Chapter
Trigonometric Functions

Topic-1: Trigonometric Ratios, Domain and Range of Trigonometric Functions, Trigonometric Ratios of Allied Angles

1 MCQs with One Correct Answer

1. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as : **[2013]**

(a) $\sin A\cos A+1$ (b) $\sec A \operatorname{cosec} A+1$
(c) $\tan A+\cot A$ (d) $\sec A+\operatorname{cosec} A$

2. Given both θ and ϕ are acute angles and $\sin\theta=\frac{1}{2}$, $\cos\phi=\frac{1}{3}$, then the value of $\theta+\phi$ belongs to **[2004S]**

(a) $\left(\frac{\pi}{3},\frac{\pi}{2}\right]$ (b) $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ (c) $\left(\frac{2\pi}{3},\frac{5\pi}{6}\right]$ (d) $\left(\frac{5\pi}{6},\pi\right]$

3. If $\tan\theta=-\frac{4}{3}$, then $\sin\theta$ is **[1979]**

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$
(c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) None of these

6 MCQs with One or More than One Correct Answer

4. Which of the following number(s) is/are rational? **[1998 - 2 Marks]**

(a) $\sin 15°$ (b) $\cos 15°$
(c) $\sin 15°\cos 15°$ (d) $\sin 15°\cos 75°$

7 Match the Following

5. In this questions there are entries in columns 1 and 2. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column 2 against the entry number in column 1 in your answer book.

$\frac{\sin 3\alpha}{\cos 2\alpha}$ is **[1992 - 2 Marks]**

Column I	Column II
(A) Positive	(p) $\left(\frac{13\pi}{48},\frac{14\pi}{48}\right)$
(B) Negative	(q) $\left(\frac{14\pi}{48},\frac{18\pi}{48}\right)$
	(r) $\left(\frac{18\pi}{48},\frac{23\pi}{48}\right)$
	(s) $\left(0,\frac{\pi}{2}\right)$

10 Subjective Problems

6. Find the range of values of t for which $2\sin t=\frac{1-2x+5x^2}{3x^2-2x-1}, t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. **[2005 - 2 Marks]**

Topic-2: Trigonometric Identities, Greatest and Latest Value of Trignometric Expressions

1 MCQs with One Correct Answer

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{\pi}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then which of the following statements is TRUE? **[Adv. 2024]**

(a) $f(x) = 0$ has infinitely many solutions in the interval $\left[\dfrac{1}{10^{10}}, \infty\right)$.

(b) $f(x) = 0$ has no solutions in the interval $\left[\dfrac{1}{\pi}, \infty\right)$

(c) The set of solutions of $f(x) = 0$ in the interval $\left(0, \dfrac{1}{10^{10}}\right)$ is finite.

(d) $f(x) = 0$ has more than 25 solutions in the interval $\left(\dfrac{1}{\pi^2}, \dfrac{1}{\pi}\right)$

2. Let $\dfrac{\pi}{2} < x < \pi$ be such that $\cot x = \dfrac{-5}{\sqrt{11}}$. Then

$$\left(\sin\frac{11x}{2}\right)(\sin 6x - \cos 6x) + \left(\cos\frac{11x}{2}\right)(\sin 6x + \cos 6x)$$

is equal to **[Adv. 2024]**

(a) $\dfrac{\sqrt{11}-1}{2\sqrt{3}}$ (b) $\dfrac{\sqrt{11}+1}{2\sqrt{3}}$

(c) $\dfrac{\sqrt{11}+1}{3\sqrt{2}}$ (d) $\dfrac{\sqrt{11}-1}{3\sqrt{2}}$

3. The value of $\displaystyle\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to **[Adv. 2016]**

(a) $3-\sqrt{3}$ (b) $2(3-\sqrt{3})$

(c) $2(\sqrt{3}-1)$ (d) $2(2-\sqrt{3})$

4. Let $\theta \in \left(0, \dfrac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then **[2006 - 3M, –1]**

(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$

(c) $t_3 > t_1 > t_2 > t_4$ (d) $t_2 > t_3 > t_1 > t_4$

5. The values of $\theta \in (0, 2\pi)$ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, are **[2006 - 3M, –1]**

(a) $\left(0, \dfrac{\pi}{6}\right) \cup \left(\dfrac{5\pi}{6}, 2\pi\right)$ (b) $\left(\dfrac{\pi}{8}, \dfrac{5\pi}{6}\right)$

(c) $\left(0, \dfrac{\pi}{8}\right) \cup \left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$ (d) $\left(\dfrac{41\pi}{48}, \pi\right)$

6. If $\alpha + \beta = \pi/2$ and $\beta + \gamma = \alpha$, then $\tan\alpha$ equals **[2001S]**

(a) $2(\tan\beta + \tan\gamma)$ (b) $\tan\beta + \tan\gamma$

(c) $\tan\beta + 2\tan\gamma$ (d) $2\tan\beta + \tan\gamma$

7. The maximum value of $(\cos\alpha_1).(\cos\alpha_2)\ldots(\cos\alpha_n)$, under the restrictions

$0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \dfrac{\pi}{2}$ and $(\cot\alpha_1).(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$

is **[2001S]**

(a) $1/2^{n/2}$ (b) $1/2^n$

(c) $1/2n$ (d) 1

8. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ is **[2000S]**

(a) ≥ 0 only when $\theta \ge 0$ (b) ≤ 0 for all real θ

(c) ≥ 0 for all real θ (d) ≤ 0 only when $\theta \le 0$

9. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) =$ **[1995S]**

(a) 11 (b) 12

(c) 13 (d) 14

10. Let $0 < x < \dfrac{\pi}{4}$ then $(\sec 2x - \tan 2x)$ equals **[1994]**

(a) $\tan\left(x - \dfrac{\pi}{4}\right)$ (b) $\tan\left(\dfrac{\pi}{4} - x\right)$

(c) $\tan\left(x + \dfrac{\pi}{4}\right)$ (d) $\tan^2\left(x + \dfrac{\pi}{4}\right)$

11. Given $A = \sin^2\theta + \cos^4\theta$ then for all real values of θ **[1980]**

(a) $1 \le A \le 2$ (b) $\dfrac{3}{4} \le A \le 1$

(c) $\dfrac{13}{16} \le A \le 1$ (d) $\dfrac{3}{4} \le A \le \dfrac{13}{16}$

12. If $\alpha+\beta+\gamma=2\pi$, then **[1979]**

(a) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

(c) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(d) None of these

2 Integer Value Answer/ Non-Negative Integer

13. Let α and β be real numbers such that $-\frac{\pi}{4}<\beta<0<\alpha<\frac{\pi}{4}$. If $\sin(\alpha+\beta)=\frac{1}{3}$ and $\cos(\alpha-\beta)=\frac{2}{3}$, then the greatest integer less than or equal to $\left(\frac{\sin\alpha}{\cos\beta}+\frac{\cos\beta}{\sin\alpha}+\frac{\cos\alpha}{\sin\beta}+\frac{\sin\beta}{\cos\alpha}\right)^2$ is ________. **[Adv. 2022]**

14. The maximum value of the expression $\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$ is ________. **[2010]**

3 Numeric/ New Stem Based Questions

15. Let $f:[0,2]\to\mathbf{R}$ be the function defined by

$$f(x)=(3-\sin(2\pi x))\sin\left(\pi x-\frac{\pi}{4}\right)-\sin\left(3\pi x+\frac{\pi}{4}\right).$$

If $\alpha,\beta\in[0,2]$ are such that $\{x\in[0,2]:f(x)\geq 0\}=[\alpha,\beta]$, then the value of $\beta-\alpha$ is ______ **[Adv. 2020]**

4 Fill in the Blanks

16. If $A>0, B>0$ and $A+B=\pi/3$, then the maximum value of $\tan A\tan B$ is ________. **[1993 - 2 Marks]**

17. If $K=\sin(\pi/18)\sin(5\pi/18)\sin(7\pi/18)$, then the numerical value of K is ________. **[1993 - 2 Marks]**

18. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is equal to ________ **[1991 - 2 Marks]**

19. Suppose $\sin^3 x\sin 3x=\sum_{m=0}^{n}C_m\cos mx$ is an identity in x, where $C_0, C_1, \ldots C_n$ are constants, and $C_n\neq 0$. then the value of n is ________ **[1981 - 2 Marks]**

5 True / False

20. If $\tan A=(1-\cos B)/\sin B$, then $\tan 2A=\tan B$. **[1983 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

21. Let $f(x)=x\sin\pi x, x>0$. Then for all natural numbers n, $f'(x)$ vanishes at **[Adv. 2013]**

(a) A unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(b) A unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(c) A unique point in the interval $(n, n+1)$

(d) Two points in the interval $(n, n+1)$

22. Let $\theta,\varphi\in[0,2\pi]$ be such that $2\cos\theta(1-\sin\varphi)=\sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\varphi-1, \tan(2\pi-\theta)>0$ and $-1<\sin\theta<-\frac{\sqrt{3}}{2}$, then φ cannot satisfy **[2012]**

(a) $0<\varphi<\frac{\pi}{2}$ (b) $\frac{\pi}{2}<\varphi<\frac{4\pi}{3}$

(c) $\frac{4\pi}{3}<\varphi<\frac{3\pi}{2}$ (d) $\frac{3\pi}{2}<\varphi<2\pi$

23. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then **[2009]**

(a) $\tan^2 x=\frac{2}{3}$ (b) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$

(c) $\tan^2 x=\frac{1}{3}$ (d) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

24. For a positive integer n, let $f_n(\theta)=\left(\tan\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$. Then **[1999 - 3 Marks]**

(a) $f_2\left(\frac{\pi}{16}\right)=1$ (b) $f_3\left(\frac{\pi}{32}\right)=1$

(c) $f_4\left(\frac{\pi}{64}\right)=1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

25. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where α,β,γ are real numbers satisfying $\alpha+\beta+\gamma=\pi$ is **[1995]**

(a) positive (b) zero
(c) negative (d) -3

26. Let $2\sin^2 x+3\sin x-2>0$ and $x^2-x-2<0$ (x is measured in radians). Then x lies in the interval **[1994]**

(a) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (b) $\left(-1, \frac{5\pi}{6}\right)$

(c) $(-1,2)$ (d) $\left(\frac{\pi}{6}, 2\right)$

27. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to **[1986 - 2 Marks]**

(a) 0 (b) 1
(c) 3 (d) $\sin 4\alpha + \cos 6\alpha$
(e) none of these

28. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to **[1984 - 3 Marks]**

(a) $\frac{1}{2}$ (b) $\cos\frac{\pi}{8}$ (c) $\frac{1}{8}$ (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

10 Subjective Problems

29. In any triangle ABC, prove that **[2000 - 3 Marks]**

$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

30. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n}=-\frac{n}{2}$, where $n \geq 3$ is an integer. **[1997 - 5 Marks]**

31. Prove that the values of the function $\frac{\sin x \cos 3x}{\sin 3x \cos x}$ do not lie between $\frac{1}{3}$ and 3 for any real x. **[1997 - 5 Marks]**

32. Find the smallest positive number p for which the equation $\cos(p \sin x) = \sin(p \cos x)$ has a solution $x \in [0,2\pi]$. **[1995 - 5 Marks]**

33. Determine the smallest positive value of x (in degrees) for which $\tan(x+100°)=\tan(x+50°)\tan(x)\tan(x-50°)$. **[1993 - 5 Marks]**

34. If exp $\{(\sin^2x + \sin^4x + \sin^6x + \ldots\ldots\ldots\ldots\ \infty)\ \ln 2\}$ satisfies the equation $x^2 - 9x + 8 = 0$, find the value of $\frac{\cos x}{\cos x+\sin x}, 0<x<\frac{\pi}{2}$. **[1991 - 4 Marks]**

35. ABC is a triangle such that

$\sin(2A+B)=\sin(C-A)=-\sin(B+2C)=\frac{1}{2}$.

