k})=5$

which gives $\lambda=0$.

Therefore, the point of intersection of line and the plane is $(2,-1,2)$ and the other given point is $(-1,-5,-10)$. Hence the distance between these two points is

$\sqrt{[2-(-1)]^2+[-1+5]^2+[2-(-10)]^2}$, i.e. 13

9. A plane meets the co-ordinates axis in $A,\ B,\ C$ such that the centroid of the ΔABC is the point $(\alpha,\ \beta,\ \gamma)$. Show that the equation of the plane is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$

Sol. Let the equation of the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Then the co-ordinate of $A,\ B,\ C$ are $(a,0,0),\ (0,b,0)\ and\ (0,0,c)$ respectively.

Centroid of the ΔABC is

$$\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3} \text{ i.e., } \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

But co-ordinates of the centroid of the ΔABC are $(\alpha,\ \beta,\ \gamma)$ (given).

Therefore, $\alpha=\frac{a}{3},\beta=\frac{b}{3},\gamma=\frac{c}{3},\ i.e., a=3\alpha, b=3\beta, c=3\gamma$

Thus, the equation of plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$$

10 Find the angle between the lines whose direction cosines are given by the equations $3l+m+5n=0$ and $6mn-2nl+5lm=0$.

Sol. Eliminating m from the given two equations, we get

$\Rightarrow 2n^2+3ln+l^2=0$

$\Rightarrow (n+l)(2n+l)=0$

$\Rightarrow either\ n=-l\ or\ l=-2n$

Now if $l=-n,$ then $m=-2n$

and if $l=-2n,$ then $m=n$.

Thus, the direction ratios of two lines are proportional to $-n,\ -2n,\ n$ and $-2n,\ n,\ n,$

i.e. $1, 2, -1$ *and* $-2, 1, 1$.

So, vectors parallel to these lines are

$\vec{a}=\hat{i}+2\hat{j}-\hat{k}$ *and* $\vec{b}=-2\hat{i}+\hat{j}+\hat{k}$, respectively.

If θ is the angle between the lines, then

$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$=\frac{(\hat{i}+2\hat{j}-\hat{k})\cdot(-2\hat{i}+\hat{j}+\hat{k})}{\sqrt{1^2+2^2+(-1)^2}\sqrt{(-2)^2+1^2+1^2}}=-\frac{1}{6}$$

Hence, $\theta=\cos^{-1}\left(-\frac{1}{6}\right)$.

11. Find the co-ordinates of the foot of perpendicular drawn from the point $A(1,\ 8,\ 4)$ to the line joining the points $B(0,\ -1,\ 3)$ and $C(2,\ -3,\ -1)$.

Sol. Let L be the foot of perpendicular drawn from the points $A(1,\ 8,\ 4)$ to the line passing through B and C as shown in the Fig. 11.2. The equation of line BC by using formula $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$, the equation of the line BC is

$$\vec{r}=(-\hat{j}+3\hat{k})+\lambda(2\hat{i}-2\hat{j}-4\hat{k})$$

$$\Rightarrow x\hat{i}+y\hat{i}+z\hat{k}=2\lambda\hat{i}-(2\lambda+1)\hat{i}+\lambda(3-4\lambda)\hat{k}$$

Comparing both sides, we get

$$x=2\lambda,\ y=-(2\lambda+1),\ z=3-4\lambda\ \ \ldots(1)$$

Thus, the co-ordinate of L are $(2\lambda,\ -(2\lambda+1),\ (3-4\lambda),$

so, that the direction ratios of the line AL are $(1-2\lambda),\ 8+(2\lambda+1),\ 4-(3-4\lambda),$ i.e.

$1-2\lambda,\ 2\lambda+9,\ 1+4\lambda$

Since AL is perpendicular to BC, we have,

$$(1-2\lambda)(2-0)+(2\lambda+9)(-3+1)+(4\lambda+1)(-1-3)=0$$

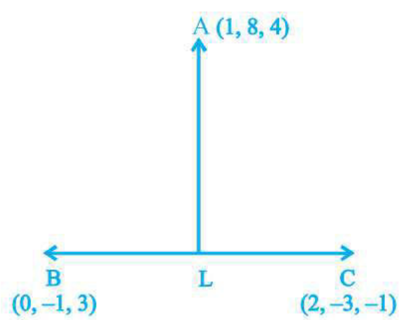


Fig.11.2

$$\Rightarrow \lambda=\frac{-5}{6}$$

The required point is obtained by substituting the value of λ, in (1), which is $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.

12. Find the image of the point $(1,\ 6,\ 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Sol. Let $P(1,\ 6,\ 3)$ be the given point and let L be the foot of perpendicular from P to the given line.

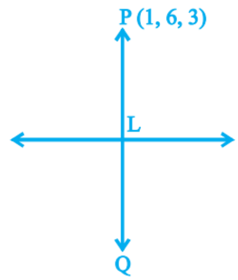


Fig.11.3

The coordinates of a general point on the given line are

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \ , i.e.,\ x = \lambda,\ y = 2\lambda + 1,\ z = 3\lambda + 2$$

If the coordinates of L are $(\lambda,\ 2\lambda+1,\ 3\lambda+2)$, then the direction ratios of PL are $\lambda - 1,\ 2\lambda - 5,\ 3\lambda - 1$.

But the direction ratios of given line which is perpendicular to PL are $1, 2, 3$. Therefore, $(\lambda-1)1 + (2\lambda-5)2 + (3\lambda-1)3 = 0$, which gives $\lambda = 1$. Hence coordinates of L are $(1,\ 3,\ 5)$.

Let $Q(x_1,\ y_1,\ z_1)$ be the image of $P(1,\ 6,\ 3)$ in the given line. Then L is the mid-point of PQ. Therefore, $\frac{x_1+1}{2} = 1, \frac{y_1+6}{2} = 3, \frac{z_1+3}{2} = 5$

$\Rightarrow x_1 = 1,\ y_1 = 0,\ z_1 = 7$

Hence, the image of $(1,\ 6,\ 3)$ in the given line is $(1,\ 0,\ 7)$.

13. Find the image of the point having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\hat{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0.$

Sol. Let the given point be $P(\hat{i} + 3\hat{j} + 4\hat{k})$ and Q be the image of P in the plane $\hat{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$ as shown in the Fig. 11.4.

Then PQ is the normal to the plane. Since PQ passes through P and is normal to the given plane, so the equation of PQ is given by

$$\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

Since Q lies on the line PQ, the position vector of Q can be expressed as

$$(\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \ \ i.e., (1+2\lambda)\hat{i} + (3-\lambda)\hat{j} + (4+\lambda)\hat{k}$$

Since R is the mid point of PQ, the position vector of R is

$$\frac{[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(4+\lambda)\hat{k}]+[\hat{i}+3\hat{j}+4\hat{k}]}{2}$$

i.e., $(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}$

Again, since R lies on the plane $\vec{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)+3=0$, we have

$$\left\{(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}\right\}.(2\hat{i}-\hat{j}+\hat{k})+3=0$$

$\Rightarrow \lambda=-2$

Hence, the position vector of Q is $\left(\hat{i}+3\hat{j}+4\hat{k}\right)-2(2\hat{i}-\hat{j}+\hat{k})$,

i.e., $-3\hat{i}+5\hat{j}+2\hat{k}$.

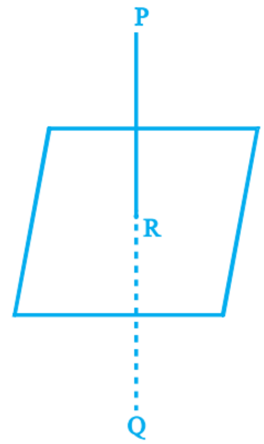


Fig.11.4

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 14 to 19.

14. **The coordinates of the foot of the perpendicular drawn from the point $(2,\ 5,\ 7)$ on the $x-axis$ are given by**

(A) $(2,\ 0,\ 0)$

(B) $(0,\ 5,\ 0)$

(C) $(0,\ 0,\ 7)$

(D) $(0,\ 5,\ 7)$

Sol. (A) is the correct Answer.

15. P **is a point on the line segment joining the points** $(3,2,-1)$ *and* $(6,2,-2)$**. If** $x-coordinate$ **of** P **is 5, then its** $y-coordinate$ **is**

(A) 2

(B) 1

(C) –1

(D) –2

Sol. (A) is the correct answer. Let P divides the line segment in the ratio of $\lambda:\ 1$, $x-coordinate$ of the point P may be expressed as $x=\frac{6\lambda+3}{\lambda+1}$ giving $\frac{6\lambda+3}{\lambda+1}=5$ so that $\lambda=2$. Thus, $y-coordinate$ of P is $\frac{2\lambda+2}{\lambda+1}=2$.

