ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર–ક્રમાંક
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MATHEMATICS

Standard 11

(Semester II)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place before the sudents this textbook of **Mathematics** for **Standard 11 (Semester II)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestions from people interested in education, to improve the quality of the textbook.

Dr. Bharat Pandit
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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) **to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;**

(B) **to cherish and follow the noble ideals which inspired our national struggle for freedom;**

(C) **to uphold and protect the sovereignty, unity and integrity of India;**

(D) **to defend the country and render national service when called upon to do so;**

(E) **to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;**

(F) **to value and preserve the rich heritage of our composite culture;**

(G) **to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;**

(H) **to develop the scientific temper, humanism and the spirit of inquiry and reform;**

(I) **to safeguard public property and to abjure violence;**

(J) **to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;**

(K) **to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.**

*Constitution of India : Section 51 A

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About This Textbook...

We have created a background in the book of Mathematics for standard 11, semester I about formation of new syllabus and writing textbooks following curriculum of NCERT.

First of all, this book was written in English. It was reviewed by teachers and professors teaching in English medium schools and colleges. According to the suggestions made by experts, necessary amendments were made and the manuscript was translated in Gujarati. It was again reviewed by experts teaching in Gujarati medium; considering their suggestions, the necessary changes were made.

Thus, the manuscript prepared was compeletely read by the authors in workshops and the authors gave final touches to the manuscript.

In chapter 1, mathematical induction which is a tool to prove many properties about statements related to natural numbers is studied. Also, we have shown the use of mathematical induction in various fields using various formats. Chapter 2 gives an introduction to complex number system. Fundamental theorem of algebra, square roots and cube roots of complex numbers, Argand diagrams, inequalities etc. have been presented in a very lucid manner in this chapter. Any algebraic n degree equation with real coefficients can be solved using complex numbers and thus complex numbers are very useful. Chapter 3 introduces binomial theorem which is an extension of expansions of the squares and the cubes studied at secondary school level. Binomial theorem for positive index is useful while using polynomials. Chapters 4, 5 and 6 advance the study of trigonometry studied in semester I. These chapters are useful to study properties of triangles and for studying general solution of trigonometric equations.

In chapter 7, there are arithmetic progression, geometric progression and power series (index 1, 2 and 3). In chapter 8, elementary study of conics and primary information have been given. We mention intersection of cones and general second degree curves. In chapter 9, there is a study of three dimensional geometry. To study this, vector is an important tool. So in the beginning of the chapter, we have given introduction of vectors. The study of three dimensional geometry is limited to section of a line segment.

Chapter 10 and 11 suggest the beginning of the calculus. Only intuitive concept of limit has been taken and then limit has been defined. We have stressed how to obtain limit using lemmas and theorems. The concept of limit has been explained with the help of graphs but students are not supposed to draw the graphs. Having defined differentiation, we have explained how to obtain derivatives of elementary functions. There are ample number of examples so that a student can understand all the concepts by himself / herself and a teacher can lead a student to self study. At the end of every chapter enough number of multiple choice questions have been given so that understanding of concept can be evaluated. We intend to render a student enough study material from the textbook itself. Attractive four colour printing is an additional attraction of the book. We have given some information about contribution of Indian Mathematicians at the end of some chapters.

Enough care has been taken to make the textbook maximally interesting and errorfree. However all constructive suggestions regarding further improvement in the textbook are most welcome.

We hope teachers and students both will find this book useful and valuable.

– Authors

Please consider following points while teaching textbook.

**Following is necessary for study by students and teachers.
But it will not be asked in the board examination.**

Chapter	Exercise	Examples
Chapter 1	Exercise 1 : Ex. No. 21	21, 24
Chapter 2	Exercise 2 : Ex. No. 16	–
Chapter 5	Exercise 5 : Ex. No. 19 to 22	–
Chapter 8	–	13, 14, 19, 32
Chapter 10	Article 10.3 Exercise 10 : From statements of examples 1, 2, 3 remove the word 'definition'.	14, 15, 16
Chapter 11	Exercise 11 : Ex. No. 6, 20(4)	17, 26 In Example 19 Let P(n) : $\frac{d}{dx} sin^n x = n sin^{n-1} x \ cos x$

**Following is useful for higher studies and competitive examinations,
but not for board examination.**

Chapter	Exercise	Examples
Chapter 1	Exercise 1 : Ex. No. 9, 24, 29	23
Chapter 2	Exercise 2.3 : Ex. No. 3	–
Chapter 8	Exercise 8.3 : Ex. No. 3, 4 Exercise 8.4 : Ex. No. 8, 9 Exercise 8 : Ex. No. 6	–
Chapter 10	Exercise 10 : Ex. No. 9	
Chapter 11	Exercise 11 : Ex. No.20(23)	

Chapter 1

PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematics is the queen of science and number theory is the queen of mathematics.

– Gauss

Mathematics passes not only truth but also supreme beauty !

– Bertrand Russell

1.1 Introduction

We have studied one method of reasoning, deductive reasoning.

For example, consider the following statements :

(1) $1 + 2 + 3 + ... + 100 = 5050$

(2) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(3) Let $n = 100$ in (2). $1 + 2 + 3 + ... + 100 = \frac{(100)(101)}{2} = (50)(101) = 5050$

Here we want to prove that sum of all integers from 1 to 100 is 5050. We have a general result $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. We take $n = 100$ in it and get the required result. Here, we apply a general principle to deduce a particular result.

Consider (1) If 3 divides product ab, then 3 divides a or 3 divides b. (2) If p is a prime and p divides ab then p divides a or p divides b. (3) Let $p = 3$ in (2) as 3 is a prime. Hence, if 3 divides product ab, then 3 divides a or 3 divides b.

Here also we apply a general principle to deduce a particular result.

(1) Amitabh Bachchan is a good actor.

(2) Actors are awarded national *Padma* honour in their category, if selected.

(3) Amitabh Bachchan was selected and got *Padma* honour.

Here also a similar situation occurs.

But consider the following against this deductive reasoning,

$4 - 1 = 3$ is divisible by 3.

$4^2 - 1 = 15$ is divisible by 3.

$4^3 - 1 = 63$ is divisible by 3.

Here we observe a pattern and we make a conjecture that for every positive integer n, $4^n - 1$ is divisible by 3. So from a particular case, we conjecture a general result. This is not a proof. This inductive assumption has to be proved. All conjectures may not be true. For example, $n^2 - n + 41$ is a prime for $n = 1, 2, 3,...39$. But for $n = 41$, $41^2 - 41 + 41 = 41^2$ is obviously not a prime. Hence we cannot deduce that $n^2 - n + 41$ is a prime by observing values for $n = 1, 2, 3,...39$.

So, inductive argument starts from a particular case and by rigorous deduction the conjecture is proved.

The history of this dates back to **Plato**. In 370 B.C. Plato's *parmenides* (Discussions or Dialogues) contained an early example of implicit inductive proof. The early traces of mathematical induction can be found in Euclid's proof that number of primes is infinite. Bhaskara II's *cyclic* method (*Chakravala*) also introduces mathematical induction.

Sorites paradox used the method of descent. He said 10,00,000 grains of sand form a heap. Removing one grain from the heap does not change the situation. So continuing the argument even one grain or no grain also forms a heap !

Around 1000 A.D., **Al-Karaji** introduced mathematical induction for arithmetic sequences in **Al-Fakhri** and proved the binomial theorem and properties of **Pascal's** triangle.

The first explicit formulation of the principle of mathematical induction was given by **Pascal** in *Traité-du-triangle arithmetique* (1665). French mathematician **Fermat** and Swiss mathematician **Jacob Bernoulli** used the principle. The modern rigorous and systematic treatment came only in 19th century with **George Boole, Sanders Peirce, Peano** and **Dedekind.**

1.2 Induction Principle

We start with following principle :

Principle of Induction : If a statement $P(n)$ of natural variable n is true for $n = 1$ and if $P(k)$ is true $\Rightarrow P(k + 1)$ is true, $k \in N$, then $P(n)$ is true, $\forall n \in N$.

Let us be given a statement $P(n)$ involving a natural variable to be true for all natural numbers n. We prove it in two stages :

(1) The basis : We prove it for $n = 1$ (or 0 or the lowest value).

(2) Inductive step : Assuming that the statement holds for some natural number k, prove it for $n = k + 1$.

Then $P(n)$ is true for all $n \in N$.

Domino effect : We are presented with a 'long' row of dominos such that,

(1) The first domino will fall.

(2) Whenever a domino falls, its next neighbour will fall.

So it is concluded that all of the dominos will fall.

So the proof is like this. The first statement in an infinite sequence of statements is true and if it is true for some $k \in$ N, it is true for the next value of the variable, then the given sequence of statements is true for all $n \in$ N.

