Chapter 3

Fluid Kinematics and Dynamics

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- *Lagrangian Description of Fluid Flow*
- *Eulerian Description of Fluid Flow*
- *Scalar, Vector and Flow Fields*
- *Tangential and Normal Acceleration*
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Fluid Dynamic Flow

The description of the motion of fluids (or fluid flows) without necessarily considering the forces and moments that cause the motion is called *fluid kinematics*. The flow of fluid can be described by two ways: (*a*) Lagrangian description and (*b*) Eulerian description.

Lagrangian Description of Fluid Flow

Here, individual fluid particles are identified (usually by specifying their initial spatial position of a given time) and the motion of each particle is observed as a function of time. Let the position of a fluid particle identified by $\vec{r}_0$, the position vector at any time 't' shall be $\vec{r} = r(\vec{r}_0, t)$,

Where $\vec{r}$ is the position vector of the fluid particle with respect to a fixed reference point at time t. Considering Cartesian coordinates,

We have

$$\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$$

$$\vec{r} = x(r_0,t)\hat{i} + y(r_0,t)\hat{j} + z(r_0,t)\hat{k} = x\hat{i} + y\hat{j} + z\hat{k}$$

Here, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the x, y, z directions respectively and r_0 denotes the point (x_0, y_0, z_0).

The velocity vector $\vec{v}$ having the scalar components u, v and w in the x, y and z directions respectively are given as follows:

$$\vec{v} = \left.\frac{\partial \vec{r}}{\partial t}\right|_{r_0}$$

$$= \hat{i}\left.\frac{\partial x}{\partial t}\right|_{r_0} + \hat{j}\left.\frac{\partial y}{\partial t}\right|_{r_0} + \hat{k}\left.\frac{\partial z}{\partial t}\right|_{r_0}$$

$$= u\hat{i} + v\hat{j} + w\hat{k}$$

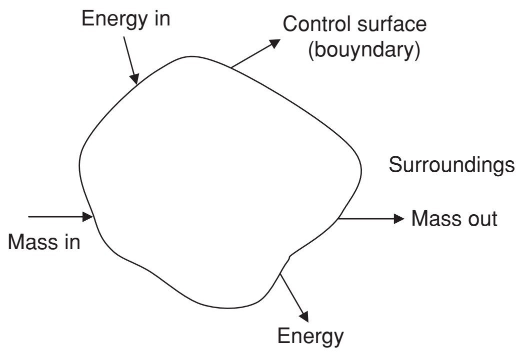
The acceleration vector $\vec{a}$ having the scalar components a_x, a_y and a_z in the x, y and z directions respectively are given as follows:

$$\vec{a} = \left.\frac{\partial^2 \vec{r}}{\partial t^2}\right|_{r_0}$$

$$= \hat{i}\left.\frac{\partial^2 x}{\partial t^2}\right|_{r_0} + \hat{j}\left.\frac{\partial^2 y}{\partial t^2}\right|_{r_0} + \hat{k}\left.\frac{\partial^2 z}{\partial t^2}\right|_{r_0} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Eulerian Description of Fluid Flow

Control Volume



A control volume is an open system with a boundary called the control surface. Transfer of mass and energy takes place across the control surface. A deforming control volume has a changing volume while the volume of a non-deforming control volume is fixed.

If $\vec{v}_{cv}$ and $\vec{v}_{cs}$ are the velocities of a control volume and its control surface respectively, then for a fixed non-deforming control volume: $\vec{v}_{cv} = \vec{v}_{cs} = 0$ and for a moving non-deforming control volume: $\vec{v}_{cv} = \vec{v}_{cs}$. A deforming control volume not only involves a changing volume but also involves control surface movement.

For a deforming control volume, $\vec{v}_{cs}$ need not necessarily be uniform and if the control volume is also moving, then $\vec{v}_{cs}$ need not necessarily be identical to $\vec{v}_{cv}$.

In this flow description, a control volume (flow domain) is defined within the fluid flow region where the flow properties of interest are described as fields within the control volume. For each field, a field variable that is a function of space and time is defined.

Scalar, Vector and Flow Fields

A *scalar field* is a region where at every point, a scalar function (scalar field variable) has a defined value e.g., pressure field of a fluid flow. A *vector field* is a region where at every point, a vector function (vector field variable) has a defined value, e.g., velocity field of a fluid in motion.

A *flow field* is a region in which the flow properties, i.e., velocity, pressure, etc., are defined at each and every point at any time instant. Two basic and important vector field variables of a flow are the velocity and acceleration fields.

Velocity Field

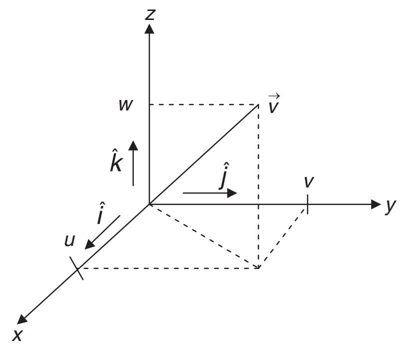
For a general three dimensional fluid flow in Cartesian coordinates, the velocity vector is given by:

$$\vec{v} = \vec{v}_{(x,y,z,t)}$$

$$\mathbf{u(x,y,z,t)\hat{i} + v(x,y,z,t)\hat{j} + w(x,y,z,t)\hat{k}}$$

The *speed* of the fluid,

$$v = |\vec{v}| = \sqrt{u^2 + v^2 + w^2}$$



A point in the fluid flow field where the velocity vector is zero is called a *stagnation point*.

Fluid Acceleration

Acceleration Field

For a general three dimensional fluid flow in Cartesian coordinates, if $\vec{v}$ is the velocity field, then the *acceleration field* is given by:

$$\vec{a}(x,y,z,t) = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} \qquad (1)$$

The scalar components of the acceleration vector are:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Magnitude of the acceleration vector,

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Equation (1) can be rewritten as

$$\vec{a}(x,y,z,t) = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{D\vec{v}}{Dt} \qquad (2)$$

The gradient (or del) operator, $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

and the operator $(\vec{v}\cdot\vec{\nabla}) = \frac{u\partial}{\partial x} + \frac{v\partial}{\partial y} + w\frac{\partial}{\partial z}$

The components of the acceleration vector in cylindrical coordinates are:

$$a_r = \frac{\partial v_r}{\partial_t} + v_r\frac{\partial v_r}{\partial_r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial_z}$$

$$a_\theta = \frac{\partial v_\theta}{\partial_t} + v_r\frac{\partial v_\theta}{\partial_r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{dv_\theta}{\partial \theta}$$

$$a_z = \frac{\partial vz}{\partial t} + v_r\frac{\partial vz}{\partial r} + \frac{v_\theta}{r}\frac{\partial vz}{\partial \theta} + v_z\frac{\partial vz}{\partial z}$$

Local and convective derivative

In equation (2), the operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})$ is called as the *total (of material or substantial derivative*. The operator

$\frac{\partial}{\partial t}$ is called the *local (or temporal or unsteady) derivative*, while the operator $(\vec{v}\cdot\vec{\nabla})$ is called the *convective derivative*. The local derivative represents the effect of unsteadiness while the convective derivative represents the variation due to the change in position of the fluid particle as it moves through a field with gradient (spatial change).

Local, Convective and Total Acceleration

In equation (2), the term $\frac{\partial \vec{v}}{\partial t}$ is called the *local (or temporal or unsteady) acceleration* whereas the term $(\vec{v}\cdot\vec{\nabla})\vec{v}$ is called the *convective (advective) acceleration.* Equation (2) elucidates that fluid particles experience acceleration due to (*a*) change in velocity with time (local acceleration) (*b*) change in velocity with space (convective acceleration). The acceleration vector $\vec{a}$ is called as the *total (or material) acceleration*.

Total acceleration = Local acceleration + convective acceleration

Solved Example

Example 1: The velocity field of a two dimensional flow is given by $\vec{v} = 2xt\hat{i} + 2yt\hat{j}$, where t is in seconds. At $t = 1$ second, if the local and convective accelerations at any point (x, y) are denoted by $\vec{a}_l$ and $\vec{a}_c$ respectively, then:

(A) $\vec{a}_\ell = 2\vec{a}_c$ (B) $\vec{a}_c = \vec{a}_\ell$
(C) $\vec{a}_c = \vec{a}_\ell = 0$ (D) $\vec{a}_c = 2\vec{a}_\ell$

Solution:
From the velocity field description
$u = 2xt$
$v = 2yt$

x – component of the local acceleration, $a_{\square,x} = \frac{\partial u}{\partial t} = 2x$

y – component of the local acceleration,

$$a_{\ell,y} = \frac{\partial v}{\partial t} = 2y$$

$$\mathbf{\vec{a}}_\ell = \mathbf{a}_{\ell,}\mathbf{x}\hat{\mathbf{i}} + \mathbf{a}_{\ell},\mathbf{y}\hat{\mathbf{j}}$$

$$= 2x\hat{i} + 2y\hat{i} \qquad (1)$$

x – component of the convective acceleration,

$$a_{c,x} = \frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y}$$
$$= 2xt \times 2t + 2yt \times 0$$
$$= 4xt^2$$

y – component of the convective acceleration,

$$a_{c,y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$
$$= 2xt \times 0 + 2yt \times 2t$$
$$= 4yt^2$$

$$\vec{a}_c = a_{c,}x\hat{i} + a_c,y\hat{j}$$
$$= 4xt^2\hat{i} + 4yt^2\hat{j}$$

At $t = 1$ second,

$$\vec{a}_c = 4x\hat{i} + 4y\hat{j} \qquad (2)$$

From equations (1) and (2), we have

$$\vec{a}_c = 2\vec{a}_l.$$

Example 2: A two-dimensional velocity field is given by $\vec{v} = xy\hat{i} + 3xt\hat{j}$, where x and y are in metres, t is in seconds and $\vec{v}$ is in metres per second. The magnitude of the acceleration at $x = 1$ m, $y = 0.5$ m and $t = 2$ secs is

(A) 6.25 m/s^2 (B) 8.663 m/s^2
(C) 12.25 m/s^2 (D) 6 m/s^2

Solution:
From the velocity field description,
$u = xy$
$v = 3xt$

Now, $a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$
$= 0 + xy \times y + 3xt + x$
$= xy^2 + 3x^2t$

Now, $a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$
$= 3x + xy \times 3t + 3xt \times 0$
$= 3x + 3xyt$

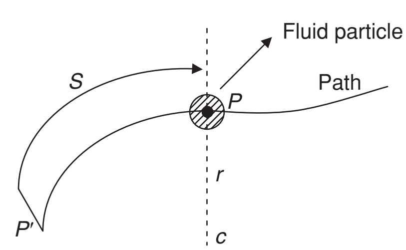
At $x = 1$ m, $y = 0.5$ m and $t = 2$ sec,
$a_x = 1 \times (0.5)^2 + 3 \times 1 \times 2$
$= 6.25$ m/s^2
$a_y = 3 \times 1 + 3 \times 1 \times 0.5 \times 2 = 6$

Magnitude of the acceleration,

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2}$$
$$= \sqrt{(6.25)^2 + 6^2} = 8.663\, m/s^2.$$

Tangential and Normal Acceleration

Consider a fluid particle moving along a path as shown in the following figure.

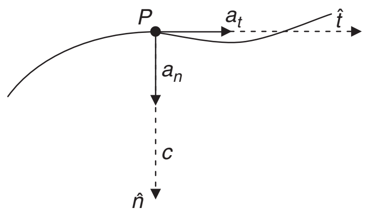


Let S denote the distance travelled by the particle along the path line relative to the reference point P', t denote time and v $(= f(s, t)$ denote the speed of the particle. Let $\hat{t}$ be a unit vector tangential to the path at point P and let $\hat{n}$ be a unit vector normal to the path at point P and pointing inward towards the centre pf curvature C. Let r denote the radius of curvature at point P.

The acceleration vector,

$$\boxed{\vec{a} = a_t\hat{t} + a_n\hat{n}}$$

$$\boxed{= \left(v\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right)\hat{t} + \frac{v^2}{r}\hat{n}}$$



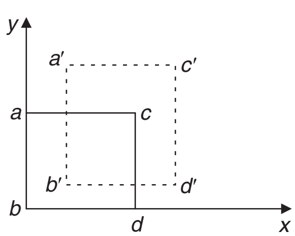
The *tangential component* of the acceleration vector, $a_t = \left(v\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right)$ and the *normal component*, $a_n = \frac{v^2}{r}$.

The component a_n is also called as the **centripetal acceleration.** The component a_n will be present anytime a fluid particle is moving on a curved path (velocity direction is changing) while the component a_t will be present whenever the fluid particle is changing speed (velocity magnitude is changing)

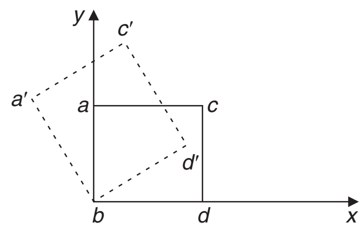
Table 1 *Translation, deformation and rotation of a fluid element*

Fluid flow Scenario (only steady flows	Tangential acceleration or deceleration	Normal Acceleration or deceleration
Flow in a straight constant diameter pipe	Not present	Not present
Flow in a straight non-constant diameter pipe	Present	Not present
Flow in a curved constant diameter pipe	Not present	Present
Flow in a curved non-constant diameter pipe	Present	Present

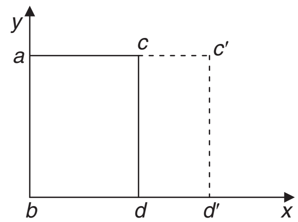
When a fluid element moves in space, several things may happen to it. Surely the moving fluid element undergoes translation, i.e., and a linear displacement from one location to another. The fluid element in addition may undergo rotation, linear deformation or angular deformation.



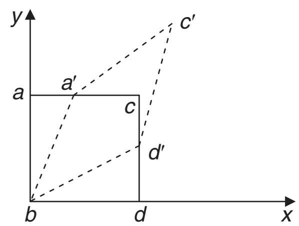
Translation



Rotation



Linear deformation



Angular deformation

In a two-dimensional flow field in Cartesian coordinates, translation without deformation and rotation is possible if the velocity components u and v are neither a function of x nor of y. When a velocity component is a function of only one space coordinate along which that velocity component is defined, e.g., $u = u(x)$ and $v = v(y)$, then translation with linear deformation is possible.

When $u = u(x, y)$ and $v = v(x, y)$, translation with angular and linear deformations is possible. It is also observed that when $u = u(x, y)$ and $v = v(x, y)$, rotation and angular deformation of a fluid element exists simultaneously. When $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$, no angular deformation takes place and the situation is known as pure rotation. When $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, the fluid element has angular deformation but no rotation about the z- axis.

Types of Fluid Flow

Steady and Unsteady Flow

In a *steady* fluid flow, fluid properties (such as density, pressure, etc.) and the flow characteristics (such as velocity, acceleration, etc.) at any point in the flow do not change with time. In a steady flow, the local derivative of the fluid property or fluid characteristic ϕ is zero, i.e., $\frac{\partial \phi}{\partial t} = 0$.

Local acceleration is zero for steady flows

Fluid flow through a pipe at a constant rate of discharge is a steady flow.

In an *unsteady* fluid flow, some of the fluid properties or flow characteristics at any point in the flow change with time. Fluid flow through a pipe at a varying rate of discharge is an unsteady flow.

Uniform and Non-uniform Flows

In a uniform fluid flow, the fluid properties or flow characteristics of any given time do not change with respect to space, i.e., from one point to another in the flow. Since for a uniform flow, there is no gradient (spatial change) the convective derivative of any fluid property of flow characteristic ϕ is zero, i.e., $(\vec{v}.\vec{\nabla})\phi = 0$.

> Convective acceleration is zero for uniform flows.

> In uniform flows, the streamlines are straight and parallel.

Fluid flow through a straight pipe of constant diameter is a uniform flow.

In a non-uniform fluid flow, some of the fluid properties or flow characteristics at any given time changes with respect to space. Flow through a straight pipe *pf* varying diameter is a non-uniform flow.

> Total acceleration is zero for steady uniform flows.

Flow combinations

Type	Example
Steady uniform flow	Flow at a constant rate through a constant diameter pipe
Steady non-uniform flow	Flow at a constant rate through a non-constant diameter pipe
Unsteady uniform flow	Flow at a varying rate through a constant diameter pipe
Unsteady non-uniform flow	Flow at a varying rate through a pipe of varying cross-section

One-, Two- and Three-dimensional Flows

A flows is said to be *one, two-* or *three-dimensional.* If one, two or three spatial dimensional are required to describe the velocity field.

Inviscid and Viscous Flow

A fluid flow in which the effects of viscosity (frictional effects) are absent is called as *inviscid (nonviscous)* fluid flow, whereas if the viscosity effects are present, then the fluid flow is called a *viscous* fluid flow. Flow of ideal fluids is inviscid flows while flow of real fluids are viscous flows.

Rotational and Irrotational Flows

A fluid flow is said to be *rotational* if the fluid particles while moving in the direction of flow rotate about their mass centres. If the fluid particle does not rotate, then the fluid flow is called as *irrational* fluid flow. Fluid flow in a rotating tank is a rotational flow while fluid flow above a wash basin or drain hole of a stationary tank is an irrotational flow.

> For an irrotational flow, the curl of the velocity vector is zero, i.e., $\vec{\nabla}\times\vec{v} = 0$ or $\text{curl}(\vec{v}) = 0$

Compressible and Incompressible Flows

If for a fluid flow, the density remains constant throughout the flow, i.e., $\frac{\partial\rho}{\partial t} = 0$ then the fluid flow is an *incompressible* fluid flow else it is a compressible fluid flow.

Example 3: The velocity field of a two dimensional irrotational flow is represented by $\vec{v} = \left(\frac{-x^2y^3}{3} + 2x - my\right)\hat{i} + \left(px - 2y - \frac{x^3y^2}{3}\right)\hat{j}$, where P and m are constants. If the value of P is equal to one, then the value of m for a streamline passing through the point (1, 2) is

(A) $\frac{-2}{3}$ (B) 0 (C) 3 (D) –1

Solution:

From the velocity field relationship,

$$u = \frac{-x^2y^3}{3} + 2x - \text{my}$$

$$v = Px - 2y - \frac{x^2}{3}y^3$$

Since the flow is irrotational

$$\vec{\nabla}\times\vec{v} = 0$$

i.e., $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

or $P - x^2y^2 = -x^2\,y^2 - m$

or $m = -P = -1$.

Streamline

A streamline is a curve that is everywhere tangent to the instantaneous local velocity vector. At a given instant of time, the tangent to a streamline at a particular point gives the direction of the velocity at that point. The fluid flow will always be along the streamlines and never cross it. At non-stagnation points, a streamline cannot interest itself nor can two streamlines cross each other. However, the two scenarios can be present at stagnation points. The differential equation of a streamline in a three-dimensional flow $(\vec{v} = u\hat{i} + v\hat{j} + w\hat{k})$ is:

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}}$$

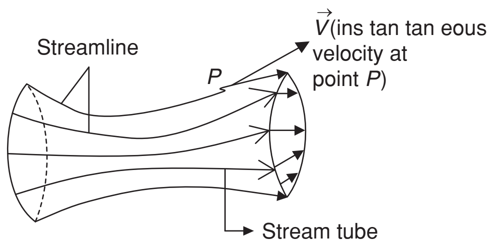
For a two – dimensional flow $(\vec{v} = u\hat{i} + v\hat{j})$, the slope of the streamline is given as:

$$\boxed{\frac{dy}{dx} = \frac{v}{u}}$$

The pattern of streamline will be fixed in space for steady flows but need not be in the case of unsteady flows.

Streamtube

An imaginary passage through which fluid flows and which is bounded by a bundle of streamline is called a *streamtube*. Fluid can enter or leave a streamline only through its ends but never across the streamtube's surface. At any instant in tube, the mass flow rate passing through any cross-sectional cut of a given stream tube will always be the same.



