Ch-6 Sampling and Sampling Theorem

* Sampling Process

- Sampling is a process of converting continuous time signal into discrete time signal under certain conditions and continuous time signal can be completely recovered by its sample sequence.

* Sampling Theorem

- According to sampling theorem, a band limited signal of finite energy signal can be completely reconstructed from its samples taken uniformly at rate $\underline{\omega_s \geq 2\omega_m}$
  ↓
  samples/rad/sec

$\underline{f_s \geq 2 f_m}$
↓
samples/sec.

$$x(t) \xleftarrow[\text{Sampling}]{t = nT_1} \longrightarrow x(n)$$

$f_s \geq 2 f_m$

f_s ↓ sampling frequency

f_m ↓ message frequency

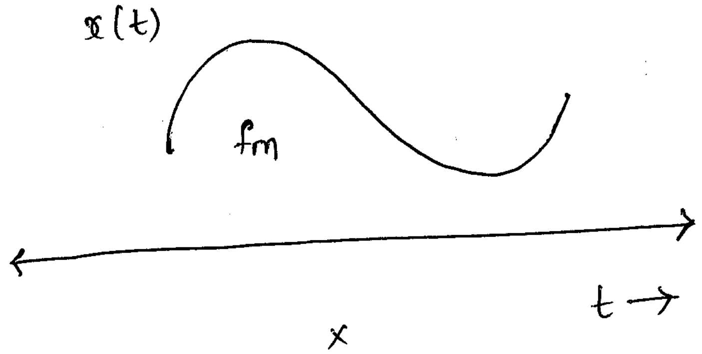
$$x(t) * \delta(t - t_0) = x(t - t_0)$$

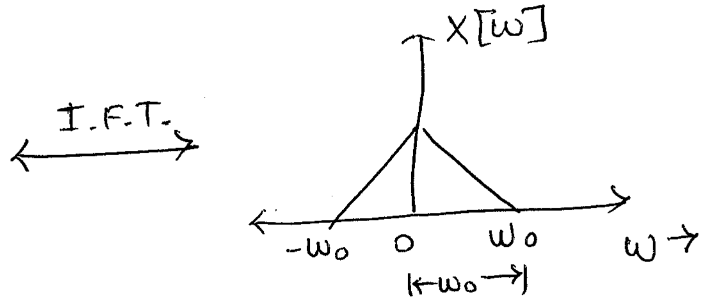
$$x_s(nT_s) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(nT_s) = \sum_{n=-\infty}^{\infty} x(nT_1) \cdot \delta(t - nT_s)$$

$$X_s(\omega) = \frac{1}{2\pi}\left[X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$$X_s[\omega] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X[\omega - n\omega_s] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X[f - f_s n]$$





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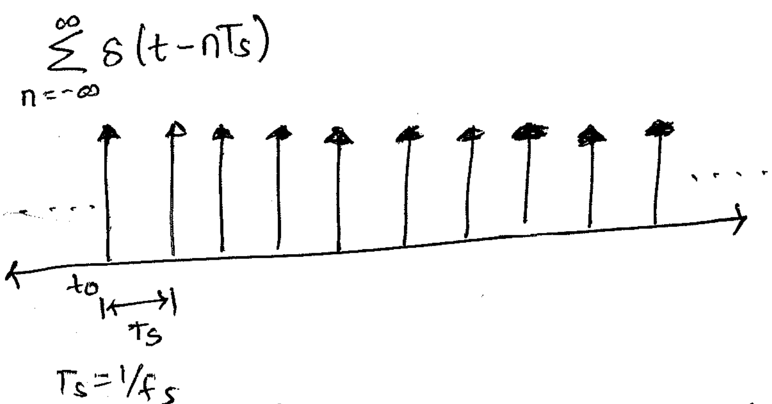
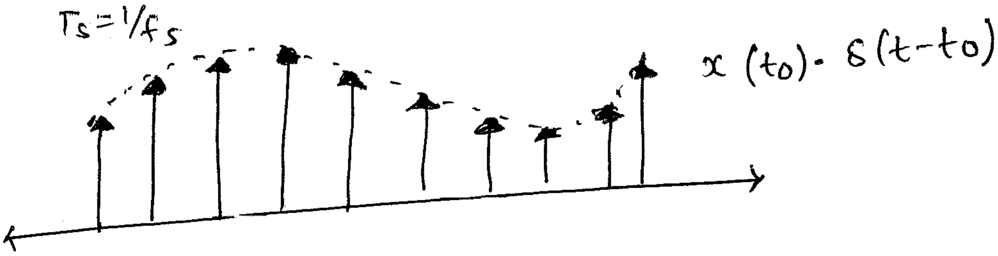




