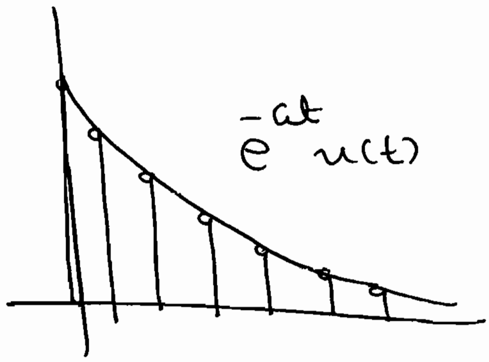
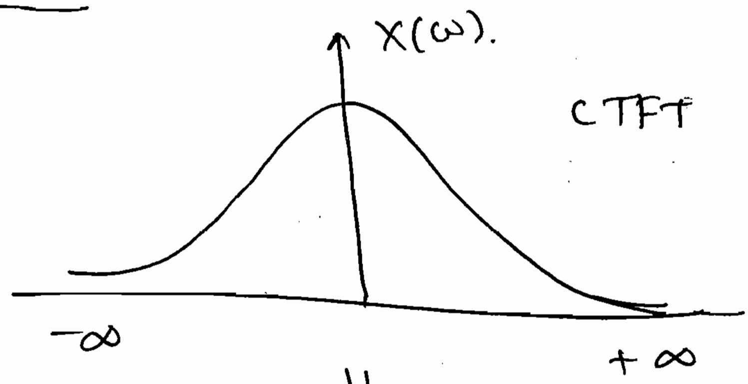
Ch- _ - DTFT

- The DTFT describes the spectrum of discrete signals & formalizes that discrete signals have periodic spectra.

→ The freq. range for a discrete signal is unique over $(-\pi, +\pi)$ (or) $(0, 2\pi)$.



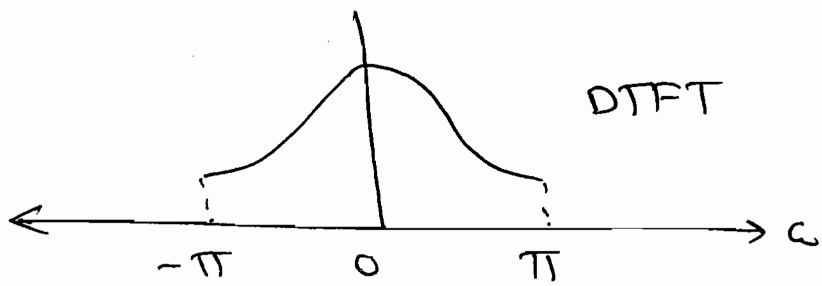
$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}.$$



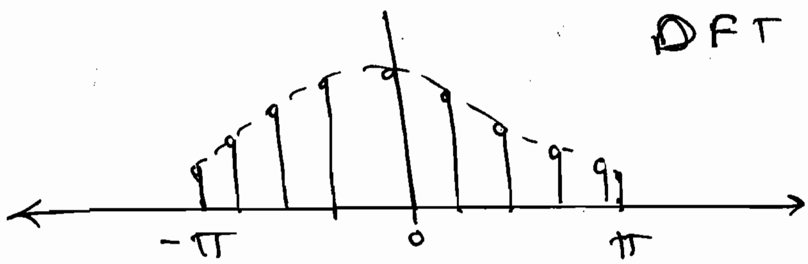
$\Downarrow$

$x[n] \longleftrightarrow X(e^{j\omega})$

(or) $X(\omega)$.



$\Downarrow$



⇒ CT FT	D.T. F.T.
ω: $-\infty$ to $+\infty$	ω : $-\pi$ to $+\pi$ (or) 0 to 2π
non-periodic	periodic

⇒ D.T.F.T

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}.}$$

⇒ I.D.T.F.T.

$$\boxed{x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j\omega n} \, d\omega.}$$

⇒

$$\boxed{\text{F.S.} \xleftrightarrow{\text{Dual}} \text{D.T.F.T}}$$

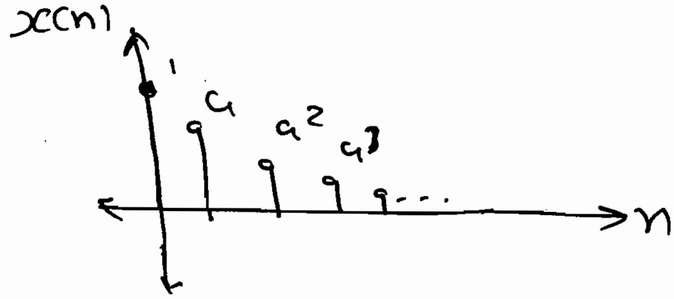
* Convergence of DTFT.

⇒ A sufficient condition for existance of DTFT is $\boxed{\sum_{n=-\infty}^{+\infty} |x(n)| < \infty.}$

- Some sequences are not absolutely summable, but they are square summable.
- There are signals that are neither absolutely summable nor have finite energy, but still have DTFT.

e.g. $x[n] = a^n \cdot u[n]$; $|a|<1$.

$\Rightarrow$



$$\therefore \quad X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=+\infty}^{+\infty} a^n \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} (a \cdot e^{-j\omega})^n$$

$$= 1 + ae^{-j\omega} + (ae^{-j\omega})^2 + \dots$$

$$\therefore \boxed{X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}}.$$

$$\Rightarrow \quad X_1(e^{j\omega}) = \frac{1}{1-a(\cos\omega - j\sin\omega)}.$$

$$X_1(e^{j\omega}) = \frac{1}{(1-a\cos\omega) + ja\sin\omega}.$$

$$|X(\omega)| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + a^2\sin^2\omega}}.$$

$$= \frac{1}{\sqrt{1-2a\cos\omega + a^2}}$$

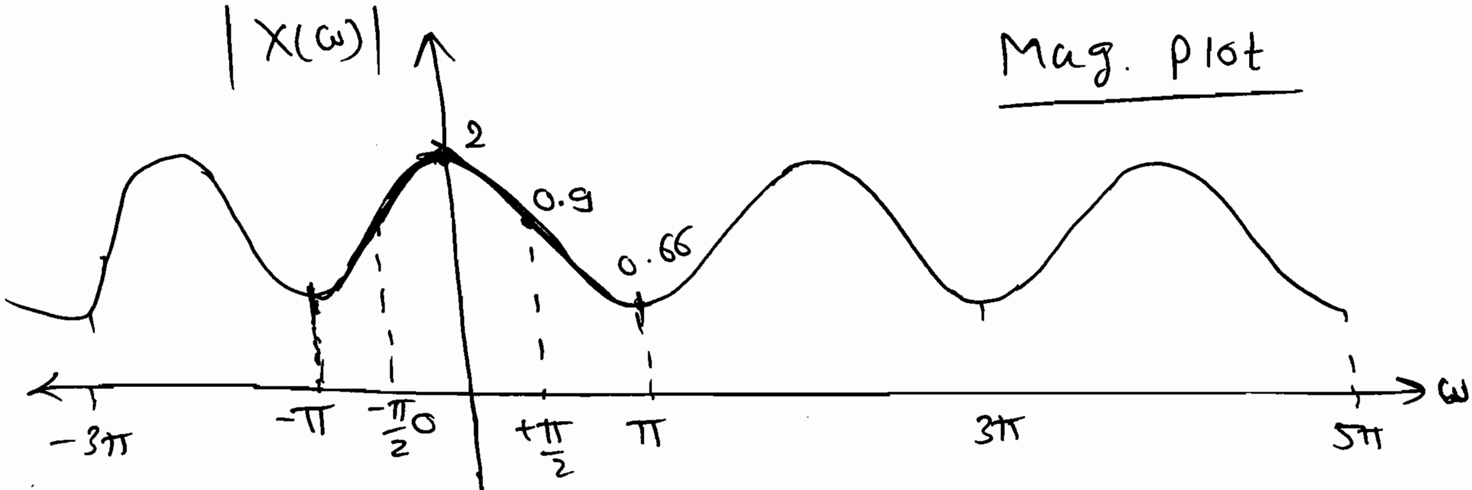
let, $a = \frac{1}{2}$.