On steady flows, the shape and position of a stream tube does not change.

Streakline

It is the locus of the fluid particles that have passed sequentially through a chosen point in the flow. It is also the curve generated by a tracer fluid, such as a dye, continuously injected in the flow field at the chosen point. An example of a streakline is the continuous smoke emitted by a chimney.

Pathline

It is the path followed by a fluid particle in motion. A pathline can intersect with itself or two pathlines can intersect with each other.

Streamline indicates the motion of bulk mass of fluid whereas the path line indicates the motion of a single fluid particle. A streakline indicates the motion of the entire fluid particle along its length.

In a steady flow, the streamline, streakline and pathline coincide if they pass through the same point.

Example 4: For a three dimensional flow, if the velocity field is given by $\vec{v} = 4x\hat{i} + 6y\hat{j} - 10z\hat{k}$, and then an equation for a streamline passing through the point (1, 4, and 5) is:

(A) $xyz = \frac{5}{4}$ (B) $xyz = \frac{1}{20}$

(C) $xyz = \frac{4}{5}$ (D) $xyz = 20$

Solution:
From the velocity field representation, we have:
$u = 4x$
$v = 6y$
$w = -10z$
For a streamline,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Taking $\frac{dx}{u} = \frac{dy}{v}$, we have:

$$\frac{dx}{4x} = \frac{dy}{6y}$$

Integrating, we get

$x^{\frac{6}{4}} = y \times C_1$, where C_1 is an integration constant.

Considering the point (1, 4, 5), we get:

$$(1)^{\frac{6}{4}} = 4 \times C_1$$

i.e., $C_1 = \frac{1}{4}$

$$\therefore \quad x^{\frac{6}{4}} = \frac{y}{4} \qquad (1)$$

Taking $\frac{d_x}{u} = \frac{d_z}{w}$, we have:

$$\frac{dx}{4x} = \frac{dz}{-10z}$$

Integrating, we get

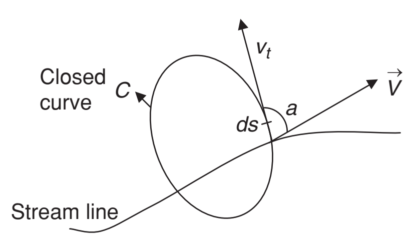
$zx^{\frac{10}{4}} = C_2$, where C_2 is an integration constant.

Considering the point (1, 4, 5), we get $5 \times (1)^{\frac{10}{4}} = C_2$
i.e., $C_2 = 5$

$$\therefore \quad zx^{\frac{10}{4}} = 5 \qquad (2)$$

Substituting equation (1) in equation (2), we get $zxy = 20$ as the equation of the streamline.

Circulation



Circulation Γ is defined as the counterclockwise line integral, around a closed curve C, of arc length ds times the velocity component tangent to the curve (v_t) in the flow field.

i.e., $\boxed{\Gamma = \oint_C v_t ds = \oint_C v \cos\alpha ds}$

or $\boxed{\Gamma = \oint_C \vec{v} \cdot \vec{ds} = \oint_C (udx + vdy + wdz)}$

For a three-dimensional flow in Cartesian coordinates:

Vorticity

The vorticity vector $\vec{\zeta}$ is defined as:

$$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{v} = \text{curl}(\vec{v})}$$

i.e., for a three-dimensional flow in Cartesian coordinates,

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

In terms of circulation, vorticity is defined as the circulation per unit of the enclosed area

i.e., $\boxed{\zeta = \frac{\Gamma}{A}}$

Vorticity vector is equal to twice the rate of rotation (or just rotation) of angular velocity vector $\vec{\omega}$,

i.e., $\boxed{\vec{\zeta} = 2\vec{\omega}}$

Vorticity is a measure of the rotation of a fluid. In a fluid flow field, points occupied by rotating or non-rotating fluid particles have respectively non-zero or zero (negligibly small) vorticities.

Vorticity is zero (negligibly small) everywhere for an irrotational flow and non-zero everywhere for a rotational flow.

Example 5: A two-dimensional irrotational flow has the velocity filed:

$\vec{v} = ay\hat{i} + bx\hat{j}$. The angle made by the velocity vector at the point (1, 1) with the horizontal is

(A) 0° (B) 45° (C) 30° (D) 60°

Solution:

From the velocity field representation, we have

$$u = ay, v = bx$$

Since the flow is irrotational,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\text{i.e., } b = a \quad (1)$$

Let the angle made by a velocity vector at point (x, y) is the flow field be θ.

$$\therefore \quad \tan\theta = \frac{v}{u} \text{ (from slope of streamline)}$$

$$= \frac{bx}{ay} \quad (2)$$

Equation (1) in equation (2) gives $\tan\theta = \frac{x}{y}$

At point (1,1), $\tan\theta = \frac{1}{1} = 1$

$\therefore \quad \theta = 45°.$

Control Volume Analysis of Mass, Momentum and Energy

Discharge (flow Rate) and Mass Flow Rate

Discharge (flow rate) is the amount of a fluid passing a cross-section of a stream in unit time. If A is the area of the cross-section and v_{avg} is the average fluid velocity over the cross-section, then:

$$\boxed{Q = A \times V_{avg}}$$

Where Q is the discharge (flow rate) or *volumetric (or volume) flow rate* over the cross-section.

Mass flow rate is the amount of mass flowing through a cross section of a stream per unit time. If $\overset{\circ}{m}$ is the mass flow rate over the cross-section, then:

$$\boxed{\overset{\circ}{m} = \rho \times Q}$$

Where ρ is the bulk average density of the fluid over the cross-section.

Control Volume Analysis of Mass

Conservation of Mass Principle

The conservation of mass principle states that the net mass transfer to or from a control volume during a given finite time interval is equal to the net change of the total mass within the control volume during that time interval.

Conservation of Mass Relation or Continuity Equation

Consider a differential area dA on the control surface (CS) of a control volume (CV) through which mass flow into or out of the control volume. Let $\vec{n}$ be the outward unit vector of dA normal to dA and let ρ be the density of the fluid. If $\vec{v}_r$ is a relative fluid velocity at dA, then the conservation of mass relation for a control volume can be written as:

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho dv + \int_{CS} \rho(\vec{v}_r \cdot \vec{n})dA = 0} \quad (1)$$

Where dv is a differential volume within the control volume. Equation (1) is called as the *continuity equation*.

Case A: Control volume is fixed

$\vec{v}_r = \vec{v}$, where $\vec{v}$ is the fluid's absolute velocity, i.e, the fluid velocity relative to a fixed point outside the control volume.

Case B: Control volume is moving but not deforming.

$$\vec{v}_r = \vec{v} - \vec{v}_{cv}$$

Case C: Control volume is deforming.

$$\vec{v}_r = \vec{v} - \vec{v}_{cs}$$

Equation (1) can be rewritten using mass flow rates as: (assuming well-defined inlets and outlets)

$$\frac{\partial m_w}{\partial t} = \sum_{\text{in}} \overset{\circ}{m} - \sum_{\text{out}} \overset{\circ}{m}, \quad (2)$$

Where the total mass within the control volume at any instant in time t, $m_w = \int_{CV} \rho dv$ and the net mass flow rate through the control surface, $\sum_{\text{out}} m - \sum_{\text{in}} m = \int_{cs} \rho(\vec{v}_r \cdot \vec{n})dA$. Here $\Sigma_{\text{out}}\overset{\circ}{m}$ and $\Sigma_{\text{in}}\overset{\circ}{m}$ correspond to the sum of the mass flow rates of all the respective outlet and inlet fluid streams of the control volume.

Mass Conservation for Steady incompressible Flows

For a steady fluid flow, conservation of mass relation (equation (2)) becomes

$$\boxed{\sum_{in} \overset{\circ}{m} - \sum_{out} \overset{\circ}{m}} \quad (3)$$

If a simple stream of a specific fluid is considered and if the subscripts 1 and 2 denote the inlet and outlet states respectively, then equation (3) becomes:

$$\boxed{\overset{\circ}{m}_1 = \overset{\circ}{m}_2}$$

or

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \quad (4)$$

If the flow is incompressible, then $\rho_1 = \rho_2$ and hence for a steady and in compressible fluid flow, equation (4) becomes

$$V_1 A_1 = V_2 A_2$$

or

$$Q1 = Q_2 \ (Q = VA)$$

Control Volume Analysis of Momentum

Principle of Conservation of Momentum

The net force acting on a mass of fluid (or a body) is equal to the change of momentum of flow (or the body) per unit time in that direction.

Forces Acting on a Control Volume

Forces acting on a control volume are classified into:

Body force These are forces that act throughout the entire body of the control volume. For example, gravity, electric and magnetic forces.

Surface forces These are forces that act on the control surface of a control volume. For example, pressure and viscous forces and reaction forces at contact points.

It should be noted that in the control volume analysis of momentum, only forces external to the control volume are considered. If $\Sigma_{\text{in}}\vec{F}$ denotes the sum of all the external forces acting on a control volume at particular instant of time,

$$\boxed{\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}}}$$

Linear Momentum Equation

The general form of the linear momentum (or simply momentum) equation that applies to a fixed, moving or deforming control volume is

$$\boxed{\sum \vec{F} = \frac{\partial}{\partial t}\int_{cv} \rho\vec{v}dv + \int_{cs} \rho\vec{v}(\vec{v}_r \cdot \vec{n})dA}$$

Here the term $\frac{\partial}{\partial t}\int_{cv} \rho\vec{v}dv$ represents the time rate of change of the linear momentum of the contents of the control volume and the term $\int_{cs} \rho\vec{v}(\vec{v}_r \cdot \vec{n})dA$ represents the net flow rate of linear momentum through the control surface by mass flow.

For a fixed and non-deforming control volume, the linear momentum equation is

$$\boxed{\sum \vec{F} = \frac{\partial}{\partial t}\int \rho\vec{v}dv + \int_{cs} \rho\vec{v}(\vec{v} \cdot \vec{n})dA} \quad (1)$$

The algebraic form of equation (1) can be written as

$$\boxed{\sum \vec{F} = \frac{\partial}{\partial t}\int_{cv} \rho\vec{v}dv + \sum_{out} \beta \overset{\circ}{m} \vec{v}_{avg} - \sum_{in} \beta \overset{\circ}{m} \vec{v}_{avg}} \quad (2)$$

Where $\vec{v}_{avg}$ is the average velocity across the inlet or outlet and β is the momentum flux correction factor. Here, $\beta \overset{\circ}{m} \vec{v}_{avg} =$

$$\int_{cs} \rho\vec{v}(\vec{v} \cdot \vec{n}\vec{v})dA$$

For a steady flow, equation (2) reduces to:

$$\boxed{\sum \vec{F} = \sum_{\text{out}} \beta \overset{\circ}{m} \vec{v}_{\text{avg}} - \sum_{\text{in}} \beta \overset{\circ}{m} \vec{v}_{avg}} \quad (3)$$

If only a single stream of a single fluid is considered and if subscripts 1 and 2 denote respectively the inlet and outlet states, then equation (3) can be written as:

$$\boxed{\sum \vec{F} = \overset{\circ}{m}(\beta_2 \vec{v}_{avg,2} - \beta_1 \vec{v}_{avg,1})}$$

Force Exerted by Flowing Fluid on a Pipe Bend

As per **Impulse-momentum theorem**, the impulse of a force on a body is equal to the change in linear momentum of the body in the duration of time for which the force acts.

i.e., $\quad \vec{F}dt = d\vec{p} = d(m\vec{v})$

This can also be applied to forces acting on fluids. Consequently,

$$\bar{F} = \frac{d\bar{p}}{dt} = \frac{d}{dt}(m\bar{v}) = \text{rate of change of linear momentum.}$$

For fluids, rate of change of linear momentum,

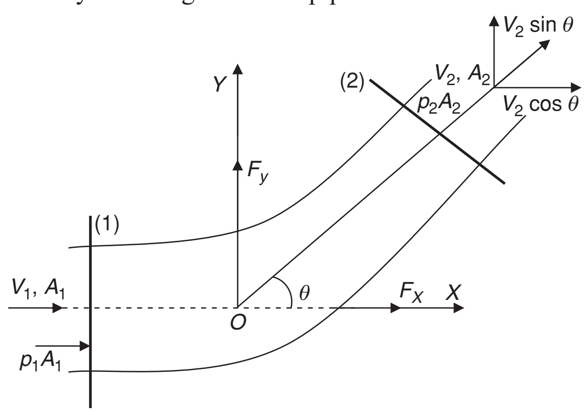
$$\frac{d\bar{p}}{dt} = \frac{d}{dt}(m\bar{v}) = \dot{m}(d\bar{v})$$

= (mass per second) × (change of velocity)

= (density × discharge) × change of velocity

$= \rho Q(d\bar{v})$; $\boxed{F = \rho Q d\bar{v}}$

This equation can be used to determine the net force exerted by a flowing fluid on a pipe bend.



Consider a reducing elbow as shown in figure. At the inlet section (1), pressure intensity = p_1, velocity of flow = V_1, along x-direction, area of cross-section = A_1. At the exit section (2), pressure intensity = p_2, velocity of flow = V_2 at an angle θ with x-axis and area of cross-section A_2. Let $\bar{F}$ be the force exerted by the flowing fluid on the bend, which can be resolved as $\bar{F}_x$ and $\bar{F}_y$ along the x and y directions respectively. As per Newton's third law of motion, the bend exerts an equal and opposite force $-\bar{F}$ on the fluid, which can be resolved as $-\bar{F}_x$ and $-\bar{F}_y$ in the x and y directions. The minus (–) sign shows that the direction of force exerted by the bend on fluid is opposite to corresponding force exerted by fluid on bend.

Along the x and y-directions, the forces on the fluid due to pressure of fluid and force exerted by bend, can be equated to the rate of change of momentum in that direction.

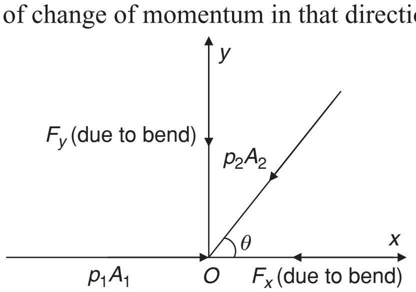


Figure 1 Forces on fluid due to pressures and due to bend

Net force on fluid is x direction is (let us call this P_x)

$$\boxed{P_x = p_1A_1 - p_2A_2\cos\theta - F_x}$$

Net force on fluid in y-direction (let us call this P_y)

$$\boxed{P_y = -p_2A_2\sin\theta - F_y}$$

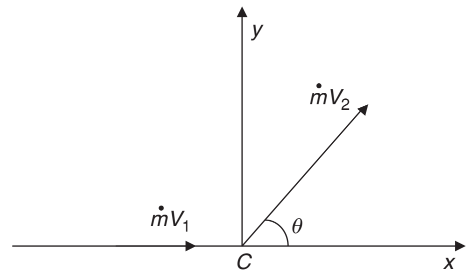


Figure 2 Linear momentum of fluid at inlet and outlet

Time rate of change of linear momentum of fluid along x-axis

$$= \dot{m}V_2\cos\theta - \dot{m}V_1$$

$$= \dot{m}(V_2\cos\theta - V_1)$$

$= \rho Q(V_2 \cos\theta - V_1)$ [Q = discharge in m^3/s
ρ = density in kg/m^3)

Time rate of change of linear momentum of fluid along y-axis,

$$= \dot{m}V_2\sin\theta - 0$$

$$= \dot{m}V_2\sin\theta$$

$$= \rho Q\, V_2 \sin\theta$$

Equate the net force on fluid in the x direction to the time rate of change of linear momentum in the x direction

$\therefore P_x = p_1A_1 - p_2A_2 \cos\theta - F_x = \rho Q(V_2\cos\theta - V_1)$

$\Rightarrow F_x = p_1 A_1 - p_2A_2 \cos\rho - \rho Q\,(V_2 \cos\theta - V_1)$

$\boxed{\Rightarrow F_x = p_1A_1 - p_2A_2\cos\theta - \rho Q\,(V_1 - V_2\cos\theta)}$ is the X-component of the force exerted by fluid on bend. Similarly, equating the net force on fluid is the y direction to the time rate of change of linear momentum is the y direction,

$P_y = -p_2A_2 \sin\theta - F_y = \rho Q\, V_2 \sin\theta$

$\therefore \quad F_y = -p_2 A_2 \sin\theta - \rho Q\, V_2 \sin\theta$

$\quad = -(p_2A_2 + \rho QV_2)\sin\theta$

$\boxed{\therefore \quad F_y = -(p_2A_2 + \rho QV_2)\sin\theta}$ is the y-component of the force exerted by fluid on bend.

The net force (F) exerted by fluid on bend is given by

$$\boxed{F = \sqrt{F_x^2 + F_y^2}}$$

The angle (α) mode by the net force exerted by fluid on bend is given by

$$\boxed{\tan\alpha = \frac{F_y}{F_x}}$$

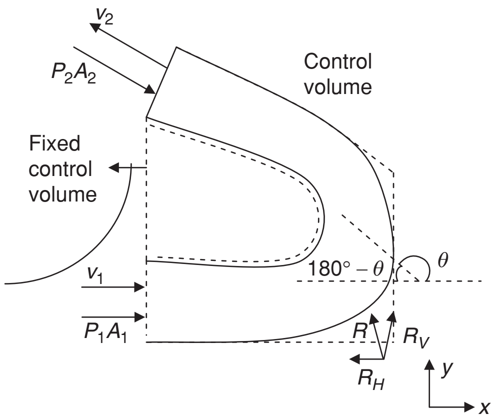
Direction for questions 6 and 7: The volumetric flow rate of a liquid of density 900 kg/ m^3, flowing through a bent pipe, as shown in the following figure, is 400 litres per second at the inlet of the pipe. The pipe which is bending by an angle θ has a constant diameter of 500 mm. The liquid is flowing in the pipe with a constant pressure of 500 kn/m^2. The horizontal component of the resultant force on the bend has a magnitude of 148325.358 N.

Example 6: The value of the angle θ is approximately:

(A) 60° (B) 120°
(C) 30° (D) 45°

Solution:

Let the subscripts 1 and 2 denote the inlet and outlet of the pipe respectively.



Diameter of the pipe, $d = 0.5$ m

Density of the fluid, $\rho = 900$ kg/m^3

Cross-sectional areas of the pipe,

$$A_1 = A_2 = \frac{\pi d^2}{4} = \frac{\pi \times (0.5)^2}{4}$$

$$= 0.1963 \text{ m}^3$$

Given, pressures $p_1 = p_2$

$$= 500 \times 10^3 \text{ N/m}^2$$

Let $\vec{R}$ be the reaction force exerted by the bend on the control volume.

Now $\vec{R}$ would be equal and opposite in direction to the resultant force exerted in the bend. Let R_H and R_r be the magnitude of the respective horizontal and vertical components of $\vec{R}$

Given, $R_H = 148325.358$ N

Now, mass flow rate:

$$\overset{\circ}{m} = \rho Q_1 = 900 \times 0.4 = 360 \text{ kg/s}$$

The flow is assumed to be steady flow. Also the weight of the pipe and the water in it is neglected. From the continuity equation, we can write:

$$A_1V_1 = A_2V_2$$

Where v_1 and v_2 are the (incompressible) liquid average velocities assuming uniform flow at inlet and outlet. Given, volumetric flow rate:

$$Q_1 = A_1V_1 = 0.4 \text{ m}^3\text{/s}$$

$$\therefore \quad v_1 = v_2 = \frac{0.4}{0.1963} = 2.0377 \frac{m}{s}$$

The change in momentum in the direction of flow can be equated to:

$$P_1A_1 + P_2A_2 \cos (180° - \theta) - R_H$$

∴ Therefore it becomes

$$P_1A_1 + P_2A_{2\,\cos} (180° - \theta) - R_H$$
$$= (-v_2\cos (180° - \theta) - v_1)\ \overset{\circ}{m}$$

∴ Cos (180°– θ) = (148325.358 – 360 × 2.0377 – 500 × 10^3 × 0.1763)

$$\therefore \quad \cos(180° - \theta) = \frac{(148325.358 - 360 \times 2.0377 - 500 \times 10^3 \times 0.1963}{(500 \times 10^3 \times 0.1963 + 360 \times 2.0377)}$$

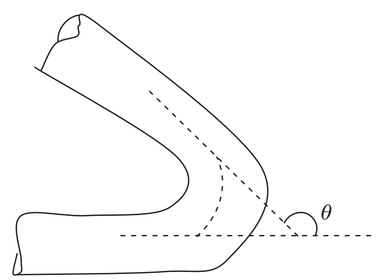
i.e., $\cos (180° - \theta) = 0.5$

or $\cos 180° - \theta = 60°$

$\theta = 120°$.