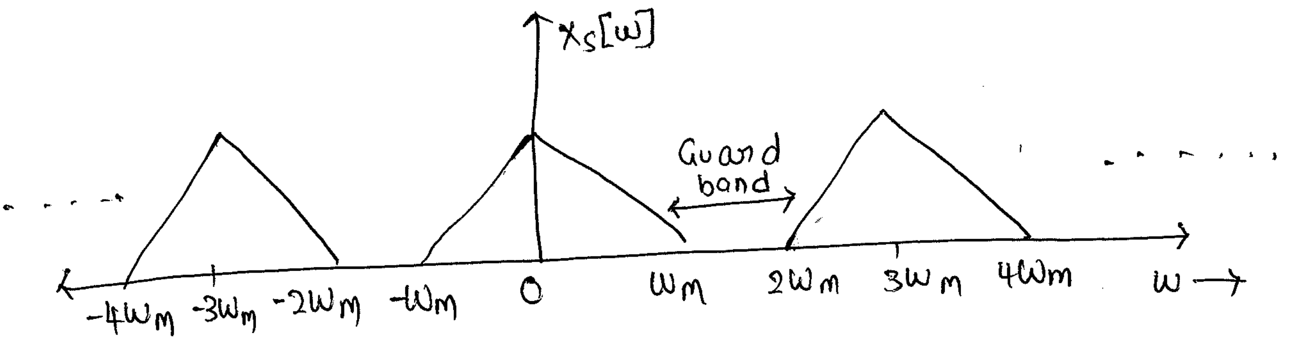
Now,

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{F.T.} \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

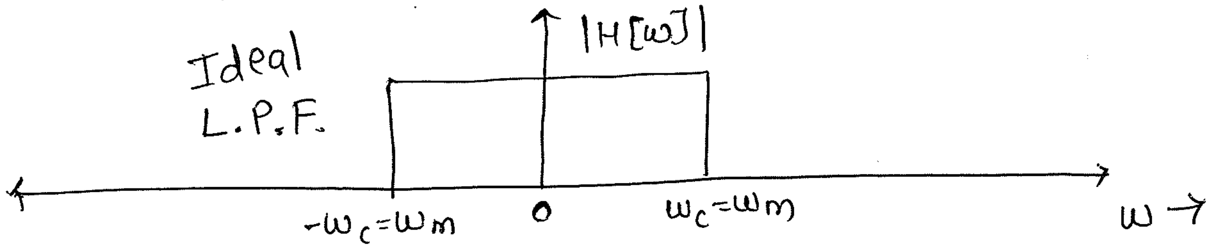
Case: 1

(1) $\omega_s > 2\omega_m$ (over sampling)

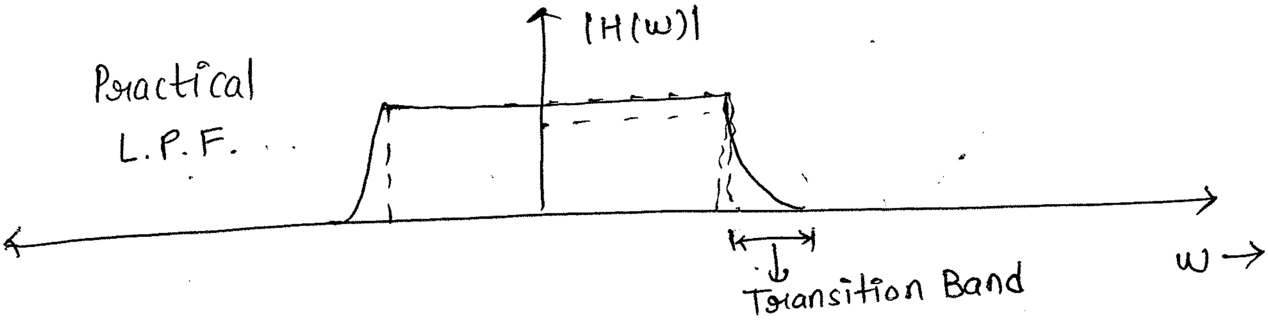
$\omega_s = 3\omega_m$; $X_s[\omega] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X[\omega - 3\omega_m n]$