$$\therefore |X(\omega)| = \frac{1}{\sqrt{1.25 - \cos\omega}}$$

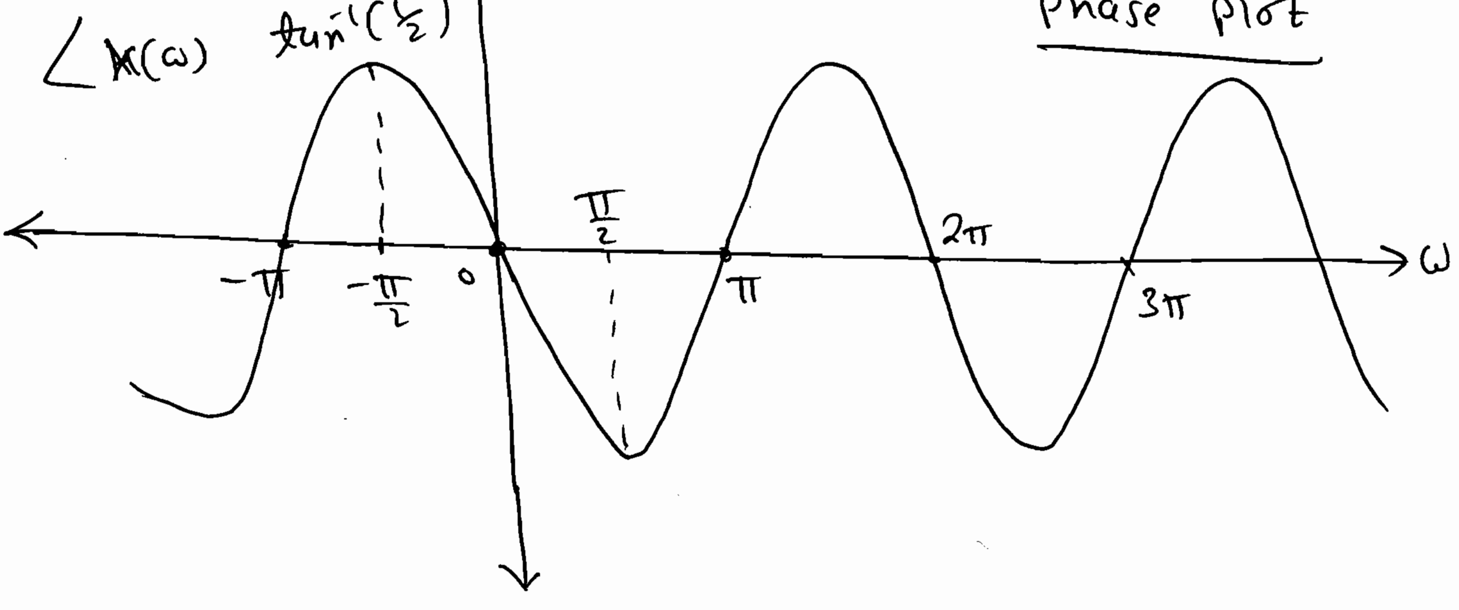
$$\angle X(\omega) = -\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right].$$

$$\angle X(\omega) = -\tan^{-1}\left[\frac{\frac{1}{2}\sin\omega}{1 - \frac{1}{2}\cos\omega}\right].$$

Mag. plot



Phase plot



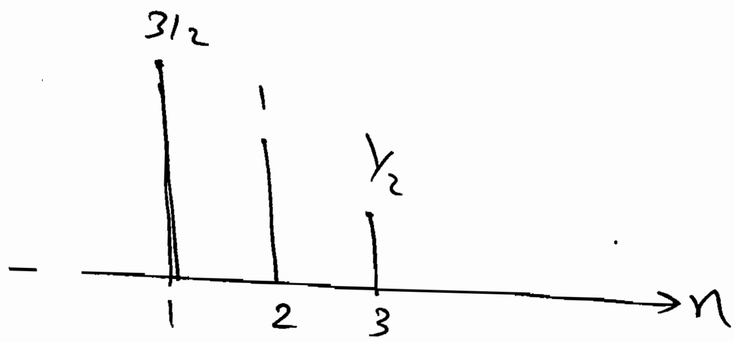
* Periodicity Property:

$$\Rightarrow \boxed{X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right).}$$

...er or not the n F.T. of the signal

frequency response ?

$x_1(n)$



$$\frac{\sin\left(\omega\left(\overset{N_1}{3}+\frac{1}{2}\right)\right)}{\sin(\omega/2)}$$

$$\sum_{-\infty}^{\infty} x[n]\cdot e^{-j\omega n}$$

...causal.

$$\sum x(n)\cdot e^{-j\omega n}$$

$$+\, e^{j\omega 3}\Big] + 1\left[e^{-j\omega\cdot 2} + e^{j\omega 2}\right]$$

then

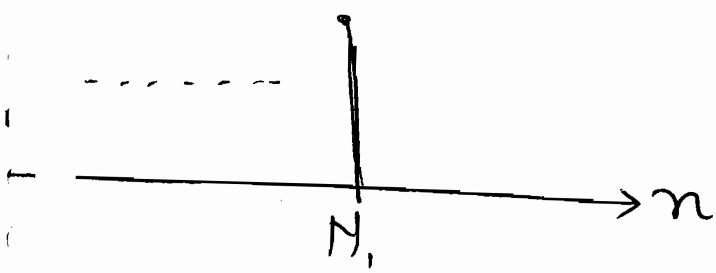
$$\frac{3}{2}\left[e^{-j\omega} + e^{j\omega}\right] + 2.$$

$\boxed{(e^{j\omega})}$

$$\boxed{+\, 2\cos 2\omega + 3\cos\omega + 2}$$

then

$\boxed{e^{j(\omega-\omega_0)}}$



$\ldots e^{-j\omega}$.

Sol^n:

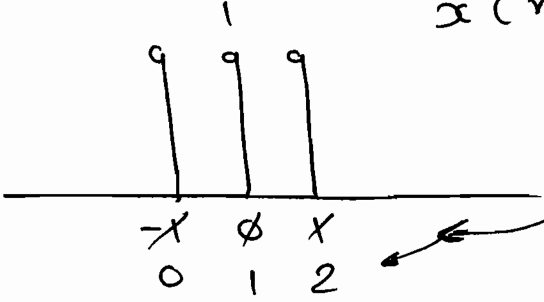
$$H(\omega) = \frac{\sin\left(\omega\left(\overset{N_1=1}{1}+\frac{1}{2}\right)\right)}{\sin(\omega/2)} \cdot \cancel{e^{+j\omega(-1)}} \cdot e^{-j\omega(\underset{N_0=1}{1})}$$

$\Big\downarrow$ I.D.T.F.T

$N_0 = 1$

$x(n-1)$.

$\therefore$



So, given system is Causal.

(c) $H(\omega) = e^{-j3\omega} \cdot 1 + e^{+j2\omega} \cdot 1$.

Sol^n: $\delta(n) \xleftrightarrow{\text{D.T.F.T}} 1$.

$\therefore \quad h(n) = \delta(n-3) + \delta(n+2)$.

$\Downarrow$

given system is Non-Causal.

$\boxed{\text{P 6.1.2.}}$ (a) Let $x(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$, $y(n) = x^2(n)$.

& $Y(e^{j\omega})$ be the F.T. of $y(n)$. Then

$Y(e^{j0})$ is ______

Sol^n: $$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x^2(n) \cdot e^{-j\omega n}.$$

$\Rightarrow \quad x(n) = \left(\frac{1}{2}\right)^n . u(n).$

$x^2(n) = \left(\frac{1}{2}\right)^{2n} . u(n) = \left(\frac{1}{4}\right)^n . u(n).$

$\therefore \; Y(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n . e^{-j\omega n}.$

$\therefore \; Y(e^{j0}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n . e^{-j\omega(0)}.$

$\therefore \; Y(e^{j0}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n.$

$\Rightarrow \; Y(e^{j0}) = \frac{1}{1-1/4} = \frac{4}{3}.$

$\therefore \; \boxed{Y(e^{j0}) = \frac{4}{3}.}$

(b) Given $X(e^{j\omega}) = \cos^3(3\omega)$, the find the sum $S = \sum_{n=-\infty}^{+\infty} (-1)^n . x(n).$

Soln:

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) . e^{-j\omega n}.$

let, $\omega = \pi$

$\therefore \; X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x(n) . e^{-j\pi n}$

$\therefore \; X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x(n) . (-1)^n = S.$

$$\therefore \quad S = \sum_{n=-\infty}^{\infty} (-1)^n . x(n) = X(e^{j\pi})$$

$$= \cos^3(3\pi)$$

$$\therefore \quad \boxed{S=-1} \qquad = (-1)^3 = -1.$$

(c) What is the d.c. & high-frequency gain of the filter described by $h(n) = \{1, 2, 3, 4\}$.

Soln:

$$H(\omega) = \sum_{n=-\infty}^{+\infty} h(n) . e^{-j\omega n}.$$

$$\therefore H(\omega) = \sum_{n=0}^{3} h(n) . e^{-j\omega n}.$$

$$\therefore H(\omega) = 1 + 2e^{-j\omega(1)} + 3e^{-j2\omega} + 4e^{-j3\omega}.$$

for d.c. gain $\omega = 0$.

$$\therefore H(0) = \sum_{n=0}^{3} h(n) . (1). = 1+2+3+4 = 10.$$

for high freq. gain $\omega = \pi$

$$\therefore H(e^{j\pi}) = \sum_{n=0}^{3} h(n) . (-1)^n. = 1-2+3-4 = -2.$$

$$\boxed{\begin{aligned} &\text{D.C. gain} = H(0) = \sum_{n=-\infty}^{\infty} x(n). \\ &\text{H.F. gain} = H(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x(n) . (-1)^n \end{aligned}}$$

P 6.1.3. Find the F.T. of

(i) $x_1(n) = (1/4)^n \cdot u(n-3)$.

Soln:

$$x_1(n) = \left(\frac{1}{4}\right)^{(n-3)+3} \cdot u(n-3).$$

$$\downarrow F.T \qquad = \frac{1}{64}\cdot\left[\left(\frac{1}{4}\right)^{n-3} \cdot u(n-3)\right]. \qquad n_0 = 3$$

$$\boxed{Y(\omega) = \frac{1}{64}\cdot\frac{e^{-j3\omega}}{1-\frac{1}{4}\cdot e^{-j\omega}}}$$

(ii) $y_2(n) = \delta(n-n_0)$.