Example 7: The magnitude of the resultant force on the bend is:

(A) 148325.358 N (B) 85633. 17 N
(C) 0 N (D) 171270.11 N



Solution:

Now cos (180° – θ) = 0.5

$$\sin(180° - \theta) = \sqrt{1 - \cos^2(180° - \theta)}$$
$$= 0.8660$$

The linear momentum equation in the y-direction

$$\sum F_y = \overset{\circ}{m}(v_{2,y} - v_{1,y}) \qquad (2)$$

Here, $v_{1,y} = 0$

$$v_{2,y} = v_2 \sin (180° - \theta)$$

$$\sum F_y = -P_2A_2 \sin (180° - \theta) + R_v$$

∴ Equation (2) becomes:

$$R_v - P_2A_2 \sin (180° - \theta)$$
$$= \overset{\circ}{m} v_2 \sin (180° - \theta)$$

or $R_v = 360 \times 2.0377 \times 0.8660$
$+ 500 \times 10^3 \times 0.1963 \times 0.8660$
$= 85633.17$ N

∴ Magnitude of the resultant force:

$$|\vec{R}| = \sqrt{R_H^2 + R_V^2}$$

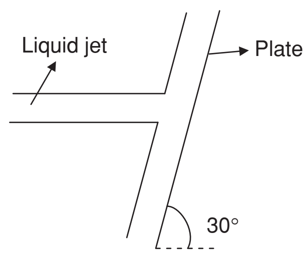
$$= \sqrt{148325.358^2 + 85633.17^2}$$

$$= 171270.11 \text{ N.}$$

Example 8: A 3.57 m diameter jet of liquid (density = 1100 kg/m^3 from a nozzle steadily strikes a flat plate, inclined at an angle of 30° to the horizontal, as shown in the following figure.

If a horizontal force of 275.27 kN is applied on the plate to hold it stationary than the velocity of the liquid jet is

(A) 9.52 m/s (B) 3.37 m/s
(C) 90.63 m/s (D) 4.76 m/s

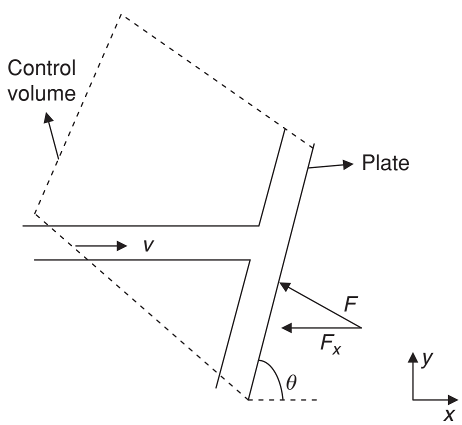


Solution:

Let F be the force applied normally on the plate to hold it stationary. Let Fx be the horizontal component of the force F.

Given $Fx = 275.27 \times 10^3$ N

Linear momentum equation in the



Direction normal to the plate yields:

$$-F = \overset{\circ}{m}\,(0 - v\cos(90 - \theta))$$

$$\text{or } F = \overset{\circ}{m}\, v \sin\theta$$

$$= \rho A v^2 \sin\theta \qquad (1)$$

Now here,

$$F_x = F\cos(90 - \theta) = F\sin\theta \qquad (2)$$

Comparing equations (1) and (2), we get

$$Fx = \rho A v^2 \sin^2\theta$$

So $275.27 \times 10^3 = 1100 \times \dfrac{\pi}{4} \times (3.75)^2 \times v^2 \times (\sin 30°)^2$

∴ $v = 9.52$ m/s .

Moment of Momentum Principle

The resulting torque acting on a rotating fluid is equal to the rate of change of moment of momentum.

Angular Momentum Equation

The general form of the angular momentum (or moment of momentum) equation that applies to a fluid, moving or deforming control volume is

$$\sum \vec{m} = \frac{\partial}{\partial t}\int_{cv} (\vec{r} \times \vec{v})\rho dv + \int_{cs} (\vec{r} \times \vec{v})\rho(\vec{v}_r \cdot \vec{n}) dA \qquad (1)$$

Here, $\sum \overset{\circ}{\vec{m}} = \sum (\vec{r} \times \vec{F})$ is the vector sum of the moment of all the forces acting on the control volume.

The term $\dfrac{\partial}{\partial_t}\int_{cv} (\vec{r} \times \vec{v})\rho dv$ represents the time rate of change of the angular momentum of the contents of the control volume and the term $\int_{cs} (\vec{r} \times \vec{v})\rho(\vec{v} \cdot \vec{n}) dA$ represents the net flow rate of angular momentum out of the control surface by mass flow

For a fixed and non-deforming control volume, the angular momentum equation is

$$\sum \vec{m} = \frac{\partial}{\partial t}\int_{cv} (\vec{r} \times \vec{v})\rho dv + \int_{cs} (\vec{r} \times \vec{v})\rho(\vec{v}_r \cdot \vec{n}) dA$$

An approximate form the angular momentum equation written in terms of average properties becomes

$$\sum \vec{m} = \frac{\partial}{\partial t}\int_{cv} (\vec{r} \times \vec{v})\rho dv + \sum_{out} (\vec{r} \times \overset{\circ}{m}\, \vec{v}_{avg}) - \sum_{in} (\vec{r} \times \overset{\circ}{m}\, \vec{v}_{avg}) \qquad (2)$$

For a steady flow, equation (2) reduces to

$$\sum \vec{m} = \sum_{out} (\vec{r} \times \dot{m} v_{avg}) - \sum_{in} (\vec{r} \times \overset{\circ}{m}\, \vec{v}_{avg}) \qquad (3)$$

Note that the term $\sum \vec{m}$ also represents the net torque acting on the control volume.

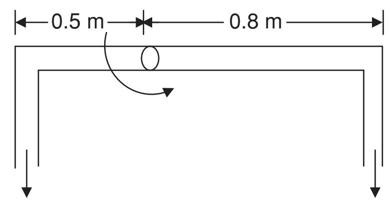
If the significant forces and momentum flows are in the same plane, then they would give rise to moments in the same plane. For such cases, equation (3) can be expressed in a scalar form as:

$$\sum \overline{m} = \sum_{\text{out}} r\, \overset{\circ}{m} v - \sum_{\text{in}} r\, \overset{\circ}{m} v$$

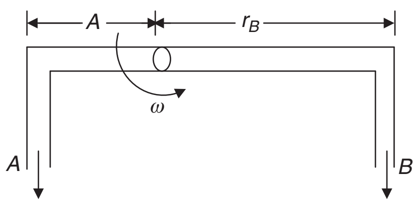
Where r represents the average normal distance between the point about which moments are taken and the line of action of the force or velocity provided that the same convention is followed for the moments. Moments in the counter clockwise position are positive and moments in the clockwise direction are negative.

Example 9: The sprinkler, shown in the following figure, has a frictionless shaft with equal flow in both the nozzles. If the water jets from the nozzles have a velocity of 10 m/s relative to the nozzles then the sprinkler rotates at an r. p. m of:

(A) 32.19 (B) 318.31
(C) 139.48 (D) 73.46



Solution:



Given $r_A = 0.5$ m

$r_B = 0.8$ m

Relative velocities, $v_{r,A} = 10$ m/s and

$V_{r,B} = 10$ m/s

Let ω be the angular velocity of the sprinkler.

Absolute fluid velocity of A,

$$v_{a,A} = v_{r,A} + \omega r_A$$
$$= 10 + 0.5\omega$$

Absolute fluid velocity of B,

$$v_{a,B} = v_{r,B} - \omega r_B$$
$$= 10 - 0.8\omega$$

NOTE

The jets of water coming out from the nozzle will exert a force in the opposite direction. So torque at B will be in the anticlockwise direction and torque at A will be in the clockwise direction. Since torque at B is greater than the torque at A, hence the sprinkler, if free, will rotate in the anticlockwise direction.

Since there is no friction and no external torque is applied on the sprinkler, $\sum m = 0$

Since the moment of momentum of the water entering the sprinkler is zero,

$$\sum_{in} r\overset{\circ}{m}v = 0$$

$\therefore$ Equation (1) becomes

$$\sum_{out} r\overset{\circ}{m}v = 0$$

or $-\overset{\circ}{m}_A r_A v_{a,A} + \overset{\circ}{m}_B r_B v_{a \cdot B} = 0$

Given $\overset{\circ}{m}_A = \overset{\circ}{m}_B$

$\therefore$ –0.5 (10 + 0.5 w) + 0.8 (10 – 0.8ω) = 0

or ω = 3.3708 rad/sec

If N is the speed of rotation of the sprinkler in rpm, then

$$\frac{2\pi N}{60} = 10$$

or $$N = \frac{60 \times 3.3708}{2 \times \pi} = 32.19\,rpm.$$

Bernoulli's Equation

Bernoulli's equation is stated as follows:

$$\frac{P}{\rho} + \frac{v^2}{2} + gz = C$$

Where C is a constant. This equation is applicable only for a steady incompressible flow along a streamline and only in the inviscid regions (regions where viscous or frictional effects are negligibly small compared to inertial, gravitational and pressure effects) of flow. For point 1 and 2 along the same streamline, Bernoulli's equation can be written as:

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2$$

Bernoulli's equation is not applicable in a flow section that involves a pump, turbine, from or any other machine or impeller since these devices destroy streamlines and transfer or extract energy to or from the fluid particles. This equation should also not be used for flow sections where significant temperature changes occur through heating or cooling sections.

NOTE

For a fluid flow, in general, the value of the constant C is different for different streamlines. However, if the flow is irrotational, constant C has the same value for all the streamlines in the flow. In other words, for irrotational flows, Bernoulli's equation becomes applicable across streamlines, i.e. between any two points in the flow region.

Bernoulli's equation and conservation of mechanical energy

The mechanical energy of a flowing fluid expressed on a unit-mass basis is

$$e_{\text{meh}} = \frac{P}{\rho} + \frac{v^2}{2} + gz$$

Where $\frac{P}{\rho}$ is the flow or pressure energy $\frac{v^2}{2}$ is the kinetic energy and gz is the potential energy of the fluid, all per unit mass.

From Bernoulli's equation the following equation can be written

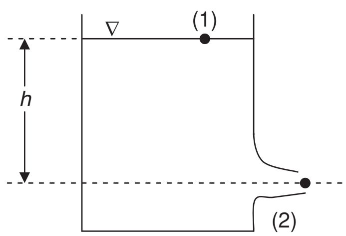
$$\boxed{E_{\text{mech}} = \text{constant}}$$

Where, E_{mech} is the mechanical energy (sum of the kinetic, potential and flow energies) of a fluid particle is constant along a streamline in a steady, incompressible and inviscid flow. Hence Bernoulli's equation can be taken as a "conservation of mechanical energy principle".

It is to be noted that the mechanical energy remains constant in an irrotational flow field.

Liquid discharge from a large tank

A large tank open to the atmosphere is filled with a liquid to a height of h metres from the nozzle as shown in the following figure.



The flow is assumed to be incompressible and irrotational. The draining of the water is slow enough that the flow can be assumed to be steady (quasi- steady). Any losses in the nozzle are neglected. Point 1 is taken to be at the free surface of water and so $p_1 = p_{\text{atm}}$ and point 2 is taken to be at the centre of the outlet area of the nozzle and so $P_2 = P_{\text{atm}}$

If A_1 and A_2 are the cross- sectional areas of the tank and nozzle respectively, then from the continuity equation, we have:

$$A_1V_1 = A_2V_2 \quad (1)$$

Since the tank is very large compared to the nozzle, we have A_1>>>>A_2. Hence from equation (1), we have

$$V_1 \approx 0$$

From the Bernoulli's equation, We have

$$\boxed{\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2}$$

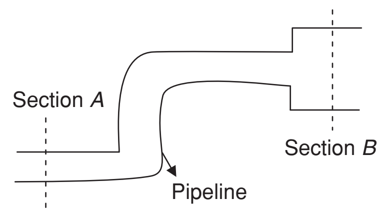
or $$V_2^2 = 2g\,(z_1 - z_2)$$

or $$\boxed{V_2 = \sqrt{2gh}} \quad (2)$$

Equation (2) is called the *Torricelli equation.*

Example 10: Section A of the pipeline, shown in the figure below, has a diameter of 20 cm and a gauge pressure (p_A) of 40 kPa. The section is at an elevation of 120 m. The section B of the pipeline has a diameter of 40 cm and is at an elevation of 125 m. The volumetric flow rate of the liquid (density = 1100 kg/m^3) through the pipeline is 70 litre/sec. If the frictional losses in the pipeline can be neglected and if p_B denotes the pressure of section B, then,

(A) Flow is from B to A and $p_A - p_B = 51.395$ kPa
(B) Flow is from A to B and $p_A - p_B = 51.395$ kP
(C) Flow is from A to B and $p_A - p_B = 28.605$ kPa
(D) Flow is from B to A and $p_A - p_B = 28.605$ kPa



Solution:

At section A, velocity of flow,

$$v_A = \frac{Q}{A_A} = \frac{\frac{70}{1000}}{\frac{\pi}{4} \times \left(\frac{20}{100}\right)^2} = 2.228 \text{ m/s}$$

At section B, velocity of flow,

$$V_B = \frac{Q}{A_B} = \frac{\frac{70}{1000}}{\frac{\pi}{4} \times \left(\frac{40}{100}\right)^2} = 0.557 \text{ m/s}$$

Assuming the flow to be steady, Bernoulli's equation application between the two sections gives:

$$\frac{p_A}{\rho} + \frac{v_A^2}{2} + gz_A = \frac{p_B}{\rho} + \frac{v_B^2}{2} + gz_B \quad (1)$$

Here $P_A = 40 \times 10^3$ *Pa* (gauge pressure)

$z_A = 120$ m

$z_B = 125$ m

$\rho = 1100$ kg/m^3

Hence equation (1) gives

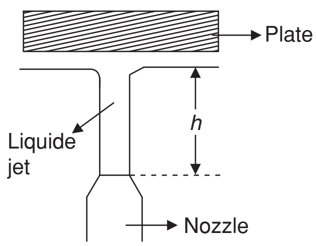
$$\frac{40 \times 10^3}{1100} + \frac{(2.228)^2}{2} + 9.81 \times 120$$

$$= \frac{P_B}{1100} + \frac{(0.557)^2}{2} + 9.81 \times 125$$

or $P_B = -11.395$ kPa (gauge pressure)
Since $p_A > p_B$, flow is from A to B and $p_A - p_B = 40 - (-11.395) = 51.395$ kPa.

Example 11: A vertical jet of liquid (density = 850 kg/m^3) is issuing upward from nozzle of exit diameter 70 mm at a velocity of 15 m/s. A flat plate weighing 250 N is supported only by the jets impact. If all losses are neglected then the equilibrium height h of the plate above the nozzle exit is:

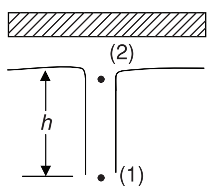
(A) 11.468 m (B) 6.434 m
(C) 9.682 m (D) 10.145 m



Solution:
Mass flow rate,

$$\mathring{m} = \rho A v$$

$$= 850 \times \frac{\pi}{4} \times \left(\frac{70}{1000}\right)^2 \times 15 = 49.068 \text{ kg/s}$$



Applying Bernoulli's equation between points (1) and (2), we get:

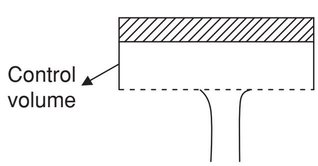
$$\frac{p_1}{\rho} + \frac{{v_1}^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{{v_2}^2}{2} + gz_2$$

Here $p_1 = p_2 = p_{\text{atm}}$

$$z_2 - z_1 = h$$

$$\therefore \quad v_2 = \sqrt{{v_1}^2 - 2gh}$$

$$= \sqrt{(15)^2 - 2 \times 9.81 \times h}$$



Applying the linear momentum balance equation for the control volume shown above, we get $-250 = \mathring{m}(0 - v_2)$ (momentum correction factor is assumed to be unity)

$$= -49.068 \times \sqrt{(15)^2 - 2 \times 9.81 \times h}$$

$h = 10.145$ m.

Different Types of Head of a Fluid in Motion

The Bernoulli's equation can be rewritten as:

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

Each term on the LHS of the above equation has the dimension of length and represents some kind of head of a flowing fluid.

Pressure Head It is the term $\frac{p}{\rho g}$ and it represents the height of a fluid column that is needed to produce the pressure p.

Velocity Head It is the term $\frac{v^2}{2g}$ and it represents the elevation needed for the fluid to reach the velocity v from rest during a frictionless free fall.

Elevation Head It is term z and it represents the potential energy of the fluid. The sum of the pressure head and the elevation head, i.e., $\frac{p}{\rho g} + z$, is known as the piezometric head.

Static, Dynamic, Hydrostatic, Total and Stagnation Pressures

The Bernoulli's equation can be rewritten as:

$$p + \frac{\rho v^2}{2} + \rho g z = \text{constant}$$

Each term on the LHS of the above equation has the units of pressure and represents some kind of pressure.

Static pressure It is the term p and it represents the actual thermodynamic pressure of the fluid as it flows.

Dynamic pressure It is the term $\frac{\rho v^2}{2}$ and it represents the pressure rise when the fluid is brought to a stop isentropically.

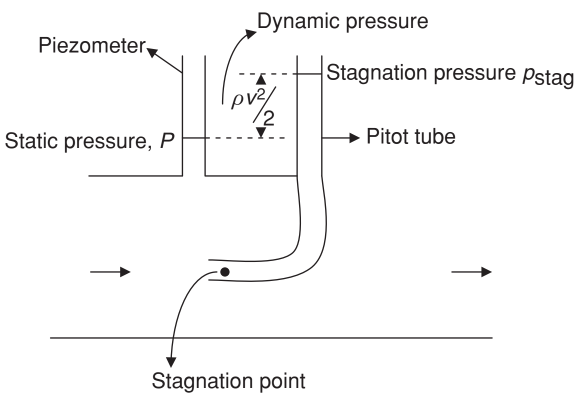
Hydrostatic pressure: It is the term $\rho g z$. It is actually not a pressure although it does represent the pressure change possible due the potential energy variation of the fluid as a result of elevation changes.

Total pressure = Static + dynamic + hydrostatic pressures

Stagnation pressure = Static + dynamic pressure

Stagnation pressure (p_{stag}) represents the pressure at a point where the fluid is brought to a complete stop isentropically.

$$\boxed{p_{stag} = p + \frac{\rho v^2}{2}}$$



Control Volume Analysis of Energy

Conservation of Energy Principle

The conservation of energy principle states that energy can neither be created nor destroyed during a process but it can be converted from one form to another.

Energy equation The general form of the energy equation that applies to a fixed, mass or deforming control volume is

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{\partial}{\partial t}\int_{cv} e\rho dv + \int_{cs}\left(\frac{p}{\rho}+e\right)\rho(\vec{v}_r\cdot\vec{n})dA \tag{1}$$

Where the total energy, $e = u + ke + \rho e$

$= u + \frac{v^2}{2} + gz$ with u, ke and ρe being the internal (u), kinetic (kc) and potential (pc) energies all being per unit mass.

