Exercise 14.5

Answer 1E.

Given that $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t$

Using chain rule $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$

$$= (2x + y).\cos t + (2y + x).e^t$$

Answer 2E.

Given that $z = \cos(x + 4y),\ x = 5t^4,\ y = \frac{1}{t}$

Using chain rule , $\frac{dz}{dt} = -\sin(x + 4y).20t^3 - 4\sin(x + 4y).\left(\frac{-1}{t^2}\right)$

or $\frac{dz}{dt} = -20\sin(x + 4y)t^3 + 4\sin(x + 4y).\left(\frac{1}{t^2}\right)$

Answer 3E.

Given that $z = \sqrt{1 + x^2 + y^2}$, $x = \ln t,\ y = \cos t$

using chain rule

$$\frac{dz}{dt} = \frac{d}{dx}\left(\sqrt{1 + x^2 + y^2}\right).\frac{\partial}{\partial t}(\ln t) + \frac{\partial}{\partial y}\left(\sqrt{1 + x^2 + y^2}\right).\frac{\partial}{\partial t}(\cos t)$$

or $\frac{dz}{dt} = \frac{2x}{2\sqrt{1 + x^2 + y^2}}.\frac{1}{t} + \frac{2y}{2\sqrt{1 + x^2 + y^2}}.(-\sin t)$

or $\frac{dz}{dt} = \frac{x}{\sqrt{1 + x^2 + y^2}}.\frac{1}{t} - \frac{y}{\sqrt{1 + x^2 + y^2}}.\sin t$

Answer 4E.

Given that $z = \tan^{-1}\left(\frac{y}{x}\right), x = e^{t}, y = 1 - e^{-t}$

using chain rule, $\frac{dz}{dt} = \frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right).\frac{\partial}{\partial t}\left(e^{t}\right) + \frac{d}{dy}\tan^{-1}\left(\frac{y}{x}\right).\frac{\partial}{\partial t}\left(1 - e^{-t}\right)$

or $\frac{dz}{dt} = \frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(\frac{-y}{x^2}\right).e^{t} + \frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(\frac{1}{x}\right).\left(0 + e^{-t}\right)$

or $\frac{dz}{dt} = \frac{1}{\frac{x^2+y^2}{x^2}}.\left(\frac{-y}{x^2}\right)e^{t} + \frac{1}{\frac{x^2+y^2}{x^2}}.\frac{1}{x}e^{-t}$

or $= \left(\frac{-y}{x^2+y^2}e^{2t} + \frac{x}{x^2+y^2}\right).e^{-t}$

Answer 5E.

Consider the function $w = xe^{y/z}$, where $x = t^2, \; y = 1 - t, \; z = 1 + 2t$

The objective is to use chain rule to find $\frac{dw}{dt}$.

Chain rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt}$$

Differentiating *w* partially with respect to *x*,

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}xe^{y/z}$$

$$= e^{y/z}\frac{\partial}{\partial x}x$$

$$= e^{y/z}.1$$

$$= e.^{y/z}$$

Differentiating *w* partially with respect to *y*,

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}x\,e^{y/z}$$

$$= x\frac{\partial}{\partial y}e^{y/z}$$

$$= x.e^{y/z}.\frac{1}{z}$$

$$= \frac{x}{z}e^{y/z}$$

Differentiating *w* partially with respect to *z*,

$$\frac{\partial w}{\partial z}=\frac{\partial}{\partial z}.xe^{y/z}$$

$$=x\frac{\partial}{\partial z}.e^{y/z}$$

$$=x\,e^{y/z}y\left(-\frac{1}{z^2}\right)$$

$$=-\frac{xy}{z^2}e^{y/z}$$

Differentiating *x* with respect to t,

$$\frac{dx}{dt}=\frac{d}{dt}.t^2$$

$$=2t.$$

Differentiating *y* with respect to t,

$$\frac{dy}{dt}=\frac{d}{dt}(1-t)$$

$$=-1$$

Differentiating *z* with respect to t,

$$\frac{dz}{dt}=\frac{d}{dt}(1+2t)$$

$$=0+2$$

$$=2$$

Now,

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$=e^{y/z}.2t+\frac{x}{z}.e^{y/z}(-1)+\left(\frac{-xy}{z^2}e^{y/z}\right)2$$

$$=e^{y/z}\left[2t-\frac{x}{z}-\frac{2xy}{z^2}\right]$$

Hence, the derivative of *w* is,

$$\boxed{\frac{dw}{dt}=e^{y/z}\left[2t-\frac{x}{z}-\frac{2xy}{z^2}\right]}.$$

Answer 6E.

Consider the following function:

$$w = \ln\sqrt{x^2+y^2+z^2}$$

Suppose that $w = f(x,y,z)$ is a differentiable function of $x=\sin t,\ y=\cos t,$ and $z=\tan t$ are each differentiable functions of *t*. Then w is a differentiable function of t.

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt} \quad ......(1)$$

Find the individual components separately and then substitute into formula (1).

Find the partial derivative of $\frac{\partial w}{\partial x}$:

$$\frac{\partial w}{\partial x}=\frac{\partial}{\partial x}\left(\ln\sqrt{x^2+y^2+z^2}\right)$$

$$=\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$=\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$=\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot(2x)$$

$$=\frac{x}{x^2+y^2+z^2}$$

Find the derivative of $\frac{dx}{dt}$:

$$\frac{dx}{dt}=\frac{d}{dt}(\sin t)$$

$$=\cos t$$

Find the partial derivative of $\frac{\partial w}{\partial y}$:

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(\ln\sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot (2y)$$

$$= \frac{y}{x^2+y^2+z^2}$$

Find the derivative of $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{d}{dt}(\cos t)$$

$$= -\sin t$$

Find the partial derivative of $\frac{\partial w}{\partial z}$:

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(\ln\sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot (2z)$$

$$= \frac{z}{x^2+y^2+z^2}$$

Find the derivative of $\frac{dz}{dt}$:

$$\frac{dz}{dt} = \frac{d}{dt}(\tan t)$$

$$= \sec^2 t$$

Substitute the results of $\frac{\partial w}{\partial x}$, $\frac{dx}{dt}$, $\frac{\partial w}{\partial y}$, $\frac{dy}{dt}$, $\frac{\partial w}{\partial z}$, and $\frac{dz}{dt}$ into formula (1) we get:

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$=\left[\frac{x}{x^2+y^2+z^2}\right][\cos t]+\left[\frac{y}{x^2+y^2+z^2}\right][-\sin t]+\left[\frac{z}{x^2+y^2+z^2}\right][\sec^2 t]$$

$$=\boxed{\frac{x\cos t-y\sin t+z\sec^2 t}{x^2+y^2+z^2}}$$

Answer 7E.

Given that $z=x^2y^3$, $x=\text{scost}$, $y=\text{ssint}$

Using chain rule, $\frac{\partial z}{\partial s}=\frac{\partial}{\partial x}\left(x^2y^3\right).\frac{\partial}{\partial s}\left(\text{scost}\right)+\frac{\partial}{\partial y}\left(x^2y^3\right).\frac{\partial}{\partial s}\left(\text{ssint}\right)$

$$=2xy^3.\text{cost}+3x^2y^2.\text{sint}$$

$$=xy^2\left(2\text{ycost}+3\text{xsint}\right)$$

and $\frac{\partial z}{\partial t}=\frac{\partial}{\partial x}\left(x^2y^3\right).\frac{\partial}{\partial t}\left(\text{scost}\right)+\frac{\partial}{\partial y}\left(x^2y^3\right)\frac{\partial}{\partial t}\left(\text{ssint}\right)$

$$=2xy^3\left(-\text{ssint}\right)+\left(3x^2y^2\right).\text{scost}$$

$$=xy^2s\left(-2\text{ysint}+3\text{xcost}\right)$$

Answer 8E.