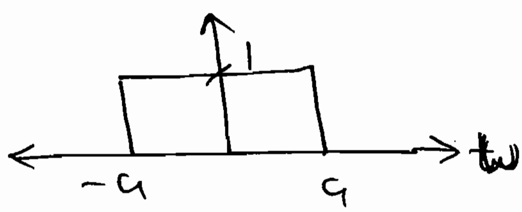
Soln:

$$y_2(n) = \delta(n-n_0).$$

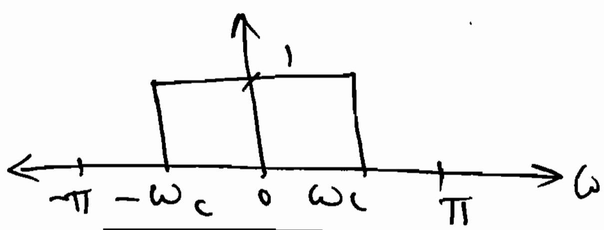
$$\downarrow F.T$$

$$\boxed{Y(\omega) = e^{-j\omega n_0} \cdot 1.}$$

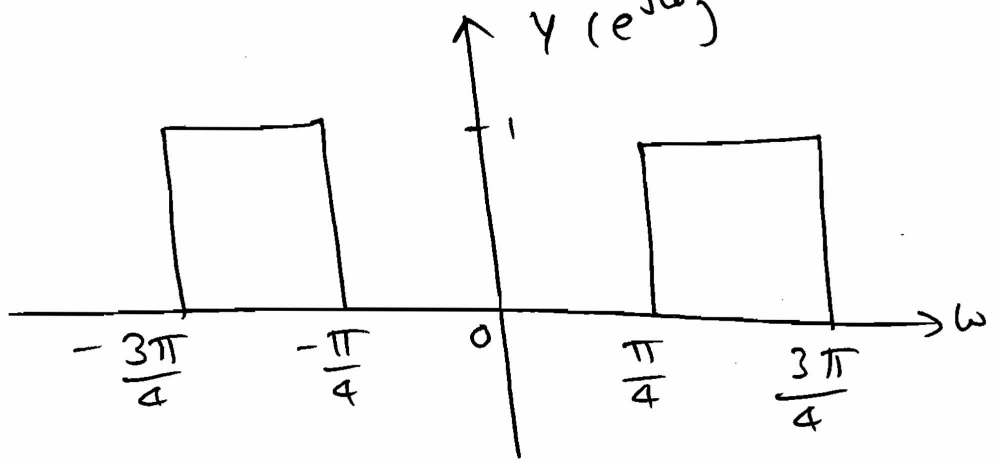
* cont^n: $\dfrac{\sin at}{\pi t} \xleftrightarrow{F.T.}$



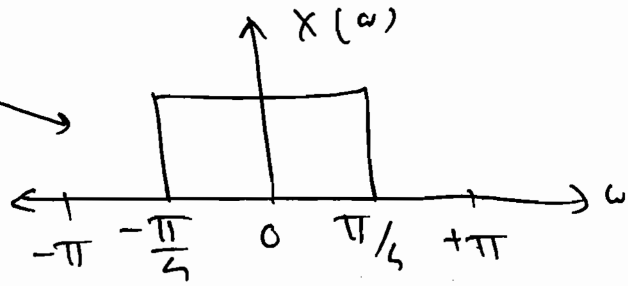
dis^r: $\boxed{\dfrac{\sin \omega_c n}{\pi n} \xleftrightarrow{F.T.}}$



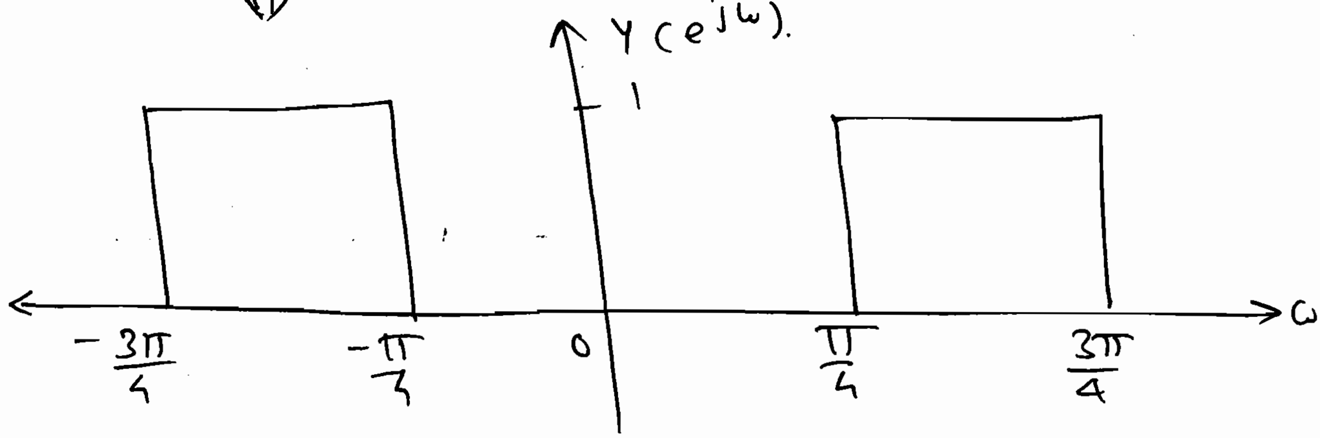
P 6.1.4 Find the signal corresponding to the spectrum shown in figure?



Soln: let, $x(n) = \frac{\sin \frac{\pi}{4} n}{\frac{\pi}{4} n} \longleftrightarrow$



$\Downarrow$



$$\therefore \quad Y(e^{j\omega}) = X\left[e^{+j(\omega - \frac{\pi}{2})}\right] + X\left[e^{j(\omega + \frac{\pi}{2})}\right].$$

$\downarrow$ I.F.T

$$y(n) = x(n) \cdot e^{j\frac{\pi}{2}n} + x(n) \cdot e^{-j\frac{\pi}{2}n}.$$

$$\therefore y(n) = x(n) \cdot e^{+j\frac{\pi}{2}n} + x(n) \cdot e^{-j\frac{\pi}{2}n}$$

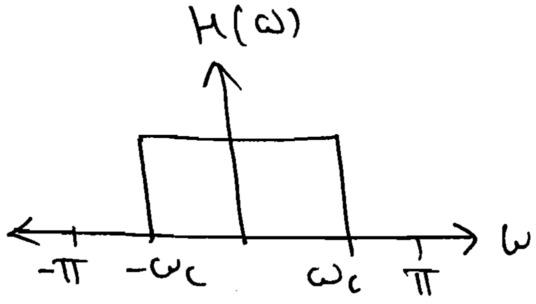
$$\boxed{y(n) = 2x(n) \cdot \cos\left(\frac{\pi}{2}n\right).}$$

$\boxed{P\ 6.1.5}$ (a) Let $h(n)$ is the impulse response of ideal L.P.F. with cutoff frequency ω_c, what type of filter has unit impulse response as $g(n) = (-1)^n \cdot h(n)$.

Soln:

$h(n) \longleftrightarrow H(\omega)$.

$\therefore h(n) = \dfrac{\sin \omega_c n}{n\pi} \xleftrightarrow{F.T.}$



$\Rightarrow g(n) = (-1)^n \cdot h(n)$.

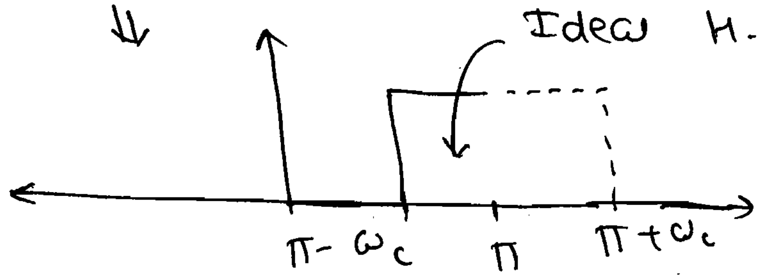
$g(n) = e^{jn\pi} \cdot h(n)$.

$\downarrow$ F.T.

$G(e^{j\omega}) = H(e^{j(\omega-\pi)})$.

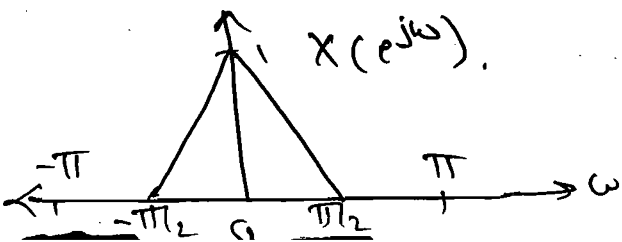
$\Downarrow$

$=$



$\Rightarrow$ Type of filter is Ideal HPF.

(b) A discrete system with input $x(n)$ & output $y(n)$ are related as $y(n) = x(n) + (-1)^n x(n)$. If the input spectrum $X(e^{j\omega})$ is shown below the o/p spectrum at $\omega = 0$ & $\omega = \pi$ are.

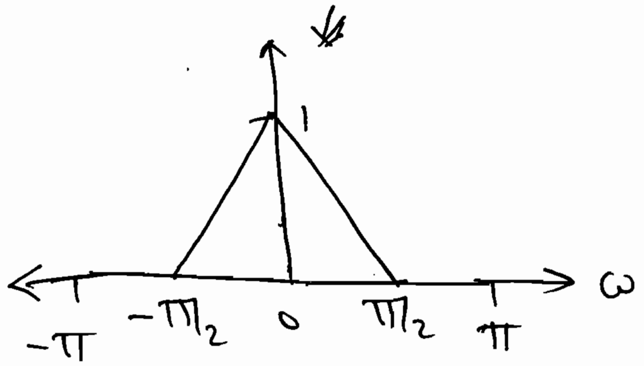


Sol^n: $y(n) = x(n) + (-1)^n . x(n)$.

$\Rightarrow y(n) = x(n) + e^{jn\pi} . x(n)$.