The term $\frac{\rho}{\rho}$ represents the flow work, i.e, work associated with passing a fluid into or out of a control volume, per unit mass. The term $\dot{Q}_{\text{net in}} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{out}}$ is the net rate of heat transfer to the system. The term $\dot{W}_{\text{shaft, net in}} = \dot{W}_{\text{shaft, in}} - \dot{W}_{\text{shaft, out}}$ is the net power input to the system. W_{shaft} is the work transfer associated with the devices such as pumps, turbines, fans or compressors whose shaft protrudes through the control surface. Instead of $\dot{W}, \dot{W}_{\text{shaft}}$ is used, since in most cases work is transferred across the control surface by a moving shaft.

The LHS of the equation (1) represents the net rate of energy transfer into a control volume by heat and work transfer. The first term on the RHS of equation (1) represents the time rate of change of the energy content of the control volume while the second term represents the net flow rate of energy out of the control surface by mass flow.

For a fixed control volume ($\vec{v}_r = \vec{v}$), the energy equation is:

$$\boxed{\begin{aligned}&\dot{Q}_{\text{net in}} + \dot{W}_{shaft,\,net\,in}\\ &= \frac{\partial}{\partial t}\int_{cv} e\rho dv + \int_{cs}\left(\frac{p}{\rho}+e\right)\rho(\vec{v}\cdot\vec{n})dA\end{aligned}} \tag{2}$$

Assuming that the term $\left(\frac{p}{\rho}+e\right)$ is nearly uniform across an inlet or outlet and using the relation $\dot{m} = \int \rho(\vec{v}\cdot\vec{n})dA$, the energy equation (2) becomes:

$$\boxed{\begin{aligned}&\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}}\\ &= \frac{\partial}{\partial t}\int_{cv} e\rho dv + \sum_{\text{out}}\dot{m}\left(\frac{p}{\rho}+e\right) - \sum_{\text{in}}\dot{m}\left(\frac{p}{\rho}+e\right)\end{aligned}}$$

If the flow is steady and considering a single inlet and single outlet scenario, the above equation becomes:

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \dot{m}\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right. \\ \left. + u_2 - u_1 + \frac{v_2^2}{2} - \frac{v_1^2}{2} + g(z_2 - z_1)\right]$$

If we consider $\dot{W}_{\text{shaft, net in}}$

$= \dot{W}_{\text{pump}} - \dot{W}_{\text{turbine}}$ and $\dot{E}_{\text{mech loss}} = \dot{m}(u_2 - u_1) - \dot{Q}_{\text{net in}}$

Then the above equation can be written as:

$$\dot{m}\left(\frac{p_1}{\rho_1} + \frac{v_1^2}{2} + gz_1\right) + \dot{W}_{\text{pump}} \\ = \dot{m}\left(\frac{p_2}{\rho_2} + \frac{v_2^2}{2} + gz_2\right) + \dot{W}_{\text{turbine}} + \dot{E}_{\text{mech loss}} \tag{3}$$

Where $\dot{W}_{\text{pump}}$ is the shaft power input through the pump's shaft, $\dot{W}_{\text{turbine}}$ is the shaft power output through the turbine's shaft and $\dot{E}_{\text{mech loss}}$ is the total mechanical power loss consisting of the pump and turbine losses and also including the frictional losses in the piping system, i.e.,

$$\dot{E}_{\text{mech loss}} = \dot{E}_{\text{mech loss, pump}} + \\ \dot{E}_{\text{mech loss , turbine}} + \dot{E}_{\text{mech loss ,piping}}$$

In terms of heads, the energy equation (3) can be written as:

$$\frac{p_1}{\rho_1 g}+\frac{v_1^2}{2g}+z_1+h_p=\frac{p_2}{\rho_2 g}+\frac{v_2^2}{2g}+z_2+h_t+h_L \quad (4)$$

Where, $h_p\left(=\frac{\eta_{pump}\overset{\circ}{W}_{pump}}{\overset{\circ}{m}g}\right)$ is the useful head delivered to the fluid by the pump, $h_t\left(=\frac{\overset{\circ}{W}\text{ turbine}}{\eta_{turbine}\overset{\circ}{m}g}\right)$ is the extracted head removed from the fluid flow in the piping system.

Direction for questions 12 and 13: The velocity profile for flow in a circular pipe is given as:

$v=v_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$ where v is the velocity of any radius r, v_{max} is the velocity of the pipe axis and R is the radius of the pipe.

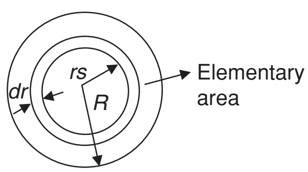
Example 12: The average velocity of flow is given by:

(A) v_{max} (B) $\frac{3}{4}v_{max}$

(C) $\frac{v_{max}}{4}$ (D) $\frac{v_{max}}{2}$

Solution:

In a cross-section of the circular pipe, consider an elementary area dA in the form of a ring at a radius r and of thickness dr.



Then, $dA=2\pi rdr$

Flow rate through the ring

$= dQ =$ elemental area × local velocity

$= 2\pi rdr \times v$

Total flow, $Q=\int_0^R 2\pi rdr\cdot v$

$$=\int_0^R 2\pi r v_{max}\left[1-\left(\frac{r}{R}\right)^2\right]dr$$

$$Q=pv_{max}\left(\frac{R^2}{2}\right) \quad (1)$$

Let v_{avg} be the average velocity,

Then $Q=\pi R^2\times v_{avg}$

From equation (1) we have

$$\pi v_{max}\left(\frac{R^2}{2}\right)=\pi R^2 v_{avg}$$

$$V_{avg}=\frac{v_{max}}{2}.$$

Example 13: The value of the kinetic energy correction factor is:

(A) 2 (B) 1.11

(C) 1.04 (D) 1

Solution:

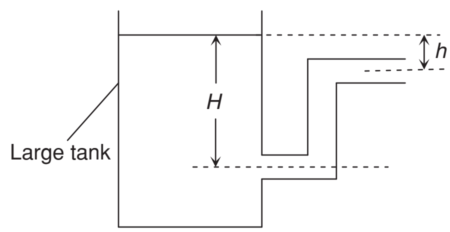
$$\alpha=\frac{1}{A}\int\left(\frac{v}{v_{avg}}\right)^3 dA$$

$$=\frac{1}{\pi R^2}\frac{8}{(V_{max})^3}\int_O^R V^3 2\pi rdr$$

$$=\frac{16}{R^2}\int_O^R\left(1-\left(\frac{r}{R}\right)^2\right)^3 rdr$$

$$=\frac{16}{R^2}\times\left(\frac{R^2}{8}\right)=2.$$

Example 14: If the head losses in the pipe shown in the figure is h_2 metres, then the discharge velocity at the pipe exit is:



(A) $\sqrt{2g(h-h_L)}$

(B) 0

(C) $\sqrt{2g(H-h_2)}$

(D) $\sqrt{2_g(H+h-h_L)}$

Solution:

Let the height of the water surface from the bottom of the tank (chosen as the datum level) be L.

Consider point 1 to be the water surface of the tank and point 2 to be at the pipe exit.

Now, $P_1=P_2=P_{atm}$

The tank is considered to be very large such that $V_1\approx 0$

Assuming the flow to be steady applying the energy equation between the two points we have:

$$\frac{P_1}{\rho g}+\alpha_1\frac{V_1^2}{2g}+Z_1+h_P=\frac{P_2}{\rho g}+\frac{\alpha_2 V_2^2}{2g}+Z_2+h_t+h_L \quad (1)$$

Since no pump and turbin is involved,

$$h_p = h_t = 0$$

The kinetic correction factor are considered to be unity, i.e., $\alpha_1 = \alpha_2 = 1$
The equation (1) can be written now as:

$$L = \frac{V_2^2}{2g} + (L - h) + hL$$

$$V_2 = \sqrt{2g(h - h_L)}.$$

Example 15: A hydrautic turbine is supplied with 5 M^3/s water at 420 kPa (guage). A vacumm gauge fitted in the turbine discharge 4 on below the turbine inlet centre line shows a readingof 200 mm Hg. If the turbine shaft output power is 1200 kW and if the internal diameters of the supply and discharge pipe are identically 100 mm, then the power loss through the turbine is:

(A) 2429.62 kW (B) 962.78 kW
(C) 1229.62 kW (D) 2162.78 kW

Solution:
Let the subscripts S and D denote points in the suction and the discharge pipe respectively.
Given $Ps = 420$ kPa
$Zs = 4$ m
$Z_D = 0$ m (discharge pipe taken at the dotum plane.)

$$W_{\text{turbine}} = 1200 \times 10^{3} \text{ W}$$

The energy equation applied between the points S and D is as follows.

$$M\left(\frac{Ps}{\rho} + \alpha_s \frac{V_s^2}{2} + gZ_s\right) + W_{\text{pump}}$$

$$= M\left(\frac{P_D}{\rho} + \alpha_D \frac{V_D^2}{2} + gZ_D\right) + W_{\text{turbine}} + E_{\text{mech loss}} \quad (1)$$

Since no pump is involved, Wpump$p = 0$. The kinetic energy correction factors are assumed to be unity, i.e., $\alpha_S = \alpha_D = 1$
Here $Q = 5$ $m^{3/S}$

$$\therefore \quad \overset{\circ}{m} = \rho Q = 1000 \times 5 = 5000 \ kg/s$$

Now $P_D = -200$ mm kg

$$= \frac{-200}{1000} \times 13600 \times 9.81$$
$$= -26.6832 \text{ kPa}$$

Since the supplies are discharge pipe have identical internal diameters, we have:

$$V_S = V_0$$

∴ Equation (1) becomes

$$5000 \times \left(\frac{420 \times 10^3}{1000} + 9.81 \times 4\right)$$
$$= 5000\left(\frac{-26.6832 \times 10^3}{1000}\right) + 1200 \times 10^3 + E_{\text{mech loss}}$$
$$= E_{\text{mech loss}} = 1229.62 \text{ kW}.$$

Differential Analysis of Mass and Momentum

Continuity Equation

The general differential equation for conservation of mass or the continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0$$

or
$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{V} = 0$$

or
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

The continuity equation in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

Special Cases of the Continuity Equation

(a) For steady compressible flow, the continuity equation reduces to

$$\vec{\nabla} \cdot (\rho \vec{V}) = 0$$

or
$$\frac{\partial}{\partial r}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial 2}(\rho w) = 0$$

(b) For incompressible (steady or unsteady) flow, continuity equation reduces to

$$\vec{\nabla} \cdot \vec{V} = 0$$

or
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Since $\vec{\nabla} \cdot \vec{V} = 0,$ velocity field $\vec{V}$ is said to be a divergence free or divergence less field in this case.

Stream Function

For an incompressible two dimensional planar flow, the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

A function $\psi(x, y)$, called the *stream function* can be defined such that whenever the velocity components are defined in terms of the stream function as shown below, the continuity equation (1) will always satisfied.

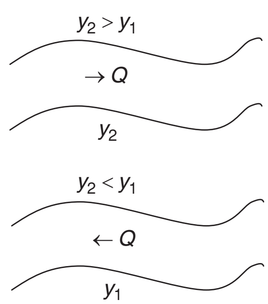
$$u = \frac{\partial \psi}{\partial y}, v = \frac{-\partial \psi}{\partial x} \qquad (2)$$

Equation (2) holds for rotational and irrotational regions of flow.

The volume rate of flow, Q, between two streamlines such as ψ_1 and ψ_2 is given by

$$\boxed{Q = \psi_2 - \psi_1}$$

The relative value of ψ_2 with respect to ψ_1 will determine the flow direction as shown below.



Flow stream lines are curves of constant ψ

Navier Stokes Equation

The Navier stokes equation is obtained when the conservation relation is applied to momentum. For an incompressible and isothermal flow, the equation is

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\overrightarrow{\nabla \rho} + \rho \vec{g} + \mu \nabla^2 \vec{V}}$$

The above equation is valid only for Newtonium fluids with constant properties such as viscocity, thermal conductivity etc.

The scalar operator $= \nabla^2 = (= \vec{\nabla} \cdot \vec{\nabla})$ is called as the Laplacian operator and is equal to $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Navier Stokes Equation (approximation) for Creeping Flow

A creeping flow is a flow in which the Reynolds number is very low (Re <<1). Reynolds number is defined as $Re = \frac{\rho V L}{\mu}$ where V and L are the characteristic speed and length. The approximate equation for creeping flow, assuming negligible gravitational effects and steady or oscillating flow is

$$\boxed{\overrightarrow{\nabla p} \cong \mu \nabla^2 \vec{V}}$$

Navier Stokes Equation (approximation) for Inviscid Regions of Flow

The inviscid regions of flow or regions of flow with negligible net viscous forces are regions of high Reynolds number. In such a region, the Navier Stokes equation reduces to

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\overrightarrow{\nabla \rho} + \rho \vec{g}}$$

The above equation is called the Euler equation which is the Navier stokes equation without the viscous term. Euler equation is approximate only in regions of flow with large Reynolds numbers and where the net viscous forces are negligible compared to the inertial and/or pressure forces.

NOTE

An irrotational region of flow is a region where net viscous forces are negligible compared to inertial and for pressure forces because of the irrotational approximation. All irrotational regions of flow are also inviscid but all inviscid regions of flow need not be irrotational. A uniform flow field is an example of an irrotational flow.

Velocity Potential Function

If the curl of a vector is zero the vector can be expressed as the gradient of a scaler function ϕ called the potential function. This is possible since the unit of the gradient of any scalar function (as long as ϕ is a smooth function) is zero.

For an irrotational flow, we have $\vec{\nabla} \times \vec{V} = 0$ and therefore the velocity vector $\vec{V}$ can be expressed as the gradient of a scalar function ϕ, called the *velocity potential function* (or just *velocity potential*) as follows:

$$\vec{V} = \vec{\nabla} \phi$$

Therefore the existence of a velocity potential implies that the fluid is irrotational.

Regions of irrotational flow are also called regions of *potential flow*. Sometimes a potential flow specifically refers to an inviscid incompressible and irrotational flow.

Substituting equation (1) in the incompressible continuity equation, we obtain for irrotational flows the following equation.

$$\nabla^2 \phi = 0$$

The above equation is called the *Laplace equation.*

Thus for incompressible, irrotational planar regions of flow, the following are applicable

(a) $\nabla^2\phi = 0$

(b) $\nabla^2\psi = 0$

(c) $u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}$

(d) $V = \dfrac{\partial\phi}{\partial y} = \dfrac{-\partial\psi}{\partial x}$

The equation $\boxed{u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}}$ and $\boxed{v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}}$ are called Cauchy Riemann equation. These equations give the relations between velocity potential function and stream function.

Curves having constant values of ϕ are called as *equipotential lines*. The slope of an equipotential line, $\dfrac{dy}{dx} = \dfrac{-u}{v}$

Potential function exists for irrotational flow only. The stream function applies to both the rotational and irrotational flows.

In a flow field streamlines intersect equipotential lines at right angles or orthogonally at all points of intersection except at stagnation points where the components vanish simultaneously.

Example 16: The velocity potential function of a two dimensional incompressible and irrotational flow is $\phi = ax^3y - y^3x$. The value of a is:

(A) 0 (B) 1 (C) 1/6 (D) 6

Solution:

For an incompressible and irrotational flow, we have $\nabla^2\phi = 0$

$$\text{or } \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\phi = ax^3y - y^3x \quad (1)$$

$$\frac{\partial\phi}{\partial x} = 3ax^2y - y^3$$

$$\frac{\partial^2\phi}{\partial x^2} = 6axy \quad (2)$$

$$\frac{\partial\phi}{\partial y} = ax^3 - 3y^2x$$

$$\frac{\partial^2\phi}{\partial y^2} = -6yx \quad (3)$$

Substituting equations (2) and (3) in equation (1) we get:

$6axy - 6yx = 0$

or $a = 1$.

Example 17: A steady threedimensional velocity field is given by: $\vec{V} = axy^3\hat{i} + (10b - 3cy^4)\hat{j} + x^2y^2\hat{k}$. The condition under which the flow field will be incompressible is:

(A) $a = 4$ c (B) $a = 0$

(C) $a = 12$ c (D) $b =$ c

Solution:

If the field is incompressible, then from the continuity equation we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

From the velocity field description,

$$u = axy^3$$
$$v = 10b - 3cy^4$$
$$w = x^2y^2$$

Substituting the above three equations in equation (1) we have:

$$ay^3 - 12cy^3 + 0 = 0$$

Or $a = 12$ c.

Example 18: An incompressible flow is represented by the velocity potential function $\phi = 4x^2 + 4y^2 + 17t$. For the flow, which one of the combinations of the following statement holds true?

(i) Flow is physically possible

(ii) Flow is physically not possible.

(iii) Flow satisfies the continuity equation

(iv) Flow does not satisfy the continuity equation

(A) (i) and (iv) (B) (i) and (iii)

(C) (ii) and (iii) (D) (ii) and (iv)

Solution:

$$\phi = 4x^2 + 4y^2 + 17t$$

$$u = \frac{\partial\phi}{\partial x} = 8x$$

$$V = \frac{\partial\phi}{\partial y} = 8y$$

The incompressible equation is:

$$\frac{\partial u}{\partial n} + \frac{\partial v}{\partial y} = 0$$

Here $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 8 + 8 = 16 \neq 0$

Hence the continuity equation is not satisfied and this implies that the flow is physically not possible.

Example 19: Persons *A*, *B* and *C* claim that the functions $\phi = 5x^2 - 5y^2$, $\phi = 10\sin x$ and $\phi = 27xy$ respectively are valid potential functions. Which one of the following statements is ONLY correct regarding the claims?

(A) The claims of persons *A* and *B* are true.

(B) The claims of persons *B* and *C* are true

(C) The claims of persons *A* and *C* are true.

(D) The claims of person *A* are false.

Solution:
For ϕ to be a valid potential function.

$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}$ should be equal to zero.

For $\phi = 5x^2 - 5y^2$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=10-10=0$$

Person A's claim is true.
For $\phi = 10 \sin x$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-10\sin x+0$$
$$=-10\sin x\neq 0$$

Person B's claim is not true.
For $\phi = 27\,xy$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0+0=0$$

Person C's claim is true.

Example 20: The stream function representing a two dimensional flow is given by: $\psi=\frac{ax^2y^2}{2}-2xy-\frac{ax^4}{12}-\frac{y^4}{6}$

If the flow is irrotational then the value of a is
(A) 0 (B) 2 (C) 0.5 (D) 12

Solution:
If the flow is irrotational,

Then $\frac{\partial^2\psi}{ax^2}+\frac{\partial^2\psi}{\partial y^2}=0$ (1)

$$\frac{\partial\psi}{\partial x}=\frac{2axy^2}{2}-2y-\frac{4x^3a}{12}$$

$$\frac{\partial^2\psi}{\partial x^2}=ay^2-x^2a \quad (2)$$

$$\frac{\partial\psi}{\partial y}=\frac{2ax^2y}{2}-2x-\frac{4y^3}{6}$$

$$\frac{\partial^2\psi}{\partial y^2}=ax^2-2y^2 \quad (3)$$

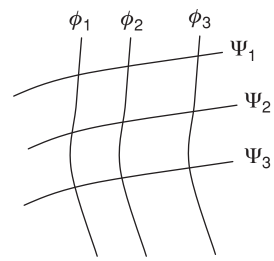
Substituting equations (2) and (3) in equation (1) we get

$$ay^2 - ax^2 + ax^2 - 2y^2 = 0$$

Or $a = 2$.

Flow Nets

A flow net is a grid obtained by drawing a set of streamlines and equipotential lines.



