CROSS, OR VECTOR, PRODUCT OF VECTORS (XII, R. S. AGGARWAL)

EXERCISE 24 [Pg. No.: 1057]

1. Find $\left(\vec{a}\times\vec{b}\right)$ and $\left|\vec{a}\times\vec{b}\right|$, when

(i) $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-4\hat{k}$

(ii) $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$

(iii) $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

(iv) $\vec{a}=4\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=3\hat{i}+\hat{k}$

(v) $\vec{a}=3\hat{i}+4\hat{j}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

Sol. **(i)** Let $\vec{a}=\left(\hat{i}-\hat{j}+2\hat{k}\right)$ and $\vec{b}=\left(2\hat{i}+3\hat{j}-4\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 2 & 3 & -4\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & 2\\ 3 & -4\end{vmatrix}-\hat{j}\begin{vmatrix}1 & 2\\ 2 & -4\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix}$$

$$=\hat{i}\left(4-6\right)-\hat{j}\left(-4-4\right)+\hat{k}\left(3+2\right)=\left(-2\hat{i}+8\hat{j}+5\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{\left(-2\right)^2+\left(8\right)^2+\left(5\right)^2}=\sqrt{4+64+25}=\sqrt{93}$$

(ii) Let $\vec{a}=\left(2\hat{i}+\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(3\hat{i}+5\hat{j}-2\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 3\\ 3 & 5 & -2\end{vmatrix}=\hat{i}\begin{vmatrix}1 & 3\\ 5 & -2\end{vmatrix}-\hat{j}\begin{vmatrix}2 & 3\\ 3 & -2\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 1\\ 3 & 5\end{vmatrix}$$

$$=\hat{i}\left(-2-15\right)-\hat{j}\left(-4-9\right)+\hat{k}\left(10-3\right)=\left(-17\hat{i}+13\hat{j}+7\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{\left(-17\right)^2+\left(13\right)^2+\left(7\right)^2}=\sqrt{289+169+49}=\sqrt{507}=13\sqrt{3}$$

(iii) Let $\vec{a}=\left(\hat{i}-7\hat{j}+7\hat{k}\right)$ and $\vec{b}=\left(3\hat{i}-2\hat{j}+2\hat{k}\right)$, $\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}=\hat{i}\begin{vmatrix}-7 & 7\\ -2 & 2\end{vmatrix}-\hat{j}\begin{vmatrix}1 & 7\\ 3 & 2\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -7\\ 3 & -2\end{vmatrix}$

$$=\hat{i}\left(-14+14\right)-\hat{j}\left(2-21\right)+\hat{k}\left(-2+21\right)=\left(19\hat{j}+19\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{\left(19\right)^2+\left(19\right)^2}=\sqrt{19^2\left(1+1\right)}=19\sqrt{2}$$

(iv) Let $\vec{a}=\left(4\hat{i}+\hat{j}-2\hat{k}\right)$ and $\vec{b}=\left(3\hat{i}+\hat{k}\right)$, $\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 1 & -2\\ 3 & 0 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}4 & -2\\ 3 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}4 & 1\\ 3 & 0\end{vmatrix}$

$$=\hat{i}\left(1-0\right)-\hat{j}\left(4+6\right)+\hat{k}\left(0-3\right)=\left(\hat{i}-10\hat{j}-3\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{\left(1\right)^2+\left(-10\right)^2+\left(-3\right)^2}=\sqrt{1+100+9}=\sqrt{110}$$

(v) Let $\vec{a}=\left(3\hat{i}+4\hat{j}\right)$ and $\vec{b}=\left(\hat{i}+\hat{j}+\hat{k}\right)$, $\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 0\\ 1 & 1 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}4 & 0\\ 1 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 4\\ 1 & 1\end{vmatrix}$

$$=\hat{i}\left(4-0\right)-\hat{j}\left(3-0\right)+\hat{k}\left(3-4\right)=\left(4\hat{i}-3\hat{j}-\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{(4)^2+(-3)^2+(-1)^2}=\sqrt{16+9+1}=\sqrt{26}$$

2. Find λ if $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right)=\vec{0}$

Sol. $\because\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+7\hat{k}\right)=\vec{0}$

$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 14\\ 1 & -\lambda & 7\end{vmatrix}=\vec{0}\Rightarrow\begin{vmatrix}6 & 14\\ 7 & 7\end{vmatrix}\hat{i}-\begin{vmatrix}2 & 14\\ 1 & 7\end{vmatrix}\hat{j}+\begin{vmatrix}2 & 6\\ 1 & -\lambda\end{vmatrix}\hat{k}=\vec{0}$$

$$\Rightarrow(42-98)\hat{i}(7-14)\hat{j}+(-2\lambda-6)\hat{k}=\vec{0}\Rightarrow-56\hat{i}+7\hat{j}-2(\lambda+3)\hat{k}=\vec{0}\Rightarrow-2(\lambda+3)=0\Rightarrow\lambda=\ 3\ \text{Ans.}$$

3. If $\vec{a}=\left(-3\hat{i}+4\hat{j}-7\hat{k}\right)$ and $\vec{b}=\left(6\hat{i}+2\hat{j}-3k\right)$, find $\left(\vec{a}\times\vec{b}\right)$

Verify that (i) $\vec{a}$ and $\left(\vec{a}\times\vec{b}\right)$ are perpendicular to each other

And (ii) $\vec{b}$ and $\left(\vec{a}\times\vec{b}\right)$ are perpendicular to each other