16. **If α, β, γ are the angles that a line makes with the positive direction of** $x,\ y,\ z\ axis$**, respectively, then the direction cosines of the line are.**

(A) $\sin\alpha, \sin\beta, \sin\gamma$

(B) $\cos\alpha, \cos\beta, \cos\gamma$

(C) $\tan\alpha, \tan\beta, \tan\gamma$

(D) $\cos^2\alpha, \cos^2\beta, \cos^2\gamma$

Sol. (B) is the correct answer.

17. The distance of a point $P(a, b, c)$ from $x-axis$ is

(A) $\sqrt{a^2+c^2}$

(B) $\sqrt{a^2+b^2}$

(C) $\sqrt{b^2+c^2}$

(D) b^2+c^2

Sol. (C) is the correct answer. The required distance is the distance of $P(a, b, c)$ from $Q(a, 0, 0)$, which is $\sqrt{b^2+c^2}$.

18. The equations of $x-axis$ in space are

(A) $x=0, y=0$

(B) $x=0, z=0$

(C) $x=0$

(D) $y=0, z=0$

Sol. (D) is the correct answer. On $x-axis$ the $y-coordinate$ and $z-coordinates$ are zero.

19. A line makes equal angles with co-ordinate axis. Direction cosines of this line are

(A) $\pm(1, 1, 1)$

(B) $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(C) $\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(D) $\pm\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

Sol. (B) is the correct answer. Let the line makes angle α with each of the axis. Then, its direction cosines are $cos\ \alpha,\ cos\ \alpha,\ cos\ \alpha$.

Since $\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$. Therefore, $\cos\alpha=\pm\frac{1}{\sqrt{3}}$

Fill in the blanks in each of the Examples from 20 to 22.

20. If a line makes angles $\frac{\pi}{2}, \frac{3}{4}\pi$ and $\frac{\pi}{4}$ with $x, y, z\ axis$, respectively, then its direction cosines are ________.

Sol. The direction cosines are $\cos\frac{\pi}{2}, \cos\frac{3}{4}\pi, \cos\frac{\pi}{4}$, i.e., $\pm\left(0, -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)$

21. If a line makes angles α, β, γ with the positive directions of the coordinate axes, then the value of $sin^2\alpha + sin^2\beta + sin^2\gamma$ is ________.

Sol. Note that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=(1-\cos^2\alpha)+(1-\cos^2\beta)+(1-\cos^2\gamma)$$
$$=3-(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)=2.$$

22. **If a line makes an angle of $\frac{\pi}{4}$ with each of y *and* z *axis*, then the angle which it makes with $x-axis$ is ________.**

Sol. Let it makes angle α with $x-axis$. Then $\cos^2\alpha+\cos^2\frac{\pi}{4}+\cos^2\frac{\pi}{4}=1$ which after simplification gives $\alpha=\frac{\pi}{2}$.

State whether the following statements are True or False in Examples 23 and 24.

23. **The points $(1,2,3)$, $(-2,3,4)$ *and* $(7,0,1)$ are collinear.**

Sol. Let A, B, C be the points $(1,2,3)$, $(-2,3,4)$ *and* $(7,0,1)$ respectively. Then, the direction ratios of each of the lines AB and BC are proportional to $-3,\ 1,\ 1$. Therefore, the statement is true.

24. **The vector equation of the line passing through the points $(3,5,4)$ *and* $(5,8,11)$ is $\vec{r}=3\hat{i}+5\hat{j}+4\hat{k}+\lambda(2\hat{i}+3\hat{j}+7\hat{k})$**

Sol. The position vector of the points $(3,5,4)$ *and* $(5,8,11)$ are

$\vec{a}=3\hat{i}+5\hat{j}+4\hat{k},\vec{b}=5\hat{i}+8\hat{j}+11\hat{k}$,

and therefore, the required equation of the line is given by

$\vec{r}=3\hat{i}+5\hat{j}+4\hat{k}+\lambda(2\hat{i}+3\hat{j}+7\hat{k})$.

Hence, the statement is true.

Three Dimensional Geometry
<u>Objective Type Questions</u>

Choose the correct answer from the given four options in each of the Exercises from 29 to 36.

29. Distance of the point (α, β, γ) from y-axis is

(A) β

(B) $|\beta|$

(C) $|\beta|+|\gamma|$

(D) $\sqrt{\alpha^2+\gamma^2}$

Sol. (D) Required distance $=\sqrt{(\alpha-0)^2+(\beta-\beta)^2+(\gamma-0)^2}=\sqrt{\alpha^2+\gamma^2}$

30. If the directions cosines of a line are $k, k, k,$ then

(A) $k>0$

(B) $0<k<1$

(C) $k=1$

(D) $k=\dfrac{1}{\sqrt{3}}\ or -\dfrac{1}{\sqrt{3}}$

Sol. (D) Since, direction cosines of a line are $k,\ k\ and\ k$.

$\therefore\ l=k, m=k$ and $n=k$

We know that, $l^2+m^2+n^2=1$

$\Rightarrow k^2+k^2+k^2=1$

$\Rightarrow k^2=\dfrac{1}{3}$

$\therefore\ k=\pm\dfrac{1}{\sqrt{3}}$

31. The distance of the plane $\vec{r}\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$ from the origin is

(A) 1

(B) 7

(C) $\dfrac{1}{7}$

(D) None of these

Sol. (A) The distance of the plane $\vec{r}\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$ from the origin is 1.

[since, $\vec{r}\cdot\vec{n}=d$ is the form of above equation, where d represents the distance of plane from the origin i.e., d = 1]

32. The sine of the angle between the straight line $\dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$ and the plane $2x-2y+z=5$ is

(A) $\frac{10}{6\sqrt{5}}$

(B) $\frac{4}{5\sqrt{2}}$

(C) $\frac{2\sqrt{3}}{5}$

(D) $\frac{\sqrt{2}}{10}$

Sol. (D) We have, the equation of line as

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

Now, the line passes through point $(2,\ 3,\ 4)$ and having direction ratios $(3,\ 4,\ 5)$.

Since, the line passes through point $(2,\ 3,\ 4)$ and parallel to the vector $(3\hat{i}+4\hat{j}+5\hat{k})$.

$\therefore \vec{b}=3\hat{i}+4\hat{j}+5\hat{k}$

Also, the Cartesian from of the given planes is $2x-2y+z=5$.

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})(2\hat{i}-2\hat{j}+\hat{k})=5$

$\therefore \vec{n}=(2\hat{i}-2\hat{j}+\hat{k})$

We know that, $\sin\theta=\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\cdot|\vec{n}|}=\frac{|(3\hat{i}+4\hat{j}+5\hat{k})\cdot(2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{3^2+4^2+5^2}\cdot\sqrt{4+4+1}}$

$=\frac{|6-8+5|}{\sqrt{50}\cdot 3}=\frac{3}{15\sqrt{2}}=\frac{1}{5\sqrt{2}}$

$\sin\theta=\frac{\sqrt{2}}{10}$

33. The reflection of the point (α, β, γ) in the xy–plane is

(A) $(\alpha,\ \beta,\ 0)$

(B) $(0,\ 0,\ \gamma)$

(C) $(-\alpha,\ -\beta,\ \gamma)$

(D) $(\alpha,\ \beta,\ -\gamma)$

Sol. (D) In XY-plane, the reflection of the point $(\alpha,\ \beta,\ \gamma)$ *is* $(\alpha,\ \beta,\ -\gamma)$

34. The area of the quadrilateral ABCD, where $A(0,4,1)$, $B\ (2,\ 3,\ -1)$, $C(4,\ 5,\ 0)$ **and** $D(2,\ 6,\ 2)$, **is equal to**

(A) 9 sq. units
(B) 18 sq. units
(C) 27 sq. units
(D) 81 sq. units

Sol. (A) We have,

$$\overrightarrow{AB} = (2-0)\hat{i} + (3-4)\hat{j} + (-1-1)\hat{k} = 2\hat{i} - \hat{j} - 2\hat{k}$$

$$\overrightarrow{BC} = (4-2)\hat{i} + (5-3)\hat{j} + (0+1)\hat{k} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\overrightarrow{CD} = (2-4)\hat{i} + (6-5)\hat{j} + (2-0)\hat{k} = -2\hat{i} + \hat{j} + 2\hat{k}$$

$$\overrightarrow{DA} = (0-2)\hat{i} + (4-6)\hat{j} + (1-2)\hat{k} = -2\hat{i} - 2\hat{j} - \hat{k}$$

$\therefore$ Area of quadrilateral $ABCD = |\overrightarrow{AB} \times \overrightarrow{BC}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix}$

$= |\hat{i}(-1+4) - \hat{j}(2+4) + \hat{k}(4+2)|$

$= |3\hat{i} - 6\hat{j} + 6\hat{k}|$

$= \sqrt{9+36+36} = 9$ sq units

35. The locus represented by $xy + yz = 0$ is

(A) A pair of perpendicular lines

(B) A pair of parallel lines

(C) A pair of parallel planes

(D) A pair of perpendicular planes

Sol. (D) We have, $xy + yz = 0$

$\Rightarrow \quad xy = -yz$

So, a pair of perpendicular planes.

36. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $sin^{-1}(\alpha)$ $with\ x-axis$. The value of α is equal to

(A) $\dfrac{\sqrt{3}}{2}$

(B) $\dfrac{\sqrt{2}}{3}$

(C) $\dfrac{2}{7}$

(D) $\dfrac{3}{7}$

Sol. (C) Since, $2x - 3y + 6z - 11 = 0$ makes an angle $sin^{-1}(\alpha)$ $with\ x-axis$.

$\vec{b} = (1\hat{i} + 0\hat{j} + 0\hat{k})$ and $\vec{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

We know that, $\sin\theta = \dfrac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| \cdot |\vec{n}|}$

$= \dfrac{|(1i) \cdot (2i - 3j + 6k)}{\sqrt{1}\sqrt{4+9+36}} = \dfrac{2}{7}$

Fill in the blanks in each of the Exercises 37 to 41.

37. A plane passes through the points $(2,0,0)$ $(0,3,0)$ and $(0,0,4)$. The equation of plane is ________.

Sol. We know that, equation of the plane that cut the coordinate axes at $(a,0,0)$ $(0,b,0)$ and $(0,\ 0,\ c)$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Hence, the equation of plane passes through the points $(2,0,0)$ $(0,3,0)$ *and* $(0,0,4)$ is $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1.$

38. The direction cosines of the vector $(2\hat{i}+2\hat{j}-\hat{k})$ are ____________.

Sol. Direction cosines of $(2\hat{i}+2\hat{j}-\hat{k})$ are $\frac{2}{\sqrt{4+4+1}},\frac{2}{\sqrt{4+4+1}},\frac{-1}{\sqrt{4+4+1}}$ *i.e.,* $\frac{2}{3},\frac{2}{3},\frac{-1}{3}.$

39. The vector equation of the line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ is ____________.

Sol. We have, $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$ and $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

So, the vector equation will be

$\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})-(5\hat{i}-4\hat{j}+6\hat{k})=\lambda(3\hat{i}+7\hat{j}+2\hat{k})$

$\Rightarrow (x-5)\hat{i}+(y+4)\hat{j}+(z-6)\hat{k}=\lambda(3\hat{i}+7\hat{j}+2\hat{k})$

40. The vector equation of the line through the points $(3,4,-7)$ *and* $(1,-1,6)$ is __________.

Sol. We know that, vector equation of a line passes through two points is represented by $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

Here, $\vec{r}=x\hat{i}+y\hat{j}+3\hat{k},\vec{a}=3\hat{i}+4\hat{j}-7\hat{k}$

And $\vec{b}=\hat{i}-\hat{j}+6\hat{k}$

$\Rightarrow (\vec{b}-\vec{a})=-2\hat{i}-5\hat{j}+13\hat{k}$

So, the required equation is

$$x\hat{i}+y\hat{j}+z\hat{k}=3\hat{i}+4\hat{j}-7\hat{k}=\lambda(-21\hat{i}-5\hat{j}+13\hat{k})$$

$$\Rightarrow \quad (x-3)\hat{i}+(y-4)\hat{j}+(z+7)\hat{k}=\lambda(-2\hat{i}-5\hat{j}+13\hat{k})$$

41. The Cartesian equation of the plane $\vec{r}.(\hat{i}+\hat{j}-\hat{k})=2$ is ____________.

Sol. We have, $\vec{r}.(\hat{i}+\hat{j}-\hat{k})=2$

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}-\hat{k})=2$

$\Rightarrow x+y-z=2$

Which is the required form.

State True or False for the statements in each of the Exercises 42 to 49.

42. The unit vector normal to the plane $x+2y+3z-6=0$ is $\frac{1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}.$

Sol. True

We have, $\vec{n}=\hat{i}+2\hat{j}+3\hat{k}$

$$\therefore \hat{n} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{1^2+2^2+3^2}} = \frac{\hat{i}}{\sqrt{14}} + \frac{2\hat{j}}{\sqrt{14}} + \frac{3\hat{k}}{\sqrt{14}}$$

43. The intercepts made by the plane $2x-3y+5z+4=0$ on the co-ordinate axis are $-2, -\frac{4}{3}, -\frac{4}{5}$.

Sol. Ture

We have, $2x-3y+5z+4=0$

$$\Rightarrow 2x-3y+5z=-4$$

$$\Rightarrow \frac{2x}{-4} - \frac{3y}{-4} + \frac{5z}{-4} = 1$$

$$\Rightarrow \frac{x}{-2} + \frac{y}{4} - \frac{z}{4} = 1$$

$$\Rightarrow \frac{x}{-2} + \frac{y}{4} + \frac{z}{\left(-\frac{4}{5}\right)} = 1$$

So, the intercepts are $-2, \frac{4}{3}$ and $-\frac{4}{5}$.

44. The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r}(3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

Sol. False

We have, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{n} = 3\hat{i} - 4\hat{j} + \hat{k}$

Let θ is the angle between line and plane.

Then, $\sin\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\cdot|\vec{n}|} = \frac{|(2\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}-4\hat{j}-\hat{k})}{\sqrt{6}\cdot\sqrt{26}}$

$$= \frac{|6+4-1|}{\sqrt{156}} = \frac{9}{2\sqrt{39}}$$

$$\therefore \theta = \sin^{-1}\frac{9}{2\sqrt{39}}$$

45. The angle between the planes $\vec{r}(2\hat{i} - 3\hat{j} + \hat{k}) = 1$ and $\vec{r}(\hat{i} - \hat{j}) = 4$ is $\cos^{-1}\frac{-5}{\sqrt{58}}$.

Sol. False,

We know that, the angle between two planes is given by $\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

Here, $\vec{n}_1 = (2\hat{i} - 3\hat{j} + \hat{k})$ and $\vec{n}_2 = (\hat{i} - \hat{j})$

$$\therefore \cos\theta = \frac{|(2\hat{i}-3\hat{j}+\hat{k})(\hat{i}-\hat{j})|}{\sqrt{4+9+1}\sqrt{1+1}}$$

$$\Rightarrow \cos\theta = \frac{|2+3|}{\sqrt{14}\cdot\sqrt{2}} = \frac{5}{2\sqrt{7}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{5}{2\sqrt{7}}\right)$$

46. **The line** $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ **lies in the plane** $\vec{r}(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0.$

Sol. False,

We have, $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = \hat{i}(2+\lambda) + \hat{j}(-3-\lambda) + \hat{k}(-1+2\hat{k})$

Since, $x = (2+\lambda), y = (-3-\lambda)$ and $z = (-1+2\lambda)$ are coordinates of general point which should satisfy the equation of the given plane.

$\therefore [(2+\lambda)\hat{i} + (-3-\lambda)\hat{j} + (2\lambda-1)\hat{k}]\cdot[\hat{i} + \hat{j} + \hat{k}] = 2$

$\Rightarrow (2+\lambda) - 3 - \lambda - 2\lambda + 1 = 2$

$\Rightarrow -2\lambda = 2$

$\Rightarrow \lambda = -1$

$\therefore \vec{r} = (2-1)\hat{i} + (-3+1)\hat{j} + (-2-1)\hat{k}$

$= \hat{i} - 2\hat{j} - 3\hat{k}$

Again, from the equation of the plane

$\vec{r}\cdot(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$\Rightarrow (\hat{i} - 2\hat{j} - 3\hat{k})(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$\Rightarrow (3-2+3) + 2 = 0$

$\Rightarrow 6 \neq 0$

Which is not true.

So, the line $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ does not lie in a plane.

