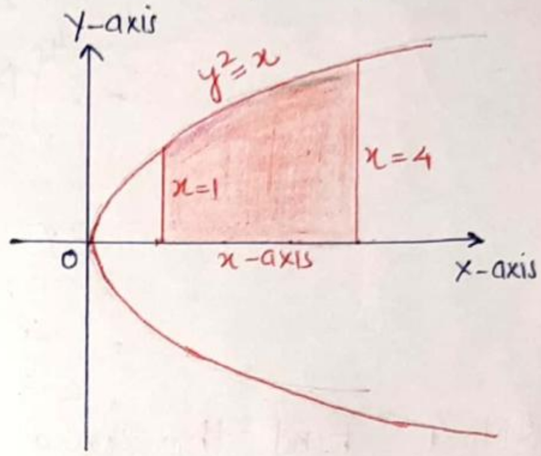
CHAPTER 8 : APPLICATION OF INTEGRALS

NCERT SOLUTIONS

EXERCISE 8.1

Q.No 1. Find the area of region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in first quadrant.

Soln. The given curve $y^2 = x$ is right hand parabola with vertex at $O(0,0)$.



Required Area (show shaded)

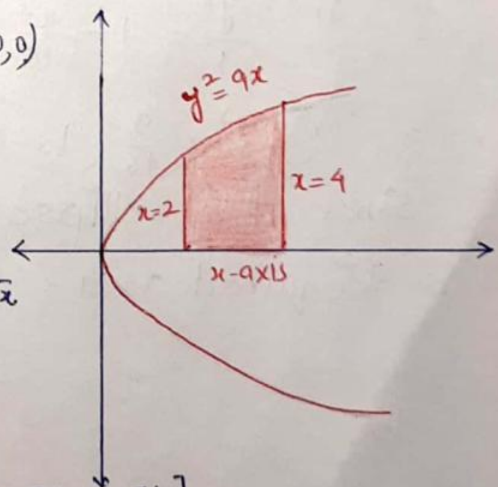
$$= \int_1^4 y\,dx = \int_1^4 \sqrt{x}\,dx \quad \left[\because y^2 = x,\ y = \sqrt{x}\right]$$

$$= \int_1^4 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4 = \frac{2}{3}\left[(4)^{3/2} - (1)^{3/2}\right] = \frac{2}{3}\left[(2^2)^{3/2} - 1\right]$$

$$= \frac{2}{3}\left[2^3 - 1\right] = \frac{2}{3}\left[8-1\right] = \frac{2}{3}(7) = \frac{14}{3} \text{ square units.}$$

Q.No 2: Find the area of region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in first quadrant.

Sol. The given curve $y^2 = 9x$ is right hand parabola with vertex at $O(0,0)$



$$\text{Required Area} = \int_2^4 y\,dx$$

$$= \int_2^4 3\sqrt{x}\,dx \quad \left[\because y^2 = 9x \Rightarrow y = \sqrt{9x} = 3\sqrt{x}\right]$$

$$= 3\int_2^4 x^{1/2}\,dx.$$

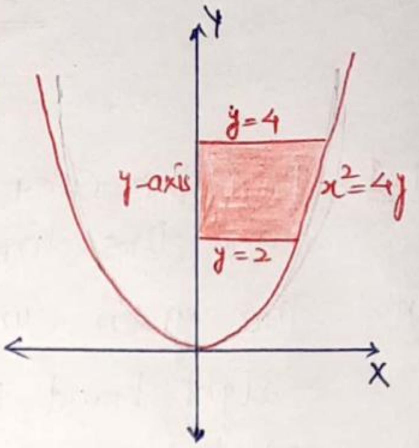
$$= 3\left[\frac{x^{3/2}}{3/2}\right]_2^4 = 3 \times \frac{2}{3}\left[4^{3/2} - 2^{3/2}\right]$$

$$= 2\left[(2^2)^{3/2} - (2)^{3/2}\right] = 2\left[2^3 - 2^{3/2}\right] = 2\left[8 - 2\sqrt{2}\right]$$

$$= 4\left[4 - \sqrt{2}\right] \text{ square units.}$$

Q.No 3: Find the area of region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Sol: The given curve $x^2 = 4y$ is an upward parabola with vertex at $O(0,0)$