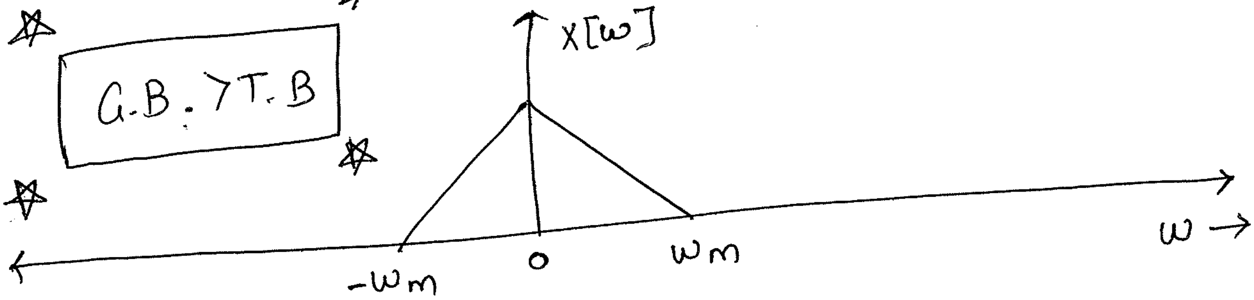
Ideal L.P.F.



Practical L.P.F.