Given that, $z = \arcsin(x - y),\ x = s^2 + t^2,\ y = 1 - 2\text{st}$

Using chain rule,

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial x}\left(\arcsin[x - y]\right).\frac{\partial}{\partial s}\left(s^2 + t^2\right) + \frac{\partial}{\partial y}\left(\arcsin[x - y]\right).\frac{\partial}{\partial s}\left(1 - 2\text{st}\right)$$

$$= \frac{1}{\sqrt{1 - (x - y)^2}}.2s + \frac{1}{\sqrt{1 - (x - y)^2}}.\left(-1\right).\left(-2t\right)$$

$$= \frac{1}{\sqrt{1 - (x - y)^2}}.\left(2s + 2t\right)$$

$$= \frac{2}{\sqrt{1 - (x - y)^2}}.\left(s + t\right)$$

and $\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left(\arcsin[x - y]\right).\frac{\partial}{\partial t}\left(s^2 + t^2\right) + \frac{\partial}{\partial y}\left(\arcsin[x - y]\right).\frac{\partial}{\partial t}\left(1 - 2\text{st}\right)$

$$= \frac{1}{\sqrt{1 - (x - y)^2}}.2t + \frac{1}{\sqrt{1 - (x - y)^2}}.\left(-1\right)\left(-2s\right)$$

$$= \frac{1}{\sqrt{1 - (x - y)^2}}.\left(2t + 2s\right)$$

$$= \frac{2}{\sqrt{1 - (x - y)^2}}.\left(t + s\right)$$

Answer 9E.

Given that $z = \sin\theta\cos\phi,\ \theta = st^2,\ \phi = s^2t$

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\right).\frac{\partial}{\partial s}\left(st^2\right) + \frac{\partial}{\partial\phi}\left(\sin\left(\theta\right)\cos\phi\right).\frac{\partial}{\partial s}\left(s^2t\right)$$

$$= \cos\theta\cos\phi.t^2 + \sin\theta\left(-\sin\phi\right)\left(2st\right)$$

$$= \cos\theta\cos\phi\, t^2 - 2\sin\theta\sin\phi st$$

$$= \cos\theta\cos\phi\, t^2 - 2\sin\theta\sin\phi st$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\right).\frac{\partial}{\partial t}\left(st^2\right) + \frac{\partial}{\partial\phi}\left(\sin\left(\theta\right)\cos\phi\right).\frac{\partial}{\partial t}\left(s^2t\right)$$

$$= \cos\theta\cos\phi.2st^{\square} + \sin\theta\left(-\sin\phi\right)\left(s^2\right)$$

$$= 2\cos\theta\cos\phi\, st^{\square} - \sin\theta\sin\phi\, s^2$$

Answer 10E.

Consider the functions

$z = e^{x+2y}, \qquad x = s/t, \qquad y = t/s$

Note that,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

And $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial t}$

First we take $z = e^{x+2y}$

Differentiating *z* partially with respect to *x*, we obtain that

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^{x+2y}\right)$$

$$= e^{x+2y} \cdot \frac{\partial}{\partial x}(x+2y)$$

$$= e^{x+2y} \cdot (1)$$

$$= e^{x+2y}$$

Differentiating *z* partially with respect to *y*, we obtain that

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(e^{x+2y}\right)$$

$$= e^{x+2y} \cdot \frac{\partial}{\partial y}(x+2y)$$

$$= e^{x+2y} \cdot (2)$$

$$= 2e^{x+2y}$$

Now take $x = s/t$

Differentiating *x* partially with respect to *s*, we obtain that

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}(s/t)$$

$$= 1/t$$

Differentiating *x* partially with respect to *t*, we obtain that

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(s/t)$$

$$= -s/t^2$$

Now take $y = t / s$

Differentiating *y* partially with respect to *s*, we obtain that

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}\left(\frac{t}{s}\right)$$

$$= -\frac{t}{s^2}$$

Differentiating *y* partially with respect to *t*, we obtain that

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left(\frac{t}{s}\right)$$

$$= \frac{1}{s}$$

Therefore,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= e^{x+2y} \cdot \left(\frac{1}{t}\right) + 2e^{x+2y} \cdot \left(\frac{-t}{s^2}\right)$$

$$= e^{x+2y}\left[\frac{1}{t} - \frac{2t}{s^2}\right]$$

And

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= e^{x+2y} \cdot \left(\frac{-s}{t^2}\right) + 2e^{x+2y} \cdot \left(\frac{1}{s}\right)$$

$$= e^{x+2y}\left[-\frac{s}{t^2} + \frac{2}{s}\right]$$

Thus,

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= e^{x+2y}\left[\frac{1}{t} - \frac{2t}{s^2}\right] \\ \frac{\partial z}{\partial t} &= e^{x+2y}\left[-\frac{s}{t^2} + \frac{2}{s}\right] \end{aligned}}$$

Answer 11E.

Given $z = e^r \cos\theta, \quad r = st, \ \theta = \sqrt{s^2+t^2}$

We know, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial r}.\dfrac{\partial r}{\partial s} + \dfrac{\partial z}{\partial \theta}.\dfrac{\partial \theta}{\partial s}$

And $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial r}.\dfrac{\partial r}{\partial t} + \dfrac{\partial z}{\partial \theta}.\dfrac{\partial \theta}{\partial t}$

Differentiating z partially with respect to r,

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}e^r \cos\theta$$

$$= \cos\theta \frac{\partial}{\partial r}e^r$$

$$= \cos\theta . e^r$$

Differentiating z partially with respect to θ,

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}.e^r \cos\theta$$

$$= e^r \frac{\partial}{\partial \theta}\cos\theta$$

$$= e^r(-\sin\theta)$$

$$= -e^r \sin\theta.$$

Differentiating r with respect to s,

$$\frac{\partial r}{\partial s} = \frac{\partial}{\partial s}(st)$$

$$= t$$

Differentiating r partially with respect to t,

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial t}(st)$$

$$= s$$

Differentiating θ partially with respect to s,

$$\frac{\partial \theta}{\partial s} = \frac{\partial}{\partial s}\sqrt{s^2+t^2}$$

$$= \frac{1}{2\sqrt{s^2+t^2}}.2s$$

$$= \frac{s}{\sqrt{s^2+t^2}}$$

Differentiating θ partially with respect to t,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t}\sqrt{s^2+t^2}$$

$$= \frac{1}{2\sqrt{s^2+t^2}}.2t$$

$$= \frac{t}{\sqrt{s^2+t^2}}$$

Therefore, $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial r}.\frac{\partial r}{\partial s}+\frac{\partial z}{\partial \theta}.\frac{\partial \theta}{\partial s}$

$$= e^r \cos\theta . t + \left(-e^r \sin\theta\right)\frac{s}{\sqrt{s^2+t^2}}$$

$$= e^r\left[t\cos\theta - \frac{s\sin\theta}{\sqrt{s^2+t^2}}\right]$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial r}.\frac{\partial r}{\partial t}+\frac{\partial z}{\partial \theta}.\frac{\partial \theta}{\partial t}$$

$$= e^r \cos\theta . s + \left(-e^r \sin\theta\right)\frac{t}{\sqrt{s^2+t^2}}$$

$$= e^r\left[s\cos\theta - \frac{t\sin\theta}{\sqrt{s^2+t^2}}\right]$$

Hence,

$$\boxed{\begin{aligned}\frac{\partial z}{\partial s}&=e^r\left[t\cos\theta-\frac{s\sin\theta}{\sqrt{s^2+t^2}}\right]\\ \frac{\partial z}{\partial t}&=e^r\left[s\cos\theta-\frac{t\sin\theta}{\sqrt{s^2+t^2}}\right]\end{aligned}}$$

Answer 12E.