$$\text{Required Area} = \int_2^4 x\,dy = \int_2^4 2\sqrt{y}\,dy$$

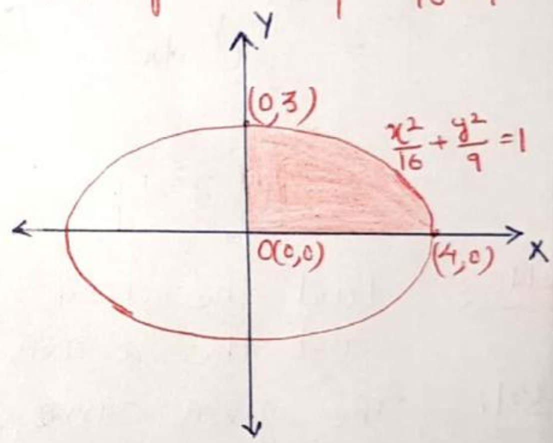
$$= 2\int_2^4 (y)^{1/2}\,dy = 2\left[\frac{y^{3/2}}{3/2}\right]_2^4$$

$$= \frac{2\times 2}{3}\left[(4)^{3/2} - (2)^{3/2}\right] = \frac{4}{3}\left[2^3 - 2^{3/2}\right]$$

$$= \frac{4}{3}\left[8 - 2\sqrt{2}\right] = \frac{8}{3}\left[4 - \sqrt{2}\right] \text{ square units.}$$

Q.No.4: Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Soln: The given curve $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is an ellipse with centre at $O(0,0)$; Major axis along x-axis and Minor axis along y-axis.



Now
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

or
$$\frac{y^2}{9} = 1 - \frac{x^2}{16}$$

or
$$y^2 = \frac{9}{16}(16 - x^2) \Rightarrow y = \sqrt{\frac{9}{16}(16-x^2)} \Rightarrow y = \frac{3}{4}\sqrt{16-x^2}$$

Since the ellipse is symmetrical about both axes.

$\therefore$ Required area = $4 \times$ area shown shaded in the first quadrant.

$$= 4\int_0^4 y\,dx = 4\int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx = 4\times\frac{3}{4}\int_0^4 \sqrt{16-x^2}\,dx$$

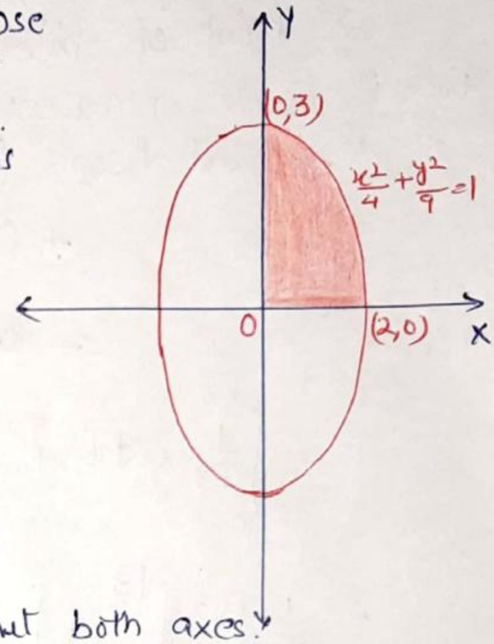
$$= 3\left[\frac{x}{2}\sqrt{4^2-x^2} + \frac{4^2}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4 \quad \left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{a}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 3\left[\left(\frac{4}{2}\sqrt{16-16} + 8\sin^{-1}(1)\right) - \left(0 + 8\sin^{-1}(0)\right)\right]$$

$$= 3\left[\left(0+8\times\frac{\pi}{2}\right)-\left(0+8\times 0\right)\right] = 3\times 8\times\frac{\pi}{2} = 12\pi \text{ square units.}$$

Q.No.5: Find the area of the region bounded by ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Soln: The given curve $\frac{x^2}{4}+\frac{y^2}{9}=1$ is an ellipse with Centre at $O(0,0)$; Major axis along Y-axis; Minor axis along x-axis



Now $\frac{x^2}{4}+\frac{y^2}{9}=1$

or $\frac{y^2}{9} = 1-\frac{x^2}{4}$ or $\frac{y^2}{9} = \frac{4-x^2}{4}$

or $y^2 = \frac{9}{4}(4-x^2)$

$\therefore$ $y = \frac{3}{2}\sqrt{4-x^2}$

Since the ellipse is symmetrical about both axes.

