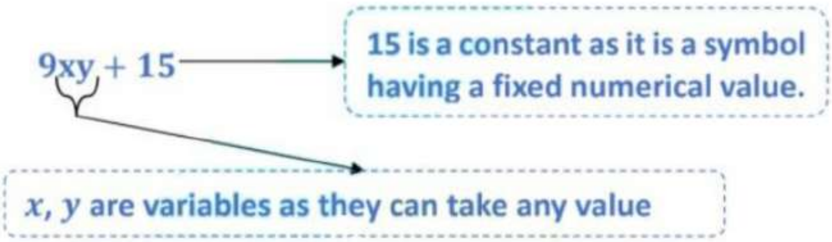
Chapter 9

Algebraic Expressions and Identities

Introduction to Algebraic Expressions and Identities

What are Algebraic Expressions?

A combination of constants and variables connected by the signs of fundamental operations of addition, subtraction, multiplication, division is called an Algebraic Expression.

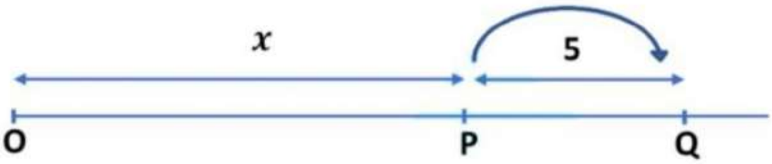


In an expression, letters (a, b, x, y, z, etc.) are used to denote variables. A variable can take various values. Its value is not fixed. A constant has a fixed value. Examples of constants are 2, 10, -6, etc.

Number line and Expressions

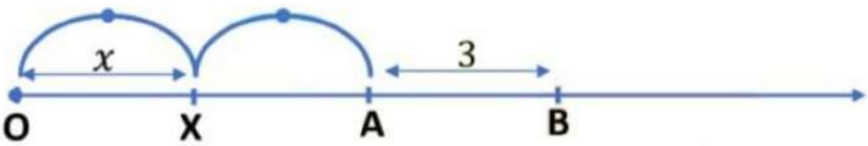
Consider the expression: $x + 5$

Let the measure of variable x be OP on the number line.



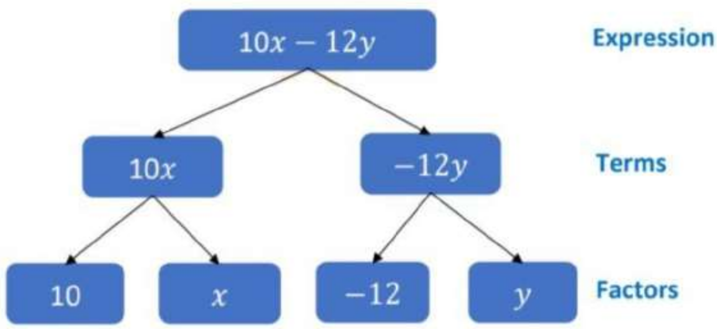
The measure of OP could be anywhere on the number line but it is definite that $x + 5$ is given by point OQ which is 5 units to the right of OP.

Similarly, the value of $x - 5$ will be 5 units to the left of x. The position of $2x + 3$ will be as follows.



The position of 2x will be point A and position B of $2x + 3$ will be 3 units to the right of A.

Terms, Factors, and Coefficients



Terms

Various parts of an algebraic expression that are separated by the sign of '+' or '–' are called the terms of the expression. Or A term in an expression that can be written as a product of constants and variables.

Factors

Terms themselves can be formed as the product of factors. Or These constants and variables are known as factors of terms. Example: 10x and $-12y$ can be factories as $2 \times 5 \times x$ and $-2 \times 2 \times 3 \times y$.

Coefficient

The numerical factor of the term is called the coefficient. Or Coefficient is a numerical factor of a term and it is written as the first factor of the term. The numerical factor of a term is called its numerical coefficient or simply coefficient.

Example: In 10x, 10 is the coefficient.

Types of Algebraic Expressions

Depending upon the number of terms, an algebraic expression can be categorized in the following manner.

a) Monomial Expressions
b) Binomial Expressions
c) Trinomial Expressions
d) Polynomial Expressions

Monomial Expressions

An algebraic expression that contains only one term is called a monomial.