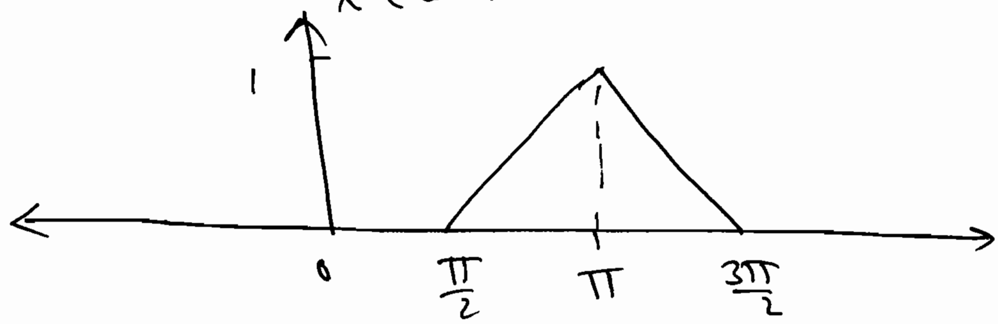
$\downarrow$ F.T.

$\therefore Y(e^{j\omega}) = X(e^{j\omega}) + X(e^{j(\omega-\pi)})$.



$X(\omega - \pi)$.

+



So, The o/p spectrum i.e.

$Y(e^{j\omega})$ at $\omega = 0$ is 1 and

$\omega = \pi$ is also 1.

* Time-Scaling :-

$\Rightarrow$ $\boxed{x(n) \longleftrightarrow X(e^{j\omega})}$, then $x[n/k] \longleftrightarrow X(e^{j\omega/k})$

then $\boxed{x(n/k) \longleftrightarrow X(e^{j\omega k})}$.

$\Rightarrow$ n is integer multiple of k.

$\boxed{\text{P 6.1.6.}}$ Find the I.F.T. of $Y(e^{j\omega})$

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} \cdot e^{-j10\omega}}$$

Soln:

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} \cdot e^{-j10\omega}}$$

$K = 10.$

$\downarrow$ I.F.T.

$$\therefore y(n) = \left(\frac{1}{2}\right)^{\frac{n}{10}} \cdot u\left(\frac{n}{10}\right), \quad n = 0, 10, 20, \ldots$$

$\boxed{\text{P 6.1.7.}}$ If the DTFT of $x(n) = \left(\frac{1}{5}\right)^n u(n+2)$ is $X(e^{j\omega})$, find the sequence that has a DTFT given $Y(e^{j\omega}) = X(e^{j2\omega})$.

Soln:

$$Y(e^{j\omega}) = X(e^{j2\omega})$$

$K = 2$

$\downarrow$ I.F.T.

$$\therefore y(n) = x(n/K) = x(n/2)$$

$$\therefore \boxed{y(n) = \left(\frac{1}{5}\right)^{n/2} \cdot u\left(\frac{n}{2} + 2\right), \quad n = 0, 2, 4, 6, \ldots}$$

* Frequency Differentiation :-

$$\Rightarrow \boxed{-jn\,x(n) \longleftrightarrow \frac{d}{d\omega} X(e^{j\omega})}$$

$$\Rightarrow \quad \boxed{n\,x(n) \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})}$$

P 6.1.8. Find the F.T. of $y(n) = n\,a^n.u(n)$?

Sol^n: Let, $x(n) = a^n.u(n)$.

$$\Big\downarrow \text{F.T.}$$

$$\therefore \; X(e^{j\omega}) = \frac{1}{1-a e^{-j\omega}}$$

$$\therefore \; n\,a^n.u(n) \longleftrightarrow j\frac{d}{d\omega}\left(X(e^{j\omega})\right).$$

$$Y(e^{j\omega}) = j\frac{d}{d\omega}\left[\frac{1}{1-a e^{-j\omega}}\right]$$

$$= j\,\frac{-a(-j).e^{-j\omega}}{(1-a e^{-j\omega})^2}$$

$$Y(e^{j\omega}) = \frac{j^2 a.e^{-j\omega}}{(1-a e^{-j\omega})^2}$$

$$\boxed{Y(e^{j\omega}) = \frac{-a.e^{-j\omega}}{(1-a e^{-j\omega})^2}}$$

P. 6.1.9. Find the value of $\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n$?

Sol^n: $$\sum_{n=0}^{\infty} n\left(\frac{1}{a}\right)^n = \left.\frac{-a.e^{j\omega}}{(1-a e^{-j\omega})^2}\right|_{\omega=0} = \frac{-a.e^{-j0}}{(1-a e^{-j(0)})^2}$$

$$\Rightarrow \; \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{-1/2}{(1-1/2)^2}.$$

$$\sum_{n=0}^{\infty} n \cdot \left(\tfrac{1}{2}\right)^n = \frac{-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = -2.$$

$\boxed{P\ 6.1.10}$ Find the F.T. of $x(n) = n e^{j\frac{n\pi}{8}} \cdot \alpha^{n-3} \cdot u(n-3)$.

Soln:

$$x(n) = n \cdot e^{j\frac{n\pi}{8}} \cdot \alpha^{(n-3)} u(n-3).$$

$\frac{\pi}{8} \to \omega_0$, $3 \to n_0$

$$= j \frac{d}{d\omega}\left[\frac{e^{-j\left(\omega-\frac{\pi}{8}\right)\cdot 3}}{1-\alpha e^{-j\left(\omega-\frac{\pi}{8}\right)}}\right]$$

ω_0

* Convolution in time :-

$\Rightarrow$ $x(n) \longleftrightarrow X(e^{j\omega})$ &

$h(n) \longleftrightarrow H(e^{j\omega})$. then

$$x(n) * h(n) \longleftrightarrow X(e^{j\omega}) \cdot H(e^{j\omega}).$$

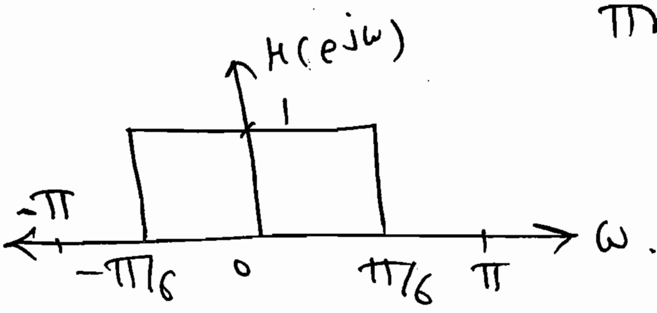
Note:-

$\Rightarrow$ F.T. of impulse response is known as frequency response.

$\boxed{P\ 6.1.11}$ Consider $x(n) = \sin\left(\frac{n\pi}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)$. if the impulse response is $h(n) = \dfrac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}$.

Soln:

$$h(n) = \frac{\sin\frac{n\pi}{6}}{\pi n} \xleftrightarrow{F.T}$$



So, only $\sin\left(\frac{\pi n}{8}\right)$ will be passed through $h(n)$.

hence o/p is $\boxed{y(n) = \sin\left(\frac{\pi n}{8}\right)}$.

P 6.1.12 An L.T.I. System is having impulse response

$$h(n) = \begin{cases} 4\sqrt{2} & ; n = 2, -2 \\ -2\sqrt{2} & ; n = 1, -1 \\ 0 & ; \text{elsewhere.} \end{cases}$$

Find the output when input applied is

$x(n) = e^{jn\pi/4}$?

Soln:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}.$$

$$H(e^{j\omega}) = 4\sqrt{2}\left[e^{-j2\omega} + e^{j2\omega}\right] + (-2\sqrt{2})\left[e^{j\omega} + e^{-j\omega}\right].$$

$$H(e^{j\omega}) = 8\sqrt{2}\cos 2\omega - 4\sqrt{2}\cos\omega.$$

Now, $x(n) = e^{jn\pi/4}$.

$\therefore$ if i/p is $x(n) = e^{j\omega n} \Rightarrow y(n) = e^{j\omega n} \cdot H(\omega)$.

$$\Rightarrow x(n) = e^{jn\pi/4} \Rightarrow y(n) = e^{jn\pi/4} \cdot H(\omega)\Big|_{\omega = \pi/4}.$$

$$= \left[0 - 4\sqrt{2} \times 1/\sqrt{2}\right] e^{\frac{jn\pi}{4}}$$

$$-4 \cdot e^{jn\pi/4}$$

$$\Rightarrow \boxed{y(n) = 4.e^{jn\pi/4}}$$

$\Rightarrow$ If $x(n) = (-1)^n = e^{jn\pi}$

$$\Rightarrow y(n) = e^{jn\pi}\left[8\sqrt{2}\cos 2\pi - 4\sqrt{2}\cos\pi\right].$$

$$= e^{jn\pi}\left[8\sqrt{2}(1) - 4\sqrt{2}(-1)\right].$$

$$\boxed{y(n) = 12\sqrt{2}.\ e^{jn\pi}}$$

P 6.1.13 Design a 3 point FIR filter with impulse response $h(n) = \{\alpha, \underset{\uparrow}{\beta}, \alpha\}$ & the magnitude response blocks the frequency $f = 1/3$ & phase the freq. $f = 1/8$ with unity gain. What is the D.C. gain of the filter?