$\therefore$ Required area = $4\times$ area shown shaded in first quadrant.

$$= 4\times\int_0^2 y\,dx = 4\times\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx$$

$$= 6\int_0^2 \sqrt{4-x^2}\,dx = 6\int_0^2 \sqrt{2^2-x^2}\,dx$$

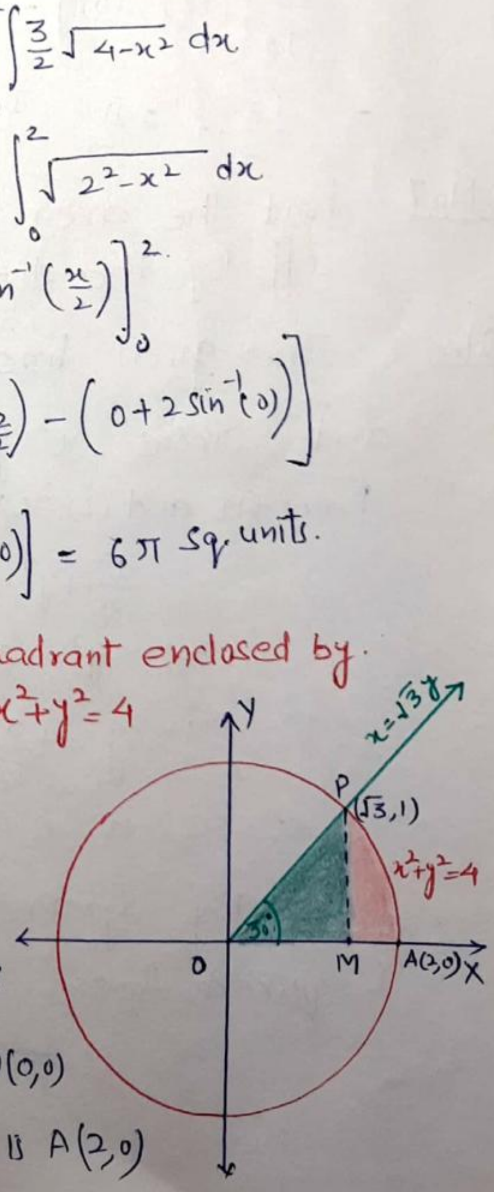
$$= 6\left[\frac{x}{2}\sqrt{4-x^2}+\frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 6\left[\left(\frac{2\sqrt{4-4}}{2}+2\sin^{-1}\frac{2}{2}\right)-\left(0+2\sin^{-1}(0)\right)\right]$$

$$= 6\left[\left(0+2\times\frac{\pi}{2}\right)-\left(0+0\right)\right] = 6\pi \text{ sq. units.}$$

Q.No.6: Find the area of region in first quadrant enclosed by x-axis, line $x=\sqrt{3}\,y$ and the circle $x^2+y^2=4$

Sol: The equation $x=\sqrt{3}\,y$ or $y=\frac{1}{\sqrt{3}}x$ —(1) represents a line through the origin having slope $\frac{1}{\sqrt{3}}$, so inclined at an angle of $30°$ with the x-axis



and $x^2+y^2=4$ —(2) is circle with centre at $O(0,0)$ and radius $= 2$ $\therefore$ $OA = 2$ so A is $A(2,0)$

From (1) and (2) we get

$$3y^2+y^2=4 \quad \text{or} \quad 4y^2=4 \quad \text{or} \quad y^2=1.$$

$\therefore$ $y=1 \Rightarrow x=\sqrt{3}$.

$\therefore$ Point of intersection of circle (2) and line (1) is $P(\sqrt{3},1)$

from P draw $PM \perp OX$

$\therefore$ Required Area = (Area under the line $y=\frac{1}{\sqrt{3}}x$ from $x=0$ to $x=\sqrt{3}$)