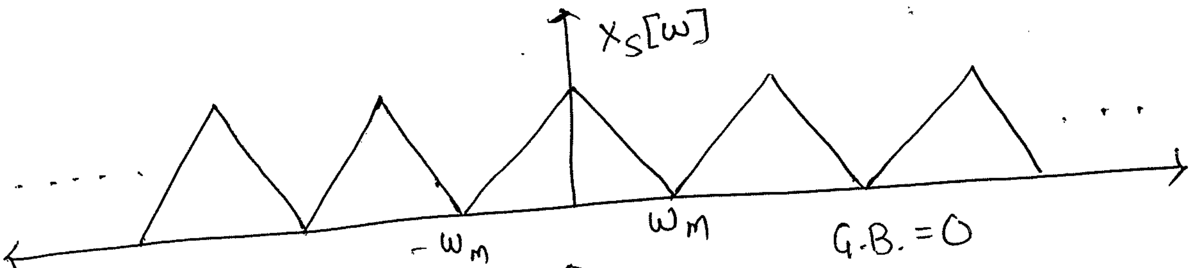


$\boxed{G.B. > T.B}$

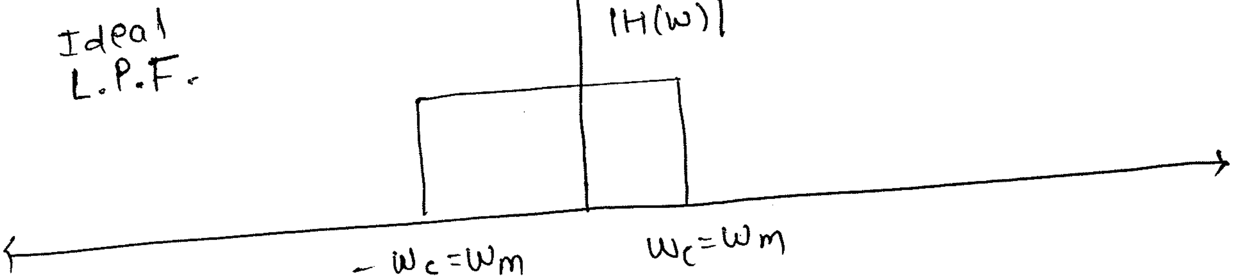


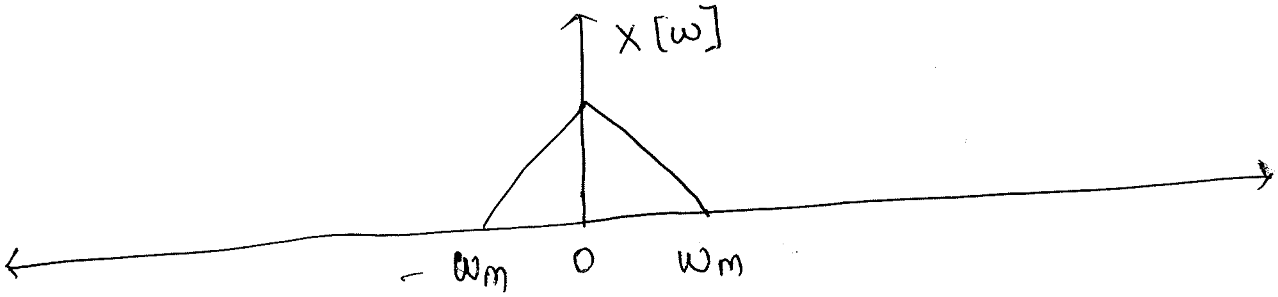
Case: 2 $\omega_s = 2\omega_m$

minimum S.R. (sampling rate) → Nyquist rate



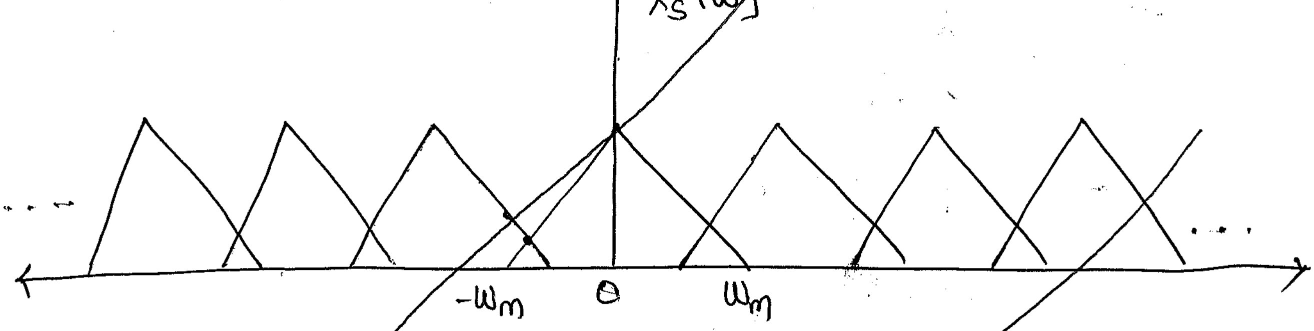
Ideal L.P.F.



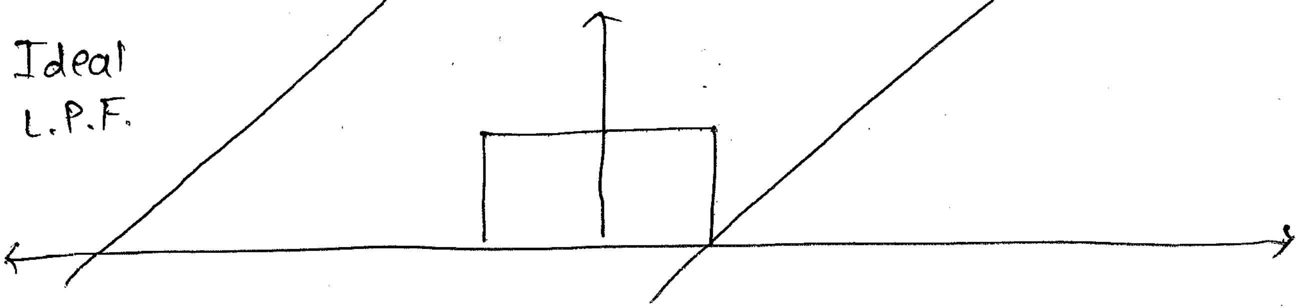


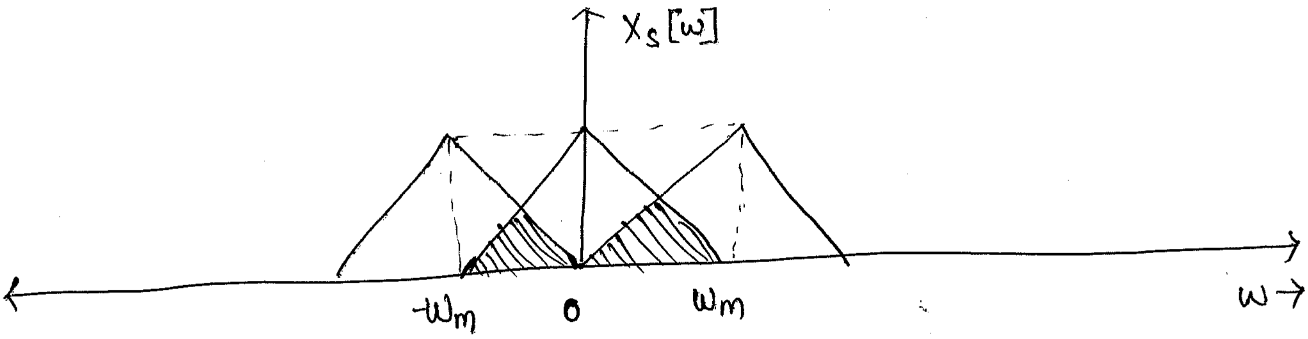
Case: 3 $\omega_s < 2\omega_m$

Let $\omega_s = \omega_m$



Ideal L.P.F.