47. **The vector equation of the line** $\frac{x-5}{3} = \frac{y-4}{7} = \frac{z-6}{2}$ **is**

$\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k}).$

Sol. True

We have, $x = 5,\ y = -4,\ z = 6$

And $a = 3,\ b = 7,\ c = 2$

$\therefore \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

48. **The equation of a line, which is parallel to** $2\hat{i} + \hat{j} + 3\hat{k}$ **and which passes through the point** $(5,-2,4)$ **is** $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}.$

Sol. False

Here, $x_1 = 5,\ y_1 = -2,\ z_1 = 4$

And $a=2,\ b=1,\ c=3$

$$\Rightarrow \frac{x-5}{2}=\frac{y+2}{1}=\frac{z-4}{3}$$

49. If the foot of perpendicular drawn from the origin to a plane is $(5,-3,-2)$, then the equation of plane is $\vec{r}.(5\hat{i}-3\hat{j}-2\hat{k})=38$.

Sol. True

Since, the required plane passes through the point P(5, -3, -2), and is perpendicular to $\overrightarrow{\mathrm{OP}}$.

$\therefore\ \vec{a}=5\hat{i}-3\hat{j}-2\hat{k}$

and $\vec{n}=\overrightarrow{\mathrm{OP}}=5\hat{i}-3\hat{j}-2\hat{k}$

Now, the equation of the plane is

$(\vec{r}\cdot\vec{a})\cdot\vec{n}=0$

$\Rightarrow \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$\Rightarrow \vec{r}\cdot(5\hat{i}-3\hat{j}-2\hat{k})=(5\hat{i}-3\hat{j}-2\hat{k})\cdot(5\hat{i}-3\hat{j}-2\hat{k})$

$\Rightarrow \vec{r}\cdot(5\hat{i}-3\hat{j}-2\hat{k})=25+9+4$

$\Rightarrow \vec{r}\cdot(5\hat{i}-3\hat{j}-2\hat{k})=38$

Three Dimensional Geometry

<u>Short Answer type Questions</u>

1. **Find the position vector of a point A in space such that $\overrightarrow{\mathrm{OA}}$ is inclined at 60^o to OX and at 45^o to OY and $|\overrightarrow{\mathrm{OA}}|=10$ units.**

Sol. Since, $\overrightarrow{\mathrm{OA}}$ is inclined at 60^o to OX and at 45^o to OY. Let $\overrightarrow{\mathrm{OA}}$ make angle α with OZ.

$\therefore \cos^2 60^o + \cos^2 45^o + \cos^2\alpha = 1$

$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\alpha = 1$ $[\because l^2+m^2+n^2=1]$

$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = 1-\left(\frac{1}{2}+\frac{1}{4}\right)$

$\Rightarrow \cos^2\alpha = 1-\left(\frac{6}{8}\right)$

$\Rightarrow \cos^2\alpha = \frac{1}{4}$

$\Rightarrow \cos\alpha = \frac{1}{2} = \cos 60^o$

$\therefore \alpha = 60^o$

$\therefore \overrightarrow{\mathrm{OA}} = |\overrightarrow{\mathrm{OA}}|\left(\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{2}\hat{k}\right)$

$=10\left(\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{2}\hat{k}\right)$ $[\because |\overrightarrow{\mathrm{OA}}|=10]$

$=5\hat{i}+5\sqrt{2}j+5\hat{k}$

2. **Find the vector equation of the line which is parallel to the vector $3\hat{i}-2\hat{j}+6\hat{k}$ and which passes through the point $(1,-2,3)$.**

Sol. Let $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$

So, vector equation of the line, which is parallel to the vector $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$ and passes through the vector $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ is $\vec{r}=\vec{b}+\lambda\vec{a}$.

$\therefore \vec{r}=(\hat{i}-2\hat{j}+3\hat{k})+\lambda(3\hat{i}-2\hat{j}+6\hat{k})$

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})-(\hat{i}-2\hat{j}+3\hat{k})=\lambda(3\hat{i}-2\hat{j}+6\hat{k})$

$\Rightarrow (x-1)\hat{i}+(y+2)\hat{j}+(z-3)\hat{k}=\lambda(3\hat{i}-2\hat{j}+6\hat{k})$

3. **Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ And $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Also, find their point of intersection.**

Sol. We have, $x_1=1,\ y_1=2,\ z_1=3$ and $a_1=2,\ b_1=3,\ c_1=4$

Also, $x_2 = 4,\ y_2 = 1,\ z_2 = 0$ and $a_2 = 5,\ b_2 = 5,\ c_2 = 1$

If two lines intersect, then shortest distance between them should be zero.

$\therefore$ Shortest distance between two given lines

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 4-1 & 1-2 & 0-3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{(3\cdot 1 - 2\cdot 4)^2 + (4\cdot 5 - 1\cdot 2)^2 + (2\cdot 2 - 5\cdot 3)^2}}$$

$$= \frac{\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{25 + 324 + 121}}$$

$$= \frac{3(2-8) + 1(2-20) - 3(4-15)}{\sqrt{470}}$$

$$= \frac{-15 - 18 + 33}{\sqrt{470}} = \frac{0}{\sqrt{470}} = 0$$

Therefore, the given two lines are intersecting.

For finding their point of intersection for first line,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$$

$$x = 2\lambda + 1,\ y = 3\lambda + 2 \text{ and } z = 4\lambda + 3$$

Since, the lines are intersecting. So, let us put these values in the equation of another line.

Thus, $\dfrac{2\lambda + 1 - 4}{5} = \dfrac{3\lambda + 2 - 1}{2} = \dfrac{4\lambda + 3}{1}$

$$\Rightarrow \frac{2\lambda - 3}{5} = \frac{3\lambda + 1}{2} = \frac{4\lambda + 3}{1}$$

$$\Rightarrow \frac{2\lambda - 3}{5} = \frac{4\lambda + 3}{1}$$

$$\Rightarrow 2\lambda - 3 = 20\lambda + 15$$

$$\Rightarrow 18\lambda = -18 = -1$$

So, the required point of intersection is

$$x = 2(-1) + 1 = -1$$

$$y = 3(-1) + 2 = -1$$

$$z = 4(-1) + 3 = -1$$

Thus, the lines intersect at $(-1, -1, -1)$.

4. Find the angles between the lines $\vec{r}=3\hat{i}-2\hat{j}+6\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=(2\hat{j}-5\hat{k})+\mu(6\hat{i}+3\hat{j}+2\hat{k})$

Sol. We have, $\vec{r}=3\hat{i}-2\hat{j}+6\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k})$

And $\vec{r}=(2\hat{j}-5\hat{k})+\mu(6\hat{i}+3\hat{j}+2\hat{k})$

Where, $\vec{a}_1=3\hat{i}-2\hat{j}+6\hat{k},\ \vec{b}_1=2\hat{i}+\hat{j}+2\hat{k}$

And $\vec{a}_2=2\hat{j}-5\hat{k},\ \vec{b}_2=6\hat{i}+3\hat{j}+2\hat{k}$

If θ is angle between the lines, then

$$\cos\theta=\frac{|\vec{b}_1\cdot\vec{b}_2|}{|\vec{b}_1|\cdot|\vec{b}_1|}$$

$$=\frac{|(2\hat{i}+\hat{j}+2\hat{k})\cdot(6\hat{i}+3\hat{j}+2\hat{k})}{|2\hat{i}+\hat{j}+2\hat{k}||6\hat{i}+3\hat{j}+2\hat{k}|}$$

$$=\frac{|12+3+4|}{\sqrt{9}\sqrt{49}}=\frac{19}{21}$$

$$\theta=\cos^{-1}\frac{19}{21}$$

5. Prove that the line through A (0, –1, –1) and B (4, 5, 1) intersects the line through $C(3,\ 9,\ 4)$ and $D(-4,\ 4,\ 4)$.

Sol. We know that, the Cartesian equation of a line that passes through two point $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Hence, the Cartesian equation of line passes through $A(0,-1,-1)\ and\ B(4,5,1)$ is

$$\frac{x-0}{4-0}=\frac{y+1}{5+1}=\frac{z+1}{1+1}$$

$$\Rightarrow \frac{x}{4}=\frac{y+1}{6}=\frac{z+1}{2}\ \ ...(i)$$

And Cartesian equation of the line passes through $C(3,9,4)\ and\ D(-4,4,4)$ is

$$\frac{x-3}{-4-3}=\frac{y-9}{4-9}=\frac{z-4}{4-4}$$

$$\Rightarrow \frac{x-3}{-7}=\frac{y-9}{-5}=\frac{z-4}{0}\ \ ...(ii)$$

If the lines intersect, then shortest distance between both of them should be zero.