Using chain rule, $\frac{\partial z}{\partial s} = \frac{\partial}{\partial u}\left(\tan\frac{u}{v}\right).\frac{\partial}{\partial s}\left(2s+3t\right)+\frac{\partial}{\partial v}\left(\tan\frac{u}{v}\right).\frac{\partial}{\partial s}\left(3s-2t\right)$

$$= \sec^2\left(\frac{u}{v}\right).\frac{1}{v}.2+\sec^2\left(\frac{u}{v}\right)\left(\frac{-u}{v^2}\right).3$$

$$= \frac{2}{v}\sec^2\frac{u}{v}-\frac{3u}{v^2}\sec^2\frac{u}{v}$$

and $\frac{\partial z}{\partial t} = \frac{\partial}{\partial u}\left(\tan\frac{u}{v}\right).\frac{\partial}{\partial t}\left(2s+3t\right)+\frac{\partial}{\partial v}\left(\tan\frac{u}{v}\right).\frac{\partial}{\partial t}\left(3s-2t\right)$

$$= \sec^2\left(\frac{u}{v}\right).\frac{1}{v}.3+\sec^2\left(\frac{u}{v}\right)\left(\frac{-u}{v^2}\right).\left(-2\right)$$

$$= \frac{3}{v}\sec^2\frac{u}{v}+\frac{2u}{v^2}\sec^2\frac{u}{v}$$

Answer 13E.

$z = f(x, y)$

As, $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

It is given that $x = g(t)$ and $y = h(t)$

Then $\dfrac{dx}{dt} = g'(t)$ and $\dfrac{dy}{dt} = h'(t)$

And $\dfrac{\partial z}{\partial x} = f_x(x, y)$, $\quad \dfrac{\partial z}{\partial y} = f_y(x, y)$

$= f_x(g(t), h(t))$, $\quad = f_y(g(t), h(t))$

Hence $\dfrac{dz}{dt} = f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t)$

At, t = 3,

$$\left(\frac{dz}{dt}\right)_{t=3} = f_x(g(3), h(3))g'(3) + f_y(g(3), h(3))h'(3)$$

$$= f_x(2,7)g'(3) + f_y(2,7)h'(3)$$

$$= (6)(5) + (-8)(-4)$$

$$= 30 + 32$$

$$= 62$$

Hence $\boxed{\left(\dfrac{dz}{dt}\right)_{t=3} = 62}$

Answer 14E.

$w(s,t) = F(u(s,t), v(s,t))$

Then $w_s(s,t) = \dfrac{\partial w}{\partial s}$

$$= \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

$$F_u(u(s,t), v(s,t))u_s + F_v(u(s,t), v(s,t))v_s$$

And $w_t(s,t) = \dfrac{\partial w}{\partial t}$

$$= \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t}$$

$$= F_u(u(s,t), v(s,t))u_t + F_v(u(s,t), v(s,t))v_t$$

Then $w_s(1,0) = F_u\left(u(1,0), v(1,0)\right)u_s(1,0) + F_v\left(u(1,0), v(1,0)\right)v_s(1,0)$

$$= F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0)$$
$$= (-1)(-2) + (10)(5)$$
$$= \boxed{52}$$

And $w_t(1,0) = F_u\left(u(1,0), v(1,0)\right)u_t(1,0) + F_v\left(u(1,0), v(1,0)\right)v_t(1,0)$

$$= F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0)$$
$$= (-1)(6) + (10)(4)$$
$$= \boxed{34}$$

Answer 15E.

$$g(u,v) = f\left(e^u + \sin v, e^u + \cos v\right)$$
$$= f(x,y)$$

Where $x(u,v) = e^u + \sin v$ and $y(u,v) = e^u + \cos v$

$$\Rightarrow x(0,0) = e^0 + \sin 0$$
$$=1$$
$$\Rightarrow y(0,0) = e^0 + \cos 0$$
$$=2$$

Then $\dfrac{\partial x}{\partial u} = e^u, \ \dfrac{\partial x}{\partial v} = \cos v$

$$\frac{\partial y}{\partial u} = e^u, \ \frac{\partial y}{\partial v} = -\sin v$$

Now $g_u(u,v) = \dfrac{\partial f}{\partial u}$

$$= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$= f_x(x,y)e^u + f_y(x,y)e^u$$

Then $g_u(0,0) = f_x\left(x(0,0), y(0,0)\right)e^0 + f_y\left(x(0,0), y(0,0)\right)e^0$

$$= f_x(1,2) + f_y(1,2)$$
$$= 2+5$$
$$= 7\left(\text{ as given in the problem that } f_x(x,y) = 2 \textit{ and } f_y(1,2) = 5\right)$$

Now $g_v(u,v) = \dfrac{\partial f}{\partial v}$

$$= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= f_x(x,y)\cos v + f_y(x,y)(-\sin v)$$

Then $g_v(0,0) = f_x\left(x(0,0), y(0,0)\right)\cos 0 + f_y\left(x(0,0), y(0,0)\right)\sin 0$

$$= f_x(1,2)(1) + f_y(1,2)(0)$$

$= 2(1)+0$ $\left(\text{as given in the problem that } f_x(x,y) = 2 \textit{ and } f_y(1,2) = 5\right)$

$= 2$

Hence $\boxed{g_u(0,0) = 7,\ g_v(0,0) = 2}$.

Answer 16E.

Consider the differentiable function $g(r,s) = f(x,y)$ where *x* and *y* are each differentiable functions of *r* and *s*. In fact, we are shown that:

$$x(r,s) = 2r - s$$

$$y(r,s) = s^2 - 4r$$

To find the derivatives $g_r(1,2)$ and $g_s(1,2)$ we use the general version of the chain rule:

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} \quad \text{......(1)}$$

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} \quad \text{......(2)}$$

We find the individual components separately and then substitute into each formula.