Flow nets are used to study 2-dimensional irrotational flow especially in cases where the stream and velocity functions are unavailable or difficult to solve.

Flow Through Orifices

A small opening of any cross-section, made on the bottom or sidewall of a tank through which a fluid can flow, is called an **orifice**.

Classification of orifices The various bases for classification of orifices are

1. Based on size of orifice as
   (i) Small orifice, if the head of liquid from the centre of orifice is more than five times the depth of orifice.
   (ii) Large orifice, if the head of liquid from the centre of orifice is less than five times the depth of orifice.
2. Based on shape of cross-sectional area as
   (i) Circular orifice
   (ii) Triangle orifice
   (iii) Square orifice
   (iv) Rectangular orifice
3. Based on shape of upstream edge of orifice as
   (i) Sharp edged orifice
   (ii) Bell-mouthed orifice
4. Based on nature of discharge as
   (i) Free discharging orifices
   (ii) Drowned or submerged orifices, which are further classified as fully submerged orifices and partially submerged orifices.

When a jet of fluid flows out of a circular orifice, the area of cross-section of the jet keeps on decreasing and becomes a minimum at the vena contracta and beyond that the jet diverges. The location of minimum cross-sectional area (i.e. Vena-contracta) is approximately at a distance of half the diameter of the orifice from the tank. If the flow through the orifice is steady at a constant head H and the cross-sectional area of the tank is very large when compared to the cross-sectional area of the jet, it can be shown using Bernoulli's theorem that the theortical velocity of flow at the vena contracta

$\boxed{V_T = \sqrt{2gH},}$ where g = acceleration due to gravity. The actual velocity of flow (V) at the vena contracta is less than this theoretical value, i.e., $V < V_T$

$$\text{The ratio } \frac{V}{V_T} = C_V = \text{coefficient of velocity}$$

Hence **coefficient of velocity (C_V)** is defined as the ratio of the actual velocity of flow at the vena contracta to the theoretical velocity of flow at the same location.

$$\therefore \quad \boxed{C_V = \frac{V}{V_T} = \frac{V}{\sqrt{2gH}}}$$

The value of C_V varies from 0.95 to 0.99 for various orifices ad this value depends on:

(i) Shape of orifice
(ii) Size of orifice and
(iii) On the head under which the flow takes place.

$$\therefore \quad C_V < 1$$

Coefficient of contraction (C_C) is defined as the ratio of area of cross-section of the jet at the vena contracta (a_c) to the cross-sectional area of orifice (a)

$$\therefore \quad \boxed{C_C = \frac{a_C}{a} < 1}$$

The value of C_C varies from 0.61 to 0.69 for various orifices and depends upon the same factors on which C_V depends.

Coefficient of discharges (C_d) is defined as the ratio of actual discharge from an orifice to the theoretically possible discharge through the orifice.

$$\therefore \quad C_d = \frac{Q_{\text{actual}}}{Q_{\text{theoretical}}}$$

$$= \frac{\text{Actual cross-sectional area} \times \text{actual velocity}}{\text{Theoretical cross-sectional area} \times \text{theoretical velocity}}$$

$$= \frac{a_C \times V}{a \times V_T} = C_C \times C_V$$

$$\therefore \quad \boxed{C_d = C_C \times C_V}$$

The value of C_d varies from 0.61 to 0.65 for different orifices and depends on shape and size of orifice and the head under which the flow occurs.

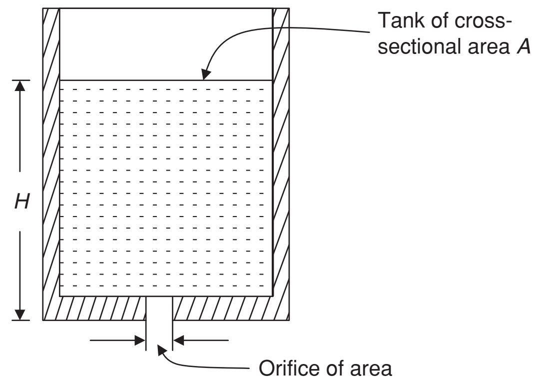


Figure 3 Time for emptying a tank of uniform cross-sectional area through an orifice at its bottom

At time $t = 0$, the height of liquid above orifice is H.

Using Bernoulli's equations, it can be shown that the theoretical time required for completely emptying the tank is $\boxed{T = \left(\frac{A}{a}\right)\sqrt{\frac{2H}{g}}}$. It may be noted that $\sqrt{\frac{2H}{g}}$ is the time needed for free fall from rest from a height of H.

If C_d is the coefficient of discharge through the nozzle, $\boxed{T_{\text{ACTUAL}} = \left(\frac{A}{a}\right)\frac{1}{C_d}\cdot\sqrt{\frac{2H}{g}}}$ is the actual time taken for emptying the tank.

Also, the time needed for emptying the same tank from an initial height of liquid H_1 above orifice to a final height of liquid H_2 above orifice is given by:

$$\boxed{T = \left(\frac{A}{a}\right)\frac{1}{C_d}\cdot\sqrt{\frac{2}{g}}(\sqrt{H_1} - \sqrt{H_2})}$$

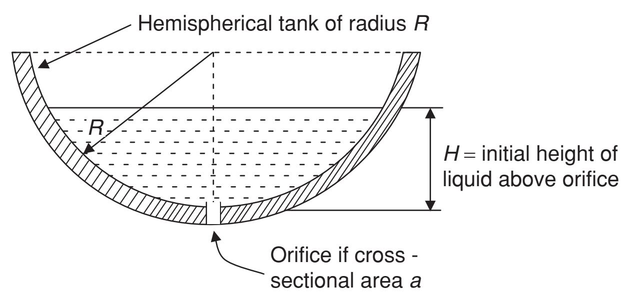


Figure 4 Time for emptying a hemispherical tank through an orifice at the bottom

If C_d is the coefficient of discharge through the orifice, it can be shown that the actual time needed for emptying the hemispherical tank is:

$$\boxed{T_{\text{actual}} = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}RH^{\frac{3}{2}} - \frac{2}{5}H^{\frac{5}{2}}\right]}$$

Where R = radius of hemispherical tank and
H = initial height of liquid above orifice
a = cross-sectional area of orifice and
g = acceleration due to gravity

If initial height of liquid above orifice is H_1 and final height of liquid above orifice is H_2, then time needed for emptying the hemispherical tank is:

$$T_{\text{actual}} = \frac{\pi}{C_d a\sqrt{2g}}\left[\frac{4}{3}R\left(H_1^{\frac{3}{2}} - H_2^{\frac{3}{2}}\right) - \frac{2}{5}\left(H_1^{\frac{5}{2}} - H_2^{\frac{5}{2}}\right)\right]$$

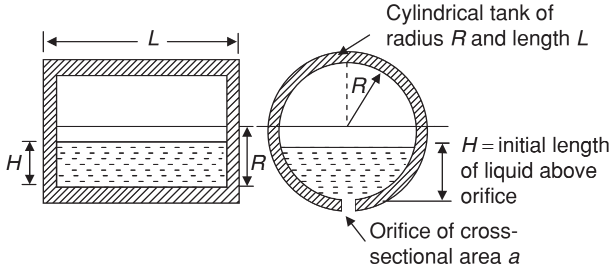


Figure 5 Time for emptying a circular horizontal tank through an orifice at its bottom

A horizontal cylindrical tank of radius R and length L is fitted with an orifice of cross-sectional area an at its bottom. The height of liquid above the nozzle is H. The coefficient of discharge through the nozzle is C_d.

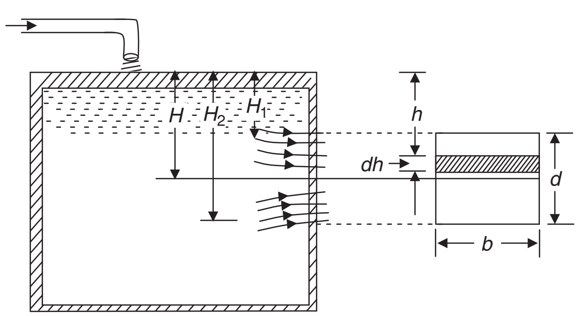
Time for emptying the horizontal cylindrical tank is:

$$T = \frac{4L}{3C_d a\sqrt{2g}}\left[(2R)^{\frac{3}{2}} - (2R-H)^{\frac{3}{2}}\right]$$

If initial height of liquid above orifice is H_1 and final height of liquid above orifice is H_2, time required for decreasing the liquid level from H_1 to H_2 (ie emptying through orifice) is:

$$T = \frac{4L}{3C_d a\sqrt{2g}}\left[(2R-H_2)^{\frac{3}{2}} - (2R-H_1)^{\frac{3}{2}}\right]$$

Discharge through large rectangular orifice In a large rectangular orifice, there is a considerable variation of effective pressure head over the height of the orifice. Hence the velocity of liquid particles through the orifice is not constant.



Consider a large rectangular orifice of with b and height d, fitted to one vertical side of a large tank, discharging freely into atmosphere, under a constant H as shown in figure.

We have H_1 = height of liquid above top edge of orifice

H_2 = height of liquid above bottom edge of orifice

$\therefore$ Height of orifice, $d = H_2 - H_1$

b = width of orifice

C_d = coefficient of discharge of orifice

Area of a strip of orifice of height dh at a depth h below the free surface of liquid in the tank is

$dA = bdh$

V = Theoretical velocity of flow through this strip = $\sqrt{2gh}$

$\therefore$ Discharge through the strip, dQ

$= C_d \times$ area of strip $\times$ velocity

$= C_d(bdh)\sqrt{2gh} = C_d b\sqrt{2gh}dh$

$\therefore$ Total discharge through orifice,

$$Q = \int dQ = \int_{H_1}^{H_2} C_d b\sqrt{2gh}\, dh$$

$\Rightarrow Q = \dfrac{2}{3}C_d b\sqrt{2g}\left[H_2^{\frac{3}{2}} - H_1^{\frac{3}{2}}\right]$ is the actual discharging through the large orifice

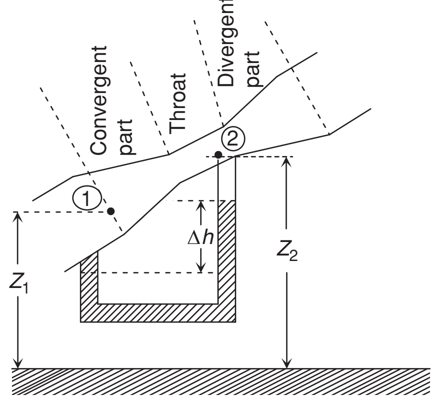
NOTE

Velocity of approach is the velocity with which the liquid approaches the orifice. In the above expression for discharge Q over the rectangular orifice, velocity of approach V_a is taken as zero. If $V_a \neq 0$, then $H_{1\text{eff}} = \left(H_1 + \dfrac{V_a^2}{2g}\right)$ and $H_{2\text{eff}} = \left(H_2 + \dfrac{V_a^2}{2g}\right)$. In the expression for Q, H_1 and H_2 will get replaced to $H_{1\text{ eff}}$ and $H_{2\text{ eff}}$.

Practical Applications of Bernoulli's Equation

Venturimeter It consists of two conical parts, the convergent part and the divergent part, with a small portion of uniform cross-section (with the minimum area), called the throat, in between the parts. The venturimeter is always used so that the upstream part of the flow takes place through the convergent part while the downstream part of the flow takes place through the divergent part.

In the convergent part, the velocity increases in the flow direction while the pressure decreases, with the velocity being maximum and pressure being is minimum at the throat. In the divergent part, velocity decreases while pressure increases.



From the Bernoulli equation and the continuity equation, the velocity at the throat is obtained as follows.

$$V_2 = \frac{A_1}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g(h_1^* - h_2^*}$$

Where h_1^* and h_2^* are the piezometric heads at section 1 and 2 respectively and are given by:

$$h_1^* = \frac{p_1}{\rho g} + z_1 \; h_2^* = \frac{p_2}{\rho g} + z_2$$

The theoretical discharge or flow rate is given by:

$$Q = A_2 V_2 = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g(h_1^* - h_2^*)}$$

Here, $h_1^* - h_2^* = \Delta h\left(\frac{\rho_m}{\rho} - 1\right)$ where r_m is the density of the manometric fluid. The actual discharge or flow rate is given by:

$$Q \text{ actual} = C_D \times Q$$

$$= C_D \times \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g\Delta h\left(\frac{\rho_m}{\rho} - 1\right)}$$

Where C_D is the coefficient of discharge or coefficient of venturimeter. C_D is always less than unity and lies between 0.95 to 0.98. The coefficient of discharge is introduced to account for the fact that the measured values of Δh for a real fluid will always be greater than that assumed for an ideal fluid due to frictional losses.

Example 21: A venturimeter with a throat diameter of 50 mm is used to measure the velocity of water in a horizontal pipe of 200 mm diameter. The pressure at the inlet of the venturimeter is 20 kPa and the vacuum pressure at the throat is 10 kPa. If frictional losses are neglected, then the flow velocity is:

(A) 28 cm/s (B) 24.2 cm/s
(C) 14 cm/s (D) 48.5 cm/s

Solution:

Given $p_1 = 20 \times 10^3 \, P_a$

$$p_2 = -10 \times 10^3 \, P_a$$

Since the venturimeter would be horizontal, $z_1 = z_2$

Now $h_1^* - h_2^* = \frac{p}{\rho g} + z_1 - \frac{p_2}{\rho g} - Z_2$

$$= \frac{20 \times 10^3 + 10 \times 10^3}{1000 \times g}$$

$$= \frac{30}{g}$$

The flow velocity, $V_1 = \frac{A_2 V_2}{A_1}$

$$= \frac{A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g \times (h_1^* - h_2^*)}$$

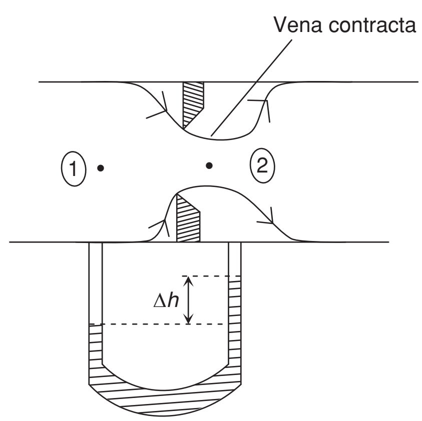
Here, $A_1 = \frac{\pi}{4}\left(\frac{200}{1000}\right)^2$

$$A_2 = \frac{\pi}{4}\left(\frac{50}{1000}\right)^2$$

$$\therefore \quad V_1 = \frac{50^2}{\sqrt{200^4 - 50^4}} \times \sqrt{60}$$

$$= 48.5 \text{ cm/s.}$$

Orificemeter An orifice meter is a thin circular plate with a sharp edged concentric circular hole in it.



The flow through the orificemeter from an upstream section contracts until a section downstream, where the vena contracta is formed, and then expands to fill the whole pipe. One of the pressure tapings is usually provided at the upstream of the orifice plate where the flow is uniform and the other is provided at the vena contracta. At the vena contracta, streamlines converge to a minimum cross section.

The velocity of flow at the vena contracta,

$$V_2 = Cv\sqrt{\frac{2\rho g\left(\frac{\rho_m}{\rho} - 1\right)\Delta h}{1 - \frac{A_2^2}{A_1^2}}}$$

Where ρ_m is the density of the manometric liquid and C_V is the *coefficient of velocity.*

C_V is always less than unity. The coefficient of velocity is introduced to account for the fact that the pressure drop for a real fluid is always more due to friction that assumed for an inviscid flow.

The volumetric flow rate is given by $Q = A_2 V_2$

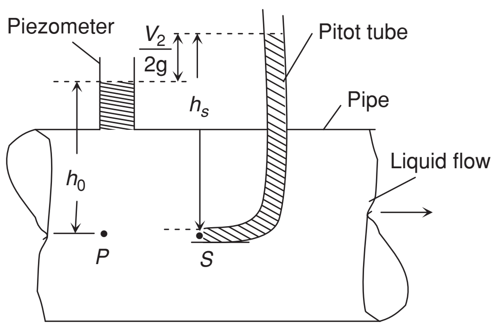
If the coefficient of contraction, C_c, is defined as $C_c = \frac{A_2}{A_0}$ where A_0 is the area of the orifice,

$$\text{Then } Q = C_d A_0 \sqrt{\frac{2g\left(\frac{\rho_m}{\rho}-1\right)\Delta h}{1 - C_c^2 \frac{A_0^2}{A_1^2}}}$$

Where the coefficient of discharge, $C_d = C_c$

The coefficient of discharge of an orificemeter lies between 0.6 to 0.65

Pitot tube It works on the principle that if the velocity of flow at a point becomes zero, the increase in the pressure at the point is due to conversion of kinetic energy into pressure energy. A pilot tube provides one of the most accurate methods for measuring the fluid velocity.



Point S is a stagnation point while point P is a point in the undisturbed flow both being at the same horizontal plane.

$$h_0 = \frac{p_0}{\rho g}$$

$$h_s = \frac{p_s}{\rho g}$$

Where p_0 is the pressure at point P, i.e., static pressure and ps is the stagnation pressure at point S.

$$\frac{p_0}{\rho g} + \frac{V^2}{2g} = \frac{p_s}{\rho g}$$

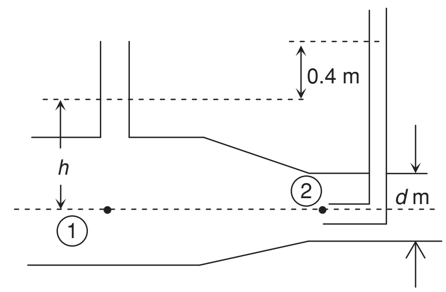
$$h_0 + \frac{V^2}{2g} = h_s$$

$$\boxed{V = \sqrt{2g(h_s - h_0)} = \sqrt{2g\Delta h}}$$

Where Δh is the dynamic pressure head which is equal to the velocity head. It is to be noted that the pitot tube measures only the stagnation pressure and so the static pressure must be measured separately by using a piezometer. A pitot static tube however measures both static and stagnation pressures.

Example 22: Water is flowing through a pipe a pipe that contracts from a diameter of 0.15 m to d meters as shown in the following figure. The difference in manometer levels is 0.4 m. If the flow rate Q in the pipe is expressed in terms of the variable d as $Q = kd^n$, then

(A) $k = 0.0495$ and $n = 0$
(B) $k = 0.0495$ and $n = 2$
(C) $k = 7.848$ and $n = 0$
(D) $k = 6.164$ and $n = 2$



Solution:

From Bernoulli's equation we have

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + Z_2$$

Here $Z_1 = Z_2$

$V_2 = 0$ (stagnation point)

$$\therefore \quad \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} \tag{1}$$

But $\frac{p_1}{\rho g} = h$

$$\frac{p_2}{\rho g} = h + 0.4$$

$$\therefore \quad \frac{p_2}{\rho g} - \frac{p_1}{\rho g} = 0.4 \tag{2}$$

Substituting equation (2) in (1),

We have:

$$\frac{V_1^2}{2g} = 0.4$$

$$V_1 = \sqrt{0.4 \times 2 \times 9.81}$$

$$= 2.801 \text{ m/s}$$

$$Q = A_1 \times V_1$$

$$= \frac{\pi}{4} \times (0.15)^2 \times 2.801$$

$$= 0.0495 \text{ m}^3\text{/s}$$

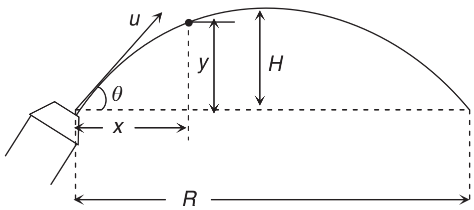
$\therefore$ In the relationship

$$Q = kd^n$$

$$k = 0.0495 \text{ and } n = 0.$$

Free Liquid Jet

A jet of liquid issuing from a nozzle in to the atmosphere is termed as a *free liquid jet*. The path traversed by a liquid jet under the action of gravity is called as its *trajectory* which would be a parabolic path.