Sol. $\vec{a}=\left(-3\hat{i}+4\hat{j}-7\hat{k}\right)$

$\vec{b}=\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 4 & -7\\ 6 & 2 & -3\end{vmatrix}$$

$$=\begin{vmatrix}4 & -7\\ 2 & -3\end{vmatrix}\hat{i}-\begin{vmatrix}-3 & -7\\ 6 & -3\end{vmatrix}\hat{j}+\begin{vmatrix}-3 & 4\\ 6 & 2\end{vmatrix}\hat{k}=(-12+14)\hat{i}-(9+42)\hat{j}+(-6-24)\hat{k}=2\hat{i}-51\hat{j}-30\hat{k}$$

4. Find the value of

(i) $\left(\hat{i}\times\hat{j}\right)\cdot\hat{k}+\hat{i}\cdot\hat{j}$ (ii) $\left(\hat{k}\times\hat{j}\right)\cdot\hat{i}+\hat{j}\cdot\hat{k}$ iii) $\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$

Sol. (i) $\left(\hat{i}\times\hat{j}\right)\cdot\hat{k}+\hat{i}\cdot\hat{j}=\hat{k}\cdot\hat{k}+\hat{i}\cdot\hat{j}=1+0=1$

(ii) $\left(\hat{k}\times\hat{j}\right)\cdot\hat{i}+\hat{j}\cdot\hat{k}=-\hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{k}=-1+0=-1$

(iii) $\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$

$$=\left(\hat{i}\times\hat{j}\right)+\left(\hat{i}\times\hat{k}\right)+\left(\hat{j}\times\hat{k}\right)+\left(\hat{j}\times\hat{i}\right)+\left(\hat{k}\times\hat{i}\right)+\left(\hat{k}\times\hat{j}\right)=\hat{k}+\left(-\hat{j}\right)+\hat{i}+\left(-\hat{k}\right)+\hat{j}+\left(-\hat{i}\right)=0$$

5. Find the unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$, when

(i) $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ (ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

(iii) $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$ (iv) $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 4\hat{j} - \hat{k}$

Sol. **(i)** Let $\vec{a} = \left(3\hat{i} + \hat{j} - 2\hat{k}\right)$ and $\vec{b} = \left(2\hat{i} + 3\hat{j} - \hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 2 & 3 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{i}(-1+6) - \hat{j}(-3+4) + \hat{k}(9-2) = \left(5\hat{i} - \hat{j} + 7\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right| = \sqrt{(5)^2 + (-1)^2 + (7)^2} = \sqrt{25+1+49} = \sqrt{75} = \pm 5\sqrt{3}$$

Hence, the required unit vector $= \dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|} = \dfrac{\left(5\hat{i} - \hat{j} + 7\hat{k}\right)}{\pm 5\sqrt{3}} = \pm\dfrac{1}{5\sqrt{3}}\left(5\hat{i} - \hat{j} + 7\hat{k}\right)$

(ii) Let $\vec{a} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$ and $\vec{b} = \left(\hat{i} + 2\hat{j} - \hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 2 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= \hat{i}(2-6) - \hat{j}(-1-3) + \hat{k}(2+2) = \left(-4\hat{i} + 4\hat{j} + 4\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right| = \sqrt{(4)^2 + (4)^2 + (4)^2} = \pm 4\sqrt{3}$$

Hence the required unit vector $= \dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|} = \dfrac{4\left(-\hat{i} + \hat{j} + \hat{k}\right)}{4\sqrt{3}} = \dfrac{\left(-\hat{i} + \hat{j} + \hat{k}\right)}{\sqrt{3}}$

(iii) $\vec{a} = \left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$ and $\vec{b} = \left(-\hat{i} + 3\hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = \hat{i}\begin{vmatrix} 3 & -2 \\ 0 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= \hat{i}(9-0) - \hat{j}(3-2) + \hat{k}(0+3) = \left(9\hat{i} - \hat{j} + 3\hat{k}\right)$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right| = \sqrt{(9)^2 + (-1)^2 + (3)^2} = \sqrt{81+1+9} = \pm\sqrt{91}$$

Hence, the required unit vector $= \dfrac{\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|} = \dfrac{\left(9\hat{i} - \hat{j} + 3\hat{k}\right)}{\pm\sqrt{91}}$

(iv) Let $\vec{a} = \left(4\hat{i} + 2\hat{j} - \hat{k}\right)$ and $\vec{b} = \left(\hat{i} + 4\hat{j} - \hat{k}\right)$

$$\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & -1 \\ 1 & 4 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 2 & -1 \\ 4 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & -1 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & 2 \\ 1 & 4 \end{vmatrix}$$

$= \hat{i}(-2+4) - \hat{j}(-4+1) + \hat{k}(16-2) = (2\hat{i}+3\hat{j}+14\hat{k})$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{(2)^2+(3)^2+(14)^2} = \sqrt{4+9+196} = \pm\sqrt{209}$

Hence, the required unit vector $= \dfrac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} = \dfrac{(2\hat{i}+3\hat{j}+14\hat{k})}{\pm\sqrt{209}}$

6. Find the unit vectors perpendicular to the plane of the vectors $\vec{a} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - 2\hat{k}$.

Sol. Let $\vec{a} = (2\hat{i} - 6\hat{j} - 3\hat{k})$, $\vec{b} = (4\hat{i} + 3\hat{j} - \hat{k})$

$$(\vec{a}\times\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} -6 & -3 \\ 3 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -6 \\ 4 & 3 \end{vmatrix}$$

$= \hat{i}(6+9) - \hat{j}(-2+12) + \hat{k}(6+24) = (15\hat{i} - 10\hat{j} + 30\hat{k})$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{(15)^2+(-10)^2+(30)^2} = \sqrt{225+100+900} = \sqrt{1225} = 35$

Hence, the required unit vector $= \dfrac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} = \dfrac{5(3\hat{i}-2\hat{j}+6\hat{k})}{35} = \pm\dfrac{1}{7}(3\hat{i}-2\hat{j}+6\hat{k})$

7. Find a vector of magnitude 6 which is perpendicular to both the vectors $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$.