Solving for $\frac{\partial g}{\partial x}$ given that $g = f(x,y)$ we get:

$$\begin{aligned}\frac{\partial g}{\partial x} &= f_x(x,y)\\ &= f_x(x(r,s), y(r,s))\end{aligned}$$

Evaluating this partial derivative when $(r,s) = (1,2)$ we get:

$$\begin{aligned}f_x(x(1,2), y(1,2)) &= f_x(2(1)-2, 2^2-4(1))\\ &= f_x(2-2, 4-4)\\ &= f_x(0,0)\\ &= 4\end{aligned}$$

Solving for $\frac{\partial x}{\partial r}$ given $x(r,s) = 2r - s$ we get:

$$\begin{aligned}\frac{\partial x}{\partial r} &= \frac{\partial}{\partial r}(2r-s)\\ &= 2\end{aligned}$$

Solving for $\frac{\partial g}{\partial y}$ given that $g = f(x,y)$ we get:

$$\begin{aligned}\frac{\partial g}{\partial y} &= f_y(x,y)\\ &= f_y(x(r,s), y(r,s))\end{aligned}$$

Evaluating this partial derivative when $(r,s) = (1,2)$ we get:

$$\begin{aligned}f_y(x(0,0), y(0,0)) &= f_y(2(1)-2, 2^2-4(1))\\ &= f_y(0,0)\\ &= 8\end{aligned}$$

Solving for $\frac{\partial y}{\partial r}$ given $y(r,s) = s^2 - 4r$ we get:

$$\begin{aligned}\frac{\partial y}{\partial r} &= \frac{\partial}{\partial r}(s^2-4r)\\ &= -4\end{aligned}$$

Substituting the evaluations into formula (1) we get:

$$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} \\ &= [4][2] + [8][-4] \\ &= 8 - 32 \\ &= -24\end{aligned}$$

Therefore, $\boxed{g_r(1,2) = -24}$.

Solving for $\frac{\partial x}{\partial s}$ given $x(r,s) = 2r - s$ we get:

$$\begin{aligned}\frac{\partial x}{\partial s} &= \frac{\partial}{\partial s}(2r - s) \\ &= -1\end{aligned}$$

Solving for $\frac{\partial y}{\partial s}$ given $y(r,s) = s^2 - 4r$.

$$\begin{aligned}\frac{\partial y}{\partial s} &= \frac{\partial}{\partial s}(s^2 - 4r) \\ &= 2s\end{aligned}$$

Evaluating this partial derivative when $(r,s) = (1,2)$.

$$\begin{aligned}2s &= 2(2) \\ &= 4\end{aligned}$$

Substituting the evaluations into formula (2).

$$\begin{aligned}\frac{\partial g}{\partial s} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} \\ &= [4][-1] + [8][4] \\ &= -4 + 32 \\ &= 28\end{aligned}$$

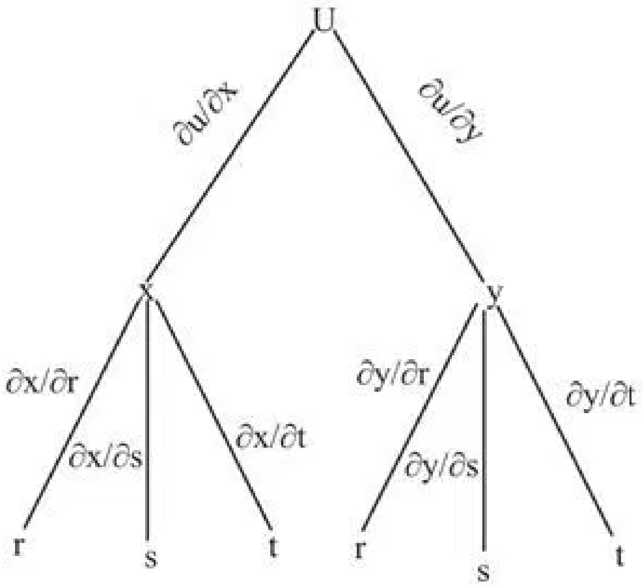
Therefore, $\boxed{g_s(1,2) = 28}$.

Answer 17E.

Given $u = f(x, y)$ where $x = x(r, s, t)$ and $y = y(r, s, t)$

Here u is a function of x and y, and x,y both are functions of r,s,t.

The tree diagram is



Now, $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

Answer 18E.

We are given that

$R = f(x, y, z, t)$

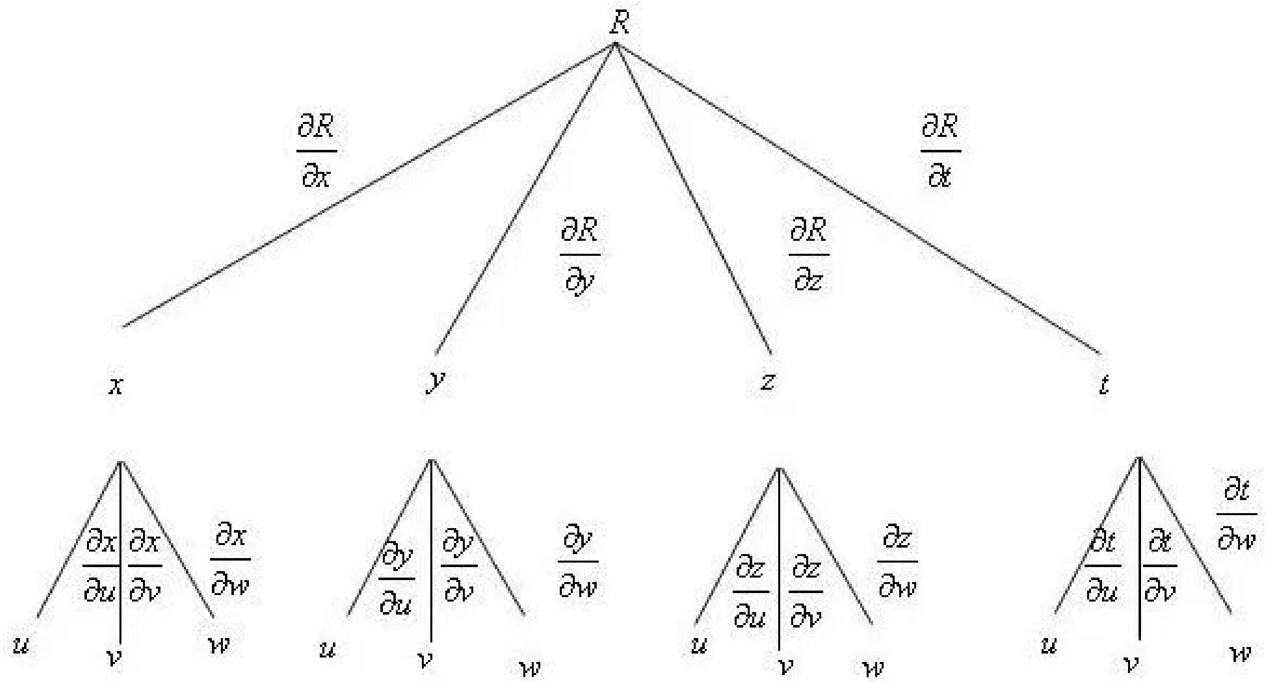
Where

$x = x(u, v, w), y = y(u, v, w)\ \ z = z(u, v, w) \text{and} t = t(u, v, w)$

Here, R is the function of $x, y, z \text{ and } t$

All $x, y, z \text{ and } t$ are the functions of $u, v \text{ and } w$

The tree diagram is



Answer 19E.