Example: $12a^2$, $3b^2$, $-5c^2$

Binomial Expressions
An algebraic expression that contains two terms is called a binomial.

Example: $12a^2 + 3b^2$, $5x^2 - 7y^2$

Trinomial Expressions

An Algebraic expression that contains three-term is called a trinomial.

Example: $12a^2 + 3b^2 - 5c^2$

Polynomial Expressions

An algebraic expression that contains one or more terms with non-zero coefficients (with variables having non-negative exponents) is called a polynomial. A polynomial can be a monomial, binomial or trinomial.

Example: $2a^2$, $2a^2 + 3b^2$, $5x^2 - y^2 - z$

Types	No. of terms	Example
Monomial	1	$12a^2,\ 3b^2, -5c^2$
Binomial	2	$12a^2 + 3b^2, 5x^2 - 7y^2$
Trinomial	3	$12a^2 + 3b^2 - 5c^2$
Polynomial	1 or more than 1	$5x^2 - y^2 - z^2$

Classify the following polynomials as monomials, binomial, trinomial. Which polynomials do not fit in any category?

a) $a + b$ b) 1000 c) $a + a^2 + a^3 + a^4$
d) $7 + a + 5b$ e) pqr f) $p^2q + pq^2$

a) Binomial
b) Monomial
c) Polynomial
d) Trinomial
e) Monomial
f) Binomial

Like and Unlike Terms

Like Terms

Terms that have the same algebraic factors are like terms. The Coefficients do not need to match.
For example,

$12a^2 + 3b^2 + 5a^2$;

$12a^2$ and $5a^2$ are like terms.

Unlike Terms

Terms which have different algebraic factors are unlike terms that means they do not have the same variables or powers.
For example: $12a^2 + 3b^2$;

$12a^2$ and $3b^2$ are unlike terms.

Operations on Algebraic Expressions - Addition & Subtraction

Addition of Algebraic Expressions

While adding algebraic expressions we collect the like terms and add them.
The sum of several like terms is the like term whose coefficient is the sum of the coefficients of these like terms.
The simplest expressions are monomials. They consist of only one term.

i) Add 8x and 6x

$8x + 6x = (8 \times x) + (6 \times x)$
$= (8 + 6) \times x$
$= 14 \times x$
$= 14x$

ii) Add 8x and 6y

$8x + 6y = (8 \times x) + (6 \times y)$
$= 8x + 6y$

The sum of two or more like terms is a like term with a numerical coefficient equal to the sum of the numerical coefficients of all the like terms.

Unlike terms cannot be added the way like terms are added.

To add two algebraic expressions, we collect different groups of like terms and find the sum of like terms in each group.

Add: $8x^2 - 5xy + 3y^2$, $2xy - 6y^2 + 3x^2$ and $y^2 + xy - 6x^2$

$$\begin{array}{r} 8x^2 - 5xy + 3y^2 \\ 3x^2 + 2xy - 6y^2 \\ -6x^2 + xy + y^2 \\ \hline 5x^2 - 2xy - 2y^2 \end{array}$$

Arrange the given expressions in descending powers of x with like terms below each other. Then add Column-wise.

Subtraction of Algebraic Expressions

While Subtracting algebraic expressions:

i) Arrange the terms of the given expressions in the same order.
ii) Write the given expressions in two rows in such a way that the like terms occur one below the other, keeping the expression to be subtracted in the second row.
iii) Change the sign of each term in the lower row from + to – and from - to +.

iv) With new signs of the terms of the lower row, add column-wise.

The simplest expressions are monomials. They consist of only one term.

i) Subtract 8x from 16x

$16x - 8x$
$= (16 \times x) - (8 \times x)$
$= (16 - 8) \times x$
$= 8 \times x$
$= 8x$

ii) Subtract 8y from 16x

$16x - 8y = (16 \times x) - (8 \times y)$
$= 16x - 8y$

The difference between two like terms is a like term with a numerical coefficient equal to the difference between the numerical coefficients of the two like terms.

Unlike terms cannot be subtracted the way like terms are subtracted.

Subtract $3x^2 - 6x - 4$ from $5 + x - 2x^2$

$$\begin{array}{r} -2x^2 + x + 5 \\ +3x^2 - 6x - 4 \\ - \quad + \quad - \\ \hline -5x^2 + 7x + 9 \end{array}$$

Arranging the terms of the given expression in descending powers of x and then subtract column-wise.