+ (Area under the circle $x^2+y^2=4$ from $x=\sqrt{3}$ to $x=2$)

$$= ar(\Delta OMP) + ar\ MAP$$

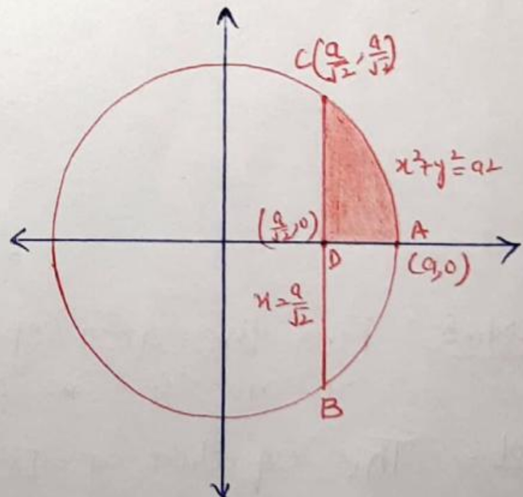
$$= \int_0^{\sqrt{3}} \frac{1}{\sqrt{3}} x\,dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx \quad \left[\because \text{For } x^2+y^2=4 \Rightarrow y^2=4-x^2 \Rightarrow y=\sqrt{4-x^2}\right]$$

$$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{2\sqrt{3}}\left[(\sqrt{3})^2-(0)^2\right] + \left[\left(0+2\sin^{-1}(1)\right)-\left(\frac{\sqrt{3}}{2}\sqrt{4-3}+2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + 2\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\left(\pi/3\right) = \pi - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq. units.}$$

Q No.7: Find the area of smaller part of circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$.



Soln: The given line is $x=\frac{a}{\sqrt{2}}$ —(1)

and circle $x^2+y^2=a^2$ —(2)

From (1) and (2) we get

$$\frac{a^2}{2}+y^2=a^2$$

$$\Rightarrow y^2=a^2-\frac{a^2}{2}=\frac{a^2}{2}$$

$$\therefore y=\pm\frac{a}{\sqrt{2}}.$$

$\therefore$ C is $C\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)$

$\therefore$ By Symmetry

Required Area = 2 × Area shown shaded.

$$= 2\int_{a/\sqrt{2}}^{a} y\,dx = 2\int_{a/\sqrt{2}}^{a}\sqrt{a^2-x^2}\,dx$$ [$\because$ Required Area lies below the circle]

$$= 2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{a/\sqrt{2}}^{a}$$

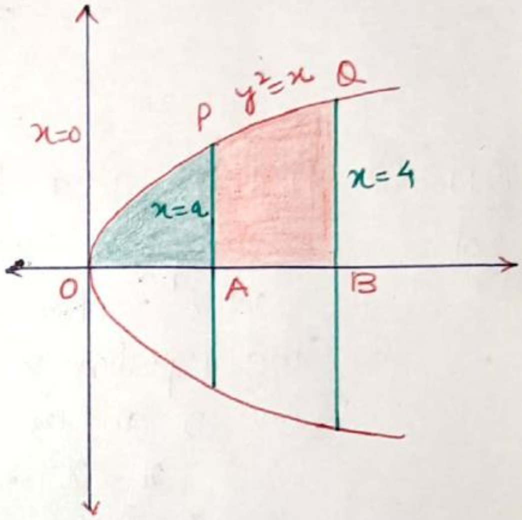
$$= 2\left[\left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - \left(\frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right]$$

$$= 2\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} + \frac{a^2}{2}(\pi/4)\right] = 2\left[\frac{\pi a^2}{4} - \frac{\pi a^2}{8} - \frac{a^2}{4}\right]$$

$$= \left(\frac{\pi a^2}{4} - \frac{a^2}{2}\right) \text{ square units.}$$

Q.No. 8: The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$. Find the value of a.

Soln: The given curve is $y^2 = x$
which is right hand parabola.



According to given condition:
Area OAP under $y^2 = x$ between $x = 0$ and $x = a$ = Area ABPQ under $y^2 = x$ between $x = a$ and $x = 4$.

$$\therefore \int_0^a y\,dx = \int_a^4 y\,dx$$

$$\Rightarrow \int_0^a \sqrt{x}\,dx = \int_a^4 \sqrt{x}\,dx \Rightarrow \left[\frac{x^{3/2}}{3/2}\right]_0^a - \left[\frac{x^{3/2}}{3/2}\right]_a^4$$