We are given that

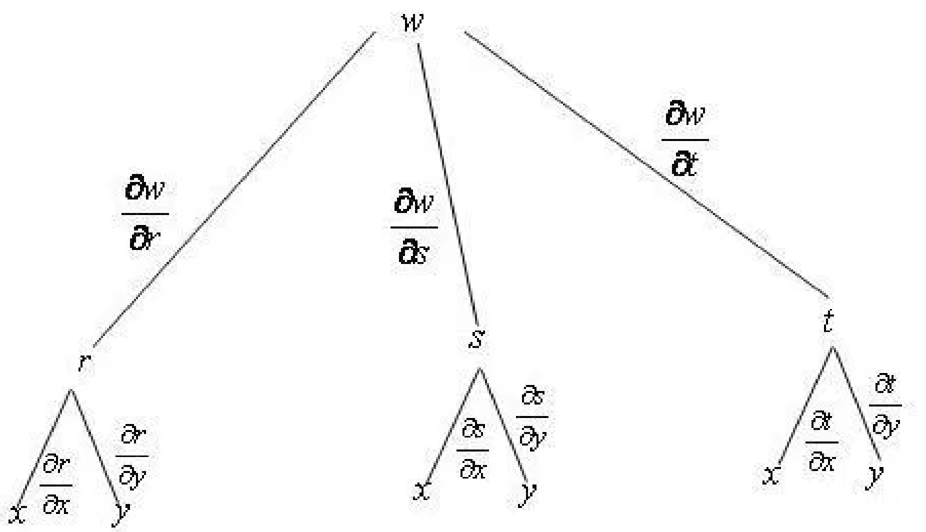
$w = f(r, s, t)$ where $r = r(x, y)$

$s = s(x, y)$ and $t = t(x, y)$

Here w is the function of r, s and t

All r, s and t are the functions of x and y

The tree diagram is



Now,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial y}$$

Answer 20E.

We are given that

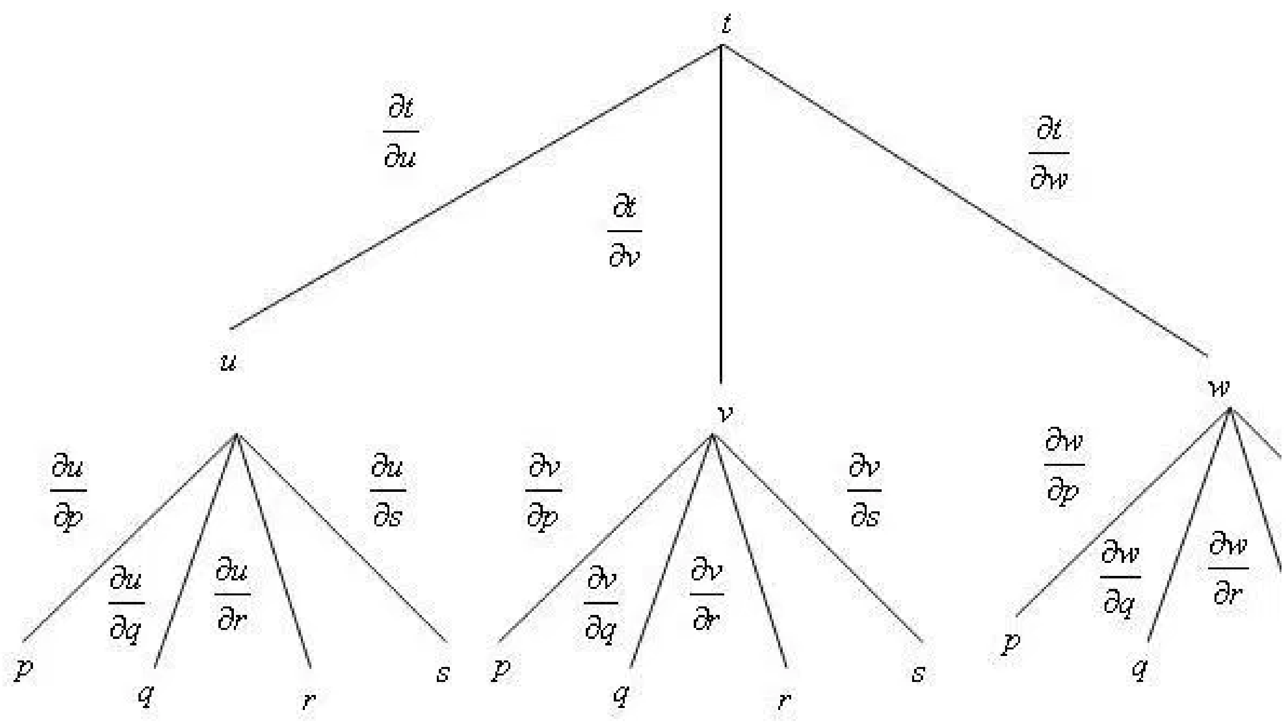
$t = f(u, v, w)$ Where

$u = u(p, q, r, s)$, $v = v(p, q, r, s)$ and $w = w(p, q, r, s)$

Here, t is the function of u, v and w

All u, v and w are the functions of p, q, r and s

The tree diagram is



Now,

$$\frac{\partial t}{\partial p} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial p} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial p} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial p}$$

$$\frac{\partial t}{\partial q} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial q} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial q} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial q}$$

$$\frac{\partial t}{\partial r} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial r} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial r}$$

$$\frac{\partial t}{\partial s} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial s} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial s}$$

Answer 21E.

If u is a differentiable function of the n variable $x_1, x_2, \ldots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots t_m$, then u is a function of $t_1, t_2, \ldots t_m$ and $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i}$ for each $i = 1, 2, \ldots, m$.

On substituting 4 for s, 2 for t, and 1 for u in the equations for x and y, we get $x = 7$ and $y = 8$.

Find $\frac{\partial z}{\partial s}$ given by $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$.

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial\left(x^4 + x^2y\right)}{\partial x}\frac{\partial\left(s + 2t - u\right)}{\partial s} + \frac{\partial\left(x^4 + x^2y\right)}{\partial y}\frac{\partial\left(stu^2\right)}{\partial s} \\ &= \left(4x^3 + 2xy\right)(1) + \left(x^2\right)\left(tu^2\right) \\ &= 4x^3 + 2xy + x^2tu^2\end{aligned}$$

Now, replace x with 7, y with 8, t with 2, and u with 1.

$$\begin{aligned}\frac{\partial z}{\partial s} &= 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2 \\ &= 1372 + 112 + 98 \\ &= \boxed{1582}\end{aligned}$$

Determine $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial t} = \frac{\partial\left(x^4 + x^2 y\right)}{\partial x}\frac{\partial\left(s + 2t - u\right)}{\partial t} + \frac{\partial\left(x^4 + x^2 y\right)}{\partial y}\frac{\partial\left(stu^2\right)}{\partial t}$$

$$= \left(4x^3 + 2xy\right)(2) + \left(x^2\right)\left(su^2\right)$$

$$= 8x^3 + 4xy + x^2 su^2$$

On substituting 7 for x, 8 for y, 4 for s, and 1 for u, we get $\frac{\partial z}{\partial t} = \boxed{3164}$.

Now, let us find $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial\left(x^4 + x^2 y\right)}{\partial x}\frac{\partial\left(s + 2t - u\right)}{\partial u} + \frac{\partial\left(x^4 + x^2 y\right)}{\partial y}\frac{\partial\left(stu^2\right)}{\partial u}$$

$$= \left(4x^3 + 2xy\right)(-1) + \left(x^2\right)(2stu)$$

$$= -4x^3 - 2xy + 2x^2 stu$$

Substituting the known values to find $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2(4)(2)(1)$$

$$= -1372 - 112 + 784$$

$$= \boxed{-700}$$

Thus, we get $\boxed{\frac{\partial z}{\partial s} = 1582,\ \frac{\partial z}{\partial t} = 3164,\ \text{and}\ \frac{\partial z}{\partial u} = -700}$.