In logical symbols, $(\forall P)\ [P(1) \wedge (\forall k \in N)\ (P(k) \Rightarrow P(k+1))] \Rightarrow (\forall n \in N)[P(n)]$

This can be proved by using **well-ordering principle** which states that every non-empty subset of N has a least element.

Proof : Let S be the set of natural numbers for which P(n) is false. $1 \notin$ S as P(1) is true. If S is non-empty, it has a least element t which is not 1. Let $t = n + 1$. Since t is the least element for which P(t) is false, P(n) is true. Also $P(n) \Rightarrow P(n+1)$. Hence $P(n+1) = P(t)$ is true, a contradiction. Hence S = $\emptyset$.

$\therefore$ P(n) is true, $\forall n \in$ N.

Sometimes paradoxes are created by misuse of the principle.

There is a famous **Polya's** proof that there is no horse of different colour.

Basis : If there is only one horse, there is only one colour and hence P(1) is true.

Induction step : Assume that in any set of n horses, all have the same colour. Consider a set of $n + 1$ horses numbered 1, 2, 3,... $n + 1$. Consider the subsets {1, 2, 3,..., n} and {2, 3, 4,...$n + 1$}. Each is a set of n horses and therefore they have the same colour and since they are overlapping sets, all $n + 1$ horses have same colour. This argument is true for 1 horse and $n \geq 3$ horses. But for 2 horses the set {1} and {2} are disjoint and the argument falls flat.

1.3 Examples

Now we will apply the principle of mathematical induction to some examples.

Example 1 : Prove $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \in$ N

Solution : Let P(n) : $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \in$ N

For $n = 1$, L.H.S. = 1 and R.H.S. = $\frac{1 \times 2}{2} = 1$. Hence, P(1) is true.

Let P(k) be true i.e. P(n) is true for $n = k$, $k \in$ N.

$\therefore \quad 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$ **(i)**

For $n = k + 1$ we have to prove,

$$1 + 2 + 3 + ... + (k+1) = \frac{(k+1)(k+2)}{2}$$

Now, $1 + 2 + 3 + ... + (k+1) = (1 + 2 + 3 + ... + k) + (k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ **by (i)**

$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$

Hence, P(k + 1) is true.

∴ P(1) is true and P(k) is true, ⇒ P(k + 1) is true.

∴ P(n) is true, $\forall n \in$ N by principle of mathematical induction.

Note : This example has historical importance.

Obviously, 1 + 2 + 3 + ... + 100 = 5050 according to this formula. When this formula was not known, Gauss, at very young age, calculated this by the following method and surprised his teacher Buttner and assistant teacher Bartels.

Let S = 1 + 2 + 3 + ... + 100 **(i)**

∴ S = 100 + 99 + 98 + ... + 1 **(ii)**

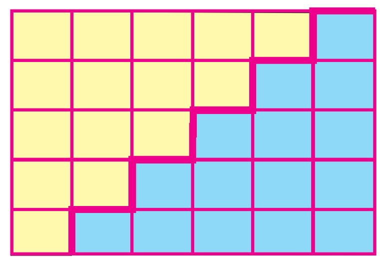
Adding (i) and (ii)

∴ 2S = (101) + (101) + ... 100 times **((i) + (ii))**

∴ S = $\frac{101 \times 100}{2}$ = 5050. This was done in no time !

Let us review a geometric 'proof'.

Consider a rectangle of sides n and n + 1 divided into subrectangles of unit sides as shown. The portion under the dark ladder has area 1 + 2 + 3 + ... + n.



By symmetry the rectangle has area

$$2(1 + 2 + 3 + ... + n) = n(n + 1)$$

∴ $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Example 2 : Prove $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in$ N

Solution : Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in$ N

Let n = 1. L.H.S. = 1^2 = 1 and R.H.S. = $\frac{1 \times 2 \times 3}{6}$ = 1.

∴ P(1) is true.

Let P(k) be true, $k \in$ N.

∴ $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Let $n = k + 1$.

∴ L.H.S. = $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= (k + 1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= (k + 1)\left(\frac{2k^2 + k + 6k + 6)}{6}\right)$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by principle of mathematical induction.

Example 3 : Prove $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$, $n \in$ N

Solution : Let P(n) : $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$, $n \in$ N

For $n = 1$, L.H.S. = $1^3 = 1$ and R.H.S. = $\frac{1^2 \times 2^2}{4} = 1$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1^3 + 2^3 + 3^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$

Let $n = k + 1$.

$$\text{L.H.S.} = 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2(k+1+1)^2}{4} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

(Now onwards we shall abbreviate Principle of Mathematical Induction as P.M.I.)

Example 4 : Prove $1 + 3 + 5 + ... + (2n - 1) = n^2$, $n \in$ N

Solution : Let P(n) : $1 + 3 + 5 + ... + (2n - 1) = n^2$, $n \in$ N

Let $n = 1$. L.H.S. = 1 and R.H.S. = $1^2 = 1$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1 + 3 + 5 + ... + (2k - 1) = k^2$

Let $n = k + 1$.

L.H.S. = $1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$

$= k^2 + 2k + 1$

$= (k + 1)^2$ = R.H.S.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 5 : Prove $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$, $n \in$ N

Solution : Let P(n) : $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$, $n \in$ N

Let $n = 1$. L.H.S. $= \frac{1}{1\cdot 2} = \frac{1}{2}$ and R.H.S. $= \frac{1}{2}$

∴ P(1) is true.

Let P(k) be true.

∴ $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Let $n = k + 1$.

$$\text{L.H.S.} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \text{R.H.S.}$$

∴ P(k + 1) is true.

∴ P(k) is true ⇒ P(k + 1) is true.

∴ P(n) is true, $\forall n \in$ N by P.M.I.

Example 6 : Prove $1\cdot 1! + 2\cdot 2! + ... + n\cdot n! = (n + 1)! - 1$, $n \in$ N

Solution : Let P(n) : $1\cdot 1! + 2\cdot 2! + ... + n\cdot n! = (n + 1)! - 1$, $n \in$ N

Let $n = 1$. L.H.S. $= 1\cdot 1! = 1$, R.H.S. $= (1 + 1)! - 1 = 2! - 1 = 1$

∴ P(1) is true.

Let P(k) be true.

∴ $1\cdot 1! + 2\cdot 2! + ... + k\cdot k! = (k + 1)! - 1$

Let $n = k + 1$.

$$\text{L.H.S.} = 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + ... + k\cdot k! + (k + 1)(k + 1)!$$

$$= (k + 1)! - 1 + (k + 1)(k + 1)!$$

$$= (k + 1)!\, [1 + (k + 1)] - 1$$

$$= (k + 1)!\, (k + 2) - 1$$

$$= (k + 2)! - 1 = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Directly, $n \cdot n! = (n + 1 - 1)\, n! = (n + 1)\, n! - n!$

$= (n + 1)! - n!$

Let n = 1, 2, 3,... etc. and add

$1 \cdot 1! + 2 \cdot 2! + 33! + ... + n \cdot n! = (2! - 1!) + (3! - 2!) + (4! - 3!) + ... + ((n + 1)! - n!)$

$= (n + 1)! - 1$

Example 7 : Prove $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \dots \left(1 + \frac{2n+1}{n^2}\right) = (n + 1)^2, \ n \in$ N

Solution : Let P(n) : $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \dots \left(1 + \frac{2n+1}{n^2}\right) = (n + 1)^2, \ n \in$ N

Let n = 1. L.H.S. = $1 + \frac{3}{1} = 4$ and R.H.S. = $(1 + 1)^2 = 2^2 = 4$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) = (k + 1)^2$

Let $n = k + 1$.

$$\text{L.H.S.} = \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$= (k + 1)^2 \times \left(\frac{k^2+2k+1+2k+3}{(k+1)^2}\right)$$

$$= k^2 + 4k + 4$$

$$= (k + 2)^2 = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Directly, $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) \dots \left(1+\frac{2n+1}{n^2}\right)$

$= \frac{4}{1} \cdot \frac{9}{4} \cdot \frac{16}{9} \dots \frac{(n+1)^2}{n^2} = (n + 1)^2$

Example 8 : Prove $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \dots + n \cdot 2^n = (n - 1)2^{n+1} + 2, \ n \in$ N

(This type of series is called **arithmetico geometric** series.)

Solution : Let P(n) : $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \dots + n \cdot 2^n = (n - 1)2^{n+1} + 2, \ n \in$ N

Let n = 1. L.H.S. = 2 and R.H.S. = 0 + 2 = 2

$\therefore$ P(1) is true.

Let P(k) be true.