Sol. Let $\vec{a} = (4\hat{i} - \hat{j} + 3\hat{k})$ and $\vec{b} = (-2\hat{i} + \hat{j} - 2\hat{k})$

$$(\vec{a}\times\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 3 \\ -2 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix}$$

$= \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4-2) = (-\hat{i}+2\hat{j}+2\hat{k})$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{(-1)^2+(2)^2+(2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

So, a unit vector $\perp$ to both $\vec{a}$ & $\vec{b}$ $= \dfrac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} = \dfrac{(-\hat{i}+2\hat{j}+2\hat{k})}{3}$

The required unit vector $= \dfrac{6(-\hat{i}+2\hat{j}+2\hat{k})}{3} = \pm 2(-\hat{i}+2\hat{j}+2\hat{k})$

8. Find a vector of magnitude 5 units, perpendicular to each of $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a} = (\hat{i}+\hat{j}+\hat{k})$ and $\vec{b} = (\hat{i}+2\hat{j}+3\hat{k})$.

Sol. Let $\vec{a} = (\hat{i}+\hat{j}+\hat{k})$ and $\vec{b} = (\hat{i}+2\hat{j}+3\hat{k})$

$(\vec{a}+\vec{b}) = (\hat{i}+\hat{j}+\hat{k}) + (\hat{i}+2\hat{j}+3\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k})$

$(\vec{a}-\vec{b}) = (\hat{i}+\hat{j}+\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k}) = (-\hat{j}-2\hat{k})$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}=\hat{i}\begin{vmatrix}3 & 4\\ -1 & -2\end{vmatrix}-\hat{j}\begin{vmatrix}2 & 4\\ 0 & -2\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 3\\ 0 & -1\end{vmatrix}$$

$$=\hat{i}(-6+4)-\hat{j}(-4-0)+\hat{k}(-2-0)=\left(-2\hat{i}+4\hat{j}-2\hat{k}\right)=2\left(-\hat{i}+2\hat{j}-\hat{k}\right)$$

$$\Rightarrow \left|\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{(-2)^2+(4)^2+(-2)^2}=\sqrt{4+16+4}=\sqrt{24}=2\sqrt{16}$$

So, *a* unit vector $\perp$ to both $\vec{a}\,\&\,\vec{b}=\dfrac{\left[\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right]}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}$

The required vector $=\dfrac{5.2\left(-\hat{i}+2\hat{j}-\hat{k}\right)}{2\sqrt{6}}=\dfrac{5}{\sqrt{6}}\left(-\hat{i}+2j-\hat{k}\right)=\dfrac{5\sqrt{6}}{6}\left(-\hat{i}+2\hat{j}-\hat{k}\right)$

9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $\left|\vec{a}\times\vec{b}\right|=\sqrt{3}$.

Sol. Let, θ = Angle between $\vec{a}$ and $\vec{b}$.

Given:- $|\vec{a}|=1, |\vec{b}|=2$ and $|\vec{a}\times\vec{b}|=\sqrt{3}$

We have, $|\vec{a}\times\vec{b}|=|\vec{a}|\ |\vec{b}|\cdot\sin\theta \Rightarrow \sqrt{3}=1\times 2\times\sin\theta \Rightarrow \sin\theta=\dfrac{\sqrt{3}}{2}\Rightarrow\theta=\dfrac{\pi}{3}$ Ans.

10. Let $\vec{a}=\left(\hat{i}-\hat{j}\right), \vec{b}=\left(3\hat{j}-\hat{k}\right)$ and $\vec{c}=\left(7\hat{i}-\hat{k}\right)$. Find a vector $\vec{d}$ such that it is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d}=1$.

Sol. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\vec{d}\perp\vec{a}$, $\vec{d}.\vec{a}=0 \quad \Rightarrow\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\left(\hat{i}-\hat{j}\right)=0$

$\Rightarrow a_1-a_2=0 \qquad \ldots$(i)

$\vec{d}\perp\vec{b}$, $\vec{a}.\vec{b}=0 \quad \Rightarrow\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\left(3j-\hat{k}\right)=0$

$\Rightarrow 3a_2-a_3=0 \qquad \ldots$(ii)

$\vec{d}.\vec{c}=1 \quad \Rightarrow\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\left(7\hat{i}-\hat{k}\right)=1$

$\Rightarrow 7a_1-a_3=1 \qquad \ldots$(iii)

Solving equation (i) and (ii) we get $3a_1-a_3=0 \qquad \ldots$(iv)

Again solving equation (iii) & (iv) we get $a_1=\dfrac{1}{4}$

From equation (i), $a_1-a_2=0$ or $a_1=a_2=\dfrac{1}{4}$

From equation (ii), $3a_2-a_3=0 \ \Rightarrow 3.\dfrac{1}{4}=a_3 \ \Rightarrow a_3=\dfrac{3}{4}$. Hence $\vec{d}=\dfrac{1}{4}\hat{i}+\dfrac{1}{4}\hat{j}+\dfrac{3}{4}\hat{k}$.