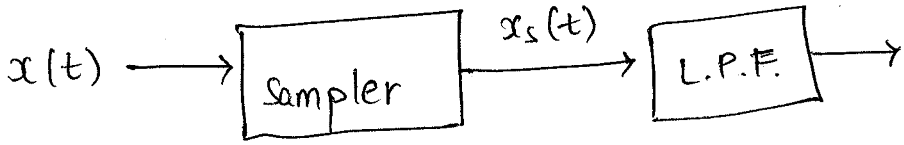


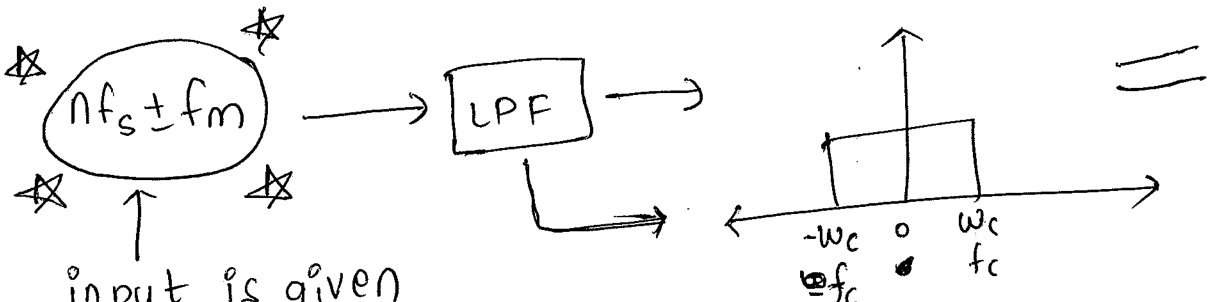
sample fold

Aliasing effect

- Signal cannot be reconstructed again

Reconstruction :-





= Required spectrum

input is given

[so that each freq. component is recovered]

n	$nf_s \pm f_m$
0	$\pm f_m$
1	$f_s \pm f_m$
-1	$-f_c \pm f_m$

NOTE:-

$x(t)$ $\quad x(t) = x_1(t) + x_2(t) + \dots$

f_m $\quad \downarrow f_{m_1} \quad \downarrow f_{m_2} \quad \downarrow f_{m_3} \dots$

f_s $\boxed{f_s = 2f_m}$

$f_s = 2\max(f_{m_1}, f_{m_2}, f_{m_3}, \dots)$

Q:- Find Nyquist rate

(1) $x_1(t) = \cos(2\pi \times 10^3 t) + \cos(6\pi \times 10^3 t)$

$f_{max.} = 3\,kHz$

$f_s = 2f_{max.} = 6\,kHz.$

(2) $x_2(t) = \cos(2\pi \times 10^3 t)$

$(\cos 2\pi f t)$

$f_m = 10^3$

$\boxed{f_s = 2\,kHz}$

(3) $x_3(t) = \left[\cos(2\pi \times 10^3 t) + \cos(6\pi \times 10^3 t)\right]^2$

$\downarrow$

$\frac{1 + \cos 12\pi \times 10^3 t}{2}$

$f_{max.} = 6\,kHz$

$\boxed{f_s = 12\,kHz}$

$\boxed{\text{NOTE:}}$

	N.R.
$x(t)$	ω_0
$x'(t)$	$2\omega_0$
$\vdots$	$\vdots$
$x^n(t)$	$n\omega_0$

Q:- A band limited signal with max^m frequency of 5kHz is to be sampled according to the sampling theorem. The sampling frequency which is not valid is?

(a) 5kHz ✓

(b) 12kHz

(c) 15kHz

(d) 20kHz

Sol^n:- $f_m = 5kHz > 10kHz$

Q:- Specify N.R. for each of following signal in rad/sec and Hz.

(1) $x(t) = \text{sinc}(300t)$

$$x(t) = \frac{\sin \pi 300t}{\pi \cdot 300t}$$

$\omega_m = 300\pi$

$\omega_s = 2\omega_m = 600\pi$ rad/sec.