Here u is the velocity of the liquid jet and θ is the angle made by the jet with the horizontal. The equation of the jet is:

$$y = x \tan \theta - gx^2 \sec^2 \theta / 2u^2$$

1. Maximum height attained by the jet (H)

$$\boxed{H = \frac{u^2 \sin^2 \theta}{2g}}$$

2. Time of flight (T)

$$\boxed{T = \frac{2u \sin \theta}{g}}$$

Time taken to reach the highest point is:

$$= \frac{u \sin \theta}{g}$$

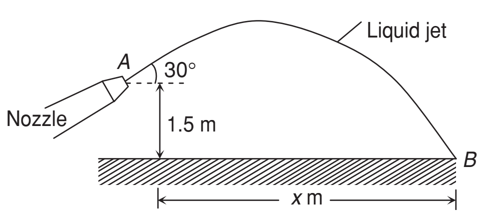
3. Horizontal range of the jet (R)

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

Range is maximum when $\theta = 45°$ and its value is $\frac{u^2}{g}$.

Example 23: The flow rate of a liquid through a nozzle of diameter 50 mm is 18.62 L/s. The nozzle is situated at a distance of 1.5 m from the ground and is inclined at an angle of 30° to the horizontal. The jet of liquid from the nozzle strikes the ground at a horizontal distance of

(A) 1.04 m (B) 1.5 m
(C) 10 m (D) 5 m



Solution:

Area of the nozzle, $A = \frac{\pi}{4} \times \left(\frac{50}{1000}\right)^2 m^2$

Flow rate $Q = 0.01862$ m^3/s

$\therefore \quad u = \frac{Q}{A} = 9.483 \frac{m}{s}$

Let the horizontal distance at which the jet strikes the ground be x.

If the co-ordinates of point A is set to (0, 0). Then the co-ordinates of point B will be $(x, -1.5)$

The equation of the jet is

$$y = x \tan \theta - \frac{gx^2 \sec^2 \theta}{2u^2}$$

i.e., $-1.5 = x \times \tan 30° - \frac{9.81 \times x^2 \times \sec^2 30°}{2 \times 9.483^2}$

$$= 0.07273\, x^2 - 0.5774x - 1.5 = 0$$

$\therefore \quad$ x = 10 m.

Vortex Flow

It is defined as the fluid flow along a curved path or the flow of a mass of fluid rotating about an axis

Plane Circular Vortex Flows

These are flows with streamlines that are concentric circles. Considering a polar coordinate system, the velocity field of such a flow is defined as

$$\boxed{V_\theta \neq 0 \text{ and } V_r = 0}$$

Where V_θ and V_r are the tangential and radial components of the velocity respectively. For such flows V_θ is a function of r only and not θ

Vortex flows can be mainly classified into two types:

1. Forced vortex flow
2. Free vortex flow

It is to be noted that a plane circular free vortex flow or a plane circular forced vortex flow will be simply referred to as respectively a free vertex flow or a forced vertex flow. Hence all the characteristics of a plane circular vortex flow will be attributed sometimes to a free or forced vortex flow.

Forced Vortex Flow

It is defined as the vortex flow in which some external torque is employed to rotate the fluid mass. The tangential velocity of a fluid particle is given by

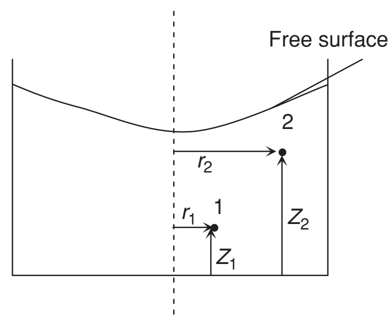
$$V_\theta = r\,\omega$$

Where r is the distance of the fluid particle from the axis of rotation and ω is the angular velocity of the fluid particle. In a forced vortex flow all fluid particles rotate with the same angular velocity like a solid body and hence this flow is termed as a *solid body rotation*. A forced vortex is also called as a *flywheel vortex* or *rotational vortex*.

A forced vortex flow is a rotational flow (vorticity = 2ω). To maintain a forced vortex flow, mechanical energy has to be spent from outside and the total mechanical energy per unit mass is not constant. In such a flow, shear stress is zero at all points in the flow field since there is no relative motion. A forced vortex flow can be generated by rotating a vessel containing a fluid so that the angular velocity is the same at all points. That is,

1. Rotation of a liquid in a centrifugal pump.
2. Rotation of a gas in a centrifugal compressor
3. Rotation of water through the turbines runner

Consider two points 1 and 2 in a fluid having a forced vortex flow as shown in the following figure.



For the two points, the following equation is applicable.

$$\boxed{p_2 - p_1 = \frac{\rho}{2}(V_2^2 - V_1^2) - \rho g(Z_2 - Z_1)} \quad (1)$$

Where $\boxed{V_1 = r_1\omega}$ and $\boxed{V_2 = r_2\omega}$

If the two points lie on the free surface of the liquid then $p_1 = p_2$ and equation (1) becomes

$$\boxed{Z_2 - Z_1 = \frac{1}{2g}(V_2^2 - V_1^2}$$

If additionally to the above case, point 1 lies on the axis of rotation.

(i.e., $v_1 = r_1 \times w = 0 \times w = 0$), then

$$Z_2 - Z_1 = \frac{V_2^2}{2g}$$

or $$\boxed{Z = \frac{\omega^2 r_2^2}{2g}} \quad (2)$$

Where $Z = Z_2 - Z_1$

Since Z varies with the square of r, equation (1) is an equation of a parabola consequently the free surface of the liquid is a paraboloid.

Cylindrical Forced Vortex

It can be generated by rotating a cylindrical vessel containing a fluid. At any horizontal plane, the tangential velocity, $V_\theta = r\,\omega$

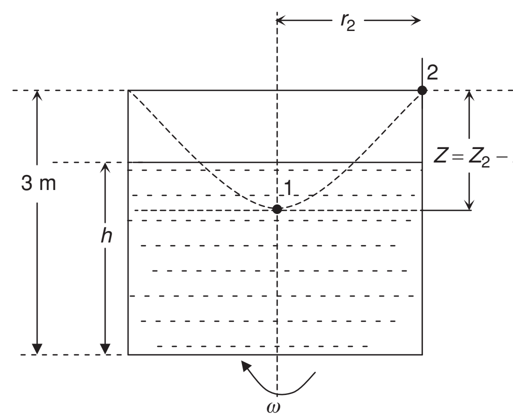
Spiral Forced Vortex

The superimposition of a purely radial flow with a plane circular forced vortex results in a spiral forced vertex flow.

Example 24: A cylindrical tank of diameter 1 m and height 3 m, which is open at the top, is filled with a liquid up to a certain depth. When the cylinder is rotated at 100 rpm. The liquid level is raised to be even with the brim. The depth of the liquid in the tank is:

(A) 1.39 m (B) 2.3 m
(C) 3 m (D) 0.5 m

Solution:



Let h be the depth of the liquid in the tank.

The points 1 and 2 are chosen as shown in the above figure.

Hence, $Z_2 - Z_1 = \dfrac{w^2 r_2^2}{2g}$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 100}{60}$$

$$r_2 = 0.5 \text{ m}$$

$$\therefore \quad Z = Z_2 - Z_1 = \frac{\left(\dfrac{2\pi \times 100}{60}\right)^2 \times (0.5)^2}{2 \times 9.81}$$

$$= 1.3973 \text{ m}$$

When the vessel is rotated a paraboloid is formed. Volume of air before rotation = volume of air after rotation

$$\Rightarrow \quad \pi r_2^2 \times 3 - \pi r_2^2 \times h$$

$$= \frac{1}{2} \times \pi \times r_2^2 \times Z$$

Or $h = 3 - \frac{Z}{2} = 3 - \frac{1.3973}{2}$

$= 2.3$ m.

Pressure Forces on the Top and Bottom of a Cylinder

Consider a cylinder of radius R and height H which is completely filled with a liquid. The cylinder is rotated about its vertical axis at a speed of ω radians/sec.

Total pressure on the top of the cylinder,

$$F_T = \frac{\rho\omega^2}{4} \times \pi R^4$$

Total pressure force on the bottom of the cylinder (F_B) = weight of the liquid in the cylinder + total pressure force on the top of the cylinder : (F_T)

That is, $\boxed{F_B = \rho g \pi R^2 H + F_T}$

Free Vortex Flow

A vortex flow in which no external torque is required to rotate the fluid mass is called a free vortex flow. The velocity field in a *free vortex flow* is described by

$$\boxed{V_\theta = \frac{c}{r}}$$

Where c (called as the *strength of the vortex*) is a constant in the entire flow field. The above equation is derived from the fact that in a free vortex flow, as the external torque is zero, the time rate of change of angular momentum, i.e., the moment of momentum is zero.

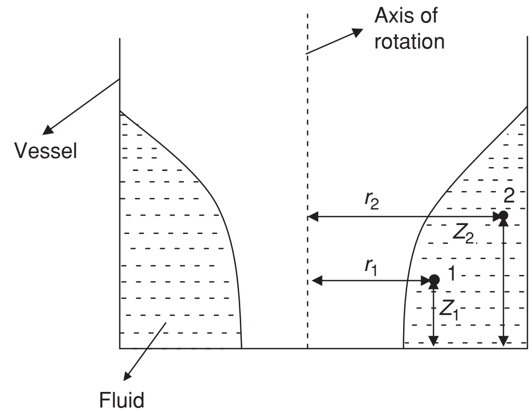
A free vortex is also called as a *potential vortex* or *irrotational vortex*.

A free vortex flow is irrotational (zero vorticity). In this type of flow, the total mechanical energy per unit mass is constant in the entire flow field with no addition or destruction of mechanical energy in the flow field. In a free vortex flow, the fluid rotates due to either some previously imparted rotation or some internal action. That is,

1. Whirlpool in a river
2. Flow around a circular bend
3. Flow of liquid through an outlet provided At the bottom of a shallow vessel. (e.g. wash tub etc.)

It is to be noted that Bernoulli's equation is applicable in the case of a free vortex flow.

Consider two points 1 and 2 in the fluid having radii r_1 and r_2 respectively from the axis of rotation and with heights Z_1 and Z_2 respectively from the bottom of the vessel as shown in the figure.



Since Bernoulli's equation is applicable for free vortex flow, we can write:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + Z_2$$

Example 25: In a free cylindrical vortex flow of air (density = 1.2 kg/m^3), point A is located at a radius of 350 mm from the axis of rotation and at a height of 200 mm from the vessel bottom. Point B is however located at a radius of 500 mm and height 300 mm. If the velocity at point A is 20 m/s then the pressure difference between the points A and B is:

(A) 121.22 P_a (B) 10.29 P_a
(C) 12.35 P_a (D) 25.62 P_a

Solution:

Given $r_A = 0.35$ m
$Z_A = 0.2$ m
$V_A = 20$ m/s
$V_B = 0.5$ m
$Z_B = 0.3$ m

For a free vortex flow

Vr = constant

$\therefore \quad V_A r_A = V_B r_B$.

Or $V_B = \frac{20 \times 0.35}{0.5} = 14 \frac{m}{s}$

From Bernoulli's equation we have:

$$\frac{p_A}{\rho g} + \frac{V_A^2}{2g} + Z_A = \frac{p_B}{\rho g} + \frac{V_B^2}{2g} + Z_B$$

$$\frac{p_A}{\rho g} + \frac{20^2}{2 \times 9.61} + 0.2 = \frac{p_B}{\rho g} + \frac{14^2}{2 \times 9.81} + 0.3$$

$$\frac{p_B}{\rho g} - \frac{p_A}{\rho g} = 10.2975$$

Or $p_B - p_A = 10.2976 \times 9.81 \times 1.2$

$= 121.22$ *Pa*.

Cylindrical Free Vortex

A cylindrical free vortex in a cylindrical coordinate system has the Z axis directly vertically upwards where at each horizontal plane, there exists a planar free vortex motion with tangential velocity given by

$$V_\theta = \frac{C}{r}$$

Spiral Free Vortex

For a plane spiral free vortex two dimensional flow, the tangential and radial velocity components at any point with respect to a polar coordinate system is inversely proportional to the radial coordinate at that point.

∴ In the flow field,

$$\boxed{\begin{aligned} V_\theta &= \frac{C_1}{r} \\ V_r &= \frac{C_2}{r} \end{aligned}}$$

Such a flow can be said to be the superimposition of a radial flow described by equation $Vr = \frac{C_2}{r}$ with a free vortex flow.

If α is the angle between the velocity vector V and the tangential component of the velocity vector V_θ at any point then:

$$\boxed{\tan\alpha = \frac{V_r}{V_\theta} = \frac{C_2}{C_1}}$$

$$\text{Now, } \frac{V_r}{V\theta} = \frac{\frac{dr}{dt}}{rw} = \frac{\frac{dr}{dt}}{r\frac{d\theta}{dt}} = \frac{dr}{rd\theta}$$

$$\boxed{\therefore \frac{dr}{rd\theta} = \tan\alpha}$$

This is the equation of the streamline in this flow. Integrating the above equation, it can be shown that:

$$r = r_0 e^{\theta x \tan\alpha} = r_0 e^{\theta \frac{c_2}{c_1}}$$

Where r_0 is the radius at $\theta = 0$. The above equation shows that the patterns of streamlines are logarithmic – spiral.

Example 26: An object, caught in a whirlpool, at a given instant is at a distance of 100 cm from the centre of the whirlpool.

The two dimensional velocity field of the whirlpool can be described by the tangential and radial components of the velocity such as V_θ and V_r respectively, where $V_\theta = -3Vr$. If after a certain period of time, the object is found to be at a distance of 4.32 m from the centre of the whirlpool, then the number of revolutions completed by the object from its original position is:

(A) 3 (B) 1.5 (C) 4.5 (D) 1

Solution:

The motion in a whirlpool can be simulated as a free vortex flow. Since $V_\theta \neq 0$ and $Vr \neq 0$ (for some finite radial location) the flow can be considered to a spiral free vortex flow.

Given $r_0 = 100$ m

$$r = 4.32 \text{ m}$$

Now for a spiral free vortex flow,

$$r = r_0 e^{\theta c_2/c_1}$$

$$= r_0 e^{\theta Vr/V_\theta}$$

i.e., $4.32 = 100 \times e^{\theta \times \left(\frac{1}{-3}\right)}$

Or $\theta = 9.425744$ radians

Now, 1 revolution $= 2\pi$ radians

∴ No. of revolution completed by the object

$$= \frac{9.425744}{2\pi}$$

$$= 1.5.$$

Exercises

Practice Problems I

Direction for questions 1 to 20: Select the correct alternative from the given choices.

1. A two-dimensional velocity field is given by $\vec{V} = xt\hat{i} - my\hat{j}$, where x and y are in meters, t is in seconds, $\vec{v}$ is in m/s and m is a constant. If at $t = 2$ secs, $x = 2$ m and $y = 1$ m, the fluid speed is 5 m/s, then the convective acceleration along the y-direction at the same values of t, x and y is:

(A) -9 m/s^2 (B) 9 m/s^2
(C) 8 m/s^2 (D) 0 m/s^2

2. A flow field is represented by the velocity field $\vec{V} = -xt\hat{i} + (6 - y)t\hat{j}$, where t is time. The equation of a streamline passing through (1, 3) is:

(A) $y + 3x = 6$
(B) $y = 3x$
(C) $6x - xy = 3$
(D) Not possible to determine

3. For a three-dimensional flow, the velocity components are given as:

$u = ax + dy + cz$, $v = dx + ey + hz$ and $w = -cx + hy + 1z$. If for this flow, the vorticity vector is $(c + 4x + 6y)\hat{i}$, then the value of c at the point (1, 1, 1) in the flow field is:

(A) −10 (B) zero
(C) 5 (D) 10

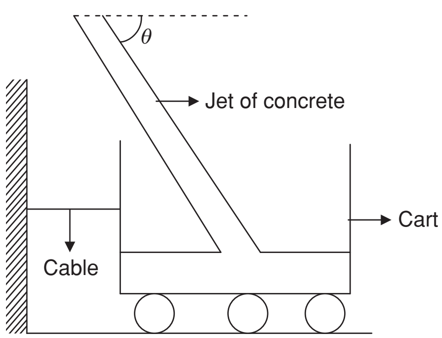
4. A horizontal jet of liquid (density = 800 kg/m^3) strikes a flat plate kept in the vertical position, with a velocity of 10m/s. The liquid then splashes off the sides in the vertical plane. A horizontal force F is applied to hold the plate stationary. If the volumetric flow rate of the liquid jet is 100 litre/sec, then the valve of F (in Newtons) is:

(A) 800 (B) 8000
(C) −8000 (D) −800

5. A 0.1 m diameter jet of concrete flows steadily at a velocity of 2 m/s into a cart which is attached to a wall by a cable as shown in the figure below.