Hence, $1 \cdot 2 + 2 \cdot 2^2 + ... + k \cdot 2^k = (k - 1)2^{k+1} + 2$

Let $n = k + 1$.

$$\begin{aligned} \text{L.H.S.} &= 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + k \cdot 2^k + (k+1)2^{k+1} \\ &= (k - 1)2^{k+1} + 2 + (k + 1)2^{k+1} \\ &= (k - 1 + k + 1)2^{k+1} + 2 \\ &= 2k \cdot 2^{k+1} + 2 \\ &= k \cdot 2^{k+2} + 2 = \text{R.H.S.} \end{aligned}$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 9 : Prove $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$ **($r \neq 1$), $n \in$ N**

Solution : Let P(n) : $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$ **($r \neq 1$), $n \in$ N**

Let $n = 1$. L.H.S. $= a$ and R.H.S. $= \frac{a(r - 1)}{r - 1} = a$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $a + ar + ar^2 + ... + ar^{k-1} = \frac{a(r^k - 1)}{r - 1}$

Let $n = k + 1$.

$$\begin{aligned} \text{L.H.S.} &= a + ar + ar^2 + ... + ar^{k-1} + ar^k \\ &= \frac{a(r^k - 1)}{r - 1} + ar^k \\ &= a\left(\frac{r^k - 1}{r - 1} + r^k\right) \\ &= a\,\frac{r^k - 1 + r^k(r - 1)}{r - 1} \\ &= a\,\frac{(r^k - 1 + r^{k+1} - r^k)}{r - 1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} = \text{R.H.S.} \end{aligned}$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 10 : Prove $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

Solution : Let P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

Let $n = 1$. $3^4 - 8 - 9 = 81 - 8 - 9 = 64$ is divisible by 8.

Let P(k) be true. Hence $3^{2k+2} - 8k - 9$ is divisible by 8.

Let $n = k + 1$.

Now, $3^{2k+4} - 8(k+1) - 9$ **$(2(k+1)+2 = 2k+4)$**

$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$= 3^{2k+2}(8+1) - 8k - 8 - 9$ **$(3^2 = 9 = 8 + 1)$**

$= 8 \cdot 3^{2k+2} + 3^{2k+2} - 8k - 8 - 9$

$= 3^{2k+2} - 8k - 9 + 8(3^{2k+2} - 1)$

Now, 8 divides $3^{2k+2} - 8k - 9$ **by $p(k)$**

Also, 8 divides $8(3^{2k+2} - 1)$

$\therefore$ 8 divides $3^{2k+2} - 8k - 9 + 8(3^{2k+2} - 1)$

$\therefore$ $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Obviously,

$3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 1 - 8n - 8$

$= (3^2 - 1)((3^2)^n + (3^2)^{n-1} + ... + 1) - 8n - 8$ **(Example 9)**

$= 8(3^{2n} + 3^{2n-2} + ... + 1) - 8n - 8$ is divisible by 8.

Another Method :

P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

For $n = 1$, $3^{2+2} - 8(1) - 9 = 64$ is divisible by 8.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $3^{2k+2} - 8k - 9$ is divisible by 8.

$\therefore$ $3^{2k+2} - 8k - 9 = 8m$ where $m \in$ N **(i)**

Now, Let $n = k + 1$,

$3^{2(k+1)+2} - 8(k+1) - 9 = 3^{2k+2} \times 3^2 - 8k - 8 - 9$

$= (8k + 9 + 8m)9 - 8k - 8 - 9$ **(From (i))**

$= 72k + 81 + 72m - 8k - 8 - 9$

$= 64k + 72m + 64$

$= 8(8k + 9m + 8)$ is divisible by 8.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 11 : Prove $2002^{2n+1} + 2003^{2n+1}$ is divisible by 4005, $n \in N$

Solution : Let P(n) : $2002^{2n+1} + 2003^{2n+1}$ is divisible by 4005, $n \in N$

Let $n = 1$.

$2002^3 + 2003^3 = (2002 + 2003)\,[(2002)^2 - (2002)(2003) + (2003)^2]$

$= (4005)\,[(2002)^2 - (2002)(2003) + (2003)^2]$

$\therefore$ $(2002)^3 + (2003)^3$ is divisible by 4005.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $2002^{2k+1} + 2003^{2k+1}$ is divisible by 4005.

Let $n = k + 1$.

Now, $2002^{2(k+1)+1} + 2003^{2(k+1)+1}$

$= 2002^{2k+3} - 2002^{2k+1}(2003)^2 + (2002)^{2k+1} \cdot (2003)^2 + (2003)^{2k+3}$

$= (2002)^{2k+1}\,[(2002)^2 - (2003)^2] + (2003)^2\,[(2002)^{2k+1} + (2003)^{2k+1}]$

$= -(4005)\,(2002)^{2k+1} + (2003)^2\,[(2002)^{2k+1} + (2003)^{2k+1}]$

Now, $(2002)^{2k+1}\ (2003)^{2k+1}$ is divisible by 4005. **(P(k))**

$\therefore$ $(2002)^{2(k+1)+1} + (2003)^{2(k+1)+1}$ is divisible by 4005.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Example 12 : Prove $x^{2n} - y^{2n}$ is divisible by $x + y$, $n \in N$

Solution : Let P(n) : $x^{2n} - y^{2n}$ is divisible by $x + y$, $n \in N$

Let $n = 1$.

Then $x^2 - y^2 = (x - y)(x + y)$ and so $x^2 - y^2$ is divisible by $x + y$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $x^{2k} - y^{2k}$ is divisible by $x + y$.

Let $n = k + 1$.

$x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - x^{2k}y^2 + x^{2k}y^2 - y^{2k+2}$

$= x^{2k}(x^2 - y^2) + y^2(x^{2k} - y^{2k})$

$= x^{2k}(x - y)(x + y) + y^2(x^{2k} - y^{2k})$

Now, $x^{2k} - y^{2k}$ is divisible by $(x + y)$. **(P(k))**

$\therefore$ $x^{2(k+1)} - y^{2(k+1)}$ is divisible by $(x + y)$.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 13 : Prove $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3}$, $n \in$ N

Solution : Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3}$, $n \in$ N

Let $n = 1$. L.H.S. $= 1^2 = 1$, R.H.S. $= \frac{1}{3}$ and $1 > \frac{1}{3}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1^2 + 2^2 + 3^2 + ... + k^2 > \frac{k^3}{3}$

Let $n = k + 1$.

Now, $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 > \frac{k^3}{3} + (k+1)^2$ **(i)**

Now, $\frac{k^3}{3} + (k+1)^2 = \frac{1}{3}(k^3 + 3k^2 + 6k + 3)$

$= \frac{1}{3}(k^3 + 3k^2 + 3k + 1 + 3k + 2)$

$> \frac{1}{3}(k^3 + 3k^2 + 3k + 1)$ as $\frac{1}{3}(3k + 2) \geq \frac{5}{3} > 0$

$\therefore$ $\frac{k^3}{3} + (k+1)^2 > \frac{1}{3}(k+1)^3$ **(ii)**

$\therefore$ $1^2 + 2^2 + 3^2 + ... + (k+1)^2 > \frac{1}{3}(k+1)^3$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6} > \frac{2n^3}{6} = \frac{n^3}{3}$, $n \in$ N

Example 14 : Prove $1 + 2 + 3 + ... + n < \frac{1}{8}(2n + 1)^2$, $n \in$ N

Solution : Let P(n) : $1 + 2 + 3 + ... + n < \frac{1}{8}(2n + 1)^2$, $n \in$ N

Let $n = 1$. L.H.S. $= 1$, R.H.S. $= \frac{1}{8}(3)^2 = \frac{9}{8}$ and $1 < \frac{9}{8}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1 + 2 + 3 + ... + k < \frac{1}{8}(2k + 1)^2$

Add $k + 1$ on both the sides.

$\therefore$ $1 + 2 + 3 + ... + k + (k + 1) < \frac{1}{8}(2k + 1)^2 + (k + 1)$ **(i)**

Now, $\frac{1}{8}(2k + 1)^2 + (k + 1) = \frac{1}{8}(4k^2 + 4k + 1 + 8k + 8)$

$= \frac{1}{8}(4k^2 + 12k + 9)$

$\therefore \quad \frac{1}{8}(2k + 1)^2 + (k + 1) = \frac{1}{8}(2k + 3)^2$ **(ii)**

$\therefore \quad 1 + 2 + 3 + ... + (k + 1) < \frac{1}{8}(2k + 3)^2$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} = \frac{4n^2 + 4n}{8} < \frac{4n^2 + 4n + 1}{8} = \frac{1}{8}(2n + 1)^2$

Example 15 : Prove $(1 + x)^n \geq 1 + nx, \quad n \in$ N **($x > -1$)**

Solution : Let P(n) : $(1 + x)^n \geq 1 + nx, \quad n \in$ N

Let $n = 1$. $(1 + x)^1 = 1 + x \geq 1 + 1 \cdot x$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad (1 + x)^k \geq 1 + kx$

Let $n = k + 1$.