$\therefore$ Shortest distance between the lines

$$=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 3-0 & 9+1 & 4+1 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{(6\cdot 0+10)^2+(-14-0)^2+(-20+42)^2}}$$

$$= \frac{\begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{100+196+484}}$$

$$= \frac{3(0+10)-10(14)+5(-20+42)}{\sqrt{780}}$$

$$= \frac{30-140+110}{\sqrt{780}} = 0$$

So, the given lines intersect.

6. Prove that the lines $x = py + q,\ z = ry + s$ and $x = p'y + q',\ z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

Sol. We have, $x = py + q \Rightarrow y = \frac{x-q}{p}$ ...(*i*)

And $z = ry + s \Rightarrow y = \frac{z-s}{r}$ ...(*ii*)

$\Rightarrow \frac{x-q}{p} = \frac{y}{1} = \frac{z-s}{r}$ [using Eqs. (i) and (ii)] ...(iii)

Similarly, $\frac{x-q'}{p'} = \frac{y}{1} = \frac{z-s'}{r'}$ ...(*iv*)

From Eqs. (iii) and (iv),

$a_1 = p,\ b_1 = 1,\ c_1 = r$

$and\ \ a_2 = p',\ b_2 = 1,\ c_2 = r'$

if these given lines are perpendicular to each other, then

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow pp' + 1 + rr' = 0$

Which is the required condition.

7. Find the equation of a plane which bisects perpendicularly the line joining the points $A(2,\ 3,\ 4)$ and $B(4,\ 5,\ 8)$ at right angles.

Sol. Since, the equation of a plane is bisecting perpendicular the line joining the points $A(2,\ 3,\ 4)$ and $B(4,\ 5,\ 8)$ at right angles.

So, mid-point of AB is $\left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2}\right)$ i.e., $(3,\ 4,\ 6)$.

Also, $\overrightarrow{N} = (4-2)\hat{i} + (5-3)\hat{j} + (8-4)\hat{k} = 2\hat{i} + 2\hat{j} + 4\hat{k}$

So, the required equation of the plane is $(\vec{r} - \vec{a})\cdot\overrightarrow{N} = 0$.

$\Rightarrow [(x-3)\hat{i}+(y-4)\hat{j}+(z-6)\hat{k}]\cdot(2\hat{i}+2\hat{j}+4\hat{k})=0$ $[\because \vec{a}=3\hat{i}+4\hat{j}+6\hat{k}]$

$\Rightarrow 2x-6+2y-8+4z-24=0$

$\Rightarrow 2x+2y+4z=38$

$\therefore x+y+2z=19$

8. Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

Sol. Since, normal to the plane is equally inclined to the coordinate axis.

Therefore, $\cos\alpha+\cos\beta=\cos\gamma=\dfrac{1}{\sqrt{3}}$

So, the normal is $\overrightarrow{N}=\dfrac{1}{\sqrt{3}}\hat{i}+\dfrac{1}{\sqrt{3}}\hat{j}+\dfrac{1}{\sqrt{3}}\hat{k}$ and plane is at a distance of $3\sqrt{3}$ units from origin.

The equation of plane is $\vec{r}\cdot\hat{N}=3\sqrt{3}$ $\left[\because \hat{N}=\dfrac{\overrightarrow{N}}{|N|}\right]$

[Since, vector equation of the plane at a distance p from the origin is $\vec{r}\cdot\hat{N}=p$]

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot\frac{\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)}{1}=3\sqrt{3}$$

$$\Rightarrow \frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}=3\sqrt{3}$$

$\therefore x+y+z=3\sqrt{3}\cdot\sqrt{3}=9$

So, the required equation of plane is $x+y+z=9$.

9. If the line drawn from the point $(-2,-1,-3)$ meets a plane at right angle at the point $(1,-3,3)$, find the equation of the plane.

Sol. Since, the line drawn from the point $(-2,-1,-3)$ meets a plane at right angle at the point $(1,-3,3)$. So, the plane passes through the point $(1,-3,3)$, and normal to plane is $(-3\hat{i}+2\hat{j}-6\hat{k})$.

$\Rightarrow \vec{a}=\hat{i}-3\hat{j}+3\hat{k}$ And $\overrightarrow{N}=-3\hat{i}+2\hat{j}-6\hat{k}$

So, the equation of required plane is $(\vec{r}-\vec{a})\cdot\overrightarrow{N}=0$

$\Rightarrow [(x\hat{i}+y\hat{j}+z\hat{k})-(\hat{i}-3\hat{j}+3\hat{k})]\cdot(-3\hat{i}+2\hat{j}-6\hat{k})=0$

$\Rightarrow [(x-1)\hat{i}+(y+3)\hat{j}+(z-3)\hat{k})]\cdot(-3\hat{i}+2\hat{j}-6\hat{k})=0$

$\Rightarrow -3x+3+2y+6-6z+18=0$

$\Rightarrow -3x+2y-6z=-27$

$\therefore 3x-2y+6z-27=0$

10. Find the equation of the plane through the points $(2,1,0)$, $(3,-2,-2)$ and $(3,1,7)$

Sol. We know that, the equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-1 & z-0 \\ 3-2 & -2-1 & -2-0 \\ 3-2 & 1-1 & 7-0 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix}=0$$

$\Rightarrow (x-2(-21+0)-(y-1)(7+2)+z(3)=0$

$\Rightarrow -21x+42-9y+9+3z=0$

$\Rightarrow -21x-9y+3z=-51$

$\therefore 7x+3y-z=17$

So, the required equation of plane is $7x+3y-z=17$.

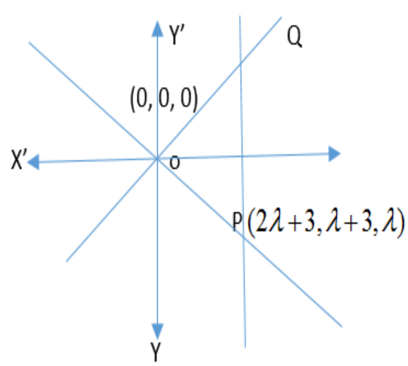
11. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

Sol. Given equation of the line is $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}=\lambda \quad ...(i)$

So, DR's of the line are $2,\ 1,\ 1$ and DC's of the given line are $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{16}}$.

Also, the required lines make angle $\frac{\pi}{3}$ with the given line.

From Eq. (i), $x=(2\lambda+3), y=(\lambda+3)$ and $z=\lambda$



$$\because \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\therefore \cos\frac{\pi}{3}=\frac{(4\lambda+6)+(\lambda+3)+(\lambda)}{\sqrt{6}\sqrt{(2\lambda+3)^2+(\lambda+3)^2+\lambda^2}}$$

$$\Rightarrow \frac{1}{2}=\frac{6\lambda+9}{\sqrt{6}\sqrt{(4\lambda^2+9+12\lambda+\lambda^2+9+6\lambda+\lambda^2)}}$$

$$\Rightarrow \frac{\sqrt{6}}{2}=\frac{6\lambda+9}{\sqrt{6\lambda^2+18\lambda+18}}$$

$\Rightarrow 6\sqrt{\lambda^2+3\lambda+3}=2(6\lambda+9)$

$\Rightarrow 36(\lambda^2+3\lambda+3)=36(4\lambda^2+9+12\lambda)$

$\Rightarrow \lambda^2+3\lambda+3=4\lambda^2+9+12\lambda$

$\Rightarrow 3\lambda^2+9\lambda+6=0$

$\Rightarrow \lambda^2+3\lambda+2=0$

$\Rightarrow \lambda(\lambda+2)+1(\lambda+2)=0 \qquad \Rightarrow (\lambda+1)(\lambda+2)=0$

$\therefore \lambda=-1,-2$

So, the DC's are $1,2,-1$ *and* $-1,1,-2$.

Also, both the required lines pass through origin.

So, the equations of required lines are $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ and $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$.

12. Find the angle between the lines whose direction cosines are given by the equation $l+m+n=0,\ l^2+m^2-n^2=0$.

Sol. Eliminating n from both the equations, we have

$l^2+m^2-(l-m)^2=0$

$\Rightarrow l^2+m^2-l^2-m^2+2ml=0$

$\Rightarrow 2lm=0 \Rightarrow lm=0$

$\Rightarrow (-m-n)m=0\ [\because l=-m-n]$

$\Rightarrow (m+n)m=0$

$\Rightarrow m=-n \Rightarrow m=0$

$\Rightarrow l=0,\ l=-n$

Thus, Dr's two lines are proportional to $0,-n,n$ and $-n,0,n$ i.e., $0,-1,1$ and $-1,0,1$.