If A, B and C are in arithmetic progression, determine the values of A, B and C. **[1990 - 5 Marks]**

36. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$ **[1988 - 2 Marks]**

37. Show that $16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right)=1$ **[1983 - 2 Marks]**

38. Without using tables, prove that

$(\sin 12°)(\sin 48°)(\sin 54°)=\frac{1}{8}$. **[1982 - 2 Marks]**

Topic-3: Solutions of Trignometric Equations

1 MCQs with One Correct Answer

1. Let $S=\left\{x\in(-\pi,\pi): x\neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to **[Adv. 2016]**

(a) $-\frac{7\pi}{9}$ (b) $-\frac{2\pi}{9}$ (c) 0 (d) $\frac{5\pi}{9}$

2. For $x\in(0,\pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has **[Adv. 2014]**

(a) infinitely many solutions
(b) three solutions
(c) one solution
(d) no solution

3. The number of solutions of the pair of equations

$2\sin^2\theta - \cos 2\theta = 0$
$2\cos^2\theta - 3\sin\theta = 0$

in the interval $[0, 2\pi]$ is **[2007 - 3 Marks]**

(a) zero (b) one (c) two (d) four

4. $\cos(\alpha-\beta)=1$ and $\cos(\alpha+\beta)=1/e$ where $\alpha, \beta \in [-\pi, \pi]$. Pairs of α, β which satisfy both the equations is/are **M[2005S]**

(a) 0 (b) 1 (c) 2 (d) 4

5. The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k+1$ has a solution is **[2002S]**

(a) 4 (b) 8 (c) 10 (d) 12

6. In a triangle PQR, $\angle R=\pi/2$. If $\tan(P/2)$ and $\tan(Q/2)$ are the roots of the equation $ax^2+bx+c=0\ (a\neq 0)$ then. **[1999 - 2 Marks]**

(a) $a+b=c$ (b) $b+c=a$
(c) $a+c=b$ (d) $b=c$

7. $\sec^2\theta=\frac{4xy}{(x+y)^2}$ is true if and only if **[1996 - 1 Mark]**

(a) $x+y\neq 0$ (b) $x=y, x\neq 0$

(c) $x = y$ (d) $x \neq 0, y \neq 0$

8. The general values of θ satisfying the equation $2\sin^2\theta - 3\sin\theta - 2 = 0$ is **[1995S]**

(a) $n\pi + (-1)^n \pi/6$ (b) $n\pi + (-1)^n \pi/2$

(c) $n\pi + (-1)^n 5\pi/6$ (d) $n\pi + (-1)^n 7\pi/6$

9. Let n be a positive integer such that

$\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{n}}{2}$. Then **[1994]**

(a) $6 \leq n \leq 8$ (b) $4 < n \leq 8$

(c) $4 \leq n \leq 8$ (d) $4 < n < 8$

10. Number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is : **[1993 - 1 Mark]**

(a) 0 (b) 1 (c) 2 (d) 3

11. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ In the variable x, has real roots. Then p can take any value in the interval **[1990 - 2 Marks]**

(a) $(0, 2\pi)$ (b) $(-\pi, 0)$ (c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $(0, \pi)$

12. The general solution of $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is **[1989 - 2 Marks]**

(a) $n\pi + \frac{\pi}{8}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$

(c) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$ (d) $2n\pi + \cos^{-1}\frac{3}{2}$

13. The value of the expression $\sqrt{3}$ cosec $20° -$ sec $20°$ is equal to **[1988 - 2 Marks]**

(a) 2 (b) 2 sin 20°/sin 40°

(c) 4 (d) 4 sin 20°/sin 40°

14. The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by : **[1981 - 2 Marks]**

(a) $x = 2n\pi$; $n=0, \pm1, \pm2...$

(b) $x = 2n\pi + \pi/2$; $n = 0, \pm1, \pm2...$

(c) $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$, where $n = 0, \pm1, \pm2...$

(d) none of these

15. The equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2}$; $0 < x \leq \frac{\pi}{2}$ has **[1980]**

(a) no real solution

(b) one real solution

(c) more than one solution

(d) none of these

2 Integer Value Answer/ Non-Negative Integer

16. The number of distinct solutions of the equation

$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

in the interval $[0, 2\pi]$ is **[Adv. 2015]**

17. The positive integer value of $n > 3$ satisfying the equation

$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is **[2011]**

18. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k > 0$, then the value of $[k]$ is **[2010]**

[**Note :** $[k]$ denotes the largest integer less than or equal to k]

19. The number of values of θ in the interval, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm1, \pm2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is **[2010]**

20. The number of all possible values of θ where $0 < \theta < \pi$, for which the system of equations

$(y + z)\cos 3\theta = (xyz)\sin 3\theta$

$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$

$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is **[2010]**

3 Numeric/ New Stem Based Questions

21. Let a, b, c be three non-zero real numbers such that the equation : $\sqrt{3}a\cos x + 2b\sin x = c, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is _______. **[Adv. 2018]**

4 Fill in the Blanks

22. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$ are ..., ..., and ________ . **[1997 - 2 Marks]**

23. General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is________. **[1996 - 1 Mark]**

24. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right)$, $\cos\left(\beta + \frac{\pi}{2}\right)$ and $\cos\left(\gamma + \frac{\pi}{2}\right)$ is equal to ________ **[1987 - 2 Marks]**

25. The set of all x in the interval $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1 \geq 0$, is ________. **[1987 - 2 Marks]**

26. The solution set of the system of equations $x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$, where x and y are real, is ________. **[1987 - 2 Marks]**

5 True / False

27. There exists a value of θ between 0 and 2π that satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1 = 0$. **[1984 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

28. Let α and β be non–zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true? **[Adv. 2017]**

(a) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(b) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

(c) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

29. The number of points in $(-\infty\ \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **[Adv. 2013]**

(a) 6 (b) 4 (c) 2 (d) 0

30. For $0 < \theta < \frac{\pi}{2}$, the solution (s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

is (are) **[2009]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

31. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is **[1998 - 2 Marks]**

(a) 0 (b) 5 (c) 6 (d) 10

32. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2\cos(2x) + a_3\sin^2(x) = 0$ for all x is **[1987 - 2 Marks]**

(a) zero (b) one (c) three (d) infinite

7 Match the Following

33. Consider the following lists: **[Adv. 2022]**

List-I	List-II
(I) $\left\{x \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] : \cos x + \sin x = 1\right\}$	(P) has two elements
(II) $\left\{x \in \left[-\frac{5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$	(Q) has three elements
(III) $\left\{x \in \left[-\frac{6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos(2x) = \sqrt{3}\right\}$	(R) has four elements
(IV) $\left\{x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$	(S) has five elements
	(T) has six elements

The correct option is:

(a) (I) → (P); (II) → (S); (III) → (P); (IV) → (S)
(b) (I) → (P); (II) →(P); (III) → (T); (IV) → (R)
(c) (I) → (Q); (II) → (P); (III) → (T); (IV) → (S)
(d) (I) → (Q); (II) → (S); (III) → (P); (IV) → (R)

34. Let $f(x) = \sin(\pi\cos x)$ and $g(x) = \cos(2\pi\sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order.

$X = \{x : f(x) = 0\}, Y = \{x : f'(x) = 0\}$

$Z = \{x : g(x) = 0\}, W = \{x : g'(x) = 0\}$

Column - I contains the sets X, Y, Z and W. Column - II contains some information regarding these sets. **[Adv. 2019]**

Column I	Column II
(I) X	(p) $\supseteq\left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II) Y	(q) an arithmetic progression
(III)Z	(r) NOT an arithmetic progression

(IV) W   (s) $\supseteq\left\{\frac{\pi}{6},\frac{7\pi}{6},\frac{13\pi}{6}\right\}$

(t) $\supseteq\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi\right\}$

(u) $\supseteq\left\{\frac{\pi}{6},\frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination?

(a) (IV), (p), (r), (s)   (b) (III), (p), (q), (u)
(c) (III), (r), (u)   (d) (IV), (q), (t)

35. Let $f(x)=\sin(\pi\cos x)$ and $g(x)=\cos(2\pi\sin x)$ be two functions defined for $x>0$. Define the following sets whose elements are written in the increasing order.

$X=\{x: f(x)=0\}, Y=\{x: f'(x)=0\}$

$Z=\{x: g(x)=0\}, W=\{x: g'(x)=0\}$

Column - I contains the sets X, Y, Z and W. Column - II contains some information regarding these sets. **[Adv. 2019]**

Column I	**Column II**
(I) X	(p) $\supseteq\left\{\frac{\pi}{2},\frac{3\pi}{2},4\pi,7\pi\right\}$
(II) Y	(q) an arithmetic progression
(III) Z	(r) NOT an arithmetic progression
(IV) W	(s) $\supseteq\left\{\frac{\pi}{6},\frac{7\pi}{6},\frac{13\pi}{6}\right\}$
	(t) $\supseteq\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi\right\}$
	(u) $\supseteq\left\{\frac{\pi}{6},\frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination?

(a) (I), (q), (u)   (b) (I), (p), (r)
(c) (II), (r), (s)   (d) (II), (q), (t)

10 Subjective Problems

36. Find all values of θ in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ satisfying the equation $(1-\tan\theta)(1+\tan\theta)\sec^2\theta+2^{\tan^2\theta}=0.$ **[1996 - 2 Marks]**

37. Find the values of $x\in(-\pi,+\pi)$ which satisfy the equation $8^{(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+......)}=4^3$ **[1984 - 2 Marks]**

Answer Key

Topic-1 : Trigonometric Ratios, Domain and Range of Trigonometric Functions, Trigonometric Ratios of Allied Angles

1. (b) **2.** (b) **3.** (b) **4.** (c)
5. $(A\to r; B\to p)$

Topic-2 : Trigonometric Identities, Greatest and Least Value of Trigonometric Expressions

1. (d) **2.** (b) **3.** (c) **4.** (b) **5.** (a) **6.** (c) **7.** (a) **8.** (c) **9.** (c) **10.** (b)
11. (b) **12.** (a) **13.** (1) **14.** (2) **15.** (1) **16.** $\frac{1}{3}$ **17.** $\frac{1}{8}$ **18.** $\frac{1}{64}$ **19.** (6) **20.** (True)
21. (b, c) **22.** (a, c, d) **23.** (a, b) **24.** (a, b, c, d) **25.** (c) **26.** (d) **27.** (b) **28.** (c)

Topic-3 : Solutions of Trigonometric Equations

1. (c) **2.** (d) **3.** (c) **4.** (d) **5.** (b) **6.** (a) **7.** (b) **8.** (d) **9.** (d) **10.** (c)
11. (d) **12.** (b) **13.** (c) **14.** (c) **15.** (a) **16.** (8) **17.** (7) **18.** (3) **19.** (3) **20.** (3)
21. (0.5) **22.** $-\frac{\pi}{2},\frac{\pi}{2},0$ **23.** $n\pi, n\pi\pm\frac{\pi}{3}$ **24.** $-\frac{\sqrt{3}}{2}$ **25.** $\left[0,\frac{\pi}{6}\right]\cup\left\{\frac{\pi}{2}\right\}\cup\left[\frac{5\pi}{6},\pi\right]$ **26.** φ
27. (False) **28.** (a, c) **29.** (c) **30.** (c, d) **31.** (c) **32.** (d) **33.** (b) **34.** (a) **35.** (d)

Hints & Solutions

Topic-1: Trigonometric Ratios, Domain and Range of Trigonometric Functions, Trigonometric Ratios of Allied Angles

1. **(b)** Given expression can be written as

$$\frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A} = 1 + \sec A \operatorname{cosec} A$$

2. **(b)** Given : $\sin\theta = 1/2$ and θ is acute angle

$\therefore \theta = \pi/6$

Also given, $\cos\phi = \frac{1}{3}$ and ϕ is acute angle.