Now, $(1 + x)^{k+1} = (1 + x)^k (1 + x)$

$\geq (1 + kx)(1 + x)$ **(by P(k) and as $x > -1$)**

$\therefore \quad (1 + x)^{k+1} \geq 1 + kx + x + kx^2 \geq 1 + kx + x$ as $k \in$ N, $x^2 \geq 0$

$\therefore \quad (1 + x)^{k+1} \geq 1 + (k + 1)x$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 16 : Prove $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \quad n \in$ N

Solution : Let P(n) : $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \quad n \in$ N

Let $n = 1$, L.H.S. = 1, R.H.S. = 2 − 1 = 1

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

Add $\frac{1}{(k+1)^2}$ on both the sides.

Hence, $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ **(i)**

Now, $2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{1}{k} + \frac{1}{(k+1)^2} + \frac{1}{k+1} - \frac{1}{k+1}$

$$= 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{-k-1+k}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} - \frac{1}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} + \frac{k-k-1}{k(k+1)^2}$$

$$= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2}$$

$\therefore \quad 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$ **$\left(k \in N \text{ gives } \frac{1}{k(k+1)^2} > 0\right)$ (ii)**

$\therefore \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{(k+1)^2} < 2 - \frac{1}{(k+1)}$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true $\forall n \in$ N by P.M.I.

Note : Thus however large n, sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}$ is 'bounded' and less than < 2.

Example 17 : Prove $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n, \quad n \in N$

Solution : Let P(n) : $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n, \quad n \in N$

Let $n = 1$. L.H.S. $= \binom{1}{0} + \binom{1}{1} = 2$, R.H.S. $= 2^1 = 2$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore \quad \binom{k}{0} + \binom{k}{1} + ... + \binom{k}{k} = 2^k$

Let $n = k + 1$.

L.H.S. $= \binom{k+1}{0} + \binom{k+1}{1} + ... + \binom{k+1}{k} + \binom{k+1}{k+1}$

$$= \binom{k}{0} + \left(\binom{k}{0} + \binom{k}{1}\right) + \left(\binom{k}{1} + \binom{k}{2}\right) + ... + \left(\binom{k}{k-1} + \binom{k}{k}\right) + \binom{k}{k}$$

$$\left(\text{as } \binom{k}{0} = \binom{k+1}{0} = 1, \binom{k}{k} = \binom{k+1}{k+1} = 1, \binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}\right)$$

$$= 2\left[\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k}\right]$$

$$= 2 \cdot 2^k$$

$$= 2^{k+1} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

1.4 Some Variants of P.M.I.

Variant 1 : If P(n) is a statement involving natural variable n and if P(k_0) is true for some positive integer k_0 and if the truth of P(k) for some integer $k \geq k_0$ implies the truth of P(k + 1), then P(n) is true $\forall n \in$ N, such that $n \geq k_0$.

Example 18 : Prove $2^n > n^2$; $n \geq 5$, $n \in$ N

Solution : Let P(n) : $2^n > n^2$; $n \geq 5$, $n \in$ N

Let $n = 5$. ($k_0 = 5$), $2^5 = 32$, $5^2 = 25$ and $32 > 25$.

$\therefore$ P(5) is true.

Let P(k) be true for $k \geq 5$. Hence, $2^k > k^2$

Let $n = k + 1$.

Now, $2^{k+1} = 2 \cdot 2^k > 2k^2$ **($2^k > k^2$) (i)**

Now, $2k^2 - (k + 1)^2 = 2k^2 - k^2 - 2k - 1$

$= k^2 - 2k + 1 - 2$

$= (k + 1)^2 - 2 > 0$ as $k \geq 5$

$\therefore$ $2k^2 > (k + 1)^2$ **(ii)**

$\therefore$ $2^{k+1} > (k + 1)^2$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Variant 2 : Let P(n) be a statement of integer variable n.

If P(1) and P(2) are true and if P(k) and P(k + 1) are true for some positive integer k implies P(k + 2) is also true, then P(n) is true for all $n \in$ N.

Example 19 : Let a_n be a sequence of natural numbers with $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$. Prove $a_n = 2^n + 3^n$, $\forall n \in$ N.

Solution : Let P(n) : If $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$, $a_1 = 5$, $a_2 = 13$, then $a_n = 2^n + 3^n$, $\forall n \in$ N.

Let $n = 1$. $a_1 = 5$ and $2^1 + 3^1 = 2 + 3 = 5$. Hence, P(1) is true.

Let $n = 2$. $a_2 = 13$ and $2^2 + 3^2 = 4 + 9 = 13$. Hence, P(2) is true.

Let $a_k = 2^k + 3^k$, $a_{k+1} = 2^{k+1} + 3^{k+1}$ for $k \geq 1$

Now, $a_{k+2} = 5a_{k+1} - 6a_k$

$= 5(2^{k+1} + 3^{k+1}) - 6 \cdot 2^k - 6 \cdot 3^k$

$= 5 \cdot 2^k \cdot 2 + 5 \cdot 3^k \cdot 3 - 6 \cdot 2^k - 6 \cdot 3^k$

$= 2^k(10 - 6) + 3^k(15 - 6)$

$= 2^k \cdot 2^2 + 3^k \cdot 3^2$

$= 2^{k+2} + 3^{k+2}$

$\therefore$ P(k + 2) is true.

$\therefore$ P(k) is true and P(k + 1) is true $\Rightarrow$ P(k + 2) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $a_{n+2} = 5a_{n+1} - 6a_n$ is called a **recurrence relation**. Its solution is $a_n = A\alpha^n + B\beta^n$ where α, β are roots of $x^2 - 5x + 6 = 0$ (5 is co-efficient of a_{n+1}, -6 is co-efficient of a_n)

$\therefore$ $\alpha = 3$, $\beta = 2$

$\therefore$ $a_n = A3^n + B2^n$

$\therefore$ $a_1 = 3A + 2B = 5$; $a_2 = 9A + 4B = 13$. Hence, A = B = 1

$\therefore$ $a_n = 3^n + 2^n$. If $a_{n+2} = l \cdot a_{n+1} - m \cdot a_n$, then α and β are the roots of equation $x^2 - lx + m = 0$.

Miscellaneous Problems :

Example 20 : Prove that any payment of ₹ 4 or more can be made using ₹ 2 and ₹ 5 coins only.

Solution : Let P(n) : Any payment of ₹ 4 or more can be made using ₹ 2 and ₹ 5 coins only. $n \in$ N

For $n = 4$, we require two coins of ₹ 2 to pay ₹ 4. Let the statement be true for $k \geq 4$.

Let $n = k + 1$.

Consider two cases :

(1) If the payment for ₹ k contains a ₹ 5 coin, take it back and give 3, ₹ 2 coins. Hence $k + 6 - 5 = k + 1$ rupees are paid.

(2) If the payment for ₹ k does not contain any ₹ 5 coin, since $k \geq 4$, he must have paid at least two ₹ 2 coins. Take them back and pay one ₹ 5 coin. Hence ₹ $k + 5 - 4 = k + 1$ are paid.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) be true, for $\forall n \in$ N by P.M.I.

Example 21 : Prove that any integer $n > 23$ can be put in the form $7x + 5y = n$, where $x \in N \cup \{0\}$, $y \in N \cup \{0\}$.

Solution : Let P(n) : Any integer $n > 23$ can be put in the form $7x + 5y = n$, where $x \in N \cup \{0\}$, $y \in N \cup \{0\}$.

Let $n = 24$. Then $7 \cdot 2 + 5 \cdot 2 = 24$ is the required form with $x = y = 2$.

Let $7x + 5y = k$ for $k \geq 24$, $x \in N \cup \{0\}$, $y \in N \cup \{0\}$. **(i)**

Now, $5 \cdot 3 - 7 \cdot 2 = 1$ **(ii)**

$\therefore$ $7(x - 2) + 5(y + 3) = k + 1$ **(Adding (i) and (ii))**

Here $y + 3 \in N \cup \{0\}$ and $x - 2 \in N \cup \{0\}$ if $x \neq 0$ or 1.

Let $x = 0$. Then $5y = k \geq 24$. Thus $y \geq 5$, using (i).

$7 \cdot 3 - 5 \cdot 4 = 1$ and $5y = k$ gives on adding. **(iii)**

$7 \cdot 3 + 5(y - 4) = k + 1$

Here $x = 3 \geq 0$, $y - 4 \geq 0$ **($y \geq 5$)**

$\therefore$ $P(k + 1)$ is true, if $x = 0$

Let $x = 1$. Hence, $7 + 5y = k$, using (i).