Soln: $h(n) = \{\underset{n=-1}{\alpha}, \underset{\underset{n=0}{\uparrow}}{\beta}, \underset{n=1}{\alpha}\}$

$\downarrow$ F.T.

$$\Rightarrow H(e^{j\omega}) = \sum_{n=-1}^{+1} h(n).\ e^{-j\omega n}.$$

$$= \alpha e^{-j\omega(-1)} + \beta + \alpha e^{+j\omega(1)}.$$

$$\therefore \boxed{H(e^{j\omega}) = 2\alpha.\cos\omega + \beta.}$$

Now. given that $H(e^{j\omega})\Big|_{f=1/3} = 0.$

& $H(e^{j\omega})\Big|_{f=1/8} = 1.$

$\Rightarrow H(e^{j\omega})\Big|_{\omega=2\pi(\frac{1}{3})} = 0.$

$\Rightarrow 2\alpha.\cos\left(\frac{2\pi}{3}\right) + \beta = 0.$

$\Rightarrow -2\alpha.\cos\frac{\pi}{3} + \beta = 0.$

$\Rightarrow -2\alpha.\frac{1}{2} + \beta = 0 \Rightarrow \boxed{\alpha = \beta}.$

$\Rightarrow H(e^{j\omega})\Big|_{\omega=2\pi(\frac{1}{8})} = 1.$

$\Rightarrow 2\alpha.\cos\left(\frac{\pi}{4}\right) + \beta = 1.$

$2\alpha.\frac{1}{\sqrt{2}} + \alpha = 1.$

$$\boxed{\alpha = \frac{1}{1+\sqrt{2}} = \beta.}$$

So, $H(e^{j\omega}) = \frac{2}{1+\sqrt{2}}.\cos\omega + \frac{1}{1+\sqrt{2}}.$

$$\boxed{\text{D.C. gain} \Rightarrow H(e^{j\omega})\Big|_{\omega=0} = \frac{2}{1+\sqrt{2}} + \frac{1}{1+\sqrt{2}} = \frac{3}{1+\sqrt{2}}}$$

$\boxed{\text{P 6.1.14}}$ **Digital filters:-**

(1) $y(n) = x(n) - x(n-1).$ [HPF].

$\Rightarrow \downarrow$ F.T.

$Y(e^{j\omega}) = X(e^{j\omega}) - e^{-j\omega}.X(e^{j\omega}).$

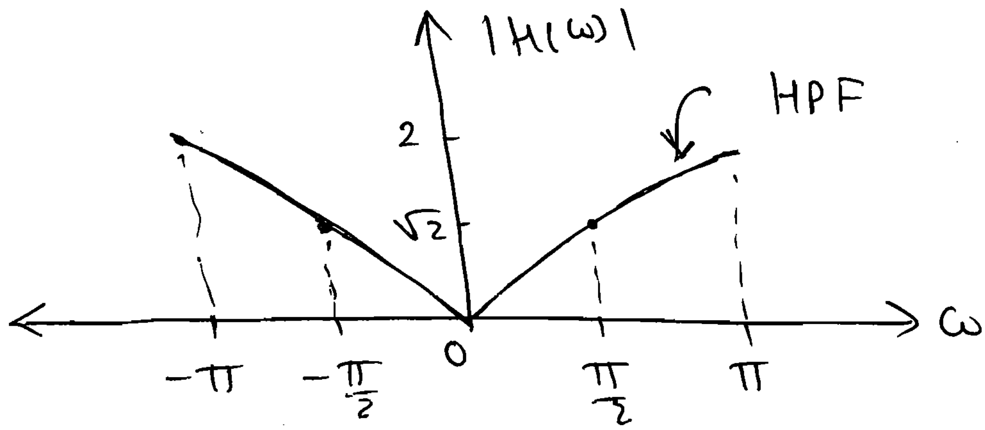
$$\Rightarrow \quad Y(\omega) = (1-e^{-j\omega})\, X(\omega).$$

$$\Rightarrow \text{T.F.} \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1-e^{-j\omega}.$$

$$\Rightarrow \omega = 0 \Rightarrow H(0) = 1-1 = 0.$$

$$\rightarrow \omega = \frac{\pi}{2} \Rightarrow H\left(\frac{\pi}{2}\right) = 1-e^{-j\frac{\pi}{2}} = 1+j = \sqrt{2}\angle 45^\circ$$

$$\rightarrow \omega = \pi \Rightarrow H(\pi) = 1-e^{-j\pi} = 1-(-1) = 2.$$



(2) $y(n) = x(n) + x(n-1).$ [LPF].

$\Rightarrow$ $\downarrow$ F.T

$$Y(\omega) = X(\omega) + e^{-j\omega} \cdot X(\omega).$$

$$\Rightarrow \text{T.F.} \quad \frac{Y(\omega)}{X(\omega)} = H(\omega) = 1+e^{-j\omega}.$$

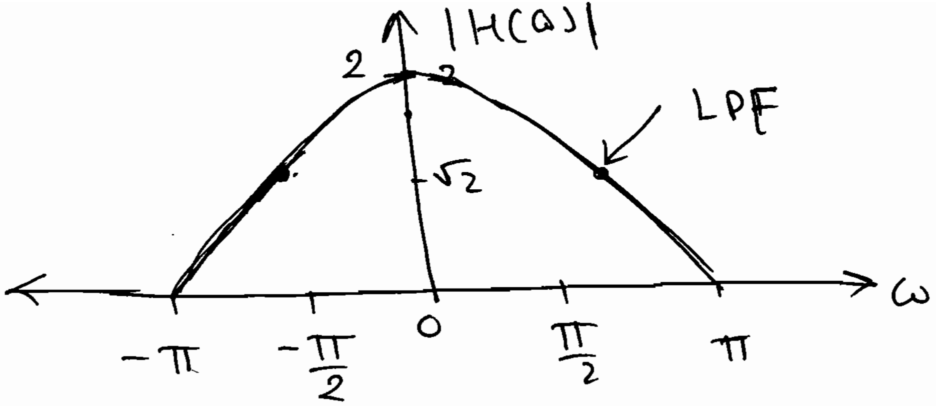
$$\rightarrow \omega = 0 \Rightarrow H(0) = 1+e^{-j0} = 2.$$

$$\rightarrow \omega = \frac{\pi}{2} \Rightarrow H\left(\frac{\pi}{2}\right) = 1+e^{-j\frac{\pi}{2}} = 1-j = \sqrt{2}\angle -45^\circ.$$

$$\rightarrow \omega = \pi \Rightarrow H(\pi) = 1+e^{-j\pi} = 1+(-1) = 0.$$

$$\Rightarrow \omega = -\pi \Rightarrow H(-\pi) = 1+e^{j\pi} = 1-1 = 0.$$

$\Rightarrow$



[3] $h(n) = \delta(n) - \delta(n-2)$. [BPF].

Soln:

$\downarrow$ F.T.

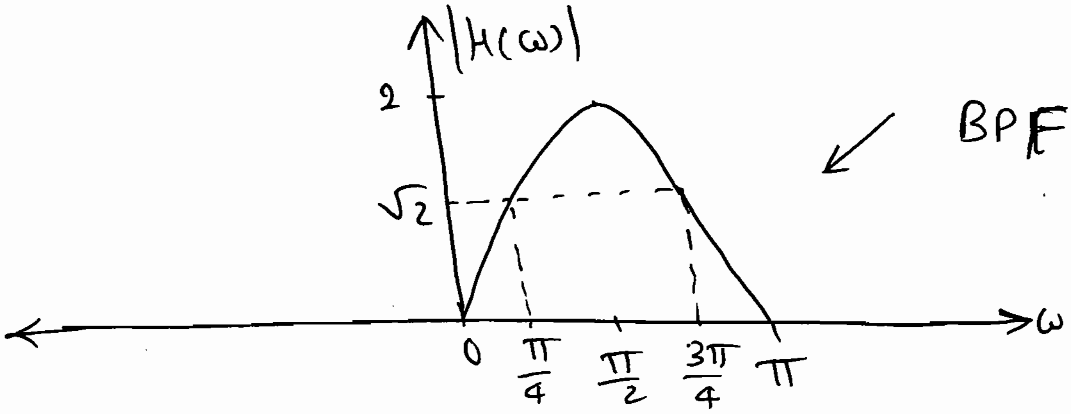
$\therefore H(\omega) = 1 - e^{-j2\omega} \cdot 1.$

$\Rightarrow H(\omega) = 1 - e^{-j2\omega}.$

$\rightarrow \omega = 0 \Rightarrow H(0) = 1 - e^{-j(0)} = 1 - 1 = 0.$

$\rightarrow \omega = \frac{\pi}{2} \Rightarrow H(\frac{\pi}{2}) = 1 - e^{-j2(\frac{\pi}{2})} = 1 - e^{-j\pi} = 1-(-1)$
$= 2.$

$\Rightarrow \omega = \pi \Rightarrow H(\pi) = 1 - e^{-j2\pi} = 1 - 1 = 0.$



[4] $h[n] = \delta[n] + \delta[n+2]$.

Soln:

$\downarrow$ F.T.