$\therefore 0 < \frac{1}{3} < \frac{1}{2}$

$\Rightarrow \cos\pi/2 < \cos\phi < \cos\pi/3$ or $\pi/3 < \phi < \pi/2$

$\therefore \frac{\pi}{3} + \frac{\pi}{6} < \theta + \phi < \frac{\pi}{2} + \frac{\pi}{6}$ or $\frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3}$

$\Rightarrow \theta + \phi \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

3. **(b)** $\tan\theta = \frac{-4}{3} \Rightarrow \theta \in$ II quad or IV quad .

$\Rightarrow 0 < \sin\theta < 1$ or $-1 < \sin\theta < 0$

$\Rightarrow \sin\theta = \frac{4}{5}$ or $-\frac{4}{5}$

4. **(c)** We know, $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (irrational)

$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ (irrational)

$\sin 15° . \cos 15° = \frac{1}{2}$ $(2 \sin 15° \cos 15°)$

$= \frac{1}{2}\sin 30° = \frac{1}{4}$ (rational)

$\sin 15° \cos 75° = \sin 15° \cos (90 - 15°)$

$= \sin 15° \sin 15° = \sin^2 15° = \frac{1}{2}(1 - \cos 30°)$

$= \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$ (irrational)

5. **(A → r; B → p)**

(p) If $\frac{13\pi}{48} < \alpha < \frac{14\pi}{48}$ then $\frac{13\pi}{16} < 3\alpha < \frac{14\pi}{16}$

and $\frac{13\pi}{24} < 2\alpha < \frac{14\pi}{24}$

$\Rightarrow 3\alpha \in$ II quad and $2\alpha \in$ II quad $\Rightarrow \sin 3\alpha = +$ ve

$\cos 2\alpha = -$ ve $\therefore \frac{\sin 3\alpha}{\cos 2\alpha} = -ve$

$\therefore$ (B) corresponds to (p).

(q) If $\alpha \in \left(\frac{14\pi}{48}, \frac{18\pi}{48}\right)$ then $\frac{14\pi}{16} < 3\alpha < \frac{18\pi}{16}$

and $\frac{14\pi}{24} < 2\alpha < \frac{18\pi}{24}$

$\Rightarrow 3\alpha \in$ II or III quad and $2\alpha \in$ II quad

$\Rightarrow$ Nothing can be said about the sign of $\frac{\sin 3\alpha}{\cos 2\alpha}$ over this interval.

(r) If $\alpha \in \left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$ then $\frac{18\pi}{16} < 3\alpha < \frac{23\pi}{16}$

and $\frac{18\pi}{24} < 2\alpha < \frac{23\pi}{24}$

$\Rightarrow$ $3\alpha \in$ III quad and $2\alpha \in$ II quad

$\Rightarrow$ $\sin 3\alpha = -$ ve, $\cos 2a = -$ ve, $\therefore \frac{\sin 3\alpha}{\cos 2\alpha} = +ve$

$\therefore$ (A) corresponds to (r)

(s) If $\alpha \in (0, \pi/2)$, then $0 < 3\alpha < 3\pi/2$ and $2 < 2\alpha < \pi$

$\Rightarrow$ Nothing can be said about the sign of $\frac{\sin 3\alpha}{\cos 2\alpha}$ over the given interval.

6. Given : $2 \sin t = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$, $t \in [-\pi/2, \pi/2]$

$\Rightarrow (6 \sin t - 5)x^2 + 2(1 - 2\sin t)x - (1 + 2\sin t) = 0$

The given equation will hold, if x be some real number, and hence, $D \geq 0$

$\Rightarrow$ $4(1 - 2\sin t)^2 + 4(6\sin t - 5)(1 + 2\sin t) \geq 0$

$\Rightarrow$ $16\sin^2 t - 8\sin t - 4 \geq 0 \Rightarrow (4\sin^2 t - 2\sin t - 1) \geq 0$

$\Rightarrow$ $4\left(\sin t - \frac{\sqrt{5}+1}{4}\right)\left(\sin t + \frac{\sqrt{5}-1}{4}\right) \geq 0$

$\Rightarrow$ $\sin t \leq -\left(\frac{\sqrt{5}-1}{4}\right)$ or $\sin t \geq \frac{\sqrt{5}+1}{4}$

$\Rightarrow$ $\sin t \leq \sin(-\pi/10)$ or $\sin t \geq \sin(3\pi/10)$

$\Rightarrow$ $t \leq -\pi/10$ or $t \geq 3\pi/10$

[Note that $\sin x$ is an increasing function from $-\pi/2$ to $\pi/2$]

$\therefore$ range of t is $[-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2]$

Topic-2: Trigonometric Identities, Greatest and Least Value of Trigonometric Expressions

1. **(d)** Given, $f(x)=\begin{cases} x^2\sin\left(\dfrac{\pi}{x^2}\right), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

$$f(x)=0\Rightarrow \sin\left(\frac{\pi}{x^2}\right)=0$$

$$\Rightarrow \frac{\pi}{x^2}=n\pi\Rightarrow x^2=\frac{1}{n}\Rightarrow x=\frac{1}{\sqrt{n}}$$

(a) If $x\in\left[\dfrac{1}{10^{10}},\infty\right)$

$$\frac{1}{\sqrt{n}}\in\left[\frac{1}{10^{10}},\infty\right)$$

$$\sqrt{n}\in\left(0,10^{10}\right]$$

$$n\in\left(0,(10^{10})^2\right]$$

Finite values of n are possible, so has finite solution.

(b) If $x\in\left[\dfrac{1}{\pi},\infty\right)\Rightarrow \dfrac{1}{\sqrt{n}}\in\left[\dfrac{1}{\pi},\infty\right)$

$$\sqrt{n}\in(0,\pi]\Rightarrow n\in\left(0,\pi^2\right]\Rightarrow n=1,2,3,....9$$

(c) If $x\in\left(0,\dfrac{1}{10^{10}}\right)\Rightarrow \sqrt{n}\in(10^{10},\infty)$

n is infinite

(d) If $x\in\left(\dfrac{1}{\pi^2},\dfrac{1}{\pi}\right)\Rightarrow\sqrt{n}\in(\pi,\pi^2)$

$$n\in(\pi^2,\pi^4)\Rightarrow n\in(9.8,97.2....)$$

More than 25 solutions

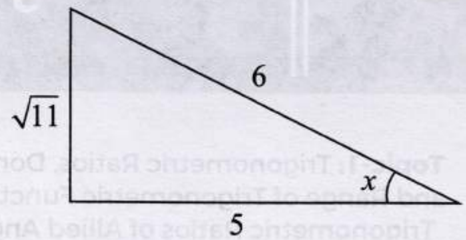
2. **(b)** Let $E=\left(\sin 6x\cos\dfrac{11x}{2}-\cos 6x\sin\dfrac{11x}{2}\right)+\left(\cos 6x\cos\dfrac{11x}{2}+\sin 6x\sin\dfrac{11x}{2}\right)$

$$=\left(\sin\left(6x-\frac{11}{2}x\right)+\cos\left(6x-\frac{11x}{2}\right)\right)$$

$$=\sin\frac{x}{2}+\cos\frac{x}{2}$$

Now, $E^2=1+\sin x$ ...(i)

$$\therefore \cot x=\frac{-5}{\sqrt{11}}$$



$$\therefore E^2=1+\frac{\sqrt{11}}{6}$$

$$\therefore E=\sqrt{\frac{6+\sqrt{11}}{6}}=\sqrt{\frac{12+2\sqrt{11}}{12}}=\frac{\sqrt{11}+1}{2\sqrt{3}}$$

3. **(c)** $\displaystyle\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$

$$=\sum_{k=1}^{13}\frac{1}{\sin\frac{\pi}{6}}\left[\frac{\sin\left\{\frac{\pi}{4}+\frac{k\pi}{6}-\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right)\right\}}{\sin\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}\right]$$

$$\sum_{k=1}^{13}2\left[\cot\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)\right]$$

$$=2\left[\left\{\cot\frac{\pi}{4}-\cot\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right\}+\left\{\cot\left(\frac{\pi}{4}+\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{2\pi}{6}\right)\right\}+.......+\left\{\cot\left(\frac{\pi}{4}+\frac{12\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{13\pi}{6}\right)\right\}\right]$$

$$=2\left[\cot\frac{\pi}{4}-\cot\left(\frac{\pi}{4}+\frac{13\pi}{6}\right)\right]=2\left[1-\cot\frac{5\pi}{12}\right]$$

$$=2\left[1-\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]=2\left[1-\left(2-\sqrt{3}\right)\right]=2\left(\sqrt{3}-1\right)$$

4. **(b)** Given : $\theta\in\left(0,\dfrac{\pi}{4}\right)\Rightarrow \tan\theta<1$ and $\cot\theta>1$

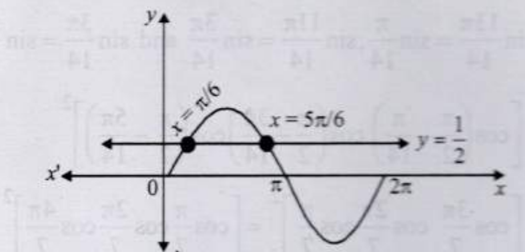
Let $\tan\theta=1-x$ and $\cot\theta=1+y$,
where $x, y>0$ and are very small, then
$\therefore\ t_1=(1-x)^{1-x},\ t_2=(1-x)^{1+y}$,
$t_3=(1+y)^{1-x},\ t_4=(1+y)^{1+y}$
Clearly, $t_4>t_3$ and $t_1>t_2$ also, $t_3>t_1$
$\therefore\ t_4>t_3>t_1>t_2$.

5. **(a)** $2\sin^2\theta-5\sin\theta+2>0$

$$\Rightarrow (\sin\theta-2)(2\sin\theta-1)>0$$

$\because -1\le\sin\theta\le 1,\Rightarrow(\sin\theta-2)<0$, so $(2\sin\theta-1)<0$

$$\therefore \sin\theta<\frac{1}{2}$$



From graph, we get $x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

6. **(c)** Given : $\alpha + \beta = \pi/2 \Rightarrow \alpha = \pi/2 - \beta$

$\Rightarrow \tan\alpha = \tan(\pi/2 - \beta) = \cot\beta = \frac{1}{\tan\beta}$

$\Rightarrow \tan\alpha \tan\beta = 1 \Rightarrow 1 + \tan\alpha\tan\beta = 2.$

Now, $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\Rightarrow \tan\gamma = \frac{\tan\alpha - \tan\beta}{2}$

$\Rightarrow 2\tan\gamma = \tan\alpha - \tan\beta \Rightarrow \tan\alpha = 2\tan\gamma + \tan\beta$

7. **(a)** Given : $(\cot\alpha_1).(\cot\alpha_2)\,....\,(\cot\alpha_n) = 1$

$\Rightarrow (\cos\alpha_1)(\cos\alpha_2)\,....\,(\cos\alpha_n)$

$= (\sin\alpha_1)\ (\sin\alpha_2)\,....(\sin\alpha_n)$ ....(i)

Let $y = (\cos\alpha_1)(\cos\alpha_2)\,....\,(\cos\alpha_n)$ (to be max.)