The density of the concrete is 2200 kg/m^3. If at instant shown in the figure, the cart and the concrete in it together weighs 3560 Newtons and the reaction force exerted by the ground on the cart is 3620 Newtons, then the tension in the cable is:

(A) 48.92 N (B) 34.31 N
(C) 11.65 N (D) 20.53 N

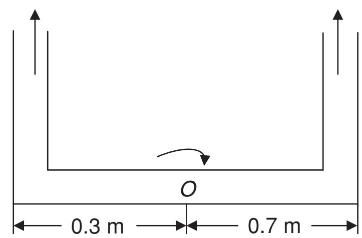


6. An incompressible fluid flows steadily through a convergent horizontal nozzle of length 100 m. where the velocities of the inlet and outlet are 10 m/s and 20 m/s respectively. If along the length of the nozzle, a one dimensional flow and a linear velocity distribution are assumed, then the fluid acceleration at a distance of 25 m from the inlet is:

(A) 0.1 m/s^2 (B) 25 m/s^2
(C) 1.25 m/s^2 (D) 12.5 m/s^2

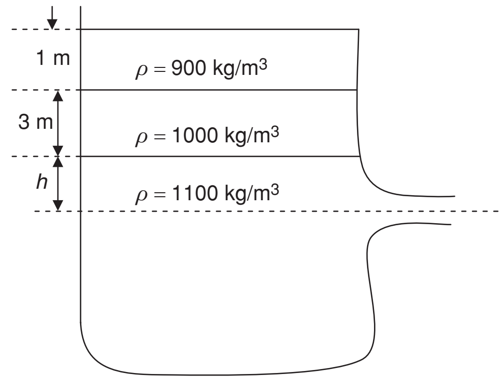
7. The nozzles of the sprinkler shown in the following figure have diameter of 7 mm. The total discharge of water from the nozzle is 4×10^{-4} m^3/s. If the friction in the sprinkler is neglected, then the torque (in Nm) required to hold the sprinkler stationary is:

(A) 1.663 (B) 4.157
(C) 1.039 (D) 0.416



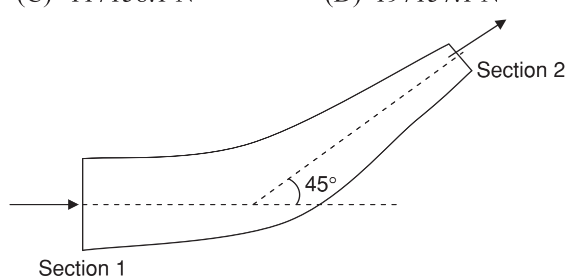
8. A large tank with nozzle attached contains three immiscible, inviscid liquids as shown in the following figure. If the changes in the heights of the liquids in the tank can be assumed to be negligible and that the instantaneous discharge velocity is 12.95 m/s, then the height h in meters is equal to:

(A) 5 (B) 1
(C) 4 (D) 9



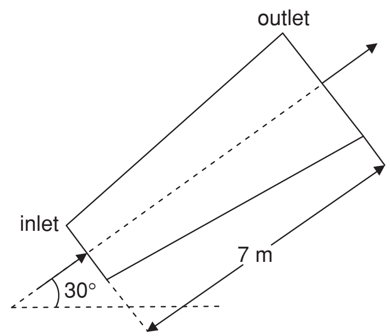
9. A duct in a horizontal plane, with a 45^0 bend as shown in the figure below, has a cross-sectional area of 2 m^2 at section 1 gradually reduced to 1.5 m^2 at section 2. The velocity of flow of the liquid (density = 950 kg/m^3) of section 1 is 15 m/s whereas the pressure at the section 1 is 90 kN/m^2. The horizontal component of the force required to hold the duct in position is:

(A) 258642.9 N (B) 459827.4 N
(C) 117158.1 N (D) 197157.1 N



10. A 7 m long pipe is inclined at an angle 30° with the horizontal as shown in the following figure. The diameters of the inlet and outlet sections are 150 mm and 300 mm respectively. The pipe is uniformly tapering. If the velocity of the liquid flowing in the pipe 0.5 m/s at the outlet section and if the differences in pressure between the inlet and outlet is 30837 N/m^2, then the density of the liquid is:

(A) 462 kg/m^3 (B) 950 kg/m^3
(C) 535 kg/m^3 (D) 602 kg/m^3



11. A closed tank is partly filled with water where air is present above the water surface. A 5 cm diameter pipe connected to the bottom of the tank discharges to an elevation of 3 m above the present level of water in the tank. If frictional losses are assumed to be absent and a discharge of 30 litre/s to be achieved, then the air in the tank is to be pressurized to a gauge pressure of:
(A) 146.2 kN/m^2 (B) 87.34 kN/m^2
(C) 116.77 kN/m^2 (D) 119.77 kN/m^2

12. A circular pipe carrying oil with specific gravity of 0.8 increases in diameter from 150 mm at section A to 450 mm at section B. The section A is 3 meters lower than section B, and the pressures at sections A and B are 50 kPa and 20 kPa respectively. If the discharge is 100 litre /sec, then:
(A) Flow is from *A* to *B* and head loss is 5.435 m
(B) Flow is from *B* to *A* and head loss is 5.435 m
(C) Flow is from *B* to *A* one head loss is 2.435 m
(D) Flow is from *A* to *B* and head loss is 2.435 m

13. If the velocity potential function for a two-dimensional flow field is given by $\phi = 10\,xy$, then the discharge between the streamlines passing through the points (2, 3) and (1, 2) is:
(A) 10 (B) 80 (C) –40 (D) 40

14. A two − dimensional flow is described by the stream function $\psi = xy$. The point in the flow field at which the velocity vector will have a magnitude of 10 units and will make an angle of 120° with the *x*-axis is:
(A) $(5\sqrt{3}, 5)$ (B) (0, 0)
(C) $(5\sqrt{3}, 5\sqrt{3})$ (D) $(5, 5\sqrt{3})$

15. A venturimeter of throat diameter 150 mm is used to measure the velocity of water flowing in a horizontal pipe of diameter 350 mm. The difference of pressures at the inlet and the throat of the venturimeter is 177 kPa. If 4% of the head is lost between the inlet and the throat, then the flow rate of water through the pipe is:
(A) 0.3382 m^3/s (B) 0.0012 m^3/s
(C) 0.3247 m^3/s (D) 0.2198 m^3/s

16. In a horizontal pipe of diameter 250 mm, water is flowing at a rate of 0.02 m^3/s through a 150 mm diameter orifice. If the coefficients C_c and C_v are 0.62 and 1.0 respectively, then the difference in pressures at the upstream section and the vena contracta section is:
(A) 1583.38 Pa (B) 1642.93 Pa
(C) 3412.84 Pa (D) 2242.36 Pa

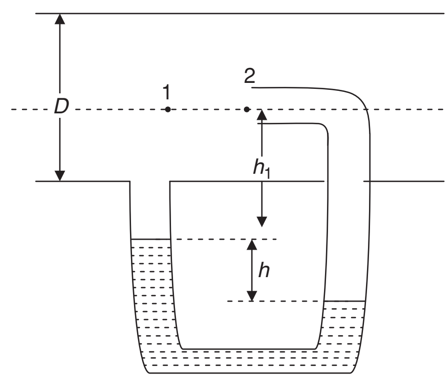
17. A liquid of density ρ is slowing in a horizontal pipe of constant diameter *D* as shown in the following figure. The manometer liquid has a density of ρ_m and the manometer reading is h. The volumetric flow rate of the liquid in the pipe is:

(A) $\frac{\pi}{4}D^2 \times \sqrt{2hg\rho_{\frac{m}{s}}}$

(B) $\frac{\pi}{4}D^2 \times \sqrt{2hg\frac{(\rho_m - \rho)}{\rho}}$

(C) $\frac{\pi}{4}D^2 \times \sqrt{2h_g\frac{\rho}{\rho_m}}$

(D) $\frac{\pi}{4}D^2 \times \sqrt{2h_g\frac{(\rho_m + \rho)}{\rho_m}}$



18. A nozzle of diameter 50 mm is inclined at an angle of 45^0 with the horizontal. The jet issuing from the nozzle strikes a point, on the ground, that is 2 m vertically beneath the nozzle and 5 m horizontally from it. If the velocity coefficient of the nozzle is 0.96, then the pressure head at the nozzle is:
(A) 1.936 m (B) 1.834 m
(C) 1.629 m (D) 2.104 m

19. A cylinder has a height 2 m and contains water upto a height of 1.5 m. When the cylinder is rotated about its vertical axis at 100 r.p.m, the actual depth becomes zero. The diameter of the cylinder is:
(A) 1.94 m (B) 1.2 m (C) 3.88 m (D) 0.6 m

20. In a tornado, the velocity and pressure of air (density = 1.2 kg/m^3) at a radius of 3 m from its axis are 100 m/s and 94.66 kPa. If the outer edge of the tornado is at a radius of 20 meters from its axis, then the pressure at the outer edge is:
(A) 88.795 kPa (B) 94.66 kPa
(C) 100.525 kPa (D) 101.325 kPa

Practice Problems 2

Direction for questions 1 to 30: Select the correct alternative from the given choices.

1. A two-dimensional velocity field is given by $\vec{v} = (x + y - m)\hat{i} + (5 + 2x - 3y)\hat{j}$, where x and y are in metres and m is a constant. If a stagnation point is found at $x = 2$ m, then the convective acceleration in the x – direction at $x = 2$ m and $y = 3$ m is
(A) 0 m/s^2 (B) 25 m/s^2
(C) 5 m/s^2 (D) –5 m/s^2

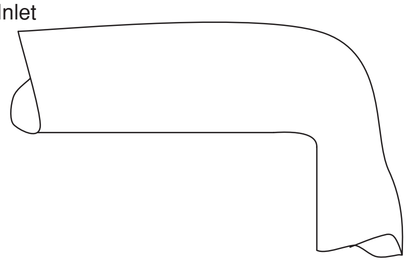
2. A two-dimensional velocity field is given by $\vec{v}$ = $t\,(m - 3)\ \hat{i}$ +2$(n - 4y + x)\,\hat{i}$, where t is in seconds and x and y are in meters. If the velocity field corresponds to a steady uniform flow, then the values of m and n, at $t = 1$ secs, $x = 2$ m and $y = 3$ m, are respectively:
(A) 3 and –10 (B) 0 and 10
(C) 3 and 10 (D) 0 and –10

3. If the velocity field for an irrotational flow is represented by $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$. Then which one of the following relationships need not necessarily be true?
(A) $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$ (B) $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$
(C) $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ (D) $\frac{\partial w}{\partial z} = \frac{\partial v}{\partial y}$

4. The equation of streamlines in a two-dimensional field is given by $yx^3 = $ c, where c is a constant. If the velocity of the flow field in the x-direction is given by $u = -2x^3$, then the velocity in the y-direction is:
(A) $v = x$
(B) $v = 6x^2y$
(C) $v = -3y$
(D) Not possible to determine

5. Fluid particle A is present at the point (2, 2) in a two-dimensional flow with the velocity field: $\vec{v} = xy^3\hat{i} + x^3y\hat{j}$, while fluid particle B is present at the point (2, 2) in another two-dimensional flow which velocity filed: $\vec{V} = 4y^2\hat{i} + 3x^2\hat{j}$ Which one of the following statements is ONLY correct?
(A) Fluid particles A and B are rotating.
(B) Fluid particles A and B are not rotating.
(C) Fluid particle B is not rotating.
(D) Fluid particle A is not rotating.

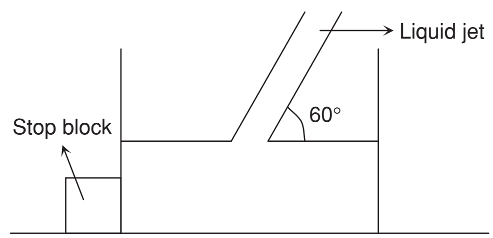
6. A liquid of density 800 kg/m^3 is flowing steadily through a 90° reducing elbow as shown in the following figure. A pressure:



gauge fitted at the inlet reads 200 KN/m^2 where the cross-sectional area is 0.05 m^2. The liquid flows into the atmosphere through the outlet, of cross-sectional area 0.01 m^2, at a velocity of 10 m/s. With respect to the horizontal, the force required to hold the elbow in place acts at an angle of:
(A) 4.5° (B) 7.25°
(C) 0.34° (D) 10.01°

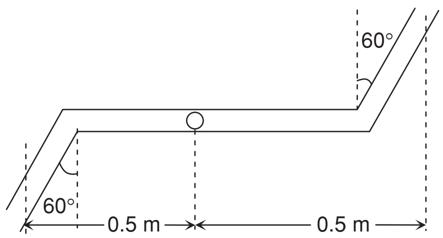
7. A 10 cm diameter horizontal jet of water having a velocity of 15 m/s impinges on a flat vertical plate and splashes at the sides in the vertical plane. If a horizontal force F is applied to hold the plate stationary then the force required to move the plate at a velocity of 10 m/s towards the water jet is:
(A) F (B) $\frac{3F}{5}$
(C) $\frac{5F}{3}$ (D) $\frac{F}{3}$

8. A jet of liquid (density = 900 kg/m^3), having a diameter of 0.2 m and speed 3 m/s, is steadily filling a tank as shown in the figure. The coefficient of friction between the tank and the ground is 0.227. If at the instant shown in the figure, a horizontal force of 100N is exerted on the stop block by the tank, and then the weight of the tank and its contents (neglecting the friction between the stop block and the ground) is:
(A) 780.656 N (B) 1001.033 N
(C) 340.352 N (D) 220.376 N



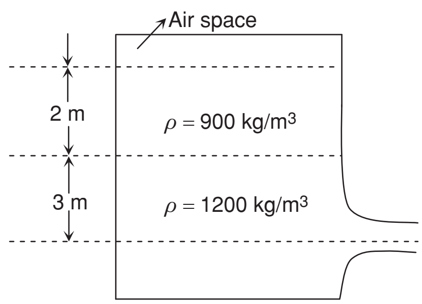
9. An incompressible fluid flows steadily through a convergent horizontal nozzle with a velocity of 2.5 m/s at the inlet. Assume a one-dimensional flow and a linear velocity distribution along the length of the nozzle. The outlet cross-sectional area is one-tenth the inlet cross-sectional area. If the difference between the fluid accelerations at distances of 30 m and 10 m from the inlet of the nozzle is 4.05 m/s^2, then the length of the nozzle is:
(A) 50 m (B) 70 m
(C) 11.8 m (D) 100.62 m

10. A sprinkler with equal arm lengths of 0.5 m, as shown in the following figure, discharges water at equal relative velocities through nozzles of equal diameters of 5 cm. The sprinkler freely rotates with no friction at a speed of 95.493 r.p.m. The torque (in Nm) required to hold the sprinkler stationary is:
(A) 98.175 (B) 49.087
(C) 61.235 (D) 22.602



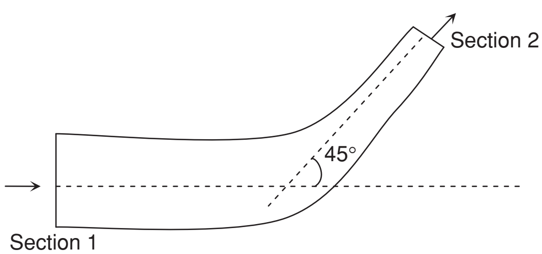
11. A large closed tank with a nozzle attached contains two immiscible inviscid liquids as shown in the following figure. The air space in the tank is pressurized to 2 atm. If the changes in the heights of the liquids in the tank are assumed to be negligible, then the instantaneous discharge velocity is:

(A) 9.9 m/s (B) 7.1 m/s
(C) 16.04 m/s (D) 9.63 m/s



12. A duct in a horizontal plane, with a 45° bend as shown in the figure below, has a cross-sectional area of 2 m^2 at section 1 gradually reduced to 1.5 m^2 at section 2. The velocity of flow of the liquid at section 1 is 10 m/s. The pressures at section 1 and 2 are 95 kN/m^2 and 52271.105 N/m^2 respectively. The vertical component of the force required to hold the duct in position is:

(A) 262808 N (B) 312456 N
(C) 101333 N (D) 200654 N

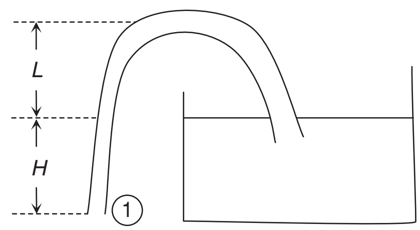


13. A liquid is flowing upwards a vertical pipe which uniformly tapers from an inlet section of diameter 600 mm to an inlet section of diameter 400 mm. The manometer fitted to the pipe, reads the pressure difference between the inlet and outlet to be 8 m in terms of the head of the liquid flowing in the pipe. If the outlet section lies above the inlet section by a height of 2 m, then the volumetric rate of flow of the liquid in the pipe is:

(A) 3.424 m^3/s (B) 0.676 m^3/s
(C) 1.522 m^3/s (D) 5.383 m^3/s

14. From a large tank of water, water is drawn steadily using a siphon as shown in the figure below. If the point 1 denotes a point at the siphon discharge exit, then the lowest pressure occurring in the siphon is given by:

(A) $P_{atm} + \rho g(L + H)$
(B) $P_{atm} - \rho g(L + H)$
(C) $P_{atm} - \rho g(L - H)$
(D) $P_{atm} + \rho g(L - H)$



15. For a flow to which the Bernoulli's equation can be applied, which one of the following quantities is definitely constant along a streamline?

(A) Sum of static and dynamic pressures
(B) Sum of dynamic and hydrostatic pressures
(C) Sum of hydrostatic and static pressures
(D) Sum of stagnation and hydrostatic pressures.

Direction for questions 16 and 17: The velocity profile for flow in a circular pipe is given as $V = V_{max}\left(\frac{r}{R}\right)^{\frac{1}{7}}$ where V is the local velocity of flow at a distance r from the pipe wall, V_{max} is the maximum velocity at the center line of the pipe and R is the pipe radius.

16. The average velocity of the flow is given by:

(A) V_{max}

(B) $\frac{49}{60}V_{max}$

(C) $\frac{V_{max}}{2}$

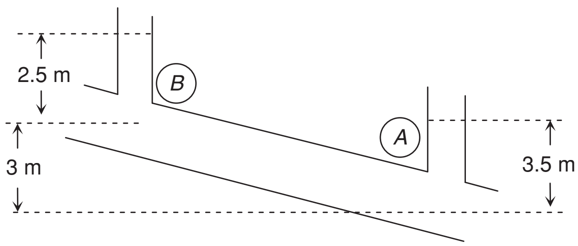
(D) $\frac{32}{49}V_{max}$

17. The value of the momentum flux correction factor is:

(A) 1.01 (B) 1.02
(C) 1 (D) 1.04

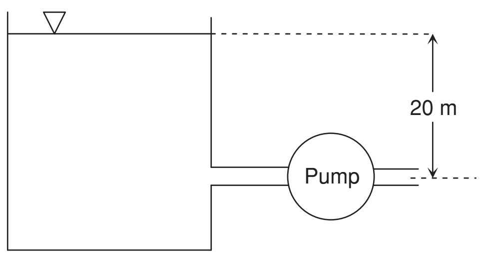
18. An incompressible liquid flows steadily along a circular pipe of constant diameter 600 mm. If the length between the sections A and B is 6 m, then between the sections, the:

(A) Flow is from A to B and head loss is 1 m.
(B) Flow is from B to A and head loss is 2 m.
(C) Flow is from B to A and head loss is 1 m.
(D) Flow is from A to B and head loss is 2 m.



19. Water is pumped from a large tank as shown in the figure below. The head loss is known to be equal to $\frac{6V^2}{2g}$, where V is the discharge velocity, and the pump head is equal to $20-5Q^2$, where Q is the discharge. If the discharge pipe has a diameter 100 mm and if only SI units are considered, then the discharge Q is equal to:

(A) 0.0914 m^3/s (B) 0.0124 m^3/s
(B) 0.0831 m^3/s (D) 0.0064 m^3/s



20. A two dimensional flow with the velocity field given by: $\vec{V} = (x+6_y)\hat{i} + (7+y)\hat{j}$ is

(A) Incompressible and rotational
(B) Compressible and rotational
(C) Incompressible and irrotational
(D) Compressible and irrotational

Direction for questions 21 and 22: In a two dimensional flow field, the point (5, 8) has been marked as point P. The velocity potential function for this flow is given by $\phi = 72xy - 48x$.

21. The respective velocity components in the x and y diameters are

(A) 48 – 72y and 72x
(B) 48 – 72y and –72x
(C) 72y – 48 and 72x
(D) 72y – 48 and –72x

22. The value in units of the stream function at point P is:

(A) 2820 (B) 1020
(C) –1020 (D) –2820

23. Water is flowing with a velocity of 5 m/s through a 0.15 meter internal diameter horizontal pipe. The velocity of the water flowing has been determined using a venturimeter, of throat diameter 0.1 m, fitted into the pipeline. The differential manometer fitted into the venturimeter shows a reading of 1.2 meter. If the venturimeter coefficient is 0.96, then the density if the manometric liquid is:

(A) 5313.73 kg/m^3 (B) 13600 kg/m^3
(C) 7000 kg/m^3 (D) 4329.67 kg/m^3

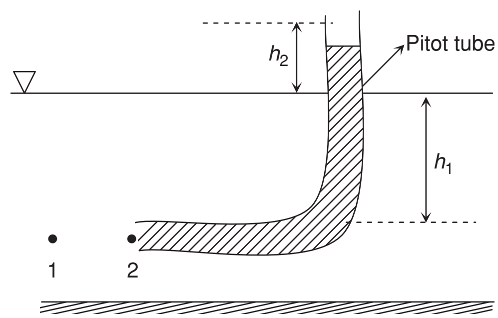
24. An orificemeter having an orifice of diameter d is present in a pipe of diameter D. Generally, the coefficient of discharge of the orificemeter:

(A) Is independent of d/D and Reynolds number of flow.
(B) Depends on d/D and Reynolds number of flow
(C) Depends only on d/D
(D) Depends only on Reynolds number of flow

25. In an open stream of flowing liquid, a pitot tube is immersed as shown in the figure below. Point 2 is a stagnation point while point 1 is located upstream of point 2. The velocity at point 1 is:

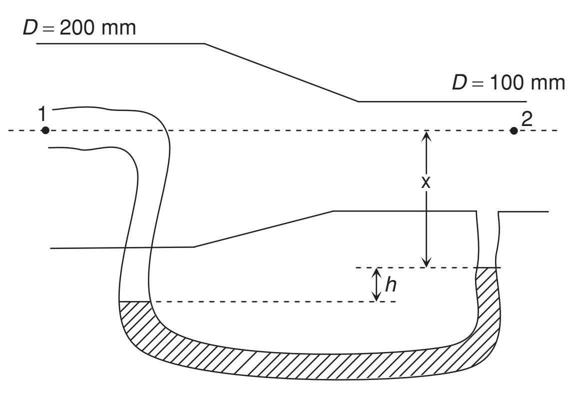
(A) $\sqrt{2gh_1}$ (B) $\sqrt{2g(h_1+h_2)}$

(C) $\sqrt{2g\left(\frac{h_1^2}{h_2}\right)}$ (D) $\sqrt{2gh_2}$



26. In a horizontal pipe converging from a diameter of 200 mm to 100 mm, air (density = 1.2 kg/m^3) is flowing at a volumetric flow rate of 1004 L/S as shown in the following figure. If the specific gravity of the manometric liquid is 0.85, then the value of the manometer reading h is

(A) 0.1 m (B) 0.0736 m
(C) 1.177 m (D) 0.00625 m



27. Water is flowing out from a tank, through an orifice at the side of the tank, as a jet. The jet strikes the ground at a horizontal distance of x metres from the tank.