So, the vector parallel to these given lines are $\vec{a}=-\hat{j}+\hat{k}$ and $\vec{b}=-\hat{i}+\hat{k}$

Now, $\cos\theta=\frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|}=\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \Rightarrow \cos\theta=\frac{1}{2}$

$\therefore \theta=\frac{\pi}{3}\left[\because \cos\frac{\pi}{3}=\frac{1}{2}\right]$

13. If a variable line in two adjacent positions has direction cosines l,m,n and $l+\delta l,\ m+\delta m,\ n+\delta n,$ show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2=\delta l^2+\delta m^2+\delta n^2$.

Sol. We have l,m,n and $l+\delta l,\ m+\delta m,\ n+\delta n,$ as direction cosines of a variable line in two different positions.

$\therefore l^2+m^2+n^2=1\ \ldots(i)$

And $(1+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1\ \ldots(ii)$

$\Rightarrow l^2+m^2+n^2+\delta l^2+\delta m^2+\delta n^2+2(l\delta l+m\delta m+n\delta n)=1$

$\Rightarrow \delta l^2+\delta m^2+\delta n^2=-2(l\delta l+m\delta m+n\delta n)\ [\because l^2+m^2+n^2=1]$

$\Rightarrow l\delta l+m\delta m+n\delta n=\frac{-1}{2}(\delta l^2+\delta m^2+\delta n^2)\ \ldots(iii)$

Now, $\vec{a}$ and $\vec{b}$ are unit vectors along a line with direction cosines l,m,n and $(l+\delta l),(m+\delta m),(n+\delta n)$ respectively.

$\therefore \vec{a}=l\hat{i}+m\hat{j}+n\hat{k}$ and $\vec{b}=(l+\delta l)\hat{i}+(m+\delta m)\hat{j}+(n+\delta n)\hat{k}$

$\Rightarrow \cos\delta\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\vec{a}\cdot\vec{b}\ [\because|\hat{a}|=|\hat{b}|=1]$

$\Rightarrow \cos\delta\theta = l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$

$=(l^2+m^2+n^2)+(l\delta l+m\delta m+n\delta n)$

$=1-\frac{1}{2}(\delta l^2+\delta m^2+\delta n^2)$ [using Eq. (iii)]

$\Rightarrow 2(1-\cos\delta\theta)=(\delta l^2+\delta m^2+\delta n^2)$

$\Rightarrow 2.2\sin^2\frac{\delta\theta}{2}=\delta l^2+\delta m^2+\delta n^2 \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$

$\Rightarrow 4\left(\frac{\delta\theta}{2}\right)^2=\delta l^2+\delta m^2+\delta n^2 \left[\sin ce, \frac{\delta\theta}{2} \text{is small, then } \sin\frac{\delta\theta}{2}=\frac{\delta\theta}{2}\right]$

$\therefore \delta\theta^2=\delta l^2+\delta m^2+\delta n^2$

14. O is the origin and A is $(a,\ b,\ c)$. Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

Sol. Since, DC's of line OA are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$.

Also, $\vec{n}=\overrightarrow{OA}=\vec{a}=a\hat{i}+b\hat{j}+c\hat{k}$

The equation of plane passes through $(a,\ b,\ c)$ and perpendicular to OA is given by

$[\vec{r}-\vec{a}]\cdot\vec{n}=0$

$\Rightarrow \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$\Rightarrow [(x\hat{i}+y\hat{j}+z\hat{k})\cdot(a\hat{i}+b\hat{j}+c\hat{k})]=(a\hat{i}+b\hat{j}+c\hat{k})\cdot(a\hat{i}+b\hat{j}+c\hat{k})$

$\Rightarrow ax+by+cz=a^2+b^2+c^2$

15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c', respectively, from the origin, prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$.

Sol. Consider OX, OY, OZ and ox, oy, oz are two system of rectangular axes.

Let their corresponding equation of plane be

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(i)

And $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1$ ...(ii)

Also, the length of perpendicular from origin to EQs. (i) and (ii) must be same.

$$\therefore \frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=\frac{\frac{0}{a'}+\frac{0}{b'}+\frac{0}{c'}-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}$$

$$\Rightarrow \sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$$

16. Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

Sol. We have, equation of line as $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$

$\Rightarrow \frac{x-4}{-2}=\frac{y}{6}=\frac{z-1}{-3}=\lambda$

$\Rightarrow x=-2\lambda+4,\ y=6\lambda$ and $z=-3\lambda+1$

Let the coordinates of L be $(4-2\lambda,\ 6\lambda,\ 1-3\lambda)$ and direction ratios of PL are proportional to $(4-2\lambda-2,\ 6\lambda-3,\ 1-3\lambda+8)$ *i.e.*, $(2-2\lambda,\ 6\lambda-3,\ 9-3\lambda)$.

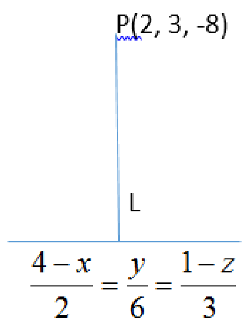
Also, direction ratios are proportional to $-2, 6, -3$. Since, PL is perpendicular to give line.

$\therefore -2(2-2\lambda)+6(6\lambda-3)-3(9-3\lambda)=0$

$\Rightarrow -4+4\lambda+36\lambda-18-27+9\lambda=0$

$\Rightarrow 49\lambda=49 \Rightarrow \lambda=1$

So, the coordinates of L are $(4-2\lambda, 6\lambda, 1-3\lambda)$ *i.e.*, $(2, 6, -2)$.



Also, length of PL $=\sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}$

$=\sqrt{0+9+36}=3\sqrt{5}$units

17. Find the distance of a point (2, 4, −1) from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$.

Sol. We have, equation of the line as $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}=\lambda$

$\Rightarrow \quad x=\lambda-5,\ y=4\lambda-3,\ z=6-9\lambda$

Let the coordinates of L be $(\lambda-5, 4\lambda-3, 6-9\lambda)$, then Dr's of PL are $(\lambda-7, 4\lambda-7, 7-9\lambda)$.]

Also, the direction ratios of given line are proportional to 1, 4, -9.

Since, PL is perpendicular to the given line.

$\therefore (\lambda-7)\cdot 1+(4\lambda-7)\cdot 4+(7-9\lambda)\cdot(-9)=0$

$\Rightarrow \lambda-7+16\lambda-28+81\lambda-63=0$

$\Rightarrow 98\lambda=98 \Rightarrow \lambda=1$

So, the coordinates of L are $(-4,1,-3)$.

$\therefore$ Required distance, $PL=\sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$

$=\sqrt{36+9+4}=7$ units

18. **Find the length and the foot of perpendicular from the point $\left(1,\frac{3}{2},2\right)$ to the plane $2x-2y+4z+5=0$.**

Sol. Equation of the given plane is $2x-2y+4z+5=0$ ...(i)

$\Rightarrow \vec{n}=2\hat{i}-2\hat{j}+4\hat{k}$

So, the equation of line through $\left(1,\frac{3}{2},2\right)$ and parallel to $\vec{n}$ is given by

$$\frac{x-1}{2}=\frac{y-3/2}{-2}=\frac{z-2}{4}=\lambda$$

$\Rightarrow x=2\lambda+1,\ y=-2\lambda+\frac{3}{2}$ and $z=4\lambda+2$

If this point lies on the given plane, then

$2(2\lambda+1)-2\left(-2\lambda+\frac{3}{2}\right)+4(4\lambda+2)+5=0$ [using Eq. (i)]

$\Rightarrow 4\lambda+2+4\lambda-3+16\lambda+8+5=0$

$\Rightarrow 24\lambda=-12 \Rightarrow \lambda=\frac{-1}{2}$

$\therefore$ Required foot of perpendicular

$=\left[2\times\left(\frac{-1}{2}\right)+1,-2\times\left(\frac{-1}{2}\right)+\frac{3}{2},4\times\left(\frac{-1}{2}\right)+2\right]$ *i.e.*, $\left(0,\frac{5}{2},0\right)$

$\therefore$ Required length of perpendicular $=\sqrt{(1-0)^2+\left(\frac{3}{2}-\frac{5}{2}\right)^2+(2-0)^2}$

$=\sqrt{1+1+4}=\sqrt{6}$ units

19. **Find the equations of the line passing through the point $(3,0,1)$ and parallel to the planes $x+2y=0$ *and* $3y-z=0$.**

Sol. Equation of the two planes are $x+2y=0$ *and* $3y-z=0$.

Let $\vec{n}_1$ and $\vec{n}_2$ are the normal to the two planes, respectively.