11. If $\vec{a}=\left(4\hat{i}+5\hat{j}-\hat{k}\right), \vec{b}=\left(\hat{i}-4\hat{j}+5\hat{k}\right)$, and $\vec{c}=\left(3\hat{i}+\hat{j}-\hat{k}\right)$, find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and for which $\vec{c}\cdot\vec{d}=21$

Sol. Given:- $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$

$\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$

and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

$\because \vec{d} \perp \vec{a}$ and $\vec{b}$

$\Rightarrow \vec{d} \parallel \left(\vec{a} \times \vec{b}\right) \Rightarrow \vec{d} = k\left(\vec{a} \times \vec{b}\right)$ (Let)

$$\Rightarrow \vec{d} = k\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix} \Rightarrow \vec{d} = k\left\{\begin{vmatrix} 5 & -1 \\ -4 & 5 \end{vmatrix}\hat{i} - \begin{vmatrix} 4 & -1 \\ -1 & 5 \end{vmatrix}\hat{j} + \begin{vmatrix} 4 & 5 \\ 1 & -4 \end{vmatrix}\hat{k}\right\}$$

$$\Rightarrow \vec{d} = k\{(25-4)\vec{i} - (20+1)\vec{j} + (-16-15)\vec{k}\} \Rightarrow \vec{d} = 21k\,\vec{i} - 21k\,\vec{j} - 21k\,\vec{k}\}$$

Now, $\vec{c} \cdot \vec{d} = 21$

$$\Rightarrow (3\hat{i} + \hat{j} - \hat{k}) \cdot (21k\hat{i} - 21k\hat{j} - 21k\hat{k}) = 21 \Rightarrow 63k - 21k + 21k = 21 \Rightarrow 63k = 21 \Rightarrow k = \frac{1}{3}$$

$$\therefore \vec{d} = 21 \times \frac{1}{3}\hat{i} - 21 \times \frac{1}{3}\hat{j} - 21 \times \frac{1}{3}\hat{k} = \vec{d} = \left(7\hat{i} - 7\hat{j} - 7\hat{k}\right)$$ Ans.

12. Prove that $\left|\vec{a} \times \vec{b}\right| = \left(\vec{a}.\vec{b}\right)\tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Sol. L.H.S $\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$ ...(i)

We know that $\vec{a}.\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta \Rightarrow |\vec{a}|\left|\vec{b}\right| = \dfrac{\vec{a}.\vec{b}}{\cos\theta}$

From equation (i), $\left|\vec{a} \times \vec{b}\right| = \dfrac{\vec{a}.\vec{b}}{\cos\theta}\sin\theta \Rightarrow \left|\vec{a} \times \vec{b}\right| = \left(\vec{a}.\vec{b}\right)\tan\theta$

13. Write the value of p for which $\vec{a} = \left(3\hat{i} + 2\hat{j} + 9\hat{k}\right)$ and $\vec{b} = \left(\hat{i} + p\hat{j} + 3\hat{k}\right)$ are parallel vectors

Sol. $\because \vec{a} \parallel \vec{b}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & p & 3 \end{vmatrix} = \vec{0}$$

$$\begin{vmatrix} 2 & 9 \\ P & 3 \end{vmatrix}\hat{i} - \begin{vmatrix} 3 & 9 \\ 1 & 3 \end{vmatrix}\hat{j} + \begin{vmatrix} 3 & 2 \\ 1 & P \end{vmatrix}\hat{k} = \vec{0}$$

$$\Rightarrow (6-9P)\hat{i} - (9-9)\hat{j} + (3P-2)\hat{k} = \vec{0} \Rightarrow 3(2-3P)\hat{i} + (3P-2)\hat{k} = \vec{0} \Rightarrow 2-3P = 0 \text{ and } 3P-2 = 0$$

$\Rightarrow P = \dfrac{2}{3}$ Ans.

14. Verify that $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times b\right) + \left(\vec{a} \times \vec{c}\right)$, when

(i) $\vec{a} = \hat{i} - \hat{j} - 3\hat{k}, \vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$

(ii) $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$

Sol. (i) $\vec{a} = \left(\hat{i} - \hat{j} - 3\hat{k}\right)$, $\vec{b} = \left(4\hat{i} - 3\hat{j} + \hat{k}\right)$ & $\vec{c} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

$\Rightarrow \left(\vec{b}+\vec{c}\right)=\left(4\hat{i}-3\hat{j}+\hat{k}\right)+\left(2\hat{i}-\hat{j}+2k\right)=\left(6\hat{i}-4\hat{j}+3\hat{k}\right)$

$$\text{L.H.S} = \left\{\vec{a}\times\left(\vec{b}+\vec{c}\right)\right\}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 6 & -4 & 3\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & -3\\ -4 & 3\end{vmatrix}-\hat{j}\begin{vmatrix}1 & -3\\ 6 & 3\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -1\\ 6 & -4\end{vmatrix}$$

$$=\hat{i}\left(-3-12\right)-\hat{j}\left(3+18\right)+\hat{k}\left(-4+6\right)=\left(-15\hat{i}-21\hat{j}+2\hat{k}\right)$$

$$\text{R.H.S} = \left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right) = \left\{\left(\hat{i}-\hat{j}-3\hat{k}\right)\times\left(4\hat{i}-3\hat{j}+\hat{k}\right)\right\}+\left\{\left(\hat{i}-\hat{j}-3\hat{k}\right)\times\left(2\hat{i}-\hat{j}+2\hat{k}\right)\right\}$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 4 & -3 & 1\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 2 & -1 & 2\end{vmatrix}$$