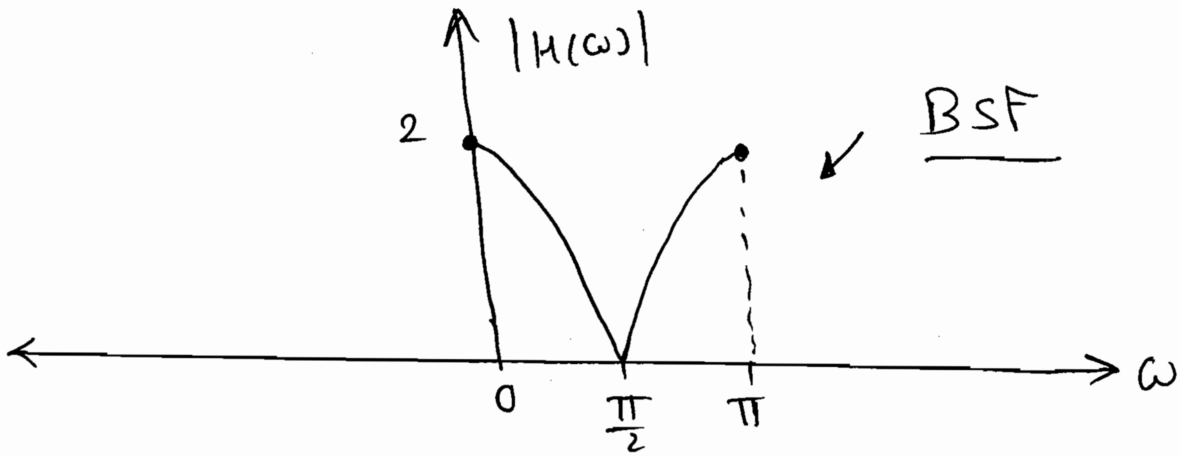
$\therefore H(\omega) = 1 + e^{-j2\omega} \cdot 1.$

$\Rightarrow \boxed{H(\omega) = 1 + e^{-j2\omega}.}$

$\rightarrow \omega = 0 \Rightarrow H(0) = 1 + e^{-j(0)} = 2.$

$\rightarrow \omega = \frac{\pi}{2} \Rightarrow H(\frac{\pi}{2}) = 1 + e^{-j \cancel{2}\pi \times \frac{\pi}{\cancel{2}}} = 1+(-1) = 0.$

$\rightarrow \omega = \pi \Rightarrow H(\pi) = 1 + e^{-j2\pi} = 1+1 = 2.$



$\boxed{P6.1.15.}$ Consider the System described by the equation $y(n) = a y(n-1) + b x(n) + x(n-1)$, Where 'a' & 'b' are real, find the relation betn a & b such that $|H(e^{j\omega})| = 1$

Soln:

$$y(n) = a y(n-1) + b x(n) + x(n-1).$$

$\downarrow$ F.T.

$$\therefore Y(\omega) = a \cdot e^{-j\omega} Y(\omega) + bX(\omega) + e^{-j\omega} X(\omega).$$

$$\therefore Y(\omega)[1 - a e^{-j\omega}] = [b + e^{-j\omega}] X(\omega).$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b + e^{-j\omega}}{1 - a \cdot e^{-j\omega}}.$$

Now, $|H(e^{j\omega})| = 1 \Rightarrow |H(\omega)|^2 = 1.$

$$\Rightarrow H(\omega) \cdot H^*(\omega) = 1.$$

$$\Rightarrow \quad \left[\frac{b+e^{-j\omega}}{1-ae^{-j\omega}}\right]\cdot\left[\frac{b+e^{j\omega}}{1-ae^{j\omega}}\right]=1.$$

$$\therefore \quad \frac{b^2+b\left[e^{j\omega}+e^{-j\omega}\right]+1}{1-a\left[e^{-j\omega}+e^{j\omega}\right]+a^2}=1.$$

$$\therefore \quad b^2+2b\cos\omega+\cancel{1}=\cancel{1}-2a\cos\omega+a^2$$

$$\therefore \quad a^2-b^2 = 2\cos\omega\,[b-a].$$

$$\therefore \quad -(a+b)=2\cos\omega.$$

$$\Rightarrow \quad \boxed{a=-b}$$

(or) $\quad |H(e^{j\omega})|_{\omega=0}=1. \qquad (\because =1\ \forall\omega).$

$$\therefore \quad \frac{b+1}{1-a}=1.$$

$$b+\cancel{1}=\cancel{1}-a \Rightarrow \boxed{b=-a}$$

DRDO

$\boxed{P6.1.16}$ An input $x(n)$ with length 3 is applied to a LTI system having an impulse response $h(n)$ of length 5, and $Y(\omega)$ is the DTFT of the o/p $y(n)$ of the system. If $|h(n)|\le L$ & $|x(n)|\le B\ \forall n$, the maximum value of $Y(0)$ can be ___.

(a) 15LB $\quad$ (B) 12LB

(C) 8LB $\quad$ (D) 7LB.

Solⁿ:

$$y(n) = x(n) * h(n).$$

$$\Rightarrow Y(\omega) = X(\omega) \cdot H(\omega).$$

$$\Rightarrow Y(0) = X(0) \cdot H(0).$$

$$\Rightarrow Y(0) = \left[\sum_{n=0}^{2} x(n)\right] \times \left[\sum_{n=0}^{4} h(n)\right]$$

$x(n) \rightarrow L = 3$

$$\rightarrow X(\omega) = \sum_{n=0}^{2} x(n) \cdot e^{-j\omega n}.$$

$$X(0) = \sum_{n=0}^{2} x(n).$$

$$\rightarrow H(\omega) = \sum_{n=0}^{4} h(n) \cdot e^{-j\omega n}$$

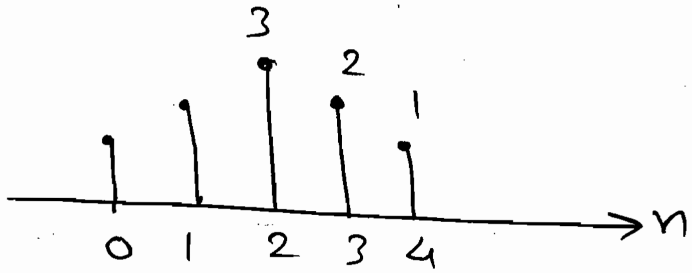
$$\rightarrow H(0) = \sum_{n=0}^{4} h(n).$$

$$\Rightarrow Y(0) = \sum_{n=0}^{2} B \times \sum_{n=0}^{4} L.$$

$$Y(0) = 3B \times 5L$$

$$\Rightarrow \boxed{Y(0) = 15LB.}$$

$\boxed{P\ 6.1.17}$ Consider a filter with I.R. shown in figure. Find the group delay of the filter?



Solⁿ: F.I.R. filters:

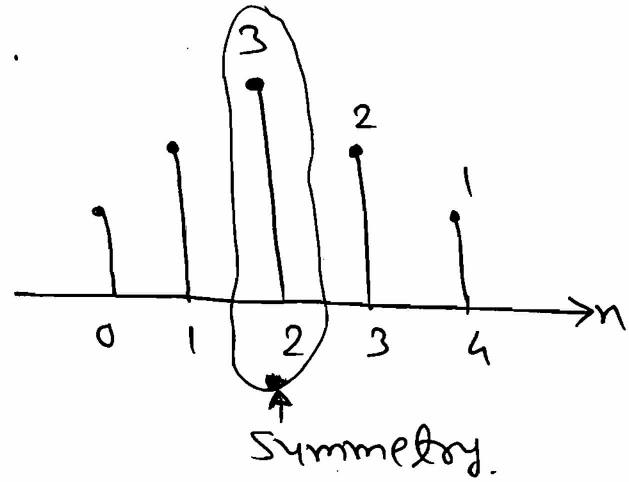
Linear phase response $\theta(\omega) = -\alpha\omega.$

$\therefore$ α is the value of n for which spectrum is symmetrical about n.

In this case, $n = 2$.

$\Rightarrow \theta(\omega) = -2\omega$

$\Rightarrow t_g(\omega) = -\dfrac{d\theta(\omega)}{d\omega}$.



$\Rightarrow \boxed{t_g(\omega) = +2}$

$\Rightarrow h[n] = \pm h[N-1-n]$. $\Rightarrow$ For symmetry.

here, N = length of I.R. $= 5$.

$\therefore h[n] = h[4-n]$.

$\Rightarrow h(0) = h(4) = 1$.

$h(1) = h(3) = 2$.

*** Note:**

Group delay $t_g(\omega)$ = Value of n about which spectrum is symmetrical.

P 6.1.18 An L.T.I. filter is described by the difference equation $y(n) = x(n) + 2x(n-1) + x(n-2)$.

(a) Obtain the magnitude & phase response ?

(b) Find the o/p when the input is

$$x(n) = 10 + 4\cos\left[\frac{\pi n}{2} + \frac{\pi}{4}\right] ?$$

Soln:

$y(n) = x(n) + 2x(n-1) + x(n-2)$.

$\xrightarrow{F.T.}$ $Y(\omega) = X(\omega) + 2e^{-j\omega} \cdot X(\omega) + 2e^{-j2\omega} \cdot X(\omega)$.

$$\Rightarrow \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$H(\omega) = (1 + 2\cos\omega + \cos 2\omega) - j(\sin\omega + \sin 2\omega)$$

<u>Mag.</u>

$$\Rightarrow \quad |H(\omega)| = \sqrt{(1+2\cos\omega + \cos 2\omega)^2 + (\sin\omega + \sin 2\omega)^2}$$

<u>Phase</u>

$$\angle H(\omega) = -\tan^{-1}\left(\frac{\sin\omega + \sin 2\omega}{1+2\cos\omega + \cos 2\omega}\right).$$

(b) $x(n) = 10 + 4\cos\left[\frac{\pi}{2}n + \frac{\pi}{4}\right]$?

$$\Rightarrow \quad H(\omega) = (1+e^{-j\omega})^2.$$

$$= e^{-j\frac{\omega}{2}(2)}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)^2$$

$$= 4e^{-j\omega}\left(\frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2}\right)^2$$

$$H(\omega) = 4e^{-j\omega}\cdot\cos^2\frac{\omega}{2}.$$

$\Rightarrow$ Mg. $|H(\omega)| = 4.\cos^2\frac{\omega}{2}.$

phase $\angle H(\omega) = \theta(\omega) = -\omega.$

Now $x(n) = \underbrace{10}_{\omega=0} + \underbrace{4\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}_{\omega=\pi/2.}$

$H(\omega)|_{\omega=0} = 4.$, $H(\omega)|_{\omega=\frac{\pi}{2}} = 4.e^{-j\frac{\pi}{2}}\cdot\cos^2\frac{\pi}{4}$

$= 2.e^{-j\pi/2}.$

$\therefore \quad y(n) = K.A.\cos(\omega_0 n + \phi).$

$\Rightarrow y(n) = 40 + 4(2).\cos\left[\frac{\pi n}{2} + \frac{\pi}{4} - \frac{\pi}{2}\right].$

$$\Rightarrow \boxed{y(n) = 40 + 8.\cos\left[\frac{\pi n}{2} - \frac{\pi}{4}\right].}$$

* Parseval's relation :-

$\Rightarrow \quad x(n) \longleftrightarrow X(e^{j\omega}).$

the $$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(e^{j\omega})|^2 d\omega.$$

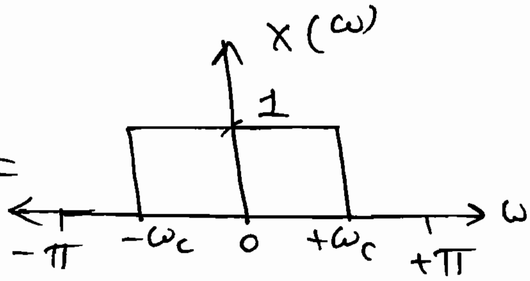
$\Rightarrow$ Parseval's relation is known as Conservation of energy theorem, because DTFT operator preserves energy, when going from time domain to freq. domain.