$f_s = 300Hz$

(2) $x(t) = \text{sinc}^2(300t)$

$\omega_m = 600\pi$

$\omega_s = 2\omega_m = 1200\pi$ rad/sec.

$f_s = 600Hz$

(3) $x(t) = \text{sinc}(300t) + \text{sinc}^2(300t)$ ✓

$\omega_m = 600\pi$

$\omega_s = 2\omega_m = 1200\pi$ rad/sec.

$f_s = 600Hz$

$\boxed{\text{NOTE:-}}$

signal $x(t)$	Nyquist rate ω_0
1. $x^2(t)$	$2\omega_0$
2. $x(t \pm t_0)$	ω_0
3. $x(t/2)$	$\omega_0/2$
4. $x(2t)$	$2\omega_0$
5. $x(\alpha t)$	$\lvert\alpha\rvert\omega_0$
6. $x(t) * x(t)$	$2\omega_0$

$\frac{d}{dt}x(t)$	ω_0
$\int_{-\infty}^{t} x(t)\,dt$	ω_0

NOTE:-

$f_s \geq 2f_m$ } exact sampling
$\omega_s \geq 2\omega_m$
$f_s = 2f_m$ ← N.R.

Q:- Let $x(t)$ be continuous signal band limited to 2f Hz. The Nyquist sampling rate in Hz for

$y(t) = x(0.5t) + x(t) - x(2t)$

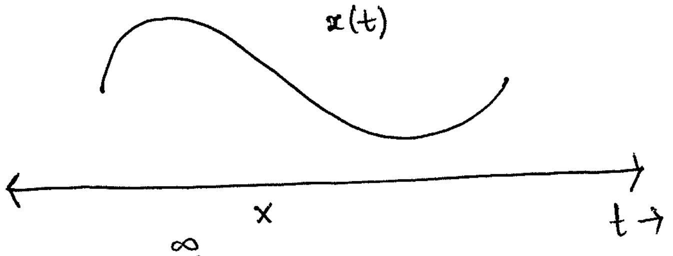
(a) F (b) 2F (c) 4F ✓ (d) 8F

$$y(t) = \underset{\substack{\downarrow \\ F}}{x(0.5t)} + \underset{\substack{\downarrow \\ 2F}}{x(t)} - \underset{\substack{\downarrow \\ 4F \checkmark (\text{max.})}}{x(2t)}$$

(c) 4F

* Sampling Techniques

(1) Ideal sampling



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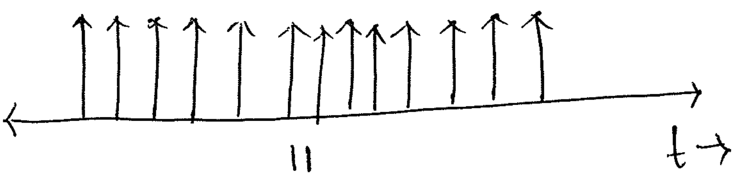
$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



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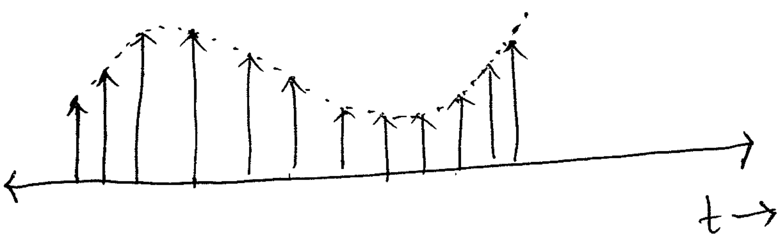
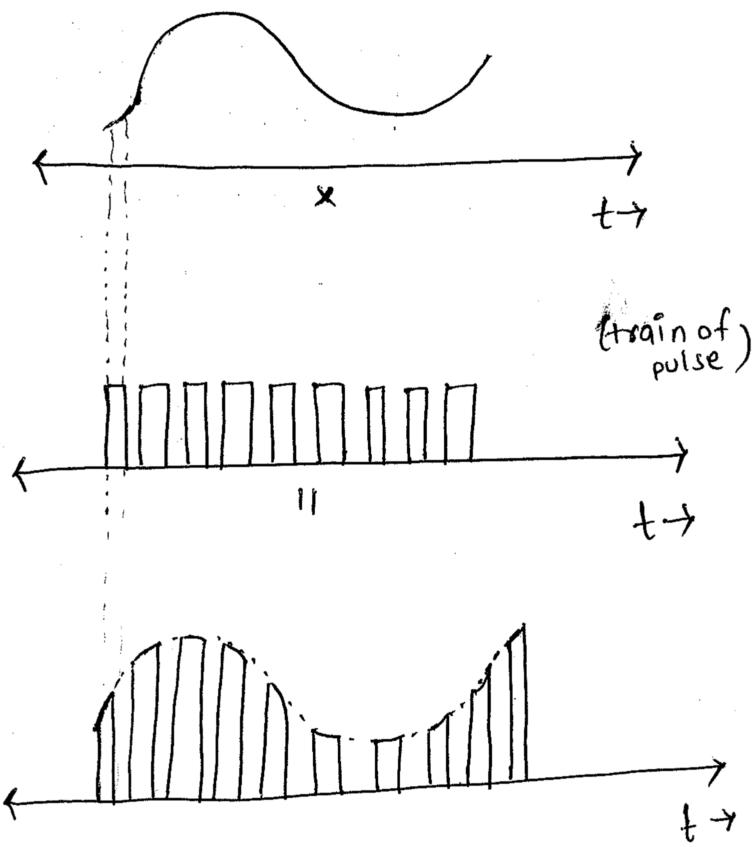
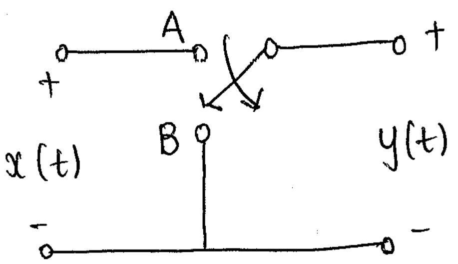