$$=\left\{\hat{i}\begin{vmatrix}-1 & -3\\ -3 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}1 & -3\\ 4 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -1\\ 4 & -3\end{vmatrix}\right\}+\left\{\hat{i}\begin{vmatrix}-1 & -3\\ -1 & 2\end{vmatrix}-\hat{j}\begin{vmatrix}1 & -3\\ 2 & 2\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}\right\}$$

$$=\left\{\hat{i}\left(-1-9\right)-\hat{j}\left(1+12\right)+\hat{k}\left(-3+4\right)\right\}+\left\{\hat{i}\left(-2-3\right)-\hat{j}\left(2+6\right)+\hat{k}\left(-1+2\right)\right\}$$

$$=\left(-10\hat{i}-13\hat{j}+\hat{k}\right)+\left(-5\hat{i}-8\hat{j}+\hat{k}\right) =\left(-15\hat{i}-21\hat{j}+2\hat{k}\right)$$

Hence, $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)$

(ii) $\vec{a}=\left(4\hat{i}-\hat{j}+\hat{k}\right)$, $\vec{b}=\left(\hat{i}+\hat{j}+\hat{k}\right)$ & $\vec{c}=\left(\hat{i}-\hat{j}+\hat{k}\right)$

$\Rightarrow \left(\vec{b}+\vec{c}\right)=\left(\hat{i}+\hat{j}+\hat{k}\right)+\left(\hat{i}-\hat{j}+\hat{k}\right)=\left(2\hat{i}+2\hat{k}\right)$

$$\Rightarrow \left(\vec{a}\times\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}4 & 1\\ 1 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}4 & -1\\ 1 & 1\end{vmatrix}$$

$$=\hat{i}\left(-1-1\right)-\hat{j}\left(4-1\right)+\hat{k}\left(4+1\right)=\left(-2\hat{i}-3\hat{j}+5\hat{k}\right)$$

$$\Rightarrow \left(\vec{a}\times\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 1 & -1 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & 1\\ -1 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}4 & 1\\ 1 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}4 & -1\\ 1 & -1\end{vmatrix}$$

$$=\hat{i}\left(-1+1\right)-\hat{j}\left(4-1\right)+\hat{k}\left(-4+1\right)=\left(-3\hat{j}-3\hat{k}\right)$$

$$\text{L.H.S} = \vec{a}\times\left(\vec{b}+\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 2 & 0 & 2\end{vmatrix}=\hat{i}\begin{vmatrix}-1 & 1\\ 0 & 2\end{vmatrix}-\hat{j}\begin{vmatrix}4 & 1\\ 2 & 2\end{vmatrix}+\hat{k}\begin{vmatrix}4 & -1\\ 2 & 0\end{vmatrix}$$

$$=\hat{i}\left(-2-0\right)-\hat{j}\left(8-2\right)+\hat{k}\left(0+2\right)=\left(-2\hat{i}-6\hat{j}+2\hat{k}\right)$$

$$\text{R.H.S} =\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\left(-2\hat{i}-3\hat{j}+5\hat{k}\right)+\left(-3\hat{j}-3\hat{k}\right)=\left(-2\hat{i}-6\hat{j}+2\hat{k}\right)$$

Hence, $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)$

15. Find the area of the parallelogram whose adjacent sides are represented by the vectors

(i) $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$ (ii) $\vec{a} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\vec{b} = (\hat{i} - \hat{j} + \hat{k})$

(iii) $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - \hat{j}$ (iv) $\vec{a} = 2\hat{i}$ and $\vec{b} = 3\hat{j}$

Sol. **(i)** Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix}$$

$$= \hat{i}(2+6) - \hat{j}(1+9) + \hat{k}(-2+6) = 8\hat{i} - 10\hat{j} + 4\hat{k}$$

Required area $= |\vec{a} \times \vec{b}| = \sqrt{(8)^2 + (-10)^2 + (4)^2} = \sqrt{64 + 100 + 16} = \sqrt{180} = 6\sqrt{5}$ sq. units.

(ii) Let $\vec{a} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\vec{b} = (\hat{i} - \hat{j} + \hat{k})$

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \hat{i}(1+4) - \hat{j}(3-4) + \hat{k}(-3-1) = (5\hat{i} + \hat{j} - 4\hat{k})$$

Required area $= |\vec{a} \times \vec{b}| = \sqrt{(5)^2 + (1)^2 + (-4)^2} = \sqrt{25 + 1 + 16} = \sqrt{42}$ sq units.

(iii) Let $\vec{a} = (2\hat{i} + \hat{j} + 3\hat{k})$, $\vec{b} = (\hat{i} - \hat{j})$

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \hat{i}(0+3) - \hat{j}(0-3) + \hat{k}(-2-1) = (3\hat{i} + 3\hat{j} - 3\hat{k})$$

Required area $= |\vec{a} \times \vec{b}| = \sqrt{(3)^2 + (3)^2 + (-3)^2} = \sqrt{3^2(1^2 + 1^2 + 1^2)} = 3\sqrt{3}$ sq. units

(iv) Let $\vec{a} = 2\hat{i}$, $\vec{b} = 3\hat{j}$

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6\hat{k}$$

Required area $= |\vec{a} \times \vec{b}| = \sqrt{(6)^2} = 6$ sq. units.