Then $5y = k - 7 \geq 17$. Thus $y \geq 4$

$\therefore$ $7 \cdot 3 - 5 \cdot 4 = 1$ and $7 + 5y = k$ gives on adding. **(iv)**

$7(4) + 5(y - 4) = k + 1$ with $y - 4 \geq 0$ and $x = 4$ **(Adding in (iv))**

$\therefore$ $P(k + 1)$ is true.

$\therefore$ $P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore$ $P(n)$ is true for $\forall n \in N$ by P.M.I.

Example 22 : (Tower of Hanoi) We have three pegs and a collection of disks of different sizes. Initially they are on top on each other according to their size on the first peg, the largest being on the bottom and the smallest on the top. A move in this game consists of moving disks from one peg to another such that larger disk can never rest on a smaller one. Prove that the number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Solution : Let $P(n)$: The number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Let $n = 1$, obviously there is only one move.

$\therefore$ $P(1)$ is true. $2^1 - 1 = 1$. **($p(k)$)**

Suppose there are $2^k - 1$ moves to transfer k disks as required.

First we move top k disks to the second peg using the third peg as the intermediate one. This will take $2^k - 1$ moves. Now move the last disk to the third peg. This is one move. Now move k disks from second peg to the third peg in $2^k - 1$ moves.

$\therefore$ The total number moves is $2^k - 1 + 1 + 2^k - 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$

$\therefore$ $P(k + 1)$ is proved.

$\therefore$ $P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore$ $P(n)$ is true, $\forall n \in N$ by P.M.I.

Example 23 : Prove $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in \mathrm{N}$, $n \in \mathrm{N}$ (to be done after chapter 3)

Solution : Let P(n) : $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in \mathrm{N}$, $n \in \mathrm{N}$

For $n = 1$, $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} = \frac{15 + 33 + 55 + 62}{165} = \frac{165}{165} = 1$

$\therefore$ P(1) is true.

Let P(k) be true. Hence, $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in \mathrm{N}$

Let $n = k + 1$.

Consider $\left(\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}\right) - \left(\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}\right)$

$$= \frac{1}{11}((k + 1)^{11} - k^{11}) + \frac{1}{5}((k + 1)^5 - k^5) + \frac{1}{3}((k + 1)^3 - k^3) + \frac{62}{165}$$

$$= \frac{1}{11}\left(1 + \binom{11}{1}k + \binom{11}{2}k^2 + ... + \binom{11}{10}k^{10}\right) + \frac{1}{5}\left(1 + \binom{5}{1}k + \binom{5}{2}k^2 + ... + \binom{5}{4}k^4\right)$$
$$+ \frac{1}{3}\left(1 + \binom{3}{1}k + \binom{3}{2}k^2\right) + \frac{62}{165}$$

$$= \frac{1}{11}\binom{11}{1}k + \frac{1}{11}\binom{11}{2}k^2 + ... + \frac{1}{11}\binom{11}{10}k^{10} + \frac{1}{5}\binom{5}{1}k + \frac{1}{5}\binom{5}{2}k^2 + ... + \frac{1}{5}\binom{5}{4}k^4$$
$$+ \frac{1}{3}\binom{3}{1}k + \frac{1}{3}\binom{3}{2}k^2 + \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} \quad \text{(i)}$$

Now, 11, 5, 3 being primes, 11 divides $\binom{11}{r}$ for $r = 1, 2, ..., 10$

5 divides $\binom{5}{r}$ for $r = 1, 2, 3, 4$

3 divides $\binom{3}{r}$ for $r = 1, 2$

and $\frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} = 1$

$\therefore$ The R.H.S. in (1) represents a natural number.

Also $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in \mathrm{N}$

$\therefore$ $\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}$

$= \frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}$ + a natural number $\in \mathrm{N}$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true for $\forall n \in \mathrm{N}$ by P.M.I.

Example 24 : There are $2n$ persons in a hall. Some persons handshake with others. There do not exist any three persons who have handshakes with each other. Prove that the number of handshakes is at most n^2.

Solution : Let P(n) : There are $2n$ persons in a hall. Some persons handshake with others. There do not exist any three persons who have handshakes with each other. Then the number of handshakes is at most n^2.

For $n = 1$, there are two persons. Hence there is at most $1 = 1^2$ handshake.

$\therefore$ P(1) is true.

Let P(k) be true.

Let $n = k + 1$.

Now there are $2k + 2$ persons. Choose two persons A and B who have had a handshake.

(If there are no two such persons, number of handshakes is zero which is at most $(k + 1)^2$.

Now the remaining $2k$ persons had at most k^2 handshakes (P(k) is true). A and B have one handshake.

Each of $2k$ persons could shake hands with A or B only as no three persons had handshakes with each other. Hence the number of handshakes is at most

$$k^2 + 1 + 2k = (k + 1)^2$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true for $\forall n \in$ N by P.M.I.

A paradox :

[**Note :** A paradox is the misinterpretation of a result to arrive at a contradictory result.]

P(n) : A thirsty man can drink n drops of water.

For $n = 1$, obviously a thirsty man would like to drink one drop of water.

If he can drink k drops of water, he can definitely drink $k + 1$ drops of water.

So he can drink any amount water to exhaust all resources of water on the earth !

Exercise 1

Prove the following by the principle of mathematical induction : **(1 to 19) ($n \in$ N)**

1. $1^2 \cdot 2 + 2^2 \cdot 3 + ... + n^2(n + 1) = \frac{n(n+1)(n+2)(3n+1)}{12}$

2. $a + (a + d) + (a + 2d) + ... + (a + (n - 1)d) = \frac{1}{2}n(2a + (n - 1)d)$

3. $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + ... + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

4. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + ... + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

5. $1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$

6. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n$

7. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + ... + \frac{1}{1+2+3+...+n} = \frac{2n}{n+1}$

8. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + ... + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

9. $1^4 + 2^4 + 3^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

10. If $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, then $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$.

11. $41^n - 1$ is divisible by 40.

12. $4007^n - 1$ is divisible by 2003.

13. $7^n - 6n - 1$ is divisible by 36.

14. $2 \cdot 7^n + 3 \cdot 5^n - 5$ is a multiple of 24.

15. $11^{n+2} + 12^{2n+1}$ is divisible by 133.

16. $n(n+1)(2n+1)$ is divisible by 6.

17. $1 \cdot 3^1 + 2 \cdot 3^2 + 3 \cdot 3^3 + ... + n \cdot 3^n = \frac{(2n-1)\,3^{n+1} + 3}{4}$

18. $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.

19. $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} \in N$

20. Prove $\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$

21. For Lucas' sequence $a_n = a_{n-1} + a_{n-2}$ $(n \geq 3)$; $a_1 = 1$, $a_2 = 3$, prove $a_n \leq (1.75)^n$.

22. Prove $2^n > n^3$, if $n \geq 10$

23. Prove a polygon of n sides has $\frac{n(n-3)}{2}$ diagonals, $n > 3$

24. If $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, then prove that

$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ (This $\{a_n\}$ is called **Fibonacci** sequence.)

25. If $f : N \rightarrow N$, $f(1) = 1$, $f(2) = 5$, $f(n+1) = f(n) + 2f(n-1)$, $n \geq 2$

then prove that $f(n) = 2^n + (-1)^n$

26. If $f : N \rightarrow N$, $f(1) = 1$, $f(n+1) - f(n) = 2^n$

then prove that $f(n) = 2^n - 1$

27. If $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$; $n \geq 3$

then prove that $a_2 + a_4 + a_6 + ... + a_{2n} = a_{2n+1} - 1$

28. If $a_1 = 1$, $a_2 = 11$ and $a_n = 2a_{n-1} + 3a_{n-2}$; $n \geq 3$

then prove that $a_n = 2(-1)^n + 3^n$ for $n \in N$

29. Prove that every integer $n \geq 12$ can be written in the form $7x + 3y = n$,

$x \in N \cup \{0\}$, $y \in N \cup \{0\}$

30. Prove that $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is even, $n \in N$.

31. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For P(n) : $2^n < n\,!$, ... is true.

(a) P(1) (b) P(2) (c) any P(n), $n \in N$ (d) P(4)

(2) For P(n) : $2^n = 0$, ... is true.

(a) P(1) (b) P(3)

(c) P(10) (d) $P(k) \Rightarrow P(k + 1)$, $k \in N$

(3) P(n) : $1 + 2 + 3 + ... + (n + 1) = \frac{(n+1)(n+2)}{2}$, $n \in N$

(a) P(1) requires L.H.S. = 7 = R.H.S.

(b) P(1) requires L.H.S. = 3 = R.H.S.

(c) $P(k) \Rightarrow P(k + 1)$ is not true for $k \in N$

(d) It is false that P(n) is true, $\forall n \in N$ by P.M.I.