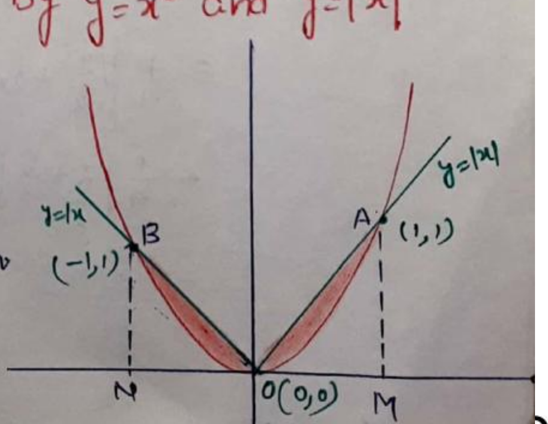
$$\Rightarrow \frac{2}{3}\left[a^{3/2} - 0\right] = \frac{2}{3}\left[(4)^{3/2} - a^{3/2}\right]$$

$$\Rightarrow a^{\frac{3}{2}} = 4^{3/2} - a^{3/2}$$

$$\Rightarrow 2a^{3/2} = 4\times\sqrt{4} \Rightarrow a^{\frac{3}{2}} = 4 \Rightarrow a = (2^2)^{2/3} \Rightarrow a = 2^{4/3}$$

Q.No 9 Find the Area of region bounded by $y = x^2$ and $y = |x|$

Soln: The given curves are
$y = x^2$ —① upward parabola.
and $y = |x|$ —②
The Required area is enclosed between two curves as shown shaded in figure.



Now from (1) and (2) $|x|^2 - |x| = 0$

$\Rightarrow |x|(|x|-1) = 0$

$\Rightarrow |x| = 0$ or $|x| = 1$

$\Rightarrow x = 0$ or $x = \pm 1$.

$\therefore$ The points $(-1,1), (0,0)$ and $(1,1)$ are points of intersection of two curves.

As ~~Area~~ Required area is symmetrical about y-axis

$\therefore$ Required area = 2 × (Area shown shaded in the first quadrant)

$$= 2\int_0^1 (|x| - x^2)\,dx = 2\left[\int_0^1 |x|\,dx - \int_0^1 x^2\,dx\right]$$

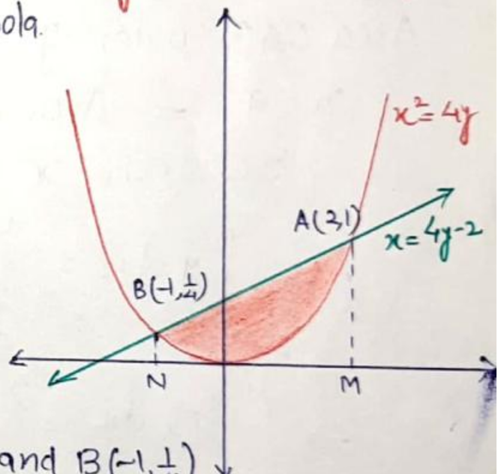
$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right] = 2\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right]$$

$$= 2\left[\left(\frac{1}{2}-0\right) - \left(\frac{1}{3}-0\right)\right] = 2\left[\frac{1}{2} - \frac{1}{3}\right] = 2\left[\frac{3-2}{6}\right] = 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq units}$$

Q.No 10. Find the area bounded by the curve $x^2 = 4y$ and line $x = 4y - 2$

Sol: The equation $x^2 = 4y$ ...(1) is an upward parabola with vertex at $(0,0)$

The equation $x = 4y - 2$ ...(2) a st. line.



From (1) and (2)

$$x = x^2 - 2 \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0 \Rightarrow x = 2, -1$$

When $x = 2$, $y = 1$

When $x = -1$, $y = \frac{1}{4}$

$\therefore$ The line meets the parabola at $A(2,1)$ and $B(-1, \frac{1}{4})$

$\therefore$ Required Area (Shown shaded) = (Area under the line $x = 4y - 2$ between $x = -1$ and $x = 2$) − (Area under parabola $x^2 = 4y$ between $x = -1$ and $x = 2$)

$$= \int_{-1}^{2} \left(\frac{x+2}{4}\right)dx - \int_{-1}^{2} \frac{x^2}{4}\,dx \quad \left[\begin{array}{l}\text{For the line } x = 4y-2 \Rightarrow y = \frac{x+2}{4} \\ \text{and for parabola } x^2 = 4y \Rightarrow y = \frac{x^2}{4}\end{array}\right]$$