$\Rightarrow y^2 = (\cos^2\alpha_1)\ (\cos^2\alpha_2)\,....\,(\cos^2\alpha_n)$

$= \cos\alpha_1 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 \,....\, \cos\alpha_n \sin\alpha_n$ (From(i))

$= \frac{1}{2^n}[\sin 2\alpha_1 \sin 2\alpha_2 \,....\, \sin 2\alpha_n]$

Now, $0 \le \alpha_1, \alpha_2, ...\alpha_n \le \pi/2$

$\therefore 0 \le 2\alpha_1, 2\alpha_2, ..., 2\alpha_n \le \pi$

$\Rightarrow 0 \le \sin 2\alpha_1, \sin 2\alpha_2, ..., \sin 2\alpha_n \le 1$

$\therefore y^2 \le \frac{1}{2^n}.1 \Rightarrow y \le \frac{1}{2^{n/2}}$

$\therefore$ Max. value of y i.e. $(\cos\alpha_1).(\cos\alpha_2)\,....(\cos\alpha_n) = \frac{1}{2^{n/2}}$.

8. **(c)** $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$= \sin\theta(\sin\theta + 3\sin\theta - 4\sin^3\theta)$

$= \sin\theta(4\sin\theta - 4\sin^3\theta) = \sin^2\theta\,(4 - 4\sin^2\theta)$

$= 4\sin^2\theta\ (1 - \sin^2\theta)$

$= 4\sin^2\theta\ \cos^2\theta = (2\sin\theta\cos\theta)^2 = (\sin 2\theta)^2 \ge 0,$

which is true for all θ.

9. **(c)** $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3(1 - \sin 2x)^2 + 6(1 + \sin 2x)$

$+ 4[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)]$

$= 3 - 6\sin 2x + 3\sin^2 2x + 6 + 6\sin 2x + 4\left[1 - \frac{3}{4}\sin^2 2x\right]$

$= 13 + 3\sin^2 2x - 3\sin^2 2x = 13$

10. **(b)** $\sec 2x - \tan 2x = \frac{1 - \sin 2x}{\cos 2x} = \frac{1 - \cos\left(\frac{\pi}{2} - 2x\right)}{\sin\left(\frac{\pi}{2} - 2x\right)}$

$= \frac{2\sin^2\left(\frac{\pi}{4} - x\right)}{2\sin\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - x\right)} = \tan\left(\frac{\pi}{4} - x\right)$

11. **(b)** $A = \sin^2\theta + \cos^4\theta = \sin^2\theta + (1 - \sin^2\theta)^2$

$= \sin^4\theta - \sin^2\theta + 1 \Rightarrow A = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4}$

Now, $0 \le \left(\sin^2\theta - \frac{1}{2}\right)^2 \le \frac{1}{4}$

$\Rightarrow \frac{3}{4} \le \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \le 1 \Rightarrow \frac{3}{4} \le A \le 1$

12. **(a)** $\alpha + \beta + \gamma = 2\pi \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi$

$\therefore \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(\pi - \frac{\gamma}{2}\right) = -\tan\frac{\gamma}{2}$

$\Rightarrow \frac{\tan\alpha/2 + \tan\beta/2}{1 - \tan\alpha/2\tan\beta/2} = -\tan\gamma/2$

$\Rightarrow \tan\alpha/2 + \tan\beta/2 + \tan\gamma/2 = \tan\alpha/2\,\tan\beta/2\,\tan\gamma/2$

13. **(1)** Rearrange the given expression

$\left(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\alpha}{\sin\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\sin\beta}{\cos\alpha}\right)^2$

$= \left(\frac{\cos(\alpha - \beta)}{\sin\beta\cos\beta} + \frac{\cos(\alpha - \beta)}{\sin\alpha.\cos\alpha}\right)^2$

$= \left(\frac{4}{3}\left\{\frac{1}{\sin 2\beta} + \frac{1}{\sin 2\alpha}\right\}\right)^2 \qquad \left[\because \cos(\alpha - \beta) = \frac{2}{3}\right]$

$= \frac{16}{9}\left[\frac{\sin 2\alpha + \sin 2\beta}{\sin 2\alpha.\sin 2\beta}\right]$

$= \frac{64}{9}\left(\frac{2\sin(\alpha + \beta).\cos(\alpha - \beta)}{2\sin 2\alpha.\sin 2\beta}\right)^2$

$= \frac{64}{9}\left(\frac{2.\frac{1}{3}.\frac{2}{3}}{\cos(2\alpha - 2\beta) - \cos(2\alpha + 2\beta)}\right)^2$

$= \frac{64}{9}\left(\frac{\frac{4}{9}}{2\cos^2(\alpha - \beta) - 1 - 1 + 2\sin^2(\alpha + \beta)}\right)^2$

$= \frac{64}{9}\left(\frac{\frac{4}{9}}{\frac{8}{9} - 2 + \frac{2}{9}}\right)^2 = \frac{64}{9}\left(\frac{1}{-2}\right)^2 = \left[\frac{16}{9}\right] = [1.7] = 1$

14. **(2)** Let $f(\theta) = \frac{1}{g(\theta)}$,

where $g(\theta) = \sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$

Clearly f is maximum when g is minimum

Now $g(\theta) = \frac{1 - \cos 2\theta}{2} + \frac{3}{2}\sin 2\theta + \frac{5}{2}(1 + \cos 2\theta)$

$$= 3+2\cos 2\theta+\frac{3}{2}\sin 2\theta \ge 3+\left(-\sqrt{4+\frac{9}{4}}\right)$$

$$\therefore\ g_{\min}=3-\frac{5}{2}=\frac{1}{2}\ \Rightarrow f_{\max}=2.$$

15. (1) Let $\pi x-\frac{\pi}{4}=\theta\in\left[\frac{-\pi}{4},\frac{7\pi}{4}\right]$

$\because f(x)\ge 0$

So, $\left(3-\sin\left(\frac{\pi}{2}+2\theta\right)\right)\sin\theta\ge\sin(\pi+3\theta)$

$\Rightarrow (3-\cos 2\theta)\sin\theta\ge-\sin 3\theta$

$\Rightarrow \sin\theta\,[3-4\sin^2\theta+3-\cos 2\theta]\ge 0$

$\Rightarrow \sin\theta\,(6-2(1-\cos 2\theta)-\cos 2\theta)\ge 0$

$\Rightarrow \sin\theta\,(4+\cos 2\theta)\ge 0\Rightarrow\sin\theta\ge 0$

$\Rightarrow \theta\in[0,\pi]\Rightarrow 0\le\pi x-\frac{\pi}{4}\le\pi\ \Rightarrow x\in\left[\frac{1}{4},\frac{5}{4}\right]$

$\Rightarrow [\alpha,\beta]=\left[\frac{1}{4},\frac{5}{4}\right];\ \therefore\beta-\alpha=\frac{5}{4}-\frac{1}{4}=1$

16. Given : $A+B=\pi/3\ \Rightarrow\tan(A+B)=\sqrt{3}$

$\Rightarrow\ \frac{\tan A+\tan B}{1-\tan A\tan B}=\sqrt{3}\Rightarrow\frac{\tan A+\frac{y}{\tan A}}{1-y}=\sqrt{3}$ [Let $y=\tan A\tan B$]

$\Rightarrow\ \tan^2 A+\sqrt{3}\ (y-1)\tan A+y=0$

For real value of tan A, $3(y-1)^2-4y\ge 0$

$\Rightarrow\ 3y^2-10y+3\ge 0\Rightarrow(y-3)\left(y-\frac{1}{3}\right)\ge 0$

$\Rightarrow\ y\le\frac{1}{3}$ or ≥ 3

But $A, B>0$ and $A+B=\pi/3\Rightarrow A, B<\pi/3$

$\Rightarrow\ \tan A\tan B<3$

$\Rightarrow\ y\le\frac{1}{3}\Rightarrow$ Max. value of y is 1/3.

17. $K=\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$

$=\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{2^3\sin\frac{\pi}{9}}.\sin\frac{8\pi}{9}$

[$\because\cos\alpha\cos 2\alpha\ \cos 2^2\alpha$ .... $\cos 2^{n-1}\alpha=\frac{1}{2^n\sin\alpha}.\sin(2^n\alpha)$]

$=\frac{1}{8\sin\pi/9}.\sin\pi/9=\frac{1}{8}$

18. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$=\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2\times 1$

$\left[\because\ \sin\frac{13\pi}{14}=\sin\frac{\pi}{14},\sin\frac{11\pi}{14}=\sin\frac{3\pi}{14}\text{ and }\sin\frac{3\pi}{14}=\sin\frac{5\pi}{14}\right]$

$=\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right]^2$

$=\left[\cos\frac{3\pi}{7}\ \cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right]^2=\left[\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\right]^2$

[$\because\cos\alpha\cos 2\alpha\ \ \cos 2^2\alpha$ .... $\cos 2^{n-1}\alpha$

$=\left[\frac{1}{2^n\sin\alpha}.\sin(2^n\alpha)\right]^2$

$=\left(\frac{1}{8\sin\pi/7}\sin\frac{8\pi}{7}\right)^2=\left(\frac{\sin(\pi+\pi/7)}{8\sin\pi/7}\right)^2$

$=\left(\frac{-\sin\pi/7}{8\sin\pi/7}\right)^2=\left(\frac{1}{8}\right)^2=\frac{1}{64}$

19. Given $\sin^3 x.\sin 3x=\sum_{m=0}^{n}C_m\cos mx$

$\sin^3 x\sin 3x=\frac{1}{4}[3\sin x-\sin 3x]\sin 3x$

$=\frac{1}{4}\left[\frac{3}{2}.2\sin x.\sin 3x-\sin^2 3x\right]$

$=\frac{1}{4}\left[\frac{3}{2}(\cos 2x-\cos 4x)-\frac{1}{2}(1-\cos 6x)\right]$

$=\frac{1}{8}\ [\cos 6x+3\cos 2x-3\cos 4x-1]$

$\therefore$ Max value of $m=6\ \ \therefore\ \ n=6$

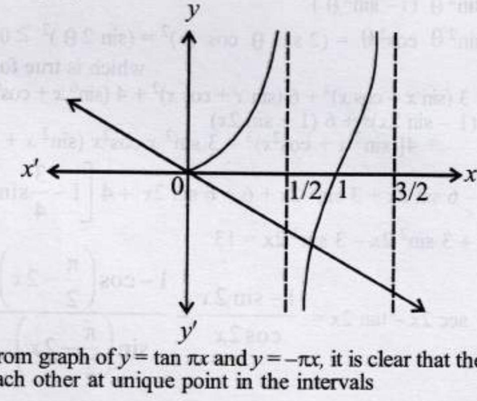
20. **(True)** $\tan A=\frac{1-\cos B}{\sin B}=\frac{2\sin^2 B/2}{2\sin B/2\cos B/2}=\tan B/2$

Now, $\tan 2A=\frac{2\tan A}{1-\tan^2 A}=\frac{2\tan B/2}{1-\tan^2 B/2}=\tan B$ Hence, statement is true.