16. Find the area of the parallelogram whose diagonals are represented by the vectors

(i) $\vec{d_1} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{d_2} = \hat{i} - 3\hat{j} + 4\hat{k}$ (ii) $\vec{d_1} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{d_2} = 3\hat{i} + 4\hat{j} - \hat{k}$

(iii) $\vec{d_1} = \hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{d_2} = -\hat{i} + 2\hat{j}$

Sol. **(i)** Let $\vec{d}_1 = (3\hat{i} + \hat{j} - 2\hat{k})$, $\vec{d}_2 = (\hat{i} - 3\hat{j} + 4\hat{k})$

$$\left(\vec{d}_1 \times \vec{d}_2\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$$

$$= \hat{i}(4-6) - \hat{j}(12+2) + \hat{k}(-9-1) = \left(-2\hat{i} - 14\hat{j} - 10\hat{k}\right)$$

$$\Rightarrow \left|\vec{d}_1 \times \vec{d}_2\right| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{4+196+100} = \sqrt{300} = 10\sqrt{3}$$

$\therefore$ Required area $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right| = \frac{1}{2} \times 10\sqrt{3} = 5\sqrt{3}$ sq. units

(ii) Let $\vec{d}_1 = \left(2\hat{i} - \hat{j} + \hat{k}\right)$, $\vec{d}_2 = \left(3\hat{i} + 4\hat{j} - \hat{k}\right)$

$$\left(\vec{d}_1 \times \vec{d}_2\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= \hat{i}(1-4) - \hat{j}(-2-3) + \hat{k}(8+3) = \left(-3\hat{i} + 5\hat{j} + 11\hat{k}\right)$$

$$\Rightarrow \left|\vec{d}_1 \times \vec{d}_2\right| = \sqrt{(-3)^2 + (5)^2 + (11)^2} = \sqrt{9+25+121} = \sqrt{155}$$

$\therefore$ Required area $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right| = \frac{1}{2}\sqrt{155}$ sq. units

(iii) Let $\vec{d}_1 = \left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$, $\vec{d}_2 = \left(-\hat{i} + 2\hat{j}\right)$

$$\left(\vec{d}_1 \times \vec{d}_2\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} -3 & 2 \\ 2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix}$$

$$= \hat{i}(0-4) - \hat{j}(0+2) + \hat{k}(2-3) = \left(-4\hat{i} - 2\hat{j} - \hat{k}\right)$$

$$\Rightarrow \left|\vec{d}_1 \times \vec{d}_2\right| = \sqrt{(-4)^2 + (-2)^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

$\therefore$ Required area $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right| = \frac{1}{2}\sqrt{21}$ sq. units.

17. Find the area of the triangle whose two adjacent sides are determined by the vectors
(i) $\vec{a} = -2\hat{i} - 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$ (ii) $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = 5\hat{i} + 7\hat{j}$

Sol. **(i)** Let $\vec{a} = \left(-2\hat{i} - 5\hat{k}\right)$, $\vec{b} = \left(\hat{i} - 2\hat{j} - \hat{k}\right)$

$$\left(\vec{a} \times \vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & -5 \\ -2 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} -2 & -5 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix}$$

$$= \hat{i}(0-10) - \hat{j}(2+5) + \hat{k}(4-0) = \left(-10\hat{i} - 7\hat{j} + 4\hat{k}\right)$$

$$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \sqrt{(-10)^2 + (-7)^2 + (4)^2} = \sqrt{100+49+16} = \sqrt{165}$$

$\therefore$ Required area $= \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{165}$ sq. units.

(ii) Let $\vec{a} = \left(3\hat{i} + 4\hat{j}\right)$, $\vec{b} = \left(-5\hat{i} + 7\hat{j}\right)$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 0\\ -5 & 7 & 0\end{vmatrix}=\hat{i}\begin{vmatrix}4 & 0\\ 7 & 0\end{vmatrix}-\hat{j}\begin{vmatrix}3 & 0\\ -5 & 0\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 4\\ -5 & 7\end{vmatrix}$$

$$=\hat{k}(21+20)=41\hat{k}\Rightarrow|\vec{a}\times\vec{b}|=\sqrt{(41)^2}=41$$

$\therefore$ Required area $=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\times 41=\frac{41}{2}$ sq. units.

18. Using vectors, find the area of ΔABC whose vertices are

(i) $A(1,1,2), B(2,3,5)$ and $C(1,5,5)$ (ii) $A(1,2,3), B(2,-1,4)$ and $C(4,5,-1)$

(iii) $A(3,-1,2), B(1,-1,-3), C(4,-3,1)$ (iv) $A(1,-1,2), B(2,1,-1)$ and $C(3,-1,2)$

Sol. (i) Given vertices are A(1, 1, 2), B (2, 3, 5) and C(1, 5, 5).

$$\overrightarrow{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$$

$$\overrightarrow{AC}=(1-1)\hat{i}+(5-1)\hat{j}+(5-2)\hat{k}=4\hat{j}+3\hat{k}$$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix}=\begin{vmatrix}2 & 3\\ 4 & 3\end{vmatrix}\hat{i}-\begin{vmatrix}1 & 3\\ 0 & 3\end{vmatrix}\hat{j}+\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix}\hat{k}=-6\hat{i}-3\hat{j}+4\hat{k}$

$$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$$

Area of ΔABC $=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{61}$ square units.