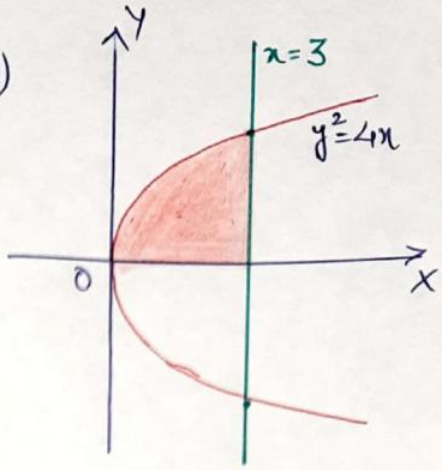
$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(\frac{2^2}{2} + 2\times 2\right) - \left(\frac{1}{2} - 2\right)\right] - \frac{1}{12}\left[2^3 - (-1)^3\right]$$

$$= \frac{1}{4}\left[6 + \frac{3}{2}\right] - \frac{1}{12} \times 9 = \frac{15}{8} - \frac{3}{4} = \frac{9}{8} \text{ square units.}$$

Q No. 11: Find the area of region bounded by curve $y^2 = 4x$ and $x = 3$

Sol: We know that $y^2 = 4x$ represents a right hand parabola with vertex at $O(0,0)$



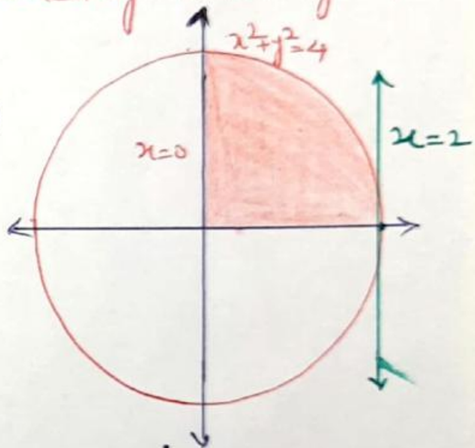
Required Area = 2 × (Area shown shaded in first quadrant) (By symmetry)

$$= 2\int_0^3 y\,dx = 2\int_0^3 2\sqrt{x}\,dx$$

$$= 2\times 2\left[\frac{x^{3/2}}{3/2}\right]_0^3 = 4\left[\frac{(3)^{3/2}}{3/2} - 0\right]$$

$$= \frac{4\times 2}{3}[3]^{3/2} = \frac{4\times 2\times 3}{3}(3)^{1/2} = 8\sqrt{3} \text{ square units.}$$

Q No 12: Find Area lying in first quadrant and bounded by circle $x^2+y^2=4$ and the lines $x=0$ and $x=2$



Soln: Required Area = Area bounded by $x^2+y^2=4$ and lines $x=0$ and $x=2$

$$= \int_0^2 \sqrt{4-x^2}\,dx$$

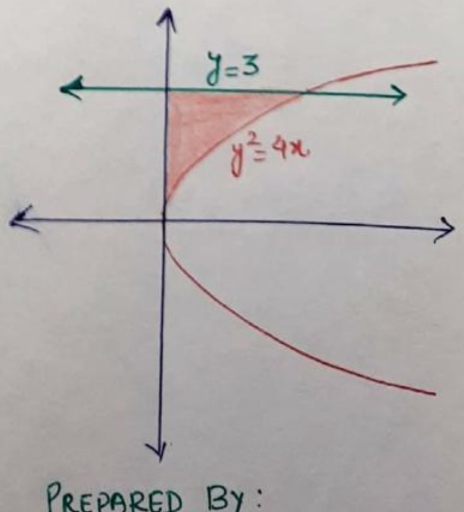
$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= (0 + 2\sin^{-1}(1)) - 0 = 2\times\frac{\pi}{2} = \pi \text{ square units.}$$

∴ Correct option is A.

Q No 13: Area of Region bounded by curve $y^2=4x$, y-axis and $y=3$ is

(A) 2 (B) 9/4 (C) 9/3 (D) 9/2.



Soln: Considering area with Y-axis

$$\text{Required Area} = \int_0^3 x\,dy = \int_0^3 \frac{y^2}{4}\,dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{4\times 3}\left[(3)^3 - 0\right]$$

$$= \frac{9}{4} \text{ square units.}$$

∴ Correct option is (B)

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