21. **(b, c)** Given : $f(x)=x\sin\pi x, x>0$

$\Rightarrow\ f'(x)=\sin\pi x+x\pi\cos\pi x$

Now, $f'(x)=0\ \Rightarrow\tan\pi x=-\pi x$



From graph of $y=\tan\pi x$ and $y=-\pi x$, it is clear that they intersect each other at unique point in the intervals

$(n, n+1)$ and $\left(n+\frac{1}{2}, n+1\right)$

22. (a, c, d)

As $\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$, $\theta \in [0, 2\pi]$

Hence $\frac{3\pi}{2} < \theta < \frac{5\pi}{3}$

Now $2\cos\theta(1 - \sin\varphi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\varphi - 1$

$\Rightarrow 2\cos\theta(1 - \sin\varphi) = 2\sin\theta\cos\varphi - 1$

$\Rightarrow 2\cos\theta + 1 = 2\sin(\theta + \varphi)$

As $\theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$, $1 < 2\sin(\theta + \varphi) < 2$

As $\theta + \varphi \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ or $(\theta + \varphi) \in \left(\frac{13\pi}{6}, \frac{17\pi}{6}\right)$

We have $\varphi \in \left(-\frac{3\pi}{2}, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \frac{7\pi}{6}\right)$

23. (a, b) Given :

$$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5} \Rightarrow 3\sin^4 x + 2\cos^4 x = \frac{6}{5}$$

$$\Rightarrow \sin^4 x + 2[\sin^4 x + \cos^4 x] = \frac{6}{5}$$

$$\Rightarrow \sin^4 x + 2[1 - 2\sin^2 x\cos^2 x] = \frac{6}{5}$$

$$\Rightarrow \sin^4 x + 2 - 4\sin^2 x(1 - \sin^2 x) = \frac{6}{5}$$

$$\Rightarrow 5\sin^4 x - 4\sin^2 x + 2 - \frac{6}{5} = 0$$

$$\Rightarrow 25\sin^4 x - 20\sin^2 x + 4 = 0$$

$$\Rightarrow (5\sin^2 x - 2)^2 = 0 \Rightarrow \sin^2 x = \frac{2}{5}$$

$$\Rightarrow \cos^2 x = \frac{3}{5} \text{ and } \tan^2 x = \frac{2}{3}$$

Also $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{625} + \frac{3}{625} = \frac{5}{625} = \frac{1}{125}$

24. (a, b, c, d) Note that multiplicative loop is very important approach in IIT Mathematics

$$\left(\tan\frac{\theta}{2}\right)(1 + \sec\theta) = \frac{\sin\theta/2}{\cos\theta/2}\left[1 + \frac{1}{\cos\theta}\right]$$

$$= \frac{(\sin\theta/2)\,2\cos^2\theta/2}{(\cos\theta/2)\cos\theta}$$

$$= \frac{(2\sin\theta/2)\cos\theta/2}{\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\therefore$ $f_n(\theta) = (\tan\theta/2)(1 + \sec\theta)$
$(1 + \sec 2\theta)(1 + \sec 2^2\theta) \ldots (1 + \sec 2^n\theta)$
$= (\tan\theta)(1 + \sec 2\theta)(1 + \sec 2^2\theta) \ldots (1 + \sec 2^n\theta)$
$= \tan 2\theta.(1 + \sec 2^2\theta) \ldots (1 + \sec 2^n\theta) = \tan(2^n\theta)$

Now, $f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2.\frac{\pi}{16}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Therefore, (a) is the correct option.

$f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3.\frac{\pi}{32}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Therefore, (b) is the correct option.

$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4.\frac{\pi}{64}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Therefore, (c) is the correct option.

$f_5\left(\frac{\pi}{128}\right) = \tan\left(2^5.\frac{\pi}{128}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Therefore, (d) is the correct option.

25. (c) $\sin\alpha + \sin\beta + \sin\gamma$

$$= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$$

$$= 2\sin\left(\frac{\pi}{2} - \frac{\gamma}{2}\right)\cos\frac{\alpha-\beta}{2} + 2\sin\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right)\cos\frac{\gamma}{2}$$

$$= 2\cos\frac{\gamma}{2}\left[\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\right]$$

$$= 2\cos(\alpha/2)\cos(\beta/2)\cos(\gamma/2)$$

$\therefore$ Each $\cos((\alpha/2), \cos(\beta/2), \cos(\gamma/2)$ lies between -1 and 1.

$\Rightarrow -1 \le \cos\alpha/2.\cos\beta/2.\cos\gamma/2 \le 1$

$\Rightarrow -2 \le 2\cos\alpha/2.\cos\beta/2.\cos\gamma/2 \le 2$

$\Rightarrow -2 \le \cos\alpha + \cos\beta + \cos\gamma \le 2$

$\therefore$ Min value of $\sin\alpha + \sin\beta + \sin\gamma = -2$.

26. (d) $2\sin^2 x + 3\sin x - 2 > 0$
$(2\sin x - 1)(\sin x + 2) > 0$
$\Rightarrow 2\sin x - 1 > 0$ $(\because -1 \le \sin x \le 1)$
$\Rightarrow \sin x > 1/2 \Rightarrow x \in (\pi/6, 5\pi/6)$ ....(i)
Also $x^2 - x - 2 < 0$
$\Rightarrow (x-2)(x+1) < 0 \Rightarrow -1 < x < 2$ ....(ii)
On combining (i) and (ii), we get $x \in (\pi/6, 2)$.

27. (b) $3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right]$

$$-2[\sin^6(\pi/2 + \alpha) + \sin^6(5\pi - \alpha)]$$

$$= 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha]$$

$$= 3\left[(\cos^2\alpha + \sin^2\alpha)^2 - 2\sin^2\alpha\cos^2\alpha\right]$$

$$-2\left[(\cos^2\alpha + \sin^2\alpha)^3 - 3\cos^2\alpha\sin^2\alpha(\cos^2\alpha + \sin^2\alpha)\right]$$

$$= 3[1 - 2\sin^2\alpha\cos^2\alpha] - 2[1 - 3\cos^2\alpha\sin^2\alpha]$$

$$= 3 - 6\sin^2\alpha\cos^2\alpha - 2 + 6\sin^2\alpha\cos^2\alpha = 1$$

28. (c) Given,

$$(1 + \cos\pi/8)(1 + \cos 3\pi/8)(1 + \cos 5\pi/8)(1 + \cos 7\pi/8)$$

$$= (1 + \cos\pi/8)(1 + \cos 3\pi/8)(1 + \cos(\pi - 3\pi/8))(1 + \cos(\pi - \pi/8))$$

$$= (1 + \cos\pi/8)(1 + \cos 3\pi/8)(1 - \cos 3\pi/8)(1 - \cos\pi/8)$$

$$= (1 - \cos^2\pi/8)(1 - \cos^2 3\pi/8) = \sin^2\pi/8\sin^2 3\pi/8$$

$$= \frac{1}{4}.[2\sin\pi/8\sin(\pi/2 - \pi/8)]^2$$

$$= \frac{1}{4}.[2\sin\pi/8\cos\pi/8)]^2 = \frac{1}{4}.\sin^2\pi/4 = \frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$$

29. $\because A+B+C=\pi$

$$\Rightarrow \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2} \Rightarrow \frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$$

$$\Rightarrow \cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \frac{\cot\frac{A}{2}.\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\tan\frac{C}{2}$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$$

30. Let $S=\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n}$

$$\Rightarrow S=(n-1)\cos\frac{2\pi}{n}+(n-2)\cos 2.\frac{2\pi}{n}+......+1\cos\frac{2(n-1)\pi}{n} \quad ...(i)$$

We know that $\cos\theta=\cos(2\pi-\theta)$
Replacing each angle θ by $2\pi-\theta$ in (i), we get

$$S=(n-1)\cos(n-1)\frac{2\pi}{n}+(n-2)\cos(n-2)\frac{2\pi}{n}+......+1.\cos\frac{2\pi}{n} \quad ...(ii)$$

On adding terms in (i) and (ii) having the same angle and taking n common, we get

$$\therefore 2S=n\left[\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+....+\cos(n-1)\frac{2\pi}{n}\right]$$

Angles are in A.P. with common difference $(d)=\frac{2\pi}{n}$

$$2S=n\left[\frac{\sin(n-1)\frac{\pi}{n}}{\sin\frac{\pi}{n}}\cos\frac{\frac{2\pi}{n}+(n-1)\frac{2\pi}{n}}{2}\right]$$

$=n.1\cos\pi=-n$, $\because \sin(\pi-\theta)=\sin\theta$, $\therefore S=-n/2$

31. Let $y=\frac{\sin x\cos 3x}{\sin 3x\cos x}=\frac{\tan x}{\tan 3x}$

$$=\frac{\tan x(1-3\tan^2 x)}{3\tan x-\tan^3 x}=\frac{1-3\tan^2 x}{3-\tan^2 x}$$

$\Rightarrow 3y-(\tan^2 x)y=1-3\tan^2 x \Rightarrow 3y-1=(y-3)\tan^2 x$

$$\Rightarrow \tan^2 x=\frac{3y-1}{y-3}=\frac{(3y-1)(y-3)}{(y-3)^2}$$

Since, $\tan^2 x>0$, $\therefore (3y-1)(y-3)>0$

$$\Rightarrow \left(y-\frac{1}{3}\right)(y-3)>0 \Rightarrow y<\frac{1}{3} \text{ or } y>3$$

$\therefore y$ cannot lie between $\frac{1}{3}$ and 3.

32. Given : $\cos\theta=\sin\phi$, where $\theta=p\sin x$, $\phi=p\cos x$

Above is possible when both $\theta=\phi=\frac{\pi}{4}$ or $\theta=\phi=\frac{5\pi}{4}$

$\therefore p\sin x=\frac{\pi}{4}$ or $p\sin x=\frac{5\pi}{4}$

and $p\cos x=\frac{\pi}{4}$ or $p\cos x=\frac{5\pi}{4}$

On squaring and adding, $p^2=\frac{\pi^2}{16}.2$ or $\frac{25\pi^2}{16}.2$

$\therefore p=\frac{\pi}{4}\sqrt{2}$ only for least positive value

33. Given : $\tan(x+100°)=\tan(x+50°)\tan x\tan(x-50°)$

$$\Rightarrow \frac{\tan(x+100°)}{\tan x}=\tan(x+50°)\tan(x-50°)$$

$$\Rightarrow \frac{\sin(x+100°)\cos x}{\cos(x+100°)\sin x}=\frac{\sin(x+50°)\sin(x-50°)}{\cos(x+50°)\cos(x-50°)}$$

$$\Rightarrow \frac{\sin(2x+100°)+\sin 100°}{\sin(2x+100°)-\sin 100°}=\frac{\cos 100°-\cos 2x}{\cos 100°+\cos 2x}$$

By componendo and dividendo,

$$\Rightarrow \frac{2\sin(2x+100°)}{2\sin 100°}=\frac{2\cos 100°}{-2\cos 2x}$$

$\Rightarrow 2\sin(2x+100°)\cos 2x=-2\sin 100°\cos 100°$
$\Rightarrow \sin(4x+100°)+\sin 100°=-\sin 200°$
$\Rightarrow \sin(4x+10°+90°)+\sin(90°+10°)=-\sin(180+20°)$
$\Rightarrow \cos(4x+10°)+\cos 10°=\sin 20°$
$\Rightarrow \cos(4x+10°)=\sin 20°-\cos 10°$
$\Rightarrow \cos(4x+10°)=\sin 20°-\sin 80°$