$\boxed{P\ 6.1.20}$ Find the energy in the signal

$$x(n) = \frac{\sin \omega_c n}{\pi n}.$$

Soln:

$$x(n) = \frac{\sin \omega_c n}{\pi n} \xleftrightarrow{F.T.} X(\omega) =$$



$$\therefore E_{x(n)} = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 . d\omega.$$

$$\Rightarrow \quad E_{x(n)} = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} (1)^2 \cdot d\omega.$$

$$= \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}.$$

$$\Rightarrow \quad \boxed{E_{x(n)} = \frac{\omega_c}{\pi}}.$$

$\boxed{\text{P 6.1.21}}$ Find the value of

$$\sum_{n=-\infty}^{+\infty} \frac{\sin\left(\frac{n\pi}{4}\right) \cdot \sin\left(\frac{n\pi}{3}\right)}{2\pi n \cdot 5\pi n}.$$

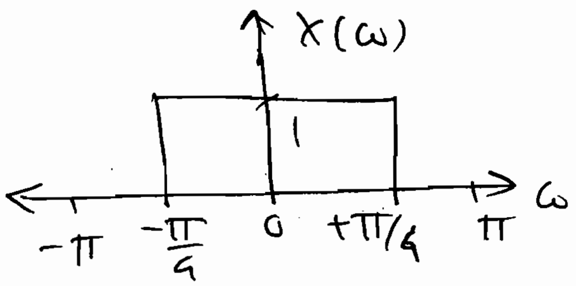
Soln:

$$\boxed{\sum_{n=-\infty}^{\infty} x[n]\cdot y[n] = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})\cdot Y(e^{j\omega})\cdot d\omega}$$

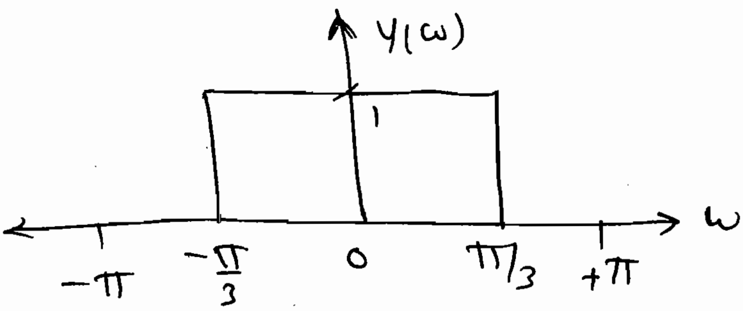
↓

Planchar's Relation.

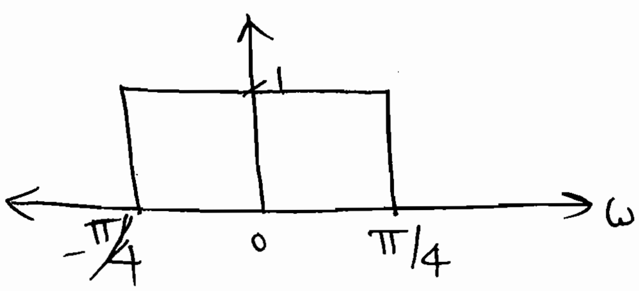
let, $x_1(n) = \dfrac{\sin\frac{n\pi}{4}}{\pi n} \longleftrightarrow$



$y(n) = \dfrac{\sin\left(\frac{n\pi}{3}\right)}{\pi n} \longleftrightarrow$



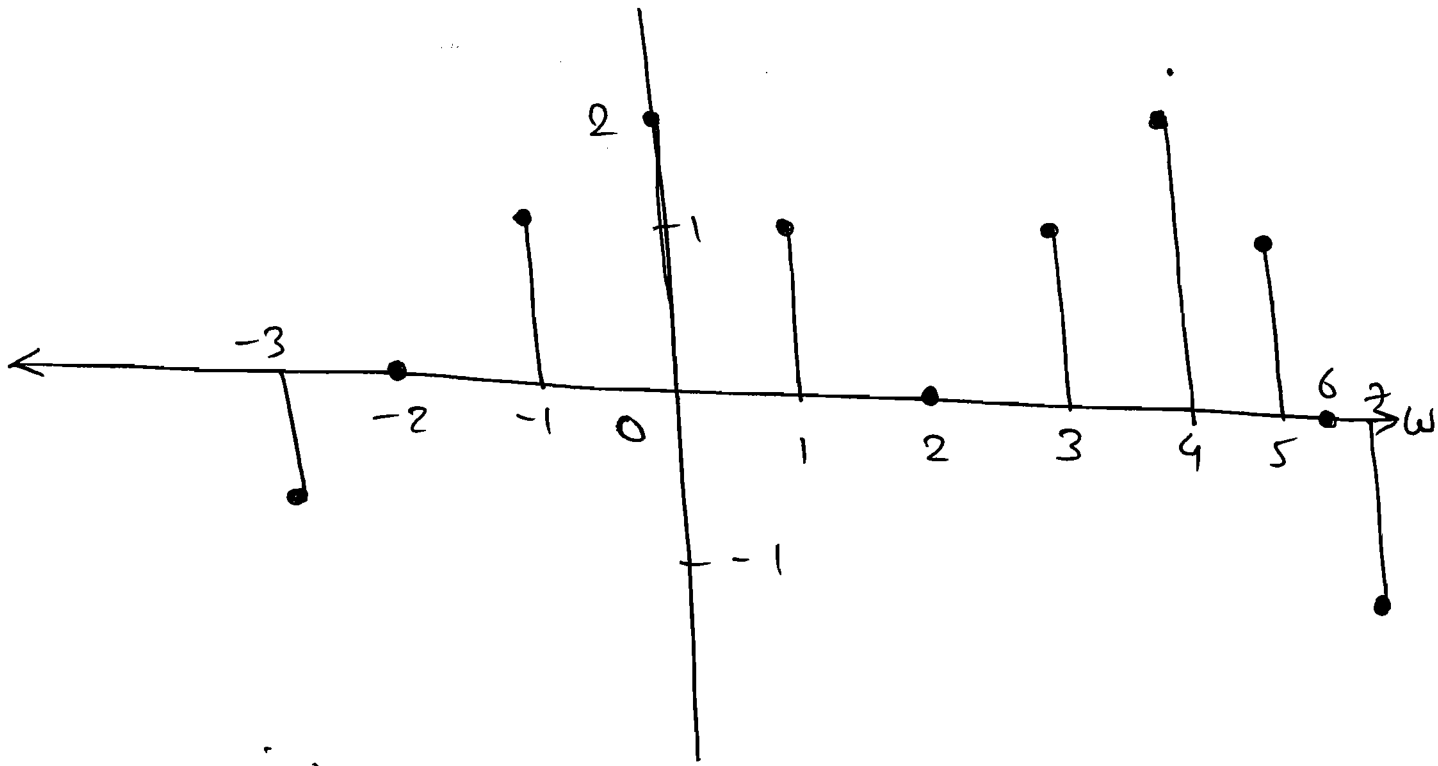
$X(\omega)\cdot Y(\omega) =$



$$\therefore \sum_{n=-\infty}^{+\infty} \frac{\sin\left(\frac{n\pi}{4}\right)}{2\pi n} \cdot \frac{\sin\left(\frac{n\pi}{3}\right)}{5\pi n} = \frac{1}{2\pi \times 10} \int_{-\pi/4}^{\pi/4} (1) \cdot d\omega.$$

$$= \frac{1}{2\cancel{\pi} \times 10} \times \frac{\cancel{\pi}}{2}. = \boxed{\frac{1}{40}}$$

$\boxed{P\ 6.1.22}$ For the signal shown in fig 6.1.22, find the following quantities without calculating DTFT?



(a) $X(e^{j0})$

<u>Soln:</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$$

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x(n) \cdot (1).$$

$$X(e^{j0}) = \sum_{n=-3}^{7} x(n). = x(-3) + x(-2) + x(-1) + x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7).$$

$\therefore X(e^{j0}) = -\cancel{1} + 0 + \cancel{1} + 2 + 1 + 0 + 1 + 2 + \cancel{1} + 0 - \cancel{1}$

$$\boxed{X(e^{j0}) = 6}$$

(b) $X(e^{j\pi})$.