(ii). Given vertices are, A(1, 2, 3), B(2, -1, 4) and C(4, 5, -1)

Now, $\overrightarrow{AB}=(2-1)\hat{i}+(-1-2)\hat{j}+(4-3)\hat{k}$

$\Rightarrow\overrightarrow{AB}=\hat{i}-3\hat{j}+\hat{k}$

again, $\overrightarrow{AC}=(4-2)\hat{i}+(5+1)\hat{j}+(-1-4)\hat{k}$

$\overrightarrow{AC}=2\hat{i}+6\hat{j}-5\hat{k}$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 2 & 6 & -5\end{vmatrix}$

$\Rightarrow\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}-3 & 1\\ 6 & -5\end{vmatrix}\hat{i}-\begin{vmatrix}1 & 1\\ 2 & -5\end{vmatrix}\hat{j}+\begin{vmatrix}1 & -3\\ 6 & 6\end{vmatrix}\hat{k}=9\hat{i}+7\hat{j}+12\hat{k}$

$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{9^2+7^2+12^2}=\sqrt{81+49+144}=\sqrt{274}$

Area of ΔABC $=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$\frac{1}{2}\sqrt{274}$ Sq. Units.

(iii). $A(3,-1,2)$, $B(1,-1,-3)$ and $C(4,-3,1)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -4 & 5 & 7\\ 4 & -5 & -7\end{vmatrix}=\hat{i}\begin{vmatrix}5 & 7\\ -5 & -7\end{vmatrix}-\hat{j}\begin{vmatrix}-4 & 7\\ 4 & -7\end{vmatrix}+\hat{k}\begin{vmatrix}-4 & 5\\ 4 & -5\end{vmatrix}=0$$

Hence, A, B and C area collinear.

(ii) Let $\vec{A}=\left(6\hat{i}-7\hat{j}-\hat{k}\right), \vec{B}=\left(2\hat{i}-3\hat{j}+\hat{k}\right), \vec{C}=\left(4\hat{i}-5\hat{j}\right)$

$\Rightarrow \overrightarrow{AB}=$ Position vector of B − position vector of A

$=\left(2\hat{i}-3\hat{j}+\hat{k}\right)-\left(6\hat{i}-7\hat{j}-\hat{k}\right)=\left(-4\hat{i}+4\hat{j}+2\hat{k}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of C − Position vector of A

$=\left(4\hat{i}-5\hat{j}\right)-\left(6\hat{i}-7\hat{j}-\hat{k}\right)=\left(-2\hat{i}+2\hat{j}+\hat{k}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -4 & 4 & 2\\ -2 & 2 & 1\end{vmatrix}=\hat{i}\begin{vmatrix}4 & 2\\ 2 & 1\end{vmatrix}-\hat{j}\begin{vmatrix}-4 & 2\\ -2 & 1\end{vmatrix}+\hat{k}\begin{vmatrix}-4 & 4\\ -2 & 2\end{vmatrix}$$

$=\hat{i}\,(4-4)-\hat{j}(-4+4)+\hat{k}(-8+8)=0$. Hence, A, B and C are collinear.

20. Show that the points A,B,C with position vectors $\left(3\hat{i}-2\hat{j}+4\hat{k}\right)$, $\left(\hat{i}+\hat{j}+\hat{k}\right)$ and $\left(-\hat{i}+4\hat{j}-2\hat{k}\right)$ respectively are collinear.

Sol. Let $\vec{A}=\left(3\hat{i}-2\hat{j}+4\hat{k}\right), \vec{B}=\left(\hat{i}+\hat{j}+\hat{k}\right), \vec{C}=\left(-\hat{i}+4\hat{j}-2\hat{k}\right)$

$\Rightarrow \overrightarrow{AB}=$ Position vector of B − position vector of A

$=\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(3\hat{i}-2\hat{j}+4\hat{k}\right)=\left(-2\hat{i}+3\hat{j}-3\hat{k}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of C − position vector of A

$=\left(-\hat{i}+4\hat{j}-2\hat{k}\right)-\left(3\hat{i}-2\hat{j}+4\hat{k}\right)=\left(-4\hat{i}+6\hat{j}-6\hat{k}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\hat{i} & \hat{j} & k\\ -2 & 3 & -3\\ -4 & 6 & -6\end{vmatrix}=\hat{i}\begin{vmatrix}3 & -3\\ 6 & -6\end{vmatrix}-\hat{j}\begin{vmatrix}-2 & 3\\ -4 & 6\end{vmatrix}+\hat{k}\begin{vmatrix}-2 & 3\\ -4 & 6\end{vmatrix}$$

$=\hat{i}(-18+18)-\hat{j}(-12+12)+\hat{k}(-12+12)=0$

Hence, proved that A,B and C are collinear.

21. Show that the points having position vectors $\vec{a},\vec{b},\left(3\vec{a}-2\vec{b}\right)$ are collinear, whatever be $\vec{a},\vec{b},\vec{c}$.

Sol. Let $\vec{A}=\vec{a}$, $\vec{B}=\vec{b}$, $\vec{C}=\left(3\vec{a}-2\vec{b}\right)$

$\Rightarrow \overrightarrow{AB}=$ position vector of B − position vector of $A=\left(b-\vec{a}\right)$

$\Rightarrow \overrightarrow{AC}=$ position vector of C − position vector of A $=\left(3\vec{a}-2\vec{b}\right)-\vec{a}=\left(2\vec{a}-2\vec{b}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ -1 & 1 & 0\\ 2 & -2 & 0\end{vmatrix}=\vec{a}\begin{vmatrix}1 & 0\\ -2 & 0\end{vmatrix}-\vec{b}\begin{vmatrix}-1 & 0\\ 2 & 0\end{vmatrix}+\vec{c}\begin{vmatrix}-1 & 1\\ 2 & -2\end{vmatrix}=\vec{c}\,(2-2)=0$$

Hence, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear.

22. Show that the points having position vectors $\left(-2\vec{a}+3\vec{b}+5\vec{c}\right)$, $\left(\vec{a}+2\vec{b}+3\vec{c}\right)$ and $\left(7\vec{a}-\vec{c}\right)$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.