(4) If ...... is true and P(k) is true $\Rightarrow$ P(k + 1) is true for $k \geq -1$, then P(n) is true for all $n \in N \cup \{0, -1\}$.

(a) P(−1) (b) P(0) (c) P(1) (d) P(2)

(5) P(n) : Every prime is of the form $2^{2^n} + 1$ is not true, for n = ...

(a) 1 (b) 2 (c) 0 (d) 5

(6) P(n) : $2^n - 1$ is a prime for n = ...

(a) 1 (b) 3 (c) 4 (d) 8

(7) P(n) : $n^2 - n + 41$ is a prime, is false for n = ...

(a) 1 (b) 2 (c) 3 (d) 41

(8) P(n) : $2n + 1$ is a prime, is false for n = ...

(a) 1 (b) 2 (c) 3 (d) 4

(9) P(n) : $4n + 1$ is a prime, is false for n = ...

(a) 1 (b) 3 (c) 7 (d) 11

(10) P(n) : $2^n > n^2$ is true for n = ...

(a) 2 (b) 3 (c) 4 (d) 5

*

Summary

We studied the following points in this chapter :

1. Principle of Induction and Examples

2. Different variants of P.M.I. and applications

Puzzle

There are n people in a room each being put on a hat from amongst at least n white hats and $n - 1$ black hats. They stand in a queue, so that every one can see the colour of the hat of the person standing in front of him. Starting from back we ask the persons in turn, 'Do you know what is the colour of your hat ?' If the first $(n - 1)$ persons say no, the person in the front will say 'Yes the colour of my hat is white.' Prove.

Solution : Let P(n) : If the first $(n - 1)$ persons say no, the person in the front will say yes.

For $n = 1$, there is no black hat $(1 - 1 = 0)$. Hence the first person will say, 'yes, my hat is white.' Suppose the statement is true for $n = k$. Let $n = k + 1$.

See how the man standing in the front would think. Suppose my hat is black. Then excluding me there are k people with at least k white hats and $k - 1$ black hats. By P(k), since the first $(k - 1)$ persons said no, the person behind me must say yes. 'I know the colour of my hat.'

But he said no. So the colour of my hat cannot be black. Hence it is white.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Explanation : If $n = 2$, there is one black hat and at least two white hats. If the last person sees a black hat put on by the person in front of him, he would definitely say, 'Yes, colour of my hat is white,' as there is only one black hat. But he is not able to answer. So the first person logically thinks he has put on a white hat and the person behind might have put on a black or a white hat.

Srinivasa Ramanujan (1887-1920) was one of India's greatest mathematical geniuses. He made substantial contributions to the analytical theory of numbers and worked on elliptical functions, continued fractions and infinite series.

In 1990 he began to work on his own on mathematics summing geometric and arithmetic series.

Ramanujan had shown how to solve cubic equations in 1902 and he went to find his own method to solve the quartic.

In 1904 Ramanujan had begun to undertake deep research. He investigated the series $\Sigma\left(\frac{1}{n}\right)$ and calculated Euler's constant to 15 decimal places.

Continuing his mathematical work Ramanujan studied continued fractions and divergent series in 1908.

Chapter 2

COMPLEX NUMBERS

A mathematician is a device for turning coffee into theorems.

– Paul Erdos

As far as the laws of mathematics refer to reality, they are not certain and as far as they are certain they do not refer to reality.

– Albert Einstein

2.1 Introduction

In previous classes, we have studied the number sets N, Z, Q and R. We know that the set of rational numbers and the set of irrational numbers constitute the set of real numbers. We also studied properties of numbers and solutions of linear equations in one variable and two variables. We also discussed the solutions of quadratic equations in one variable. We observed that if the discriminant $b^2 - 4ac < 0$, the quadratic equation $ax^2 + bx + c = 0$, $a, b, c \in$ R, $a \neq 0$ has no solution in R. For example $x^2 + 1 = 0$ has no solution in R. To allow the square root of negative numbers, the real number system has to be extended to a larger system. In fact, Greeks were the first to recognize the fact that square root of a negative number does not exist in the real number system. The Indian mathematician **Mahavira** or **Maviracharya** (850 A.D.) too mentions this difficulty in his work ***'Ganitasara Sangraha'***. The extension of real number system should be in such a way that the algebraic operations such as addition, subtraction, multiplication and division can be defined properly. This new set is called the set of **Complex Numbers** and is denoted by **C**.

2.2 The Set R × R and the Set of Complex Numbers

We begin with the set R of real numbers to obtain the set C of complex numbers. R × R is the set of all ordered pairs of real numbers.

R × R = $\{(a, b) \mid a \in$ R, $b \in$ R$\}$

We shall define the equality, addition and multiplication of two elements of R × R.

(1) Equality : **Two elements (a, b) and (c, d) of R × R are defined to be equal if $a = c$ and $b = d$. Thus $a = c, b = d \Rightarrow (a, b) = (c, d)$**

For example, $(1, 0) = (sin^2x + cos^2x, \log 1)$ but $(1, 4) \neq (4, 1)$

(2) Addition : **The sum of two elements (a, b) and (c, d) of R × R is defined as follows :**

$(a, b) + (c, d) = (a + c, b + d)$

For example, $(5, 2) + (2, 3) = (5 + 2, 2 + 3) = (7, 5)$

(3) Multiplication : **The product of two elements (a, b) and (c, d) of R × R is defined as follows :**

$(a, b)(c, d) = (ac - bd, ad + bc)$

For example, $(5, 2)(2, 3) = (5 \times 2 - 2 \times 3, 5 \times 3 + 2 \times 2) = (4, 19)$

The set R × R with these rules is called the set of complex numbers and it is denoted by C. Generally, we denote a complex number by z.

2.3 Basic Algebraic Properties of Complex Numbers

We have discussed the properties of closure, commutativity, associativity and distributivity with respect to operations of addition and multiplication on R. We shall see that these properties hold good in C too.

The operation of addition satisfies the following properties :

(1) The closure property : **The sum of two complex numbers is a complex number.**

i.e. $z_1 + z_2 \in C \quad \forall z_1, z_2 \in C$

We also say that the **addition is a binary operation on C.**

(2) The commutative property : $z_1 + z_2 = z_2 + z_1 \quad \forall z_1, z_2 \in C$

(3) The associative property : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \forall z_1, z_2, z_3 \in C$

(4) The existence of additive identity : **There exists a complex number O = (0, 0), called an additive identity or the zero complex number, such that**

$z + O = z = O + z \quad \forall z \in C$

It can be proved that **the additive identity O is unique.**

In fact if $(a, b) + (x, y) = (a, b)$ for all $(a, b) \in C$,

then $a + x = a, \quad b + y = b,$

$\therefore \quad x = 0, \quad y = 0.$

Thus, $(x, y) = (0, 0)$

Also $(a, b) + (0, 0) = (a, b)$.

(5) The existence of additive inverse : **To every complex number $z = (a, b)$, there corresponds a complex number $(-a, -b)$, denoted by $-z$, called the additive inverse (or negative) of z such that $z + (-a, -b) = (0, 0) = O$.**

We observe that, $z + (-z) = (a, b) + (-a, -b)$

$= (a + (-a), b + (-b))$

$= (0, 0)$

$= O$ (O is the additive identity.)

Also, $(-z) + z = O$

We can prove that **for $z \in C$, its additive inverse $-z$ is unique.**

Note : $(a, b) + (x, y) = (0, 0)$ requires $a + x = 0 = b + y$

$\therefore \quad x = -a, y = -b$

$\therefore$ **$(-a, -b)$ is the additive inverse of (a, b).**

The operation of multiplication satisfies following properties :

(1) **The closure property : The product of two complex numbers is a complex number.**

i.e. $z_1 z_2 \in C, \qquad \forall z_1, z_2 \in C$

We also say that the multiplication is a binary operation on C.

(2) **The commutative property : $z_1 z_2 = z_2 z_1 \quad \forall z_1, z_2 \in C$**

(3) **The associative property : $(z_1 z_2)z_3 = z_1(z_2 z_3) \quad \forall z_1, z_2, z_3 \in C$**

(4) **The existence of multiplicative identity : There exists a complex number $(1, 0)$, called a multiplicative identity such that $z(1, 0) = z = (1, 0)z, \quad \forall z \in C$**

By taking $z = (a, b)$, $z.\ (1, 0) = (a, b)(1, 0) = (a - 0, 0 + b) = (a, b) = z$

Also, $(1, 0)\, z = z\,(1, 0) = z$

The multiplicative identity $(1, 0)$ is unique.

Note : If $(a, b)(x, y) = (a, b)$. $\forall (a, b) \in C$, then $ax - by = a$ and $ay + bx = b$, $\forall a, b \in R$. In particular $a = 1, b = 0$ gives $x = 1, y = 0$. Then $(a, b)(1, 0) = (a, b)$, $\forall (a, b) \in C$.