The height of the water in the tank is 1.5 m and the orifice is situated at a distance of h metres from the free liquid surface. The value of x will be maximum is

(A) 1.33 m (B) 1.5 m
(C) 0.66 m (D) 0.75 m

28. A cylindrical vessel of diameter 0.2 m and 0.5 m height is filled with a liquid completely upto the top. The volume of the liquid that will be left in the vessel after it is rotated with a speed of 250 r.p.m. is

(A) 0.0157 m^3 (B) 0.00548 m^3
(C) 0 m^3 (D) 0.01022 m^3

29. A cylindrical vessel is closed at the top and the bottom and has a diameter of 0.4 m and height 0.5 m. The vessel is completely filled with a liquid. When the vessel is rotated about its vertical axis with an angular speed of ωrad/s, the total pressure exerted by the liquid on the bottom is twice that exerted by the liquid on the top the vessel. The value of ω is

(A) 22.14 rad/s (B) 14 rad/s
(C) 44.29 rad/s (D) 28 rad/s

30. In a free vortex flow of a fluid, at a radial location of r = 1 m, the tangential velocity is 2 m/s. At two radial locations in the same horizontal plane r_1 and r_2 ($r_2 > r_1$) in the free vortex flow, the pressure difference is determined to be $P_1 - P_2$. If for a forced vortex flow of the same fluid, having an angular velocity of 5 rad/s for the same radial locations, the pressure difference is the same, then the value of r_1 when $r_2 = 2$ m is

(A) 0.4 m (B) 0.1 m
(C) 0.5 m (D) 0.2 m

Previous Years' Questions

1. A fluid flow is represented by the velocity field $\bar{V} = ax\bar{i} + ay\bar{j}$, where a is a constant. The equation of streamline passing through a point (1, 2) is **[2004]**

(A) $x - 2y = 0$ (B) $2x + y = 0$
(C) $2x - y = 0$ (D) $x + 2y = 0$

2. For a fluid flow through a divergent pipe of length L having inlet and outlet radii and R_1 and R_2 respectively and a constant flow rate of Q, assuming the velocity to be axial and uniform at any cross section, the acceleration at the exit is **[2004]**

(A) $\dfrac{2Q(R_1 - R_2)}{\pi L R_2^3}$ (B) $\dfrac{2Q^2(R_1 - R_2)}{\pi L R_2^3}$

(C) $\dfrac{2Q^2(R_1 - R_2)}{\pi^2 L R_2^5}$ (D) $\dfrac{2Q^2(R_2 - R_1)}{\pi^2 L R_2^5}$

3. A closed cylinder having a radius R and height H is filled with oil of densityr. If the cylinder is rotated about its axis at an angular velocity of ω, the thrust at the bottom of the cylinder is: **[2004]**

(A) $pR^2\rho gH$

(B) $\pi R^2 \dfrac{\rho\varpi^2 R^2}{4}$

(C) $pR^2 (\rho w^2R^2 + \rho gH)$

(D) $\pi R^2 \left(\dfrac{\rho\omega^2 R^2}{4} + \rho gH\right)$

4. The velocity components in the x and y directions of a two dimensional potential flow are u and v, respectively, then $\dfrac{\partial u}{\partial x}$ is equal to: **[2005]**

(A) $\dfrac{\partial v}{\partial x}$ (B) $-\dfrac{\partial v}{\partial x}$

(C) $\dfrac{\partial v}{\partial y}$ (D) $-\dfrac{\partial v}{\partial y}$

5. A venturimeter of 20 mm throat diameter is used to measure the velocity of water in a horizontal pipe of 40 mm diameter. If the pressure difference between the pipe and throat sections is found to be 30 kPa then, neglecting frictional losses, the flow velocity is: **[2005]**

(A) 0.2 m/s (B) 1.0 m/s
(C) 1.4 m/s (D) 2.0 m/s

6. A leaf is caught in a whiripool. At a given instant, the leaf is at a distance of 120 m from the centre of the whirlpool. The whirlpool can be described by the following velocity distribution: $V_r = -\left(\dfrac{60\times 10^3}{2\pi r}\right)\dfrac{m}{s}$ and $V_\theta = \dfrac{300\times 10^3}{2\pi r}\dfrac{m}{s}$, where r (in meters) is the distance from the centre of the whirlpool. What will be the distance of the leaf from the centre when it has moved through half a revolution? **[2005]**

(A) 48 m (B) 64 m
(C) 120 m (D) 142 m

7. In a two-dimensional velocity field with velocities u and v along the x and y directions respectively, the convective an acceleration along the x-direction is given by: **[2006]**

(A) $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}$ (B) $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial y}$

(C) $u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial y}$ (D) $v\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y}$

8. A two-dimensional flow field has velocities along the x and y directions given by $u = x^2t$ and $v = -2xyt$ respectively, where t is time. The equation of streamlines is: **[2006]**

(A) x^2y = constant
(B) xy^2 = constant
(C) xy = constant
(D) Not possible to determine

9. In a steady flow through a nozzle, the flow velocity on the nozzle axis is given by $v = u_o (1 + 3x/L)i$, where x is the distance along the axis of the nozzle from its inlet plane and L is the length of the nozzle. The time required for a fluid particle on the axis to travel from the inlet to the exit plane of the nozzle is: **[2007]**

(A) $\frac{L}{u_o}$ (B) $\frac{L}{3u_o} in4$

(C) $\frac{L}{4u_o}$ (D) $\frac{L}{2.5u_o}$

10. Which combination of the following statements about steady incompressible forced vortex flow is correct?

P: Shear stress is zero at all points in the flow.

Q: Vorticity is zero at all points in the flow.

R: Velocity is directly proportional to the radius from the centre of the vortex.

S: Total mechanical energy per unit mass is constant in the entire flow field.

Select the correct answer using the codes given below. **[2007]**

(A) P and Q (B) R and S

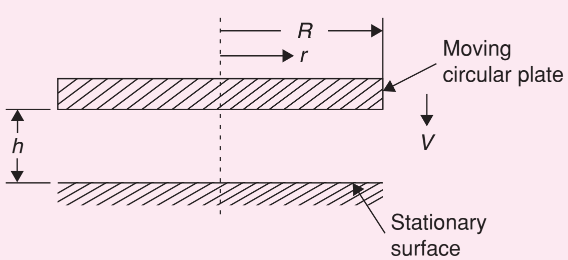
(C) P and R (D) P and S

11. For the continuity equation given by $\vec{\nabla} \cdot \vec{V} = 0$ to be valid, where $\vec{V}$ is the velocity vector, which one of the following is a necessary condition? **[2008]**

(A) Steady flow (B) irrotational flow

(C) In viscid flow (D) incompressible flow

Direction for questions 12 and 13: The gap between a moving circular plate and a stationary surface is being continuously reduced, as the circular plate comes down at a uniform speed V towards the stationary bottom surface, as shown in the figure. In the process, the fluid contained between the two plates flows out radially. The fluid is assumed to be incompressible and inviscid.



12. The radial velocity V_r at any radius r, when the gap width is h, is: **[2008]**

(A) $v_r = \frac{Vr}{2h}$ (B) $v_r = \frac{Vr}{h}$

(C) $v_r = \frac{2Vh}{r}$ (D) $v_r = \frac{Vh}{r}$

13. The radial component of the fluid acceleration at $r = R$ is: **[2008]**

(A) $\frac{3V^2R}{4h^2}$ (B) $\frac{V^2R}{4h^2}$

(C) $\frac{V^2R}{2h^2}$ (D) $\frac{V^2h}{2R^2}$

14. Consider steady, incompressible and irrotational flow through a reducer in a horizontal pipe where the diameter is reduced from 20 cm to 10 cm. The pressure in the 20 cm pipe just upstream of the reducer is 150 kPa. The fluid has a vapour pressure of 50 kPa and a specific weight of 5 kN/m^3. Neglecting frictional effects, the maximum discharge (in m^3/s) that can pass through the reducer without causing cavitation is: **[2009]**

(A) 0.05 (B) 0.16

(C) 0.27 (D) 0.38

15. You are asked to evaluate assorted fluid flows for their suitability in a given laboratory application. The following three flow choices, expressed in terms of the two-dimensional velocity fields in the xy-plane, are made available.

P. $u = 2y, v = -3x$

Q. $u = 3xy, v = 0$

R. $u = -2x, v = 2y$

Which flow(s) should be recommended when the application requires the flow to be incompressible and irrotational? **[2009]**

(A) P and R (B) Q

(C) Q and R (D) R

16. Velocity vector of a flow field is given as $\vec{V} = 2xy\hat{i} - x^2z\hat{j}$. The vorticity vector at (1, 1, and 1) is: **[2010]**

(A) $4\hat{i} - \hat{j}$ (B) $4\hat{i} - \hat{k}$

(C) $\hat{i} - 4\hat{j}$ (D) $\hat{i} - 4\hat{k}$

17. A streamline and an equipotential line in a flow field. **[2011]**

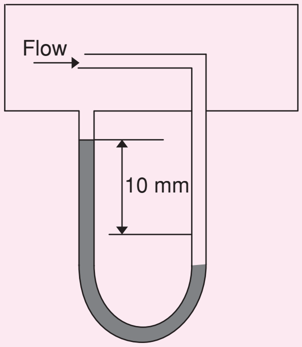
(A) Are parallel to each other

(B) Are perpendicular to each other

(C) Intersect at an acute angle

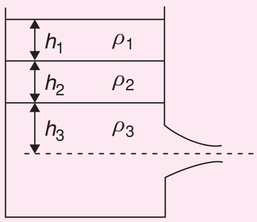
(D) Are identical

18. Figure shows the schematic for the measurement of velocity of air (density = 1.2 kg/m^3) through a constant area duct using a pitot tube and a water-tube manometer. The differential head of water (density = 1000 kg/m^3) in the two columns of the manometer is 10 mm. Take acceleration due to gravity as 9.8 m/s^2. The velocity of air in m/s is **[2011]**



(A) 6.4 (B) 9.0
(C) 12.8 (D) 25.6

19. A large tank with a nozzle attached contains three immiscible, inviscid fluids as shown. Assuming that the changes in h_1, h_2 and h_3 are negligible, the instantaneous discharge velocity is: **[2012]**



(A) $\sqrt{2gh_3\left(1+\frac{\rho_1}{\rho_3}\frac{h_1}{h_3}+\frac{\rho_2}{\rho_3}\frac{h_2}{h_3}\right)}$

(B) $\sqrt{2g(h_1+h_2+h_3)}$

(C) $\sqrt{2g\left(\frac{\rho_1 h_1+\rho_2 h_2+\rho_3 h_3}{\rho_1+\rho_2+\rho_3}\right)}$

(D) $\sqrt{2g\left(\frac{\rho_1 h_2 h_3+\rho_2 h_3 h_1+\rho_3 h_1 h_2}{\rho_1 h_1+\rho_2 h_2+\rho_3 h_3}\right)}$

20. Water is coming out from a tap and falls vertically downwards. At the tap opening, the stream diameter is 20 mm with uniform velocity of 2 m/s. Acceleration due to gravity is 9.81 m/s^2. Assuming steady, inviscid flow, constant atmospheric pressure everywhere and neglecting curvature and surface tension effects, the diameter in mm of stream 0.5 m below the tap is approximately **[2013]**

(A) 10 (B) 15
(C) 20 (D) 25

21. For an incompressible flow field, $\vec{V}$,which one of the following conditions must be satisfied? **[2014]**

(A) $\nabla\cdot\vec{V}=0$ (B) $\nabla\times\vec{V}=0$

(C) $(\vec{V}\cdot\nabla)\vec{V}=0$ (D) $\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}=0$

22. Consider the following statements regarding streamline (s):

(i) It is a continuous line such that the tangent at any point on it shows the velocity vector at that point
(ii) There is no flow across streamlines
(iii) $\frac{dx}{u}=\frac{dy}{y}=\frac{dz}{w}$ is the differential equation of a streamline, where u, v and w are velocities in directions x, y and z respectively
(iv) In an unsteady flow, the path of a particle is a streamline

Which one of the following combinations of the statements is true? **[2014]**

(A) (i), (ii), (iv)
(B) (ii), (iii), (iv)
(C) (i), (iii), (iv)
(D) (i), (ii), (iii)

23. Consider a velocity field $\vec{\mathbf{V}}=\mathbf{K}(\mathbf{y}\hat{\mathbf{i}}+\mathbf{x}\hat{\mathbf{k}})$, where K is a constant. The vorticity, Ω_z, is **[2014]**

(A) $-K$ (B) K
(C) $-K/2$ (D) $K/2$

24. Match the following pairs: **[2015]**

	Equation		Physical Interpretation
P	$\nabla\times\vec{V}=0$	I	Incompressible continuity equation
Q	$\nabla.\vec{V}=0$	II	Steady flow
R	$\frac{D\vec{V}}{Dt}=0$	III	Irrotational flow
S	$\frac{\partial\vec{V}}{\partial t}=0$	IV	Zero acceleration of fluid particle

(A) P-IV, Q-I, R-II, S-III
(B) P-IV, Q-III, R-I, S-II
(C) P-III, Q-I, R-IV, S-II
(D) P-III, Q-I, R-II, S-IV

25. The velocity field of an incompressible flow is given by

$V=(a_1x+a_2y+a_3z)i+(b_1x+b_2y+b_3z)j+(c_1x+c_2y+c_3z)k$, where $a_1=2$ and $c_3=-4$. The value of b_2 is ______. **[2015]**

26. Water (ρ = 1000 kg/m^3) flows through a venturimeter with inlet diameter 80 mm and throat diameter 40 mm. The inlet and throat guage pressures are measured to be 400 kPa and 130 kPa respectively. Assuming the venturimeter to be horizontal and neglecting friction, the inlet velocity (in m/s) is _____. **[2015]**

27. If the fluid velocity for a potential flow is given by $V(x, y)=u(x, y)i+v(x, y)j$ with usual notations, then the slope of the potential line at (x, y) is: **[2015]**

28. A Prandtl tube (Pitot-static tube with $C = 1$) is used to measure the velocity of water. The differential manometer reading is 10 mm of liquid column with a relative density of 10. Assuming $g = 9.8$ m/s^2, the velocity of water (in m/s) is ____. **[2015]**

29. The instantaneous stream-wise velocity of a turbulent flow is given as follows:

$$u(x, y, z, t) = \bar{u}\ (x, y, z) + u'\ (x, y, z, t)$$

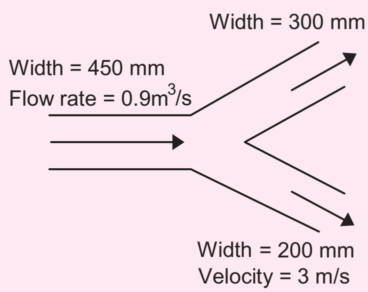
The time-average of the fluctuating velocity $u'\ (x, y, z, t)$ is: **[2016]**

(A) $u'/2$ (B) $-\bar{u}/2$
(C) zero (D) $\bar{u}/2$

30. The volumetric flow rate (per unit depth) between two streamlines having stream functions Ψ_1 and Ψ_2 is: **[2016]**

(A) $|\Psi_1 + \Psi_2|$ (B) $\Psi_1\ \Psi_2$
(C) $\Psi_1/\ \Psi_2$ (D) $|\Psi_1 - \Psi_2|$

31. A channel of width 450 mm branches into two sub-channels having width 300 mm and 200 mm as shown in figure. If the volumetric flow rate (taking unit depth) of an incompressible flow through the main channel is 0.9 m^3/s and the velocity in the sub-channel of width 200 mm is 3 m/s, the velocity in the sub-channel of width 300 mm is ______ m/s.
Assume both inlet and outlet to be at the same elevation. **[2016]**



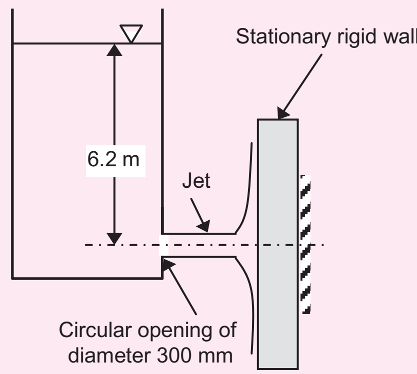
32. For a certain two-dimensional incompressible flow, velocity field is given by $2xy\hat{i} - y^2\hat{j}$. The streamlines for this flow are given by the family of curves. **[2016]**

(A) $x^2 y^2$ = constant (B) xy^2 = constant
(C) $2xy - y^2$ = constant (D) xy = constant

33. The water jet exiting from a stationary tank through a circular opening of diameter 300 mm impinges on a rigid wall as shown in the figure. Neglect all minor losses and assume the water level in the rank to remain constant. The net horizontal force experienced by the wall is _________ kN. **[2016]**

Density of water is 1000 kg/m^3.

Acceleration due to gravity $g = 10$ m/s^2.



34. For a two-dimensional flow, the velocity field is

$$\vec{u} = \frac{x}{x^2 + y^2}\hat{i} + \frac{y}{x^2 + y^2}\hat{j}$$

where $\hat{i}$ and $\hat{j}$ are the basis vectors in the x-y Cartesian coordinate system. Identify the CORRECT statements from below. **[2016]**

(1) The flow in incompressible
(2) The flow is unsteady
(3) y-component of acceleration, $a_y = \dfrac{-y}{(x^2 + y^2)^2}$
(4) x-component of acceleration, $a_x \dfrac{-(x+y)}{(x^2 + y^2)^2}$

(A) (2) and (3) (B) (1) and (3)
(C) (1) and (2) (D) (3) and (4)

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** A	**3.** D	**4.** A	**5.** B	**6.** C	**7.** D	**8.** A	**9.** D	**10.** B
11. A	**12.** D	**13.** A	**14.** D	**15.** C	**16.** A	**17.** B	**18.** A	**19.** B	**20.** C

Practice Problems 2

1. A	**2.** C	**3.** D	**4.** B	**5.** C	**6.** A	**7.** C	**8.** A	**9.** A	**10.** A
11. C	**12.** A	**13.** C	**14.** B	**15.** D	**16.** B	**17.** B	**18.** B	**19.** C	**20.** B
21. C	**22.** B	**23.** A	**24.** B	**25.** D	**26.** C	**27.** D	**28.** D	**29.** C	**30.** D

Previous Years' Questions

1. C	**2.** C	**3.** D	**4.** D	**5.** D	**6.** B	**7.** A	**8.** D	**9.** B	**10.** B
11. D	**12.** A	**13.** B	**14.** B	**15.** D	**16.** D	**17.** B	**18.** C	**19.** A	**20.** B
21. A	**22.** D	**23.** A	**24.** C	**25.** 1.9 to 2.1		**26.** 6	**27.** B	**28.** 1.30 to 1.34	
29. C	**30.** D	**31.** 1	**32.** B	**33.** 8.7–8.8		**34.** B			