$\therefore \vec{n}_1=\hat{i}+2\hat{j}$ and $\vec{n}_2=3\hat{j}-\hat{k}$

Since, required line is parallel to the given two planes.

Therefore, $\vec{b} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix}$

$= \hat{i}(-2) - \hat{j}(-1) + \hat{k}(3)$

$= -2\hat{i} + \hat{j} + 3\hat{k}$

So, the equation of the lines through the point (3, 0, 1) and parallel to the given two planes are

$(x-3)\hat{i} + (y-0)\hat{j} + (z-1)\hat{k} + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$

$\Rightarrow (x-3)\hat{i} + y\hat{j} + (z-1)\hat{k} + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$

20. Find the equation of the plane through the points $(2,1,-1)$ *and* $(-1,3,4)$, and perpendicular to the plane $x-2y+4z=10$.

Sol. The equation of the plane passing through $(2,-1,1)$ is

$a(x-2)+b(y-1)+c(z+1)=0 \quad \ldots(i)$

Since, this passes through $(-1,3,4)$.

$\therefore a(-1-2)+b(3-1)+c(4+1)=0$

$\Rightarrow -3a+2b+5c=0 \quad \ldots(ii)$

Since, the plane (i) is perpendicular to the plane $x-2y+4z=10$.

$\therefore 1\cdot a - 2\cdot b + 4\cdot c = 0$

$\Rightarrow a-2b+4c=0 \quad \ldots(iii)$

On solving Eqs. (ii) and (iii), we get

$\frac{a}{8+10} = \frac{-b}{-17} = \frac{c}{4} = \lambda$

$\Rightarrow a = 18\lambda, b = 17\lambda, \lambda = 4\lambda$

From Eq. (i),

$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$

$\Rightarrow 18x-36+17y-17+4z+4=0$

$\Rightarrow 18x+17y+4z-49=0$

$\therefore 18x+17y+4z=49$

21. Find the shortest distance between the lines given by
$\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$ **and** $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

Sol. We have, $\vec{r} = (8+3\lambda)\hat{i} + (-9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$

$= 8\hat{i} - 9\hat{j} + 10\hat{k} + 3\lambda\hat{i} - 16\lambda\hat{j} + 7\lambda\hat{k}$

$= 8\hat{i} - 9\hat{j} + 10\hat{k} + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

$\Rightarrow \vec{a}_1 = 8\hat{i} - 9\hat{j} + 10\hat{k}$ and $\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k} \quad \ldots(i)$

Also, $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

$\Rightarrow \vec{a}_2 = 15\hat{i} + 29\hat{j} + 5\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k} \quad \ldots(ii)$

Now, shortest distance between two lines is given by $\left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_1-\vec{a}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$$\therefore \vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$=\hat{i}(80-56)-\hat{j}(-15-21)+\hat{k}(24+48)$

$=24\hat{i}+36\hat{j}+72\hat{k}$

Now, $|\vec{b}_1\times\vec{b}_2|=\sqrt{(24)^2+(36)^2+(72)^2}$

$=12\sqrt{2^2+3^2+6^2}=84$

And $(\vec{a}_2-\vec{a}_1)=(15-8)\hat{i}+(29+9)\hat{j}+(5-10)\hat{k}$

$=7\hat{i}+38\hat{j}-5k$

$\therefore$ Shortest distance $=\left|\frac{(24\hat{i}+36\hat{j}+72\hat{k})\cdot(7\hat{i}+38\hat{j}-5\hat{k})}{84}\right|$

$=\left|\frac{168+1368-360}{84}\right|=\left|\frac{1176}{84}\right|=14$ units

22. Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$.

Sol. The equation of a plane through the line intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$(x+2y+3z-4)+\lambda(2x+y-z+5)=0$

$\Rightarrow x(1+2\lambda)+y(2+\lambda)+z(-\lambda+3)-4+5\lambda=0$ ...(i)

Also, this is perpendicular to the plane $5x+3y+6z+8=0$.

$\therefore 5(1+2\lambda)+3(2+\lambda)+6(3-\lambda)=0$ $[\because a_1a_2+b_1b_2+c_1c_2=0]$

$\Rightarrow 5+10\lambda+6+3\lambda+18-6\lambda=0$

$\therefore \lambda=-29/7$

From Eq. (i),

$$x\left[1+2\left(\frac{-29}{7}\right)\right]+y\left(2-\frac{29}{7}\right)+z\left(\frac{29}{7}+3\right)-4+5\left(\frac{-29}{7}\right)=0$$

$\Rightarrow x(7-58)+y(14-29)+z(29+21)-28-145=0$

$\Rightarrow -51x-15y+50z-173=0$

So, the required equation of plane is $51x+15y-50z+173=0$.

23. The plane $ax+by=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation of the plane in its new position is $ax+by\pm(\sqrt{a^2+b^2}\tan\alpha)z=0$.

Sol. Equation of the plane is $ax+by=0$ ...(i)

$\therefore$ Equation of the plane after new position is

$$\frac{ax\cos\alpha}{\sqrt{a^2+b^2}}+\frac{by\cos\alpha}{\sqrt{b^2+a^2}}\pm z\sin\alpha=0$$

$$\Rightarrow \frac{ax}{\sqrt{a^2+b^2}}+\frac{by}{\sqrt{b^2+a^2}}\pm z\tan\alpha=0 \text{ [on dividing by } \cos\alpha]$$

$$\Rightarrow ax+by\pm z\tan\alpha\sqrt{\alpha^2+b^2}=0 \text{ [on multiplying with } \sqrt{a^2+b^2}]$$

Alternate Method

Given, planes are $ax+by=0$ ...(i)

And $z=0$ ...(ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as $ax+by+k=0$ ...(iii)

Then, direction cosines of a normal to the plane (iii) are $\frac{a}{\sqrt{a^2+b^2+k^2}},\frac{b}{\sqrt{a^2+b^2+k^2}}$, $\frac{c}{\sqrt{a^2+b^2+k^2}}$ and direction cosines of the normal to the plane (i) are $\frac{a}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}},0.$

Since, the angle between the planes (i) and (ii) is α,

$$\therefore \cos\alpha=\frac{a\cdot a+b\cdot b+k\cdot 0}{\sqrt{a^2+b^2+k^2\sqrt{a^2+b^2}}}$$

$$=\sqrt{\frac{a^2+b^2}{a^2+b^2+k^2}}$$

$$\Rightarrow k^2\cos^2\alpha=a^2(1-\cos^2\alpha)+b^2(1-\cos^2\alpha)$$

$$\Rightarrow k^2=\frac{(a^2+b^2)\sin^2\alpha}{\cos^2\alpha}$$

$$k=\pm\sqrt{a^2+b^2}\tan\alpha$$

On putting this value in plane (iii), we get the equation of the plane as

$$ax+by+z\sqrt{a^2+b^2}\tan\alpha=0$$

24. Find the equation of the plane through the intersection of the planes $\vec{r}.(\hat{i}+3\hat{j})-6=0$ and $\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0$, whose perpendicular distance from origin is unity.

Sol. We have, $\vec{n}_1.(\hat{i}+3\hat{j}), d_1=6$ and $\vec{n}_2.(3\hat{i}-\hat{j}-4\hat{k}), d_2=0$

Using the relation, $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+d_2\lambda$

$$\Rightarrow \vec{r}\cdot[(\hat{i}+3\hat{j})+\lambda(3\hat{i}-\hat{j}-4\hat{k})]=6+0\cdot\lambda$$

$$\Rightarrow \vec{r}\cdot[(1+3\lambda)i+(3-\lambda)j+k(-4\lambda)]=6 \quad ...(i)$$

On dividing both sides by $\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}$, we get

$$\frac{\vec{r}\cdot[(1+3\lambda)\hat{i}+(3-\lambda)\hat{j}+\hat{k}(-4\lambda)]}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}=\frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}$$

Since, the perpendicular distance from origin is unity.

$$\therefore \frac{6}{\sqrt{(1-3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}=1$$

$$\Rightarrow (1-3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2=36$$

$$\Rightarrow 1+9\lambda^2+6\lambda+9+\lambda^2-16\lambda+16\lambda^2=36$$

$$\Rightarrow 26\lambda^2+10=36$$

$$\Rightarrow \lambda^2=1$$

$$\therefore \lambda=\pm 1$$

Using Eq. (i), the required equation of plane is

$$\vec{r}\cdot[(1\pm 3)\hat{i}+(3\mp 1)\hat{j}+(\mp 4)\hat{k}]=6$$

$$\Rightarrow \vec{r}\cdot[(1+3)\hat{i}+(3-1)\hat{j}+(-4)\hat{k}]=6$$

And $\vec{r}\cdot[(1-3)\hat{i}+(3+1)\hat{j}+4\hat{k}]=6$

$$\Rightarrow \vec{r}\cdot(4\hat{i}+2\hat{j}-4\hat{k})=6$$

And $\vec{r}\cdot(-2\hat{i}+4\hat{j}+4\hat{k})=6$

$$\Rightarrow 4x+2y-4z-6=0$$

and $-2x+4y+4z-6=0$

25. Show that the points $(\hat{i}-\hat{j}+3\hat{k})$ and $3(\hat{i}+\hat{j}+\hat{k})$ are equidistance from the plane $\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0$ and lies on the opposite side of it.