$=-2\cos 50°\sin 30°=-2\cos 50°.\frac{1}{2}$

$=-\cos 50°=\cos 130° \Rightarrow 4x+10°=130° \Rightarrow x=30°$

34. Let $y=\exp[\sin^2 x+\sin^4 x+\sin^6 x+....\infty]\ln 2$

$=e^{In}\cdot 2^{\sin^2 x+\sin^4 x+\sin^6 x+....\infty}$

$$=2^{\sin^2 x+\sin^4 x+\sin^6 x+....\infty}=\frac{\sin^2 x}{2^{1-\sin^2 x}}=2^{\tan^2 x}$$

As y satisfies the eq. $x^2-9x+8=0$
$\therefore y^2-9y+8=0$
$\Rightarrow (y-1)(y-8)=0 \Rightarrow y=1, 8$

$\Rightarrow 2^{\tan^2 x}=1$ or $2^{\tan^2 x}=8$

$\Rightarrow \tan^2 x=0$ or $\tan^2 x=3$

$\Rightarrow \tan x=0$ or $\tan x=\sqrt{3},-\sqrt{3}$
$\Rightarrow x=0$ or $x=\pi/3, 2\pi/3$
But given $0<x<\pi/2 \Rightarrow x=\pi/3$

$$\therefore \frac{\cos x}{\cos x+\sin x}=\frac{1}{1+\tan x}=\frac{1}{1+\sqrt{3}}=\frac{\sqrt{3}-1}{2}$$

35. Given : In ΔABC; A, B and C are in A.P.
$\therefore A+C=2B$
Also $A+B+C=180° \Rightarrow B+2B=180° \Rightarrow B=60°$

Also given that, $\sin(2A+B)=\sin(C-A)=-\sin(B+2C)=\frac{1}{2}$

$$\Rightarrow \sin(2A+60°)=\sin(C-A)=-\sin(60+2C)=\frac{1}{2} \quad ...(i)$$

From eq. (i), $\sin(2A+60°)=\frac{1}{2} \Rightarrow 2A+60°=30°, 150°$

But A can not be –ve

$\therefore \quad 2A + 60° = 150° \Rightarrow 2A = 90° \Rightarrow A = 45°$

Again from (i), $\sin(60° + 2C) = -\frac{1}{2}$

$\Rightarrow \quad 60° + 2C = 210°$ or $330° \Rightarrow C = 75°$ or $135°$

Also from (i), $\sin(C - A) = \frac{1}{2} \Rightarrow C - A = 30°, 150°$

For $A = 45°; C = 75°, 195°$

But $C = 195°$ is not possible.

$\therefore C = 75° \therefore A = 45°, B = 60°, C = 75°$.

36. We know $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$\Rightarrow \dfrac{1-\tan^2\alpha}{\tan\alpha} = 2\cot 2\alpha \Rightarrow \cot\alpha - \tan\alpha = 2\cot 2\alpha$ ......(i)

Now, we have to prove

$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$

LHS = $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(2\cot 2.4\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(\cot 4\alpha - \tan 4\alpha)$ [From (i)]

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4\cot 4\alpha - 4\tan 4\alpha$

$= \tan\alpha + 2\tan 2\alpha + 2(2\cot 2.2\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 2(2\cot 2\alpha - \tan 2\alpha)$ [From (i)]

$= \tan\alpha + 2\cot 2\alpha$

$= \tan\alpha + (\cot\alpha - \tan\alpha)$ [From (i)]

$= \cot\alpha$ = RHS.

37. We know,

$\cos A\cos 2A\cos 4A \ldots \cos 2^n A = \dfrac{1}{2^{n+1}\sin A}\sin.(2^{n+1}A)$

$\therefore 16\cos\frac{2\pi}{15}\cos 2\left(\frac{2\pi}{15}\right)\cos 2^2\left(\frac{2\pi}{15}\right)\cos 2^3\left(\frac{2\pi}{15}\right)$

$= 16 \cdot \dfrac{\sin(2^4 A)}{2^4\sin A}$, where $A = 2\pi/15$

$= 16 \cdot \dfrac{\sin(32\pi/15)}{16\sin 2\pi/15} = \dfrac{\sin(32\pi/15)}{\sin(2\pi + 2\pi/15)} = \dfrac{\sin(32\pi/15)}{\sin(32\pi/15)} = 1$

38. L.H.S. = $\sin 12°\sin 48°\sin 54°$

$= \frac{1}{2}[2\sin 12°\cos 42°]\sin 54°$

$= \frac{1}{2}\sin^2 54° - \frac{1}{4}\sin 54° = \frac{1}{4}[2\sin^2 54° - \sin 54°]$

$\therefore$ L.H.S $= \frac{1}{4}\left[2\left(\frac{1+\sqrt{5}}{4}\right)^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right]$

$\left[\because \sin 54° = \frac{1+\sqrt{5}}{4}\right]$

$= \frac{1}{4}\left[2\left(\frac{1+5+2\sqrt{5}}{16}\right) - \left(\frac{1+\sqrt{5}}{4}\right)\right]$

$= \frac{1}{4}\times\frac{1}{8}\left[6 + 2\sqrt{5} - 2 - 2\sqrt{5}\right] = \frac{1}{32}\times 4 = \frac{1}{8}$ = R.H.S.

Topic-3: Solutions of Trigonometric Equations

1. **(c)** $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$

$\Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos^2 x - \sin^2 x$

$\Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \cos 2x \Rightarrow x - \frac{\pi}{3} = 2n\pi \pm 2x$

$\Rightarrow x = \frac{2n\pi}{3} + \frac{\pi}{9}$ or $x = -2n\pi - \frac{\pi}{3}$

For $x \in S$, $n = 0 \Rightarrow x = \frac{\pi}{9}, -\frac{\pi}{3}$

Now, $n = 1 \Rightarrow x = \frac{7\pi}{9}$; and $n = -1 \Rightarrow x = \frac{-5\pi}{9}$

Hence, sum of all values of $x = \frac{\pi}{9} - \frac{\pi}{3} + \frac{7\pi}{9} - \frac{5\pi}{9} = 0$

2. **(d)** $\sin x + 2\sin 2x - \sin 3x = 3$

$\Rightarrow \sin x + 4\sin x\cos x - 3\sin x + 4\sin^3 x = 3$

$\Rightarrow \sin x(-2 + 4\cos x + 4\sin^2 x) = 3$

$\Rightarrow \sin x(-2 + 4\cos x + 4 - 4\cos^2 x) = 3$

$2 + 4\cos x - 4\cos^2 x = \dfrac{3}{\sin x}$ $[\because 0 \le \sin x \le 1]$

$\Rightarrow 2 - 4\left(\cos^2 x - 2\cos x\cdot\frac{1}{2} + \frac{1}{4}\right) + 1 = \dfrac{3}{\sin x}$

$\Rightarrow 3 - 4\left(\cos x - \frac{1}{2}\right)^2 = \dfrac{3}{\sin x}$ $\therefore$ L.H.S ≤ 3 and R.HS ≥ 3

Hence, the equation has no solution.

3. **(c)** $2\sin^2\theta - \cos 2\theta = 0 \Rightarrow 1 - 2\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ ....(i)

where $\theta \in [0, 2\pi]$

Also, $2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta + 2) = 0 \Rightarrow \sin\theta = \frac{1}{2}$

$[\because \sin\theta \ne -2]$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ .... (ii)

where $\theta \in [0, 2\pi]$

Combining (i) and (ii), we get $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, there are two solutions.

4. **(d)** Given : $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$,

where $\alpha, \beta \in [-\pi, \pi]$

Now, $\cos(\alpha - \beta) = 1 \Rightarrow \alpha - \beta = 0 \Rightarrow \alpha = \beta$

and $\cos(\alpha + \beta) = 1/e \Rightarrow \cos 2\alpha = 1/e$

$\because 0 < 1/e < 1$

Now $2\alpha \in [-2\pi, 2\pi]$

$\Rightarrow$ There will be two values of 2α in $[-2\pi, 0]$ satisfying $\cos 2\alpha = 1/e$ and two values in $[0, 2\pi]$.

$\Rightarrow$ There will be four values of α in $[-\pi, \pi]$ and correspondingly four values of β. Hence there are four sets of (α, β).

5. **(b)** We know, $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

$\Rightarrow -\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

$\Rightarrow -\sqrt{74} \le 2k+1 \le \sqrt{74} \Rightarrow -8.6 \le 2k+1 \le 8.6$

$\Rightarrow -4.8 \le k \le 3.8$

Hence, k can take only 8 integral values.

6. **(a)** Given : In ΔPQR, $\angle R = \pi/2$

$\Rightarrow \angle P + \angle Q = \pi/2 \Rightarrow \frac{\angle P}{2} + \frac{\angle Q}{2} = \frac{\pi}{4}$

Also tan $P/2$ and tan $Q/2$ are roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$

$\therefore \tan P/2 + \tan Q/2 = -\frac{b}{a}; \tan P/2 \tan Q/2 = c/a$

Now $\tan\left(\frac{P+Q}{2}\right) = \frac{\tan P/2 + \tan Q/2}{1 - \tan P/2 \tan Q/2}$

$\Rightarrow \tan\frac{\pi}{4} = \frac{-b/a}{1 - c/a} \Rightarrow 1 - \frac{c}{a} = -\frac{b}{a}$

$\Rightarrow a - c = -b \Rightarrow \boxed{a + b = c}$

7. **(b)** Given: $\sec^2\theta = \frac{4xy}{(x+y)^2}$

But $\sec^2\theta \ge 1 \Rightarrow \frac{4xy}{(x+y)^2} \ge 1 \Rightarrow 4xy \ge x^2 + y^2 + 2xy$

$\Rightarrow x^2 + y^2 - 2xy \le 0 \Rightarrow (x-y)^2 \le 0$

$\Rightarrow x = y$, because perfect square of real number can not be negative.

Also $x \neq 0$, otherwise $\sec^2\theta$ will become indeterminate.

8. **(d)** $2\sin^2\theta - 3\sin\theta - 2 = 0$

$\Rightarrow (2\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}$ $\quad [\because \sin\theta - 2 = 0$, is not possible$]$

$\Rightarrow \sin\theta = \sin(-\pi/6)$ and $\sin(7\pi/6)$

$\Rightarrow \theta = n\pi + (-1)^n(-\pi/6)$ and $n\pi + (-1)^n 7\pi/6$

$\Rightarrow$ Thus, $\theta = n\pi + (-1)^n 7\pi/6$

9. **(d)** $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{n}}{2}$

$\Rightarrow \sin^2\frac{\pi}{2n} + \cos^2\frac{\pi}{2n} + 2\sin\frac{\pi}{2n}\cos\frac{n}{2n} = \frac{n}{4}$

$\Rightarrow 1 + \sin\frac{\pi}{n} = \frac{n}{4} \Rightarrow \sin\frac{\pi}{n} = \frac{n-4}{4}$

For $n = 2$ the given equation is not satisfied.

Considering $n > 1$ and $n \neq 2$, $0 < \sin\frac{\pi}{n} < 1$

$\Rightarrow 0 < \frac{n-4}{4} < 1 \Rightarrow 4 < n < 8.$

10. **(c)** $\tan x + \sec x = 2\cos x$

$\Rightarrow \frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2\cos x$

$\Rightarrow \sin x + 1 = 2\cos^2 x \Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0 \Rightarrow \sin x = \frac{1}{2}, -1$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \in [0, 2\pi]$

But for $x = \frac{3\pi}{2}$, given eq. is not defined,

Hence, there are only 2 solutions.