(5) **The existence of multiplicative inverse :** To each non-zero complex number $z = (a, b)$, there corresponds a complex number $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ (denoted by z^{-1}), called a multiplicative inverse of z such that

$z \cdot z^{-1} = (1, 0) = z^{-1} \cdot z$ **((1, 0) is the multiplicative identity)**

Since $(a, b) \neq (0, 0)$, $a^2 + b^2 \neq 0$ and hence $z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) \in C$ and

$$z \cdot z^{-1} = (a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(a \cdot \frac{a}{a^2+b^2} - b \cdot \frac{-b}{a^2+b^2},\ a \cdot \frac{-b}{a^2+b^2} + b \cdot \frac{a}{a^2+b^2}\right)$$

$$= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ab}{a^2+b^2}\right) = (1, 0)$$

Also, $z^{-1} \cdot z = (1, 0)$

Note that for each non-zero $z \in C$, its **multiplicative inverse z^{-1} is unique.**

z^{-1} is also denoted in $\frac{1}{z}$.

Note : Let z' be a complex number such that $zz' = (1, 0)$

Let $z' = (x, y)$

$\therefore \quad zz' = (a, b)(x, y) = (1, 0)$

$\therefore \quad (ax - by, ay + bx) = (1, 0)$

$\therefore \quad ax - by = 1, ay + bx = 0$

Solving these equations we get $x = \frac{a}{a^2+b^2}$, $y = \frac{-b}{a^2+b^2}$

$\therefore \quad z' = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2} \right)$

As $z = (a, b) \neq (0, 0)$ we have $a^2 + b^2 \neq 0$.

$\therefore \quad z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2} \right)$

The existence of multiplicative inverse enables us to show that if a product $z_1 z_2$ is zero, then at least one of the factors z_1 and z_2 is zero. (why ?)

(6) The distributive laws : For any three complex numbers z_1, z_2, z_3

(a) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

2.4 R as a Subset of C

By definition, every complex number is an ordered pair of real numbers. Let us denote by R' the set of those complex numbers (a, b) in which $b = 0$. So, $R' = \{(a, 0) \mid a \in R\}$. Obviously $R' \subset C$. Let $(a, 0)$, $(b, 0)$ be two elements of R'. Note that,

(1) $(a, 0) = (b, 0) \Leftrightarrow a = b$

(2) $(a, 0) + (b, 0) = (a + b, 0) \in R'$

(3) $(a, 0)(b, 0) = (ab, 0) \in R'$

Thus, the sum as well as the product of two elements of R' is again an element of R'. Moreover, the first component of the sum or product of two numbers $(a, 0)$ and $(b, 0)$ is obtained merely by adding or multiplying respectively the first components a and b, while the second component remains zero. Infact R' is closed for addition and multiplication as in C. So as far as equality, sum and multiplication are concerned, the complex numbers of the form $(a, 0)$ behave exactly like real number a. Hence we identify complex numbers of the form $(a, 0)$ with a and write a for $(a, 0)$. Thus $(4, 0) = 4$, $(0, 0) = 0$ etc. In this way we look upon every real number a as the complex number $(a, 0)$, which allows us to identify R with R' and so $R' = R \subset C$. Thus we have **$N \subset Z \subset Q \subset R \subset C$**. Now $O = (0, 0) = 0$, the additive identity $(1, 0) = 1$, the multiplicative identity.

2.5 Representation of a Complex Number in the form *a* + *ib*

By writing a for $(a, 0)$ we are able to represent a complex number (a, b) in another form.

Firstly, let us get familiar with a special complex number $(0, 1)$. We use the symbol i for this complex number. Thus, $i = (0, 1)$.

Now, $i^2 = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1$. In the year 1737 Euler was the first person to introduce the symbol i for the complex number $(0, 1)$, satisfying $i^2 = -1$. $i = (0, 1)$ is called **an imaginary number**.

Now, $(a, b) = (a, 0) + (0, b)$

$= (a, 0) + (0, 1)(b, 0) \qquad$ **$((0, 1)(b, 0) = (0 - 0, 0 + b) = (0, b))$**

$= a + ib$

$\therefore \quad (a, b) = a + ib$

Hence, every complex number (a, b) can be expressed in the form $a + ib$, where $a, b \in R$ and $i^2 = -1$.

Thus, **C = {$a + ib$ | $a, b \in$ R}**

According to the commutative law for multiplication, $ib = bi$.

Hence, $a + ib = a + bi$

For example, (3, 5) = 3 + 5i, (0, 7) = 0 + 7i = 7i, (5, 0) = 5 + 0i = 5

For the complex number $z = a + bi$, ***a*** is called the **real part** of z and is denoted by $Re(z)$ and ***b*** is called the **imaginary part** of z and is denoted by $Im(z)$.

So, $z = a + ib = Re(z) + iIm(z)$. For example, if $z = 3 + 2i$, then $Re(z) = 3$ and $Im(z) = 2$.

Note that both the real and imaginary parts of a complex number are real numbers.

A complex number, whose real part is zero and whose imaginary part is non-zero is called a **purely imaginary number**. For example, $9i = 0 + 9i$ is a purely imaginary number.

Let us now revert to the algebraic operations on complex numbers which are in the form $a + bi$.

Equality of two complex numbers :

Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal i.e. $(a, b) = (c, d)$ if $a = c$ and $b = d$.

If $z = a + bi = 0$ then $a = 0$ and $b = 0$. **$(0 = 0 + 0i)$**

Example 1 : if $3x + (3x - y)i = 4 + (-6)i$, where x and y are real numbers, then find the values of x and y.

Solution : We have $3x + (3x - y)i = 4 + (-6)i$. Since $a + bi = c + di \Rightarrow a = c$ and $b = d$, we get $3x = 4$, $3x - y = -6$. On solving, we get $x = \frac{4}{3}$, $y = 10$.

Addition of two complex numbers :

Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers. Then the sum of $z_1 = (a, b)$, $z_2 = (c, d)$ is as follows :

$$z_1 + z_2 = (a, b) + (c, d) = (a + c, b + d) = a + c + (b + d)i$$

For example, $(2 + 2\sqrt{2}\,i) + (-3 + \sqrt{2}\,i) = (2 - 3) + (2\sqrt{2} + \sqrt{2})i$

$$= -1 + 3\sqrt{2}\,i$$

Difference of two complex numbers :

Let z_1 and z_2 be any two complex numbers. The difference $z_1 - z_2$ is defined by,

$$z_1 - z_2 = z_1 + (-z_2)$$

Let $z_1 = (a, b)$, $z_2 = (c, d)$

Then $-z_2 = (-c, -d)$

$$\begin{aligned} \therefore\quad z_1 - z_2 &= z_1 + (-z_2) \\ &= (a, b) + (-c, -d) \\ &= (a - c, b - d) \\ &= (a - c) + (b - d)i \end{aligned}$$

For example, $(2 + \sqrt{3}i) - (-3 + 2\sqrt{3}i) = 2 - (-3) + (\sqrt{3} - 2\sqrt{3})\,i$

$$= 5 - \sqrt{3}i$$

Multiplication of two complex numbers :

Let $z_1 = a + bi$ and $z_2 = c + di$ be any two complex numbers.

$\therefore \quad z_1 = (a, b), z_2 = (c, d)$

$\therefore \quad z_1 z_2 = (ac - bd, ad + bc)$

$\therefore \quad \boldsymbol{z_1 z_2 = (ac - bd) + (ad + bc)i}$

For example, $(2 + \sqrt{3}i)(-3 + \sqrt{3}i) = (2 \times (-3) - \sqrt{3}\sqrt{3}) + (2\sqrt{3} + \sqrt{3} \times (-3))i$

$= (-6 - 3) + (2\sqrt{3} - 3\sqrt{3})i = -9 - \sqrt{3}i$

We can open the bracket and multiply them because of the distributive laws.

Quotient of two complex numbers :

Let z_1 and z_2 be any two complex numbers where, $z_2 \neq 0$. The quotient $\frac{z_1}{z_2}$ is defined as $\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \frac{1}{z_2}$.

In fact, $\frac{1}{z} = z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$

For example, $\frac{6+3i}{10+8i} = (6 + 3i)(10 + 8i)^{-1}$

$= (6 + 3i)\left(\frac{10}{164} - \frac{8i}{164}\right)$

$= \frac{60 + 30i - 48i - 24i^2}{164}$

$= \frac{84 - 18i}{164}$ $\quad (i^2 = -1)$

$= \frac{21}{41} + \frac{(-9)}{82} i$

Powers of i :

We shall assume that the usual laws of indices hold good for integral powers of z.