Sol. To show that these given points $(\hat{i}-\hat{j}+3\hat{k})$ and $3(\hat{i}+\hat{j}+\hat{k})$ are equidistant from the plane

$\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0$, we first find out the mid-point of the points which is $2\hat{i}+\hat{j}+3\hat{k}$.

On substituting $\vec{r}$ by the mid-point in plane, we get

$$\text{LHS}=(2i+j+3k)\cdot(5\hat{i}+2\hat{j}-7\hat{k})+9$$
$$=10+2-21+9=0$$
$$=\text{RHS}$$

Hence, the two points lie on opposite sides of the plane are equidistant from the plane.

26. $\overrightarrow{AB}=3\hat{i}-\hat{j}+\hat{k}$ and $\overrightarrow{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i}+7\hat{j}+4\hat{k}$ and $-9\hat{j}+2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

Sol. We have, $\overrightarrow{AB}=3\hat{i}-\hat{j}+\hat{k}$ and $\overrightarrow{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$

Also, the position vectors of A and C are $6\hat{i}+7\hat{j}+4\hat{k}$ and $-9\hat{j}+2\hat{k}$, respectively. Since, $\overrightarrow{PQ}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

So, P and Q will be foot of perpendicular to both the lines through A and C.

Now, equation of the line through A and parallel to the vector $\overrightarrow{AB}$ is,

$\vec{r}=(6\hat{i}+7\hat{j}+4\hat{k})+\lambda(3\hat{i}-\hat{j}+\hat{k})$

And the line through C and parallel to the vector $\overrightarrow{CD}$ is given by

$\vec{r}=-9\hat{j}+2\hat{k}+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$ ...(i)

Let $\vec{r}=(6\hat{i}+7\hat{j}+4\hat{k})+\lambda(3\hat{i}-\hat{j}+\hat{k})$

and $\vec{r}=-9\hat{j}+2\hat{k}+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$ ...(ii)

Let $P(6+3\lambda,7-\lambda,4+\lambda)$ is any point on the first line and Q be any point on second line is given by $(-3\mu,-9+2\mu,2+4\mu)$.

$\therefore \overrightarrow{PQ}=(-3\mu-6-3\lambda)\hat{i}+(-9+2\mu-7+\lambda)\hat{j}+(2+4\mu-4-\lambda)\hat{k}$

$=(-3\mu-6-3\lambda)\hat{i}+(2\mu+\lambda-16)\hat{j}+(4\mu-\lambda-2)\hat{k}$

If $\overrightarrow{PQ}$ is perpendicular to the first line, then

$3(-3\mu-6-3\lambda)-(2\mu+\lambda-16)+(4\mu-\lambda-2)=0$

$\Rightarrow -9\mu-18-9\lambda-2\mu-\lambda+16+4\mu-\lambda-2=0$

$\Rightarrow -7\mu-11\lambda-4=0$ ...(iii)

If $\overrightarrow{PQ}$ is perpendicular to the second line, then

$-3(-3\mu-6-3\lambda)+(2\mu+\lambda-16)+(4\mu-\lambda+2)=0$

$\Rightarrow 9\mu+18+9\lambda+4\mu+2\lambda-32+16\mu-4\lambda-8=0$

$\Rightarrow 29\mu+7\lambda-22=0$ ...(iv)

On solving Eqs. (iii) and (iv), we get

$-49\mu-77\lambda-28=0$

$\Rightarrow 319\mu+77\lambda-242=0$

$\Rightarrow 270\mu-270=0$

$\Rightarrow \mu=1$

Using μ in Eq. (iii), we get

$-7(1)-11\lambda-4=0$

$\Rightarrow -7-11\lambda-4=0$

$\Rightarrow -11-11\lambda=0$

$\Rightarrow \lambda=-1$

$\therefore \overrightarrow{PQ}=[-3(1)-6-3(-1)]\hat{i}+[2(1)+(-1)-16]\hat{j}+[4(1)-(-1)-2]\hat{k}$

$=-6\hat{i}-15\hat{j}+3\hat{k}$

27. Show that the straight lines whose direction cosines are given by $2l+2m-n=0$ and $mn+nl+lm=0$ are at right angles.

Sol. We have, $2l+2m-n=0$ ...(i)

And $mn+nl+lm=0$ ...(ii)

Eliminating m from the both equations, we get

$m=\frac{n-2l}{2}$ [from Eq. (i)]

$\Rightarrow \left(\frac{n-2l}{2}\right)n+nl+l\left(\frac{n-2l}{2}\right)=0$

$\Rightarrow \frac{n^2-2nl+2nl+nl-2l^2}{2}=0$

$\Rightarrow n^2+nl-2l^2=0$

$\Rightarrow n^2+2nl-nl-2l^2=0$

$\Rightarrow (n+2l)(n-l)=0$

$\Rightarrow n=-2l$ and n = l

$\therefore\ m=\frac{-2l-2l}{2}, m=\frac{l-2l}{2}$

$\Rightarrow m=-2l, m=\frac{-l}{2}$

Thus, the direction ratios of two lines are proportional to $l,-2l,-2$ and $l,\frac{-l}{2},l$.

$\Rightarrow 1,-2,-2$ and $1,\frac{-1}{2},1$

$\Rightarrow 1,-2,-2$ and $2,-1,2$

Also, the vectors parallel to these lines are $\vec{a}=\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+2\hat{k}$ respectively.

$\therefore\ \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(\hat{i}-2\hat{j}-2\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k})}{3\cdot 3}$

$=\frac{2+2-4}{9}=0$

$\therefore\ \theta=\frac{\pi}{2}\ \left[\because \cos\frac{\pi}{2}=0\right]$

28. **If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1+l_2+l_3, m_1+m_2+m_3, n_1+n_2+n_3$ makes equal angles with them.**

Sol. Let

$\vec{a}=l_1\hat{i}+m_1\hat{j}+n_1\hat{k}$

$\vec{b}=l_2\hat{i}+m_2\hat{j}+n_2\hat{k}$

$\vec{c}=l_3\hat{i}+m_3\hat{j}+n_3\hat{k}$

$\vec{d}=(l_1+l_2+l_3)\hat{i}+(m_1+m_2+m_3)\hat{j}+(n_1+n_2+n_3)\hat{k}$

Also, let α,β and γ are the angles between $\vec{a}$ and $\vec{d}$, $\vec{b}$ and $\vec{d}$, $\vec{c}$ and $\vec{d}$.

$\therefore \cos\alpha = l_1(l_1+l_2+l_3)+m_1(m_1+m_2+m_3)+n_1(n_1+n_2+n_3)$

$= l_1^2 + l_1l_2 + l_1l_3 + m_1^2 + m_1m_2 + m_1m_3 + n_1^2 + n_1n_2 + n_1n_3$

$= (l_1^2 + m_1^2 + n_1^2) + (l_1l_2 + l_1l_3 + m_1m_2 + m_1m_3 + n_1n_2 + n_1n_3)$

$= 1 + 0 = 1$

$[\because l_1^2 + m_1^2 + n_1^2 = 1 \text{ and } l_1 \perp l_2, l_1 \perp l_3, m_1 \perp m_2, m_1 \perp m_3, n_1 \perp n_2, n_1 \perp n_3]$

Similarly, $\cos\beta = l_2(l_1+l_2+l_3)+m_2(m_1+m_2+m_3)+n_2(n_1+n_2+n_3)$

$= 1+0 \ \ and \ \cos\gamma = 1+0$

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow \ \alpha = \beta = \gamma$

So, the line whose direction cosines are proportional to $l_1+l_2+l_2, m_1+m_2+m_3, n_1+n_2+n_3$ make equal to angles with the three mutually perpendicular lines whose direction cosines are $l_1, m_1, n_1, l_2, m_2, n_2$ and l_3, m_3, n_3 respectively.