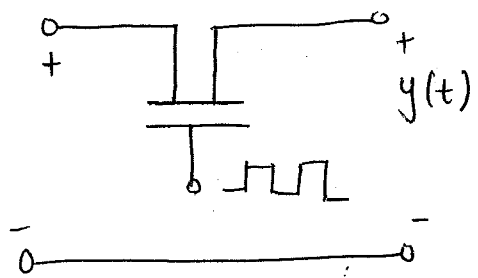
(2) Natural Sampling



I] Ideal Sampling

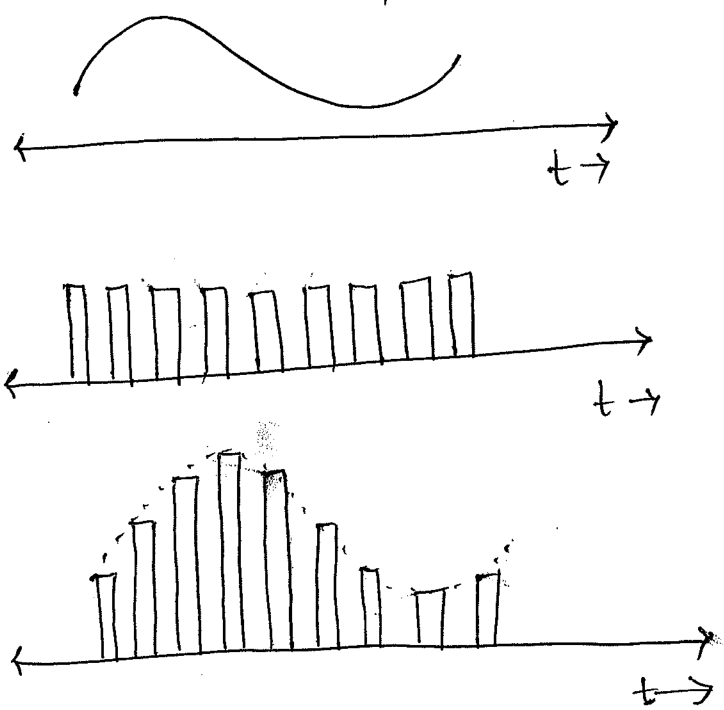


2]

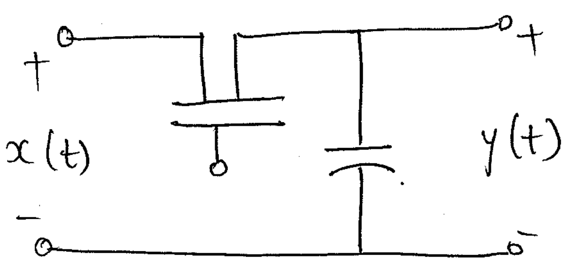


Natural Sampling

(3) Flat-top sampling



3] Flat-top sampling (sample & hold)