Soln:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$$

$$\therefore X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\pi n} = \sum_{n=-\infty}^{+\infty} x(n) \cdot (-1)^n.$$

$$\therefore X(e^{j\pi}) = -x(-3) + x(-2) - x(-1) + x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7).$$

$$= +\cancel{1} - \cancel{1} + 2 - 1 - 1 + 2 - \cancel{1} + \cancel{1}$$

$$\boxed{X(e^{j\pi}) = 2}$$

(c) $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega.$

$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n}$$

$$\therefore \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{+\infty} x(n) \cdot [\cos\omega n - j\sin\omega n]\, d\omega.$$

$$\Rightarrow x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega.$$

Put $n = 0$.

$$\Rightarrow \boxed{\int_{-\pi}^{+\pi} X(e^{j\omega}) \cdot (1) \cdot d\omega = 2\pi\, x(0)}$$

so,
$$\int_{-\pi}^{+\pi} X(e^{j\omega}) \cdot d\omega = 2\pi(2) = 4\pi.$$

(d) $\int_{-\pi}^{+\pi} X(e^{j\omega}) \cdot e^{j2\omega} \cdot d\omega.$

Soln:
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j2\omega n} \cdot d\omega.$$

here, $n=2$.

$$\therefore \int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j2\omega} \cdot d\omega = 2\pi\, x(2) = 0.$$

(e) $\int_{-\pi}^{+\pi} |X(e^{j\omega})|^2 d\omega.$

Soln:
$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

$$\Rightarrow \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \left[\sum_{n=-\infty}^{\infty} |x(n)|^2 . \right].$$

$$= 2\pi\left[1+1+4+1+1+4+1+1\right].$$

$$= 28\pi.$$

(f) $\int_{-\pi}^{\pi} \left| \frac{d}{d\omega} X(e^{j\omega}) \right|^2 d\omega.$

Soln:

$\cancel{x(n)} \longrightarrow X(e^{j\omega})$.

$$n x(n) \longleftrightarrow +j \frac{d}{d\omega} X(e^{j\omega}).$$

$$\therefore \int_{-\pi}^{+\pi} \left| \frac{d}{d\omega} X(e^{j\omega}) \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{+\infty} |-jn\, x(n)|^2.$$

$$= 2\pi \sum_{n=-\infty}^{+\infty} |n x(n)|^2.$$

$$= 2\pi [\, 9 + 0 + 1 + 0 + 1 + 0 + 1 + 64 + 25 + 49 \,].$$

$$= 300\pi$$

(g) $\angle X(e^{j\omega})$.

Soln: $\angle X(e^{j\omega}) = \theta(\omega) = -2\omega = -2\omega.$

($\because$ Symm. @ $n = 2$).

$\boxed{P\ 6.1.23}$ Given $h(n) = [1, 2, 2]$, $f(n)$ is obtained by convolving $h(n)$ with itself and $g(n)$ by correlating $h(n)$ with itself. Which one of the following statements is TRUE?

(a) $f(n)$ is causal and its maximum value is 9.

(b) $f(n)$ is non-causal.

(c) $g(n)$ is causal and its maximum

value is 9.

(d) g(n) is non causal and maximum value is 9.

$\underline{\underline{\text{Sol}^n}}$: $f(n) = h(n) * h(n)$. ← ~~correlation~~ convolution

$g(n) = h(n) * h(-n)$. ← correlation

$\Rightarrow h(n) = [1, 2, 2] \rightarrow 0 \le n \le 2.$

$n \rightarrow 0, 1, 2$

$\Rightarrow h(-n) = [1, 2, 2] \rightarrow -2 \le n \le 0.$

$n \rightarrow -2, -1, \underline{0}$

Limits

$\Rightarrow f(n) = \underset{0 \le n \le 2}{h(n)} * \underset{0 \le n \le 2}{h(n)} \Rightarrow 0 \le n \le 4. \rightarrow$ C

$\Rightarrow g(n) = \underset{0 \le n \le 2}{h(n)} * \underset{-2 \le n \le 0}{h(-n)} \longrightarrow -2 \le n \le 2. \rightarrow$ NC

Now, $f(n) = h(n) * h(n)$

$\Rightarrow f(n) = \{1, 4, 10, 8, 4\}.$

	1	2	2
1	1	2	2
2	2	4	4
2	2	4	4

$\Rightarrow g(n) = h(n) * h(-n).$

$= \{2, 6, 9, 8, 2\}.$

	1	2	2
2	2	4	4
2	2	4	4
1	1	2	2

So, Ans - (d) g(n) is N.C. and max. value is 9.

PG.1.14 A continous time signal $x(t)$ is to be filtered to remove freq. component in the range $5\ KHz \le f \le 10\ KHz$. The maximum freq. present in $x(t)$ is $20\ KHz$. Find the minimum sampling frequency & find freq. response of ideal digital filter that will remove the desired freq. from $x(t)$?

⇒ Soln: $5\ KHz \le f \le 10\ KHz$ ← Analog rang.

⇒ $-\pi \le \omega \le +\pi$ ← Digital filter.
↳ Dig. freq.

⇒ here, $f_{max} = 20\ KHz$.

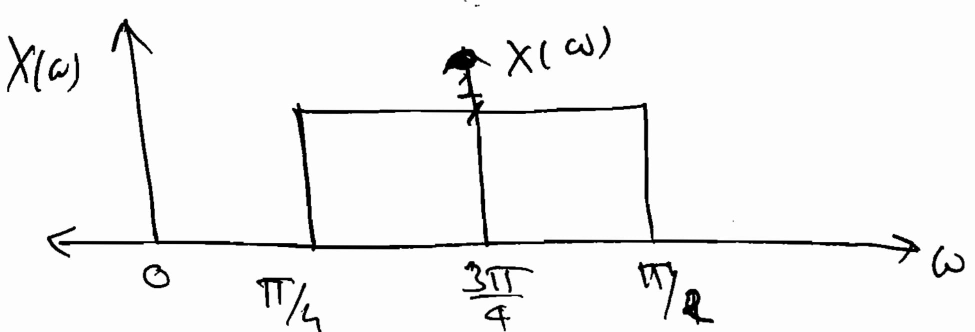
⇒ S.f. ⇒ $f_s = 2 f_m = 40\ KHz$.

**
*

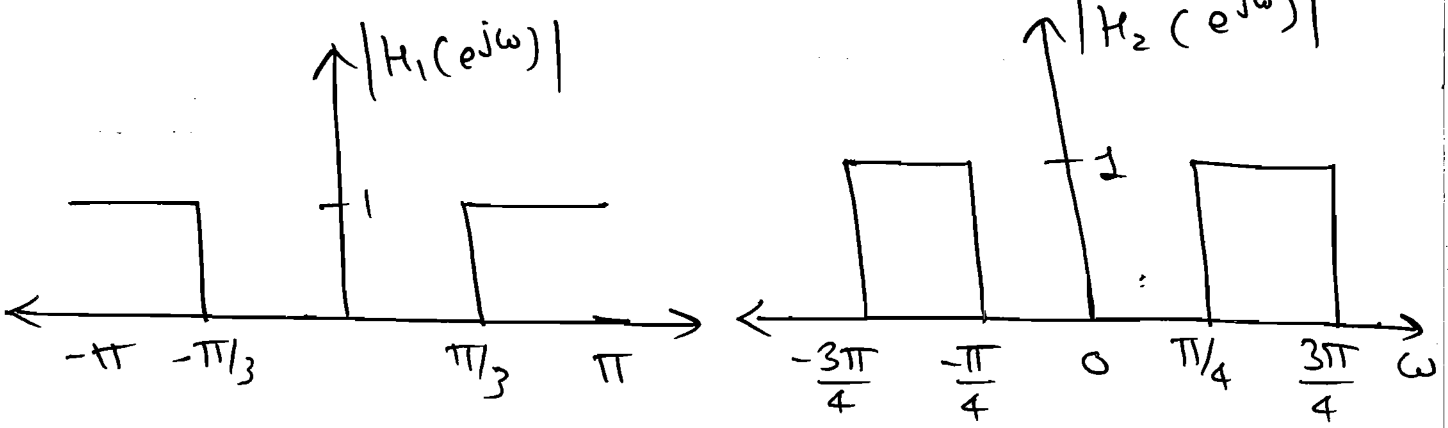
$$\boxed{\text{Digital freq. } \omega = \frac{2\pi f \rightarrow \text{Analog freq.}}{f_s \rightarrow \text{sampling freq.}}}$$

$$\therefore \frac{2\pi(5K)}{40K} \le \omega \le \frac{2\pi(10K)}{40K}.$$

$$\therefore \boxed{\frac{\pi}{4} \le \omega \le \frac{\pi}{2}}$$



P 6.1.25 — 2 ideal filters freq. response are shown in fig. For an arbitrary input $x(n)$, find the range of freq. that can be present in the o/p $y(n)$, if they are connected in (a) Cascade (b) Parallel.



Soln:

(a) Cascade:

$$|H(e^{j\omega})| = |H_1(e^{j\omega})| \cdot |H_2(e^{j\omega})|.$$

so,
$$\boxed{\pi/3 \le \omega \le \frac{3\pi}{4}}.$$

(b) Parallel.

$$|H(e^{j\omega})| = |H_1(e^{j\omega})| + |H_2(e^{j\omega})|.$$

so,
$$\boxed{\omega \geq \frac{\pi}{4}}$$

*
$$e^{-at} * e^{-at} = te^{-at} \longleftrightarrow \frac{1}{(a+j\omega)^2}.$$

$$a^n u[n] * a^n u[n] = (n+1) a^n u[n] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}.$$