11. **(d)** $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$

For real roots $D \ge 0$

$\Rightarrow \cos^2 p - 4\sin p(\cos p - 1) \ge 0$

$\Rightarrow \cos^2 p - 4\sin p\cos p + 4\sin^2 p + 4\sin p - 4\sin^2 p \ge 0$

$\Rightarrow (\cos p - 2\sin p)^2 + 4\sin p(1 - \sin p) \ge 0$

Since, $(\cos p - 2\sin p)^2 \ge 0$ and $1 - \sin p \ge 0$

$\therefore D \ge 0, \forall p \in (0, \pi)$

12. **(b)** $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\Rightarrow 2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cos x - 3\cos 2x$

$\Rightarrow \sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$

$\Rightarrow \sin 2x = \cos 2x$ $\quad [\because \cos x \neq 3/2]$

$\Rightarrow \tan 2x = 1 \Rightarrow 2x = n\pi + \pi/4 \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}$

13. **(c)** $\sqrt{3}\operatorname{cosec} 20° - \sec 20°$

$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$

$= 4\left[\frac{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°}{2\sin 20°\cos 20°}\right]$

$= 4\left[\frac{\sin 60°\cos 20° - \cos 60°\sin 20°}{\sin(2 \times 20°)}\right] = \frac{4\sin 40°}{\sin 40°} = 4$

14. **(c)** Given : $\sin x + \cos x = 1$

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4} = \sin\frac{\pi}{4}$

$\Rightarrow \sin(x + \pi/4) = \sin\pi/4$

$\Rightarrow x + \pi/4 = n\pi + (-1)^n\pi/4$, $n \in Z$ (the set of integers)

$\Rightarrow x = n\pi + (-1)^n\pi/4 - \pi/4;$

where $n = 0, \pm 1, \pm 2, \ldots$

15. **(a)** Given :

$2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$ where $0 < x \le \frac{\pi}{2}$

LHS $= 2\cos^2\frac{x}{2}\sin^2 x = (1 + \cos x)\sin^2 x$

$\because 1 + \cos x < 2$ and $\sin^2 x \le 1$ for $0 < x \le \frac{\pi}{2}$

$\therefore (1 + \cos x)\sin^2 x < 2$

And R.H.S. $= x^2 + \frac{1}{x^2} \ge 2$

Thus for $0 < x \le \frac{\pi}{2}$, given equation is not possible

16. **(8)** $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

$\Rightarrow \frac{5}{4}\cos^2 2x + 1 - \frac{1}{2}\sin^2 2x + 1 - \frac{3}{4}\sin^2 2x = 2$

$\Rightarrow \quad \frac{5}{4}(\cos^2 2x - \sin^2 2x) = 0 \Rightarrow \cos 4x = 0$

$\Rightarrow \quad 4x = (2n+1)\frac{\pi}{2}$ or $x = (2n+1)\frac{\pi}{8}$

For $x \in [0, 2\pi]$, n can take values 0 to 7
Hence, there are 8 solutions.

17. (7) $\frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} \Rightarrow \frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow \quad 2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \sin\frac{4\pi}{n} - \sin\frac{3\pi}{n} = 0$

$\Rightarrow \quad 2\cos\frac{7\pi}{2n}\sin\frac{\pi}{2n} = 0 \Rightarrow \cos\frac{7\pi}{2n} = 0$ or $\sin\frac{\pi}{2n} = 0$

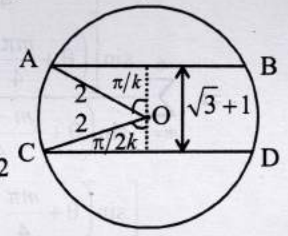
$\Rightarrow \quad \frac{7\pi}{2n} = (2k+1)\frac{\pi}{2}$ or $\frac{\pi}{2n} = 2k\pi$, where $k \in Z$

$\Rightarrow \quad n = \frac{7}{2k+1}$ or $n = \frac{1}{4k}$

($n = \frac{1}{4k}$ not possible for any integral value of k)

As $n > 3$; for $k = 0$, we get $n = 7$.

18. (3) From the figure,



$2\cos\frac{\pi}{k} + 2\cos\frac{\pi}{2k} = \sqrt{3}+1$

$\Rightarrow 2\times 2\cos^2\frac{\pi}{2k} + 2\cos\frac{\pi}{2k} - 2$

$= \sqrt{3}+1$

$\Rightarrow 4\cos^2\frac{\pi}{2k} + 2\cos\frac{\pi}{2k} - (3+\sqrt{3}) = 0$

$\Rightarrow \cos\frac{\pi}{2k} = \frac{-2 \pm \sqrt{4+16(3+\sqrt{3})}}{8} = \frac{-1 \pm \sqrt{13+4\sqrt{3}}}{4}$

$= \frac{-1 \pm (2\sqrt{3}+1)}{4} = \frac{\sqrt{3}}{2}$ or $-\left(\frac{\sqrt{3}+1}{2}\right)$

As $\frac{\pi}{2k}$ is an acute angle, $\cos\frac{\pi}{2k} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow k = 3$

19. (3) $\tan\theta = \cot 5\theta, \theta \neq \frac{n\pi}{5}$

$\Rightarrow \cos\theta\cos 5\theta - \sin 5\theta \sin\theta = 0 \Rightarrow \cos 6\theta = 0$

$\Rightarrow 6\theta = \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\Rightarrow \theta = \frac{-5\pi}{12}, \frac{-\pi}{4}, \frac{-\pi}{12}, \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}$

Again $\sin 2\theta = \cos 4\theta = 1 - 2\sin^2 2\theta$

$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0 \Rightarrow \sin 2\theta = -1, \frac{1}{2}$

$\Rightarrow 2\theta = \frac{-\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{-\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

So, common solutions are $\theta = \frac{-\pi}{4}, \frac{\pi}{12}$ and $\frac{5\pi}{12}$

$\therefore$ Number of solutions = 3.

20. (3) Given equations are

$xyz \sin 3\theta = (y+z)\cos 3\theta$ ...(i)

$xyz \sin 3\theta = 2z\cos 3\theta + 2y \sin 3\theta$ ...(ii)

$xyz \sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$ ...(iii)

On subtracting eq. (ii) from (i), we get

$(\cos 3\theta - 2\sin 3\theta)y - (\cos 3\theta)z = 0$ ...(iv)

On subtracting eq. (i) from (iii), we get

$\sin 3\theta\, y + (\cos 3\theta)z = 0$ ...(v)

Eq. (iv) and (v) from homogeneous system of linear equation.

But $y \neq 0, z \neq 0$

$\therefore \frac{\cos 3\theta - 2\sin 3\theta}{\sin 3\theta} = -\frac{\cos 3\theta}{\cos 3\theta} \Rightarrow \cos 3\theta = \sin 3\theta$

$\Rightarrow \tan 3\theta = 1 \Rightarrow 3\theta = n\pi + \frac{\pi}{4} \Rightarrow \theta = (4n+1)\frac{\pi}{12}, n \in Z$

For $\theta \in (0, \pi) \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

$\therefore$ Three such solutions are possible.

21. (0.5) Given : $\sqrt{3}$ a cos x + 2b sin x = c

which has two roots α and β, such that $\alpha + \beta = \frac{\pi}{3}$

$\therefore \sqrt{3}$ a cos α + 2b sin α = c ... (i)

and $\sqrt{3}$ a cos β + 2b sin β = c ... (ii)

On subtracting equation (ii) from (i),

$\sqrt{3}$ a (cos α – cos β) + 2b (sin α – sin β) = 0

$\Rightarrow -\sqrt{3}\, a\, 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} + 2b.2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 0$

$\Rightarrow -2\sqrt{3}\, a \sin\frac{\pi}{6} + 4b\cos\frac{\pi}{6} = 0 \quad \left(\because \sin\frac{\alpha-\beta}{2} \neq 0\right)$

$\Rightarrow -2\sqrt{3}\, a \times \frac{1}{2} + 4b\frac{\sqrt{3}}{2} = 0 \Rightarrow \frac{b}{a} = \frac{1}{2} = 0.5$

22. $\cos^7 x = 1 - \sin^4 x = (1 - \sin^2 x)(1 + \sin^2 x) = \cos^2 x(1 + \sin^2 x)$

$\therefore \cos x = 0 \Rightarrow x = \pi/2, -\pi/2$

or $\cos^5 x = 1 + \sin^2 x \Rightarrow \cos^5 x - \sin^2 x = 1$

Now maximum value of each cos x or sin x is 1.

Hence the above equation will hold when

cos x = 1 and sin x = 0.

Both these imply x = 0

$\therefore x = -\frac{\pi}{2}, \frac{\pi}{2}, 0$

23. $\tan^2\theta + \sec 2\theta = 1$

$t^2 + \frac{1+t^2}{1-t^2} = 1$, [but $t = \tan\theta$]

$\Rightarrow \quad t^2(t^2 - 3) = 0 \Rightarrow \tan\theta = 0, \pm\sqrt{3}$

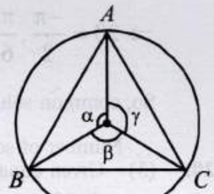
$\Rightarrow \quad \theta = n\pi$ and $\theta = n\pi \pm \pi/3$

24. We know that A.M. ≥ G.M.

⇒ Minimum value of AM. is obtained when AM = GM

⇒ The quantities whose AM is being taken are equal i.e., $\cos\left(\alpha+\frac{\pi}{2}\right)=\cos\left(\beta+\frac{\pi}{2}\right)$

$=\cos\left(\gamma+\frac{\pi}{2}\right)$



$\Rightarrow \sin\alpha=\sin\beta=\sin\gamma$

Also $\alpha+\beta+\gamma=360^\circ \Rightarrow \alpha=\beta=\gamma=120^\circ=2\pi/3$

$\therefore$ Min value of A.M.

$$=\frac{\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\cos\frac{2\pi}{3}+\frac{\pi}{2}\right)}{3}$$

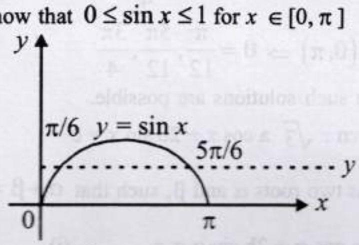
$$=\frac{-3\sin\frac{2\pi}{3}}{3}=-\frac{\sqrt{3}}{2}$$

25. Given : $2\sin^2 x-3\sin x+1\geq 0$

$\Rightarrow (2\sin x-1)\ (\sin x-1)\geq 0$

$\Rightarrow \left(\sin x-\frac{1}{2}\right)(\sin x-1)\geq 0 \Rightarrow \ \sin x\leq\frac{1}{2}\ \text{or}\sin x\geq 1$

But we know that $0\leq\sin x\leq 1$ for $x\in[0,\pi]$



$\Rightarrow \quad 0\leq\sin x\leq\frac{1}{2}$ or $\sin x=1$

$\Rightarrow \quad x\in[0,\pi/6]\cup[5\pi/6,\pi]$ or $x=\pi/2$

On combining, we get $x\in[0,\pi/6]\cup\{\pi/2\}\cup[5\pi/6,\pi]$

26. Given equations : $x+y=2\pi/3$ ... (i)

and $\cos x+\cos y=3/2$ ...(ii)

From eq. (ii), $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{3}{2}$

$\Rightarrow \quad 2\cos\frac{\pi}{3}\cos\frac{x-y}{2}=\frac{3}{2}$ [From eq. (i)]

$\Rightarrow \quad 2.\frac{1}{2}\cos\frac{x-y}{2}=\frac{3}{2}\Rightarrow \cos\frac{x-y}{2}=\frac{3}{2}$

which is not possible because $-1\leq\cos\theta\leq 1$.