Answer 22E.

If u is a differentiable function of the n variable $x_1, x_2, \ldots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots t_m$, then u is a function of $t_1, t_2, \ldots t_m$ and $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i}$ for each $i = 1, 2, \ldots, m$.

On substituting 2 for p, 1 for q, and 4 for r in the equations for u and v, we get $u = 4$ and $v = 8$.

Find $\frac{\partial T}{\partial p}$ given by $\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial p}$.

$$\begin{aligned}
\frac{\partial T}{\partial p} &= \frac{\partial\left(\frac{v}{2u+v}\right)}{\partial u}\frac{\partial\left(pq\sqrt{r}\right)}{\partial p} + \frac{\partial\left(\frac{v}{2u+v}\right)}{\partial v}\frac{\partial\left(p\sqrt{qr}\right)}{\partial p} \\
&= \left[-\frac{2v}{(2u+v)^2}\right]\left(q\sqrt{r}\right) + \left[\frac{2u}{(2u+v)^2}\right]\left(\sqrt{qr}\right) \\
&= -\frac{2vq\sqrt{r}}{(2u+v)^2} + \frac{2u\sqrt{qr}}{(2u+v)^2}
\end{aligned}$$

Now, replace u with 4, v with 8, q with 1, and r with 4.

$$\begin{aligned}
\frac{\partial T}{\partial p} &= -\frac{2(8)(1)\sqrt{4}}{(2(4)+8)^2} + \frac{2(4)\sqrt{1}(4)}{(2(4)+8)^2} \\
&= -\frac{32}{(16)^2} + \frac{32}{(16)^2} \\
&= 0
\end{aligned}$$

Determine $\frac{\partial T}{\partial q}$.

$$\begin{aligned}
\frac{\partial T}{\partial q} &= \frac{\partial\left(\frac{v}{2u+v}\right)}{\partial u}\frac{\partial\left(pq\sqrt{r}\right)}{\partial q} + \frac{\partial\left(\frac{v}{2u+v}\right)}{\partial v}\frac{\partial\left(p\sqrt{qr}\right)}{\partial q} \\
&= \left[-\frac{2v}{(2u+v)^2}\right]\left(p\sqrt{r}\right) + \left[\frac{2u}{(2u+v)^2}\right]\left(\frac{pr}{2\sqrt{q}}\right) \\
&= -\frac{2vp\sqrt{r}}{(2u+v)^2} + \frac{upr}{\sqrt{q}\,(2u+v)^2}
\end{aligned}$$

On substituting 2 for p, 4 for r, 4 for u, and 8 for v, we get $\frac{\partial T}{\partial q} = -\frac{1}{8}$.

Now, let us find $\frac{\partial T}{\partial r}$.

$$\frac{\partial T}{\partial r} = \frac{\partial\left(\frac{v}{2u+v}\right)}{\partial u}\frac{\partial\left(pq\sqrt{r}\right)}{\partial r} + \frac{\partial\left(\frac{v}{2u+v}\right)}{\partial v}\frac{\partial\left(p\sqrt{q}r\right)}{\partial r}$$

$$= \left[-\frac{2v}{(2u+v)^2}\right]\left(\frac{pq}{2\sqrt{r}}\right) + \left[\frac{2u}{(2u+v)^2}\right]\left(p\sqrt{q}\right)$$

$$= -\frac{pqv}{\sqrt{r}(2u+v)^2} + \frac{2up\sqrt{q}}{(2u+v)^2}$$

Substituting the known values to find $\frac{\partial T}{\partial r}$.

$$\frac{\partial T}{\partial r} = -\frac{(2)(1)(8)}{\sqrt{4}\left(2(4)+8\right)^2} + \frac{2(4)(2)\sqrt{1}}{\left(2(4)+8\right)^2}$$

$$= -\frac{16}{2(16)^2} + \frac{16}{(16)^2}$$

$$= -\frac{1}{32} + \frac{1}{16}$$

$$= \frac{1}{32}$$

Thus, we get $\boxed{\frac{\partial T}{\partial p} = 0,\ \frac{\partial T}{\partial q} = -\frac{1}{8},\ \text{and}\ \frac{\partial T}{\partial r} = \frac{1}{32}}$

Answer 23E.

If u is a differentiable function of the n variable $x_1, x_2, \ldots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots t_m$, then u is a function of $t_1, t_2, \ldots t_m$ and $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i}$ for each $i = 1, 2, \ldots, m$.

On substituting 2 for r and $\frac{\pi}{2}$ for θ in the equations for x, y and z, we get $x = 0, y = 2$ and $z = \pi$.

Find $\frac{\partial w}{\partial r}$ given by $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial(xy+yz+zx)}{\partial x}\frac{\partial(r\cos\theta)}{\partial r} + \frac{\partial(xy+yz+zx)}{\partial y}\frac{\partial(r\sin\theta)}{\partial r} + \frac{\partial(xy+yz+zx)}{\partial z}\frac{\partial(r\theta)}{\partial r}\\ &= (y+z)(\cos\theta) + (x+z)(\sin\theta) + (x+y)(\theta)\end{aligned}$$

Now, replace x with 0, y with 2, z with π, and θ with $\frac{\pi}{2}$.

$$\begin{aligned}\frac{\partial w}{\partial r} &= (2+\pi)\left(\cos\frac{\pi}{2}\right) + (0+\pi)\left(\sin\frac{\pi}{2}\right) + (0+2)\left(\frac{\pi}{2}\right)\\ &= (2+\pi)(0) + (\pi)(1) + (2)\left(\frac{\pi}{2}\right)\\ &= \pi + \pi\\ &= 2\pi\end{aligned}$$

Determine $\frac{\partial w}{\partial \theta}$ given by $\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$..

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= \frac{\partial(xy+yz+zx)}{\partial x}\frac{\partial(r\cos\theta)}{\partial \theta} + \frac{\partial(xy+yz+zx)}{\partial \theta}\frac{\partial(r\sin\theta)}{\partial \theta} + \frac{\partial(xy+yz+zx)}{\partial z}\frac{\partial(r\theta)}{\partial \theta}\\ &= (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (x+y)(r)\end{aligned}$$

Substitute the known values to find $\frac{\partial z}{\partial u}$.

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= (2+\pi)\left(-(2)\sin\frac{\pi}{2}\right) + (0+\pi)\left((2)\cos\frac{\pi}{2}\right) + (0+2)(2)\\ &= (2+\pi)(-2) + (\pi)(0) + (2)(2)\\ &= -4 - 2\pi + 4\\ &= -2\pi\end{aligned}$$

Thus, we get $\boxed{\frac{\partial w}{\partial r} = 2\pi \text{ and } \frac{\partial w}{\partial \theta} = -2\pi}$.

Answer 24E.

If u is a differentiable function of the n variable $x_1, x_2, \ldots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots t_m$, then u is a function of $t_1, t_2, \ldots t_m$ and $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i}$ for each $i = 1, 2, \ldots, m$.