Sampling Theorem for Band pass ~~filter~~ signal

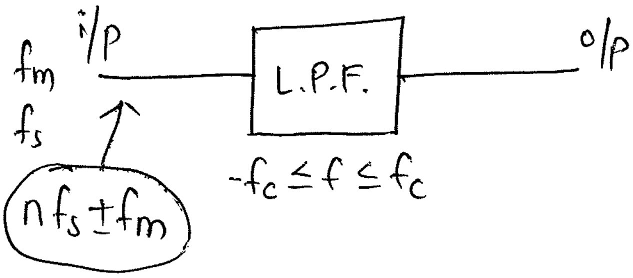
Band pass signal $x(t)$ with bandwidth B and upper cut-off frequency f_H can be sampled version $x_s(t)$ by bandpass filtering if $f_s = \frac{2f_H}{k}$, $k = \frac{f_H}{B}$ where k is integer part of $\frac{f_H}{B}$.

Q:- A band pass ~~filter~~ signal extends 4 – 6 kHz. What is the smallest sampling frequency required to retain all the information in signal.

<u>Soln:-</u> $K = \frac{f_H}{B} = \frac{6}{2} = 3$ kHz

$$f_s = \frac{2f_H}{K} = \frac{2 \cdot 6}{3} = \underline{\underline{4\,\text{kHz}}}$$

<u>Concept:-</u>



$n = 0, \pm 1, \pm 2, \ldots$

Q:- A signal represented by $x(t) = 5\cos 400\pi t$ if sampled at a rate 300 samples/sec. The resulting samples are passed through an ideal low pass filter of cut-off frequency 150Hz. Which of the following will be contained in the output of L.P.F.

(a) 100Hz

(b) 100Hz, 150Hz

(c) 50Hz, 100Hz

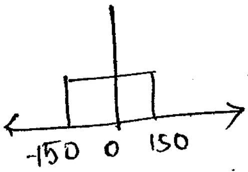
(d) 50Hz, 100Hz

$f_m = 200\,Hz$

$f_s = 300\,Hz$

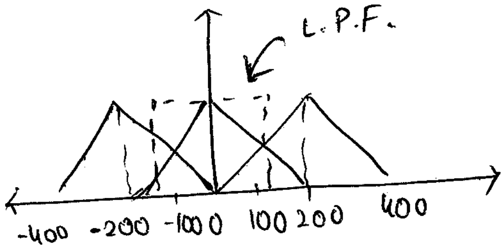
$f_c = 150\,Hz$

$\xrightarrow{} \boxed{L.P.F.} \xrightarrow{}$



$nf_s \pm f_m$

n	I/p
0	± 200
1	500, 100 ✓
-1	-500, -100
2	400, 800
-2	-400, -800



(a) ✓

Q:- A signal $x(t) = \cos 24\pi \times 10^3 t$ and sampling (interval) rate is $T_s = 50\mu sec$. (i) If sampled signal is passed through L.P.F. having cut-off frequency of 15 kHz Determine the frequency present at o/p of L.P.F.

(ii) If sampled signal is passed through B.P.F. having cut-off frequency of 50kHz and 75kHz. Determine frequency present at o/p of B.P.F.

$f_s = \frac{1}{T_s} = \frac{1}{50 \times 10^{-6}} = 0.2 \times 10^5 = 20\,kHz$

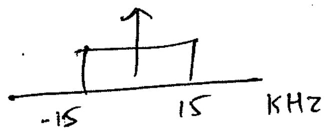
$f_m = 12\,kHz$

$nf_s \pm f_m = i/p$

$20n \pm 12$

n	i/p
0	± 12 kHz
1	32k, 8k
2	52kHz, 28kHz
3	72kHz, 48kHz
4	92kHz, 68kHz

(i) ± 8 KHz, ± 12 KHz



(ii) ± 52 KHz, ± 72 KHz, ± 68 KHz

NOTE:- $\boxed{GB > TB}$

$f_s = 2f_m + G.B.$

$f_s = 2f_m + T.B.$

Q:- A signal $x(t) = \cos 24\pi \times 10^3 t$ is sampled using appropriate frequency and sampled signal is passed through low pass filter having transition bandwidth of 4KHz. If o/p of lowpass filter must be same as the original continuous time signal. Then what is the minimum value of sampling frequency to be used.

Sol:- $f_m = 12$ KHz

T.B. $= 4$ KHz

$f_s = 2f_m + T.B.$

$= 2(12k) + 4k$

$f_s = 24k + 4k = 28$ KHz