- **Operations on Algebraic Expressions - Multiplication**

Multiplications of Algebraic Expressions

Let us have a look at the rules of sign and laws of exponents which will be useful for multiplication of two algebraic expressions.

a) The product of two factors with like signs is positive and the product of two factors with unlike signs is negative.

1. $+ \times - = -$
2. $+ \times + = +$
3. $- \times + = -$
4. $- \times - = +$

b) If x is any variable and m, n are positive integers, then

$$x^m \text{ x } x^n = x^{m+n}$$

$$(x^3 \times x^5) = x^{(3+5)} = x^8$$

Multiplication of two Monomials

When we multiply constants as we multiply rational numbers and multiply the same variables by using laws of exponents.

Product of two monomials

= (Product of their numerical coefficients) × (Product of their variable parts)

Find the product of 3ab and 5b.

$(3ab)\times(5b) = (3ab)\times(5\times b)$

$=(3ab)\times(b\times5)$ [By commutativity of multiplication. so, $5\times b=(b\times5)$]

$=(3ab\times b)\times5$ [By associativity of multiplication. so, $3ab=3a\times b$]

$=[(3a\times b)\times b]\times5$

$=[3a\times b\times b]\times5$ [By commutativity. so, $3a\times b\times b= [3a\times b\times b]$]

$=[3a\times b^2]\times5$

$=[3ab^2\times5]$

$=(3\times5)ab^2$ [By commutativity law]

$=(3\times5)\times ab^2$ [By commutativity law]

$=15ab^2$

We have the following two rules for the multiplication of two monomials.

i) The coefficient of the product of two monomials is equal to the product of their coefficients.

ii) The variable part in the product of two monomials is equal to the product of the variable parts in the given monomials.

These two rules are also applicable to the product of three or more monomials.

Find the product of 6xy and $-3x^2y^3$

$(6xy)\times(-3x^2y^3)$

$=\{6\times(-3)\}\times\{xy\times x^2y^3\}$

$=-18\times xy\times x^2y^3$

$=-18\times x^{1+2}y^{1+3}$

$=-18x^3y^4$

Find the product of $7ab^2$, $-4a^2b$, $-5abc$ and $5bc^2$.

$7ab^2 -4a^2b -5abc 5bc^2$

$= 7\times -4 -5\times 5ab^2a^2b\times abc\times bc^2$

$= (-28)\times(-25)\times(a^{(1+2+1)}b^{(2+1+1+1)}c^{(1+2)})$

$=700a^4b^5c^3$

Multiplication of a Monomial by a Polynomial

i) Multiplying a monomial by a binomial

Multiplying the monomial by each term of the binomial, using the distributive law

$a\times b+c=a\times b+a\times c$

Find the product:

$5a^2b^2(3a^2-4ab)$

$=5a^2b^2 3a^2+5a^2b^2(-4ab)$

$= 15\times a^{(2+2)}b^2-20\times a^{(2+1)}b^{(2+1)}$

$= 15a^4b^2-20a^3b^3$

ii) Multiplying a monomial by a trinomial

Multiplying the monomial by each term of the trinomial, using the distributive law

$a\times b+c+d=a\times b+a\times c+a\times d$

Find the product:

$5a^2b^2(3a^2-4ab+6b^2)$

$=5a^2b^2 3a^2+5a^2b^2(-4ab)+5a^2b^2 6b^2$

$= 15a^{(2+2)}b^2-20a^{(2+1)}b^{(2+1)}+30a^2b^{(2+2)}$

$=15a^4b^2-20a^3b^3+30a^2b^4$

Multiplication of a Polynomial by a Polynomial

i) Multiplying a binomial by a binomial

Suppose
a+b and c d are two binomials, we may find their product as given below:

$a+bc+d=a\times c+d+b\times c+d$

$= a\times c+a\times d+b\times c+c\times d$

$= ac+ad+(bc+cd)$

Find the product:

$(3x+5y)\times(5x-7y)$

$= 3x(5x-7y)+5y(5x-7y)$

$=3x\times5x-3x\times7y+5y\times5x-5y\times7y$

$=15x^{(1+1)}-21xy+25xy-35y^{(1+1)}$

$= 15x^2+4xy-35y^2$

ii) Multiplying a binomial by a trinomial

In this type of multiplication, we multiply each of the three terms in the trinomial by each of the two terms in the binomial and we get $3\times2 = 6$ terms, which may reduce to 5 or less.