Sol. Let, $\vec{A}=\left(-2\vec{a}+3\vec{b}+5\vec{c}\right)$, $\vec{B}=\left(\vec{a}+2\vec{b}+3\vec{c}\right)$, $\vec{C}=\left(7\vec{a}-\vec{c}\right)$

$\Rightarrow \overrightarrow{AB}=$ Position vector of $B-$ position vector of A

$=\left(\vec{a}+2\vec{b}+3\vec{c}\right)-\left(-2\vec{a}+3\vec{b}+5\vec{c}\right)=\left(3\vec{a}-\vec{b}-2\vec{c}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of $C-$ position vector of A

$=\left(7\vec{a}-\vec{c}\right)-\left(-2\vec{a}+3\vec{b}+5\vec{C}\right)=\left(9\vec{a}-3\vec{b}-6\vec{c}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ 3 & -1 & -2\\ 9 & -3 & -6\end{vmatrix}=\vec{a}\begin{vmatrix}-1 & -2\\ -3 & -6\end{vmatrix}-\vec{b}\begin{vmatrix}3 & -2\\ 9 & -6\end{vmatrix}+\vec{c}\begin{vmatrix}3 & -1\\ 9 & -3\end{vmatrix}$$

$$=\vec{a}(6-6)-\vec{b}(-18+18)+\vec{c}(-9+9)=0.$$

Hence, $\vec{A}, \vec{B}$ and $\vec{C}$ are collinear.

23. Find a unit vector perpendicular to the plane ABC, where the points A, B, C are $(3,-1,2)$, $(1,-1,-3)$ and $(4,-3,1)$ respectively.

Sol. Let $\vec{A}=\left(3\hat{i}-\hat{j}+2\hat{k}\right)$, $\vec{B}=\left(\hat{i}-\hat{j}-3\hat{k}\right)$, $\vec{C}=\left(4\hat{i}-3\hat{j}+\hat{k}\right)$

$\Rightarrow \overrightarrow{AB}=$ Position vector of $B-$ position vector of A

$=\left(\hat{i}-\hat{j}-3\hat{k}\right)-\left(3\hat{i}-\hat{j}+2\hat{k}\right)=\left(-2\hat{i}-5\hat{k}\right)$

$\Rightarrow \overrightarrow{AC}=$ Position vector of $C-$ position vector of A

$=\left(4\hat{i}-3\hat{j}+\hat{k}\right)-\left(3\hat{i}-\hat{j}+2\hat{k}\right)=\left(\hat{i}-2\hat{j}-\hat{k}\right)$

$$\Rightarrow \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}=\hat{i}\begin{vmatrix}0 & -5\\ -2 & -1\end{vmatrix}-\hat{j}\begin{vmatrix}-2 & -5\\ 1 & -1\end{vmatrix}+\hat{k}\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix}$$

$$=\hat{i}(0-10)-\hat{j}(2+5)+\hat{k}(4-0)=\left(-10\hat{i}-7\hat{j}+4\hat{k}\right)$$

$$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-10)^2+(-7)^2+(4)^2}=\sqrt{100+49+16}=\sqrt{165}$$

$\therefore$ Required unit vector is $=\dfrac{\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)}{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}=\dfrac{\left(-10\hat{i}-7\hat{j}+4\hat{k}\right)}{\sqrt{165}}$

24. If $\vec{a}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(\hat{i}-3\hat{k}\right)$ then find $\left|\vec{b}\times 2\vec{a}\right|$

Sol. Given :- $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$

and, $\vec{b}=\hat{i}-3\hat{k}$

Now, $2\vec{a}=2\hat{i}-4\hat{j}+6\hat{k}$

and, $\vec{b}\times 2\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 1 & 0 & -3 \end{vmatrix}$

$= \begin{vmatrix} -2 & -3 \\ 0 & -3 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}\hat{k} = 6\hat{i} - 6\hat{j} + 2\hat{k} \Rightarrow |\vec{b}\times 2\vec{a}| = \sqrt{6^2 + 6^2 + 2^2} = \sqrt{76} = 2\sqrt{19}$

25. if $\left|\vec{a}\right| = 2, \left|\vec{b}\right| = 5$ and $\left|\vec{a}\times\vec{b}\right| = 8$, find $\vec{a}\cdot\vec{b}$

Sol. Given:- $|\vec{a}| = 2,\ |\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$

$\because |\vec{a}\times\vec{b}| = 8$

$\Rightarrow |\vec{a}||\vec{b}|\cdot\sin\theta = 8 \Rightarrow 2\times 5\times\sin\theta = 8 \Rightarrow \quad \sin\theta = \frac{4}{5}$

Now, $\cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow \cos\theta = \sqrt{1-\frac{16}{25}} \Rightarrow \cos\theta = \frac{3}{5}$

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 2\times 5\times\frac{3}{5}$ $= 6$ Ans.

26. If $\left|\vec{a}\right| = 2, \left|\vec{b}\right| = 7$ and $\left(\vec{a}\times\vec{b}\right) = \left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$, find the angle between $\vec{a}$ and $\vec{b}$

Sol. Given:- $|\vec{a}| = 2,\ \ |\vec{b}| = 7$ & $\vec{a}\times\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

Let, θ = Angle between $\vec{a}$ and $\vec{b}$

Now, $|\vec{a}\times\vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = 7$

Now, $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\cdot\sin\theta \Rightarrow 7 = 2\times 7\times\sin\theta \Rightarrow \sin\theta - \frac{1}{2} \Rightarrow \theta = \frac{\pi^c}{6}$ Ans.