On substituting 0 for x and 2 for y, we get $u = 0$, $v = 2$, and $w = 1$.

Find $\frac{\partial P}{\partial x}$ given by $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial x}$.

Substitute the know values and simplify.

$$\frac{\partial P}{\partial x} = \frac{u}{\sqrt{u^2+v^2+w^2}}\left(e^y\right) + \frac{v}{\sqrt{u^2+v^2+w^2}}\left(ye^x\right) + \frac{w}{\sqrt{u^2+v^2+w^2}}\left(ye^{xy}\right)$$
$$= \frac{ue^y + vye^x + wye^{xy}}{\sqrt{u^2+v^2+w^2}}$$

Now, replace x with 0, y with 2, u with 0, v with 2, and w with 1

$$\frac{\partial P}{\partial x} = \frac{(0)e^2 + (2)(2)e^0 + (1)(2)e^{(0)(2)}}{\sqrt{0^2+2^2+1^2}}$$
$$= \frac{0+4+2}{\sqrt{5}}$$
$$= \frac{6}{\sqrt{5}}$$

Determine $\frac{\partial P}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{u}{\sqrt{u^2+v^2+w^2}}\left(xe^y\right) + \frac{v}{\sqrt{u^2+v^2+w^2}}\left(e^x\right) + \frac{w}{\sqrt{u^2+v^2+w^2}}\left(xe^{xy}\right)$$
$$= \frac{uxe^y + ve^x + wxe^{xy}}{\sqrt{u^2+v^2+w^2}}$$

Substitute the known values to find $\frac{\partial P}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{(0)(0)e^2 + (2)e^0 + (1)(0)e^{(0)(2)}}{\sqrt{0^2+2^2+1^2}}$$
$$= \frac{0+2e^0+0}{\sqrt{5}}$$
$$= \frac{2}{\sqrt{5}}$$

Thus, we get $\boxed{\frac{\partial P}{\partial y} = \frac{2}{\sqrt{5}}}$.

Answer 25E.

If u is a differentiable function of the n variable $x_1, x_2, \ldots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots t_m$, then u is a function of $t_1, t_2, \ldots t_m$ and $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i}$ for each $i = 1, 2, \ldots, m$.

On substituting 2 for u, 3 for v, and 4 for w in the equations for p, q, and r, we get $p = 14$, $q = 11$, and $r = 10$.

Find $\frac{\partial N}{\partial u}$ given by $\frac{\partial N}{\partial u} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial u} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial u}$.

$$\frac{\partial N}{\partial u} = \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial p}\frac{\partial(u+vw)}{\partial u} + \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial q}\frac{\partial(v+uw)}{\partial u} + \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial r}\frac{\partial(w+uv)}{\partial u}$$

$$= \left[-\frac{q-r}{(p+r)^2}\right](1) + \left(\frac{1}{p+r}\right)(w) + \left(-\frac{p+q}{(p+r)^2}\right)(v)$$

Now, replace p with 14, q with 11, r with 10, v with 3, and w with 4

$$\frac{\partial N}{\partial u} = \left[-\frac{11-10}{(14+10)^2}\right] + \left(\frac{1}{14+10}\right)(4) + \left(-\frac{14+11}{(14+10)^2}\right)(3)$$

$$= \left[-\frac{1}{(24)^2}\right] + \left(\frac{1}{24}\right)(4) + \left(-\frac{25}{(24)^2}\right)(3)$$

$$= -\frac{1}{576} + \frac{1}{6} - \frac{25}{192}$$

$$= \frac{5}{144}$$

Determine $\frac{\partial N}{\partial v}$.

$$\frac{\partial N}{\partial v} = \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial p}\frac{\partial(u+vw)}{\partial v} + \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial q}\frac{\partial(v+uw)}{\partial v} + \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial r}\frac{\partial(w+uv)}{\partial v}$$

$$= \left[-\frac{q-r}{(p+r)^2}\right](w) + \left(\frac{1}{p+r}\right)(1) + \left(-\frac{p+q}{(p+r)^2}\right)(u)$$

On substituting 14 for p, 11 for q, 10 for r, 2 for u, and 4 for w we get $\frac{\partial N}{\partial v} = -\frac{5}{96}$.

Now, let us find $\frac{\partial N}{\partial w}$.

$$\frac{\partial N}{\partial w} = \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial p}\frac{\partial(u+vw)}{\partial w} + \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial q}\frac{\partial(v+uw)}{\partial w} + \frac{\partial\left(\frac{p+q}{p+r}\right)}{\partial r}\frac{\partial(w+uv)}{\partial w}$$

$$= \left[-\frac{q-r}{(p+r)^2}\right](v) + \left(\frac{1}{p+r}\right)(u) + \left(-\frac{p+q}{(p+r)^2}\right)(1)$$

Substitute the known values to find $\frac{\partial N}{\partial w}$.

$$\frac{\partial N}{\partial w} = \left[-\frac{11-10}{(14+10)^2}\right](3) + \left(\frac{1}{14+10}\right)(2) + \left(-\frac{14+11}{(14+10)^2}\right)$$

$$= \left[-\frac{1}{(24)^2}\right](3) + \left(\frac{1}{24}\right)(2) + \left(-\frac{25}{(24)^2}\right)$$

$$= -\frac{1}{192} + \frac{1}{12} - \frac{25}{576}$$

$$= -\frac{1}{192} + \frac{1}{12} - \frac{25}{576}$$

$$= \frac{5}{144}$$

Thus, we get $\boxed{\frac{\partial N}{\partial u} = \frac{5}{144}, \frac{\partial N}{\partial v} = -\frac{5}{96}, \text{ and } \frac{\partial N}{\partial w} = \frac{5}{144}}$.

Answer 26E.

Consider the function:

$u = xe^{ty}, x = \alpha^2\beta, y = \beta^2\gamma$, and $t = \gamma^2\alpha$.

Apply chain rule general version to compute the partial derivatives

$\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}$ and $\frac{\partial u}{\partial \gamma}$ when $\alpha = -1, \beta = 2, \gamma = 1$.

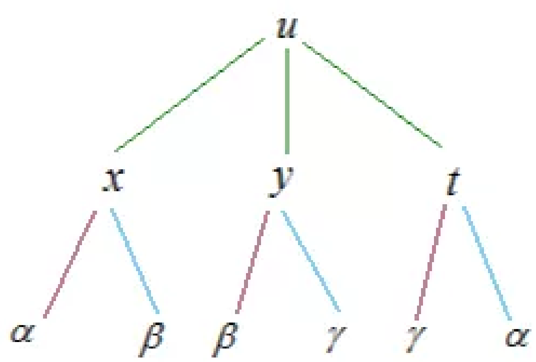
The Chain Rule:

If u is a differentiable function of the n variables $x_1, x_2, x_3, \cdots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, t_3, \cdots, t_m$ then u is a function of $t_1, t_2, t_3, \cdots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial t_i}.$$

For each $i = 1, 2, \cdots, m$.

Consider the tree diagram of the function u.