We know that $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$, $i^5 = i$, $i^6 = -1$ etc.

Remember, $\boldsymbol{i^2 = -1, i^3 = -i, i^4 = 1}$

Also, we have $i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$ etc.

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

In mathematics, **trichotomy** is the property of an order relation. For any real numbers x and y, exactly one of the following holds : $x < y$, $x = y$ or $x > y$. This law of trichotomy holds for comparison of real numbers. This property is no more valid for complex numbers as C is not ordered.

Example 2 : Evaluate (i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2$ $\quad$ (ii) $i^1 + i^2 + i^3 + i^4 + ... + i^{1000}$

Solution : (i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{16} i^3 + \left(\frac{1}{i}\right)^{24}\left(\frac{1}{i}\right)\right]^2$

$= (-i - i)^2$

$= (-2i)^2 = -4$

(ii) $i^1 + i^2 + i^3 + i^4 + ... + i^{997} + i^{998} + i^{999} + i^{1000}$

$= (i - 1 - i + 1) + (i - 1 - i + 1) + ... + (i - 1 - i + 1)$ (250 brackets)

$= 0 + 0 + ... + 0 = 0$

Conjugate of a Complex Number :

If $z = (a, b) = a + bi$, then its conjugate complex number is defined to be the complex number $a - bi = (a, -b)$ and is denoted by $\bar{z}$.

We note that $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$. So just like a surd $\bar{z}$ acts like a 'rationalising' factor. Since $z\bar{z} = a^2 + b^2$ is real, we express complex number $\frac{p}{q}$ as $\frac{p\bar{q}}{q\bar{q}}$ so that the denominator $q\bar{q}$ is real. Let us understand this concept by a few examples.

Example 3 : Express the following in the form of $a + ib$, where $a, b \in R$

(1) $\frac{(2-8i)(7+8i)}{1+i}$ (2) $(3 + 4i)^{-1}$ (3) $\frac{(1+i)^3}{4+3i}$ (4) $\frac{1}{1+\cos\theta - i\sin\theta}$

Solution : (1) $\frac{(2-8i)(7+8i)}{1+i} = \frac{14+16i-56i-64i^2}{1+i}$

$= \frac{14-40i+64}{1+i}$ **($i^2 = -1$)**

$= \frac{78-40i}{1+i}$

$= \frac{78-40i}{1+i} \times \frac{1-i}{1-i}$ **(multiply and divide by conjugate of $1 + i$)**

$= \frac{78-78i-40i+40i^2}{1-i^2}$

$= \frac{38-118i}{2}$ **($i^2 = -1$)**

$= 19 - 59i$

(2) $(3 + 4i)^{-1} = \frac{1}{3+4i} = \frac{1}{3+4i} \times \frac{3-4i}{3-4i} = \frac{3-4i}{9+16} = \frac{3}{25} + i\left(-\frac{4}{25}\right)$

or directly $(3 + 4i)^{-1} = \frac{3-4i}{3^2+4^2} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4i}{25}$ **(formula of z^{-1})**

(3) $\frac{(1+i)^3}{4+3i} = \frac{1^3+3\cdot 1^2\cdot i+3\cdot 1\cdot i^2+i^3}{4+3i}$

$= \frac{1+3i-3-i}{4+3i}$

$= \frac{-2+2i}{4+3i}$

$= \frac{(-2+2i)(4-3i)}{(4+3i)(4-3i)}$

$= \frac{-8+8i+6i-6i^2}{16+9}$

$= -\frac{2}{25} + \frac{14}{25}i$ **($i^2 = -1$)**

(4) $\dfrac{1}{1+\cos\theta - i\sin\theta} = \dfrac{1}{1+\cos\theta - i\sin\theta} \times \dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta + i\sin\theta}$

$= \dfrac{1+\cos\theta + i\sin\theta}{(1+\cos\theta)^2 + \sin^2\theta}$

$= \dfrac{1+\cos\theta + i\sin\theta}{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}$

$= \dfrac{1+\cos\theta + i\sin\theta}{2+2\cos\theta}$

$= \dfrac{1+\cos\theta}{2(1+\cos\theta)} + i\,\dfrac{\sin\theta}{2(1+\cos\theta)}$

$= \dfrac{1}{2} + i\,\dfrac{\sin\theta}{2(1+\cos\theta)}$

Note : We will see in chapter 5 that $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\dfrac{\theta}{2}$.

Example 4 : Find the real values of x and y so that

(1) $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i$ (2) $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$

Solution : (1) $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i$

$\therefore$ $[x + (x-2)i](3-i) + [2y + (1-3y)i](3+i) = (3+i)(3-i)i$

(Multiplying both sides by $(3+i)(3-i)$)

$\therefore$ $3x + (x-2) + [3(x-2) - x]i + 6y - (1-3y) + [2y + 3(1-3y)]i = (9+1)i$

$\therefore$ $(4x + 9y - 3) + (2x - 7y - 3)i = 10i$

$\therefore$ $4x + 9y - 3 = 0$ and $2x - 7y - 3 = 10$ **(Equality of two complex numbers)**

$\therefore$ $4x + 9y - 3 = 0$ and $2x - 7y - 13 = 0$

Solving the above simultaneous equations, we get $x = 3$, $y = -1$.

(2) $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$

$\therefore$ $iy(3x+y) - (3y+4i)(ix+1) = 0$ **(Multiplying both sides by $(ix+1)(3x+y)$)**

$\therefore$ $(-3y + 4x) + i(3xy + y^2 - 3xy - 4) = 0 + i0$

$\therefore$ $(-3y + 4x) + i(y^2 - 4) = 0 + i0$

$\therefore$ $-3y + 4x = 0$ and $y^2 - 4 = 0$ **($a + bi = 0 \Rightarrow a = 0, b = 0$)**

$y^2 - 4 = 0$ gives us $y = \pm 2$

For $y = 2$ we get $x = \frac{3}{2}$ and for $y = -2$ we get $x = -\frac{3}{2}$.

$\therefore$ The solution set is $\left\{\left(\frac{3}{2}, 2\right), \left(-\frac{3}{2}, -2\right)\right\}$.

Exercise 2.1

1. Express the following complex numbers in the form $a + bi$:

(1) $(\sqrt{2} - i) - i(1 - \sqrt{2}\,i)$ (2) $(2 - 3i)(-2 + i)$

(3) $(3 + i)(3 - i)\left(\frac{1}{5} + \frac{1}{10}i\right)$ (4) $\dfrac{4+i}{2-3i}$ (use $(2-3i)^{-1}$)

(5) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$ (6) $\frac{5i}{(1-i)(2-i)(3-i)}$

(7) $(1 - i)^4$ (8) $\left[i^{17} - \left(\frac{1}{i}\right)^{34}\right]^2$

(9) $\left(\frac{4i^3-1}{2i+1}\right)^2$ (10) $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

2. Find the real values of x and y, if

(1) $x + 4yi = xi + y + 3$

(2) $(4 + 5i)x + (3 - 2i)y + i^2 + 6i = 0$

(3) $\frac{x}{1-i} + \frac{y}{1+i} = 1 + 3i$

(4) $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$

(5) $(3x - 2yi)(2 + i)^2 = 10(1 + i)$

3. Find the multiplicative inverse of :

(1) $3 - 2i$ (2) $-1 + i\sqrt{3}$ (3) $\frac{4+3i}{5-3i}$ (4) $(2 - 3i)^2$ (5) $-i$

4. Show that, (1) $Re(iz) = -Im(z)$ (2) $Im(iz) = Re(z)$

5. Verify that each of the two complex numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

*

2.6 Conjugate and Modulus of a Complex Number

Complex Conjugate : We know if $z = a + bi$, $\bar{z} = a - bi$.

As an example,

(1) If $z = 3 + 5i$, then $\bar{z} = 3 - 5i$

(2) If $z = 5 - 3i$, then $\bar{z} = 5 + 3i$

(3) If $z = 3 = 3 + 0i$, then $\bar{z} = 3 - 0i = 3$

(4) If $z = 3i = 0 + 3i$, then $\bar{z} = 0 - 3i = -3i$

Here are some basic facts about conjugates.

For any three complex numbers z, z_1, z_2 we have the following properties :

1. $\overline{(\bar{z})} = z$ **2.** $\frac{z+\bar{z}}{2} = Re(z)$

3. $\frac{z-\bar{z}}{2i} = Im(z)$ **4.** $z = \bar{z}$ if and only if z is real.

5. $\bar{z} = -z$ if and only if z is purely imaginary.

6. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ **7.** $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

8. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, where $z_2 \neq 0$

The above properties are easy to verify. Let us verify some of them.

Let $z = a + ib$

1. $\bar{z} = a - ib$

$\therefore$ $\overline{(\bar{z})} = \overline{a - ib} = a + ib = z$