Hence, solution of given equations is φ.

27. **(False)** Given : $\sin^4\theta-2\sin^2\theta-1=0$

$\therefore \quad D=4+4=8\ \because \sin^2\theta=\frac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}.$

$\because$ value of $\sin^2\theta$ is +ve, $\quad \therefore \sin^2\theta=\sqrt{2}+1$

$\because \ -1\leq\sin\theta\leq 1,\ \therefore\ \sin^2\theta\neq\sqrt{2}+1$

Hence, there is no value of θ, which satisfy the given equation.

Hence, the statement is false.

28. **(a, c)** If we consider tan α/2 = x and tan β/2 = y, then

2(cos β − cos α) + cos α cos β = 1

$$\Rightarrow \quad 2\left[\frac{1-y^2}{1+y^2}-\frac{1-x^2}{1+x^2}\right]=1-\frac{\left(1-x^2\right)\left(1-y^2\right)}{\left(1+x^2\right)\left(1+y^2\right)}$$

$\Rightarrow \quad 2[(1+x^2)(1-y^2)-(1-x^2)(1+y^2)]$

$=(1+x^2)(1+y^2)-(1-x^2)(1-y^2)$

$\Rightarrow \quad 4(x^2-y^2)=2(x^2+y^2)$

$\Rightarrow \quad x^2=3y^2\Rightarrow x=\pm\sqrt{3}\,y\Rightarrow \quad \tan\frac{\alpha}{2}\pm\sqrt{3}\tan\frac{\beta}{2}=0$

29. **(c)** Let $f(x)=x^2-x\sin x-\cos x$

$\therefore \quad f'(x)=2x-x\cos x=x(2-\cos x)$

$\therefore \quad f$ is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$

Also $\lim\limits_{x\to\infty} f(x)=\infty,\quad \lim\limits_{x\to-\infty} f(x)=\infty$ and $f(0)=-1$

$\therefore \quad y=f(x)$ meets x-axis twice.

i.e., $\quad f(x)=0$ has two points in $(-\infty,\infty)$.

30. **(c, d)** $\sum\limits_{m=1}^{6}\operatorname{cosec}\left[\theta+\frac{(m-1)\pi}{4}\right]\operatorname{cosec}\left[\theta+\frac{m\pi}{4}\right]=4\sqrt{2}$

$$\Rightarrow \sum_{m=1}^{6}\frac{\sin\frac{\pi}{4}}{\sin\left[\theta+\frac{(m-1)\pi}{4}\right]\sin\left[\theta+\frac{m\pi}{4}\right]}=4$$

$$\Rightarrow \sum_{m=1}^{6}\frac{\sin\left[\left(\theta+\frac{m\pi}{4}\right)-\left(\theta+\frac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)}=4$$

$$\Rightarrow \sum_{m=1}^{6}\frac{\left[\begin{array}{c}\sin\left(\theta+\frac{m\pi}{4}\right)\cos\left(\theta+\frac{(m-1)\pi}{4}\right)\\-\cos\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\end{array}\right]}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)}=4$$

$$\Rightarrow \sum_{m=1}^{6}\left[\cot\left(\theta+\frac{(m-1)\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right]=4$$

$$\Rightarrow \left[\cot\theta-\cot\left(\theta+\frac{\pi}{4}\right)\right]+\left[\cot\left(\theta+\frac{\pi}{4}\right)-\cot\left(\theta+\frac{2\pi}{4}\right)\right]$$

$$+\ldots+\left[\cot\left(\theta+\frac{5\pi}{4}\right)-\cot\left(\theta+\frac{6\pi}{4}\right)\right]=4$$

$\Rightarrow \cot\theta-\cot\left(\theta+\frac{3\pi}{2}\right)=4\Rightarrow \cot\theta+\tan\theta=4$

$\Rightarrow \cos^2\theta+\sin^2\theta=4\sin\theta\cos\theta$

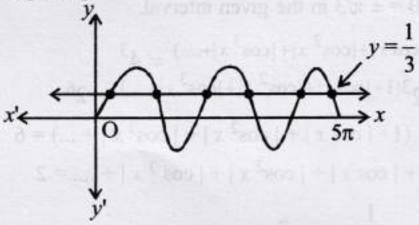
$\Rightarrow \sin 2\theta=\frac{1}{2}\Rightarrow 2\theta=\frac{\pi}{6}\text{ or }\frac{5\pi}{6}\Rightarrow \theta=\frac{\pi}{12}\text{ or }\frac{5\pi}{12}$

31. **(c)** $3\sin^2 x-7\sin x+2=0$, put $\sin x=s$

$\Rightarrow \quad (s-2)\ (3s-1)=0$

$\because \quad s=2$ is not possible, $\quad \therefore\ s=1/3\ \Rightarrow\ \sin x=\frac{1}{3}$

Number of solutions of $\sin x = \frac{1}{3}$ from the following graph is 6 between $[0, 5\pi]$



32. **(d)** $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0,\ \forall x$

For $x = 0$ and $x = \pi/2$, we get $a_1 + a_2 = 0$ ....(i)
and $a_1 - a_2 + a_3 = 0$ ....(ii)
$\Rightarrow a_2 = -a_1$ and $a_3 = -2a_1$
$\therefore$ The given equation becomes
$a_1 - a_1 \cos 2x - 2a_1 \sin^2 x = 0,\ \forall x$
$\Rightarrow a_1 (1 - \cos 2x - 2\sin^2 x) = 0, \forall x$
$\Rightarrow a_1 (2\sin^2 x - 2\sin^2 x) = 0,\ \forall x$
The above is satisfied for all values of a_1.
Hence, infinite number of triplets $(a_1, -a_1, -2a_1)$ are possible.

33. **(b)** We have (I)

$\left\{x \in \left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right] : \cos x + \sin x = 1\right\}$

$\cos x + \sin x = 1$

$\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}$

$\sin\left(\frac{\pi}{4} + x\right) = \frac{1}{\sqrt{2}}$

$\frac{\pi}{4} + x = n\pi + (-1)^n \frac{\pi}{4}$

$x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$ $\therefore x$ has 2 elements $\rightarrow (P)$

We have (II)

$\left\{x \in \left[\frac{-5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$; $\sqrt{3}\tan 3x = 1$

$\Rightarrow \tan 3x = \frac{1}{\sqrt{3}}$

$\Rightarrow x = (6n+1)\frac{\pi}{18}; n \in Z \Rightarrow 3x = n\pi + \frac{\pi}{6}$

or $x = \frac{n\pi}{3} + \frac{\pi}{18}$ $\therefore x$ has 2 elements $\rightarrow (P)$

We have III

$\left\{x \in \left[\frac{-6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos 2x = \sqrt{3}\right\}$

$2\cos 2x = \sqrt{3}$

$\Rightarrow \cos 2x = \frac{\sqrt{3}}{2} \Rightarrow 2x = 2n\pi \pm \frac{\pi}{6}; n \in Z$

or, $x = n\pi \pm \frac{\pi}{12}; n \in Z$

$\Rightarrow x \in \left\{\pm\frac{\pi}{12}, -\pi \pm \frac{\pi}{12}, \pi \pm \frac{\pi}{12}\right\}$

$\therefore$ x has 6 elements $\rightarrow$ (T)

We have (IV)

$\left\{x \in \left[\frac{-7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$

$\sin x - \cos x = 1$

$\frac{1}{\sqrt{2}}\sin(x) - \frac{1}{\sqrt{2}}\cos(x) = \frac{1}{\sqrt{2}}$

$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$x - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$

$x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}$

$x \in \left\{\frac{\pi}{2}, \frac{-3\pi}{2}, -\pi, \pi\right\}$

$\therefore$ x has 4 elements $\rightarrow$ R

For Q 34 and 35.

$f(x) = 0 \Rightarrow \sin(\pi \cos x) = 0 \Rightarrow \pi\cos x = n\pi$
$\Rightarrow \cos x = n \Rightarrow \cos x = -1, 0, 1$

$\Rightarrow x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \frac{7\pi}{2}, 4\pi........$

$\therefore X = \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \frac{7\pi}{2}, 4\pi, ...\right\}$

$\therefore$ (I) – P, Q

$f'(x) = 0 \Rightarrow \cos(\pi\cos x)(-\pi\sin x) = 0$

$\Rightarrow \cos(\pi\cos x) = 0,\ \sin x = 0$

$\Rightarrow \pi\cos x = (2n-1)\frac{\pi}{2},\ x = n\pi$

$\Rightarrow \cos x = (2n-1)\frac{1}{2},\ x = \pi, 2\pi, 3\pi, ...$

$\Rightarrow \cos x = \frac{-1}{2}, \frac{1}{2}.$

$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, ...$

$\therefore Y = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, ...\right\}.$

$\therefore$ (II) – Q, T.

$g(x) = 0 \Rightarrow \cos(2\pi\sin x) = 0$

$\Rightarrow 2\pi\sin x = (2n-1)\frac{\pi}{2} \Rightarrow \sin x = \frac{2n-1}{4}$

$\Rightarrow \sin x = \frac{1}{4}, \frac{-1}{4}, \frac{3}{4}, \frac{-3}{4}$

$\therefore Z = \left\{-\sin^{-1}\frac{3}{4}, -\sin^{-1}\frac{1}{4}, \sin^{-1}\frac{1}{4}, \sin^{-1}\frac{3}{4}\right\}.$

(III) – R.

$g'(x) = 0 \Rightarrow -\sin(2\pi\sin x).2\pi\cos x = 0$

$\Rightarrow \sin(2\pi\sin x) = 0,\ \cos x = 0$

$\Rightarrow 2\pi\sin x = n\pi,\ x = (2n-1)\frac{\pi}{2}.$

$\Rightarrow \sin x = \frac{n}{2},\ x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, ...$

$\Rightarrow \sin x = -1, \frac{-1}{2}, 0, \frac{1}{2}, 1.$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi, \frac{13\pi}{6}, ...$

$\therefore W = \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi, \frac{13\pi}{6}, ...\right\}.$

(IV) – P, R, S.

34. **(a)** Option (a) is correct.

35. **(d)** Option (d) is correct.

36. Given : $(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$

$\Rightarrow (1 - \tan^2\theta)(1 + \tan^2\theta) + 2^{\tan^2\theta} = 0$

Put $\tan^2\theta = t$

$\therefore (1 - t)(1 + t) + 2^t = 0$ or $1 - t^2 + 2^t = 0$

It is clearly satisfied by $t = 3$.

As $-8 + 8 = 0 \quad \therefore \tan^2\theta = 3$

$\therefore \theta = \pm\pi/3$ in the given interval.

37. $8^{(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+....)} = 4^3$

$\Rightarrow 2^{3(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+....)} = 2^6$

$\Rightarrow 3(1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + ...) = 6$

$\Rightarrow 1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + ... = 2$

$\Rightarrow \frac{1}{1 - |\cos x|} = 2$

$\Rightarrow 1 - \cos x = 1/2 \Rightarrow |\cos x| = \frac{1}{2}$

$\Rightarrow x = \pi/3, -\pi/3, 2\pi/3, -2\pi/3, ...$

The values of $x \in (-\pi, \pi)$ are $\pm\pi/3, \pm 2\pi/3$.