Plug in the values $\alpha=-1, \beta=2, \gamma=1$ to $x=\alpha^2\beta, y=\beta^2\gamma$, and $t=\gamma^2\alpha$.

$$x=(-1)^2(2), \quad y=(2)^2(1), \text{ and } \quad t=(1)^2(-1)$$
$$x=2, \quad y=4, \quad t=-1$$

With the help of the tree diagram, write the partial derivatives of u:

$$\frac{\partial u}{\partial \alpha}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \alpha}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \alpha}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial \alpha}$$
$$\frac{\partial u}{\partial \beta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial \beta}$$
$$\frac{\partial u}{\partial \gamma}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \gamma}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \gamma}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma}$$

Use the formula $\frac{\partial u}{\partial \alpha}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \alpha}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \alpha}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial \alpha}$ to compute $\frac{\partial u}{\partial \alpha}$.

$$\frac{\partial u}{\partial \alpha}=\frac{\partial(xe^{ty})}{\partial x}\frac{\partial(\alpha^2\beta)}{\partial \alpha}+\frac{\partial(xe^{ty})}{\partial y}\frac{\partial(\beta^2\gamma)}{\partial \alpha}+\frac{\partial(xe^{ty})}{\partial t}\frac{\partial(\gamma^2\alpha)}{\partial \alpha}$$
$$=(e^{ty})(2\alpha\beta)+(xte^{ty})(0)+(xye^{ty})(\gamma^2)$$
$$=2\alpha\beta e^{ty}+xy\gamma^2e^{ty}$$

Replace α with -1, β with 2, γ with 1, x with 2, y with 4 and t with -1.

$$\frac{\partial u}{\partial \alpha}=2(-1)(2)e^{(-1)(4)}+(2)(4)(1)^2e^{(-1)(4)}$$
$$=-4e^{-4}+8e^{-4}$$
$$=4e^{-4}$$

Thus, $\boxed{\frac{\partial u}{\partial \alpha}=4e^{-4}}$.

Use the formula $\frac{\partial u}{\partial \beta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial \beta}$ to compute $\frac{\partial u}{\partial \beta}$.

$$\frac{\partial u}{\partial \beta}=\frac{\partial\left(xe^{ty}\right)}{\partial x}\frac{\partial\left(\alpha^2\beta\right)}{\partial \beta}+\frac{\partial\left(xe^{ty}\right)}{\partial y}\frac{\partial\left(\beta^2\gamma\right)}{\partial \beta}+\frac{\partial\left(xe^{ty}\right)}{\partial t}\frac{\partial\left(\gamma^2\alpha\right)}{\partial \beta}$$
$$=\left(e^{ty}\right)\left(\alpha^2\right)+\left(xte^{ty}\right)\left(2\beta\gamma\right)+\left(xye^{ty}\right)(0)$$
$$=\alpha^2e^{ty}+2\beta\gamma xte^{ty}$$

Replace α with –1, β with 2, γ with 1, x with 2, y with 4 and t with –1.

$$\frac{\partial u}{\partial \beta}=(-1)^2e^{(-1)(4)}+2(2)(1)(2)(-1)e^{(-1)(4)}$$
$$=e^{-4}-8e^{-4}$$
$$=-7e^{-4}$$

Thus, $\boxed{\frac{\partial u}{\partial \beta}=-7e^{-4}}$.

Use the formula $\frac{\partial u}{\partial \gamma}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \gamma}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \gamma}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma}$ to compute $\frac{\partial u}{\partial \gamma}$.

$$\frac{\partial u}{\partial \gamma}=\frac{\partial\left(xe^{ty}\right)}{\partial x}\frac{\partial\left(\alpha^2\beta\right)}{\partial \gamma}+\frac{\partial\left(xe^{ty}\right)}{\partial y}\frac{\partial\left(\beta^2\gamma\right)}{\partial \gamma}+\frac{\partial\left(xe^{ty}\right)}{\partial t}\frac{\partial\left(\gamma^2\alpha\right)}{\partial \gamma}$$
$$=\left(e^{ty}\right)(0)+\left(xte^{ty}\right)\left(\beta^2\right)+\left(xye^{ty}\right)\left(2\gamma\alpha\right)$$
$$=x\beta^2te^{ty}+2\gamma\alpha xye^{ty}$$

Replace α with –1, β with 2, γ with 1, x with 2, y with 4 and t with –1.

$$\frac{\partial u}{\partial \gamma}=(2)(2)^2(-1)e^{(-1)(4)}+2(1)(-1)(2)(4)e^{(-1)(4)}$$
$$=-8e^{-4}-16e^{-4}$$
$$=-24e^{-4}$$

Thus, $\boxed{\frac{\partial u}{\partial \gamma}=-24e^{-4}}$.

Therefore, the value of the partial derivatives of u with respect to $\alpha,\beta,$ and γ when $\alpha=-1,\beta=2,\gamma=1$ is $\boxed{4e^{-4},-7e^{-4}\text{ and }-24e^{-4}}$ respectively.

Answer 27E.

We know that $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Let $F(x, y) = y\cos x - x^2 - y^2$.

Find $F_x(x, y)$ and $F_y(x, y)$.

$$F_x(x, y) = \frac{\partial}{\partial x}\left(y\cos x - x^2 - y^2\right)$$
$$= -y\sin x - 2x$$

$$F_y(x, y) = \frac{\partial}{\partial y}\left(y\cos x - x^2 - y^2\right)$$
$$= \cos x - 2y$$

Then, $\frac{dy}{dx} = -\frac{(-y\sin x - 2x)}{\cos x - 2y}$ or $\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$.

Thus, we get $\boxed{\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}}$.

Answer 28E.

We know that $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Let $F(x, y) = \cos(xy) - 1 - \sin y$.

Find $F_x(x, y)$ and $F_y(x, y)$.

$$F_x(x, y) = \frac{\partial}{\partial x}(\cos xy - 1 - \sin y)$$
$$= -y\sin xy$$

$$F_y(x, y) = \frac{\partial}{\partial y}(\cos xy - 1 - \sin y)$$
$$= -x\sin xy - \cos y$$

Then, $\frac{dy}{dx} = -\frac{(-y\sin xy)}{(-x\sin xy - \cos y)}$ or $\frac{dy}{dx} = -\frac{y\sin xy}{x\sin xy + \cos y}$.

Thus, we get $\boxed{\frac{dy}{dx} = -\frac{y\sin xy}{x\sin xy + \cos y}}$.

Answer 29E.

We know that $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Let $F(x, y) = \tan^{-1}(x^2y) - x - xy^2$.

Find $F_x(x, y)$ and $F_y(x, y)$.

$$F_x(x, y) = \frac{\partial}{\partial x}\left(\tan^{-1}(x^2y) - x - xy^2\right)$$
$$= \frac{2xy}{1+x^4y^2} - 1 - y^2$$

$$F_y(x, y) = \frac{\partial}{\partial y}\left(\tan^{-1}(x^2y) - x - xy^2\right)$$
$$= \frac{x^2}{1+x^4y^2} - 2xy$$

Substitute the known values in $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

$$\frac{dy}{dx} = -\frac{\left(\frac{2xy - 1 - x^4y^2 - y^2 - y^4x^4}{1+x^4y^2}\right)}{\left(\frac{x^2 - 2xy - 2x^5y^3}{1+x^4y^2}\right)}$$
$$= -\frac{2xy - 1 - x^4y^2 - y^2 - y^4x^4}{x^2 - 2xy - 2x^5y^3}$$
$$= \frac{-2xy + 1 + x^4y^2 + y