Find the product:

$(2x-3)(5x^2 -6x+9)$

$= 2x(5x^2-6x+9)-3(5x^2-6x+9)$

$=(10x^{(1+2)}-12x^{(1+1)}+18x)-(15x^2-18x+27)$

$= 10x^3-12x^2+18x-15x^2+18x-27$

$= 10x^3-12x^2-15x^2+18x+18x-27$

$= 10x^3-27x^2+36x-27$

Algebraic Identities

Identity

An identity is an equality which is true for all values of the variables. Or
An identity implies that the expressions on either side of the equality sign (=) are identical.

Consider the equality $(x + 2)(x + 3) = (x^2 + 5x + 6)$

Let us evaluate both sides for $x = 2$

For $x = 2$,

L.H.S.

$(x + 2)(x + 3) = (2 + 2)(2 + 3) = (4)(5) = 4 \times 5 = 20$

R.H.S.

$(x^2 + 5x + 6) = (2^2 + 5 \times 2 + 6) = 4 + 10 + 6 = 20$

Thus, L.H.S = R.H.S, for $x = 2k$

Equations and identities are different. Equations are only true for a particular value of the variable but identities are true for all values of the variable.

An unconditional equation, which is true for all values of its variable is known as identity.

An equation is true for only certain values of the variable in it. It is not true for all values of the variable.

Standard Identities

Identity I:

$(a + b)^2 = a^2 + 2ab + b^2$

Proof:
We have, $(a + b)^2 = (a + b)(a + b)$

$(a + b)^2 = a(a + b) + b(a + b)$

$(a + b)^2 = a^2 + ab + ba + b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

Identity II:

$(a - b)^2 = a^2 - 2ab + b^2$

Proof:
We have, $(a - b)^2 = (a - b)(a - b)$

$(a + b)^2 = a(a - b) - b(a - b)$

$(a + b)^2 = a^2 - ab - ba + b^2$

$(a + b)^2 = a^2 - 2ab + b^2$

Identity III:
$(a + b)(a - b) = a^2 - b^2$

Proof:

We have, $(a + b)(a - b) = a(a - b) + b(a - b)$

$(a + b)(a - b) = a(a - b) + b(a - b)$

$(a + b)(a - b) = a^2 - ab + ba - b^2$

$(a + b)(a - b) = a^2 - b^2$

A useful Identity

$(x + a)(x + b) = x^2 + (a + b)x + ab$

$(x + a)(x + b) = x(x + b) + a(x + b)$

$= x^2 + xb + ax + ab$

$= x^2 + bx + ax + ab$

$= x^2 + ax + bx + ab$

$= x^2 + (a + b)x + ab$

Applying Identities

Using the Identity, find

i) $(2x + 3y)^2$

ii) $(4p - 3q)^2$

iii) $(4x + 5y)(4x - 5y)$

i) $(2x + 3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$

$= 4x^2 + 12xy + 9y^2$

ii) $(4p - 3q)^2 = (4p)^2 - 2(4p)(3q) + (3q)^2$

$= 16p^2 - 24pq + 9q^2$

iii) $(4x + 5y)(4x - 5y) = (4x)^2 - (5y)^2$

$= 16x^2 - 25y^2$

Use the Identity

$(x + a)(x + b) = x^2 + (a + b)x + ab$ to find 401×402.

$401 \times 402 = (400 + 1) \times (400 + 2)$

$= 400^2 + (1 + 2) \times 400 + 1 \times 2$

$= 160000 + 1200 + 2$

$= 161202$

If $x - \frac{1}{x} = 9$, find the value of $x^2 + \frac{1}{x^2}$.

We have $x - \frac{1}{x} = 9$

$\Rightarrow (x - \frac{1}{x})^2 = (9)^2$

$\Rightarrow x^2 - 2 \times x \times + \frac{1}{x} (\frac{1}{x})^2 = 81$

$\Rightarrow x^2 - 2 + \frac{1}{x^2} = 81$

$\Rightarrow x^2 + 1 \frac{1}{x^2} = 81 + 2$

$\Rightarrow x^2 + \frac{1}{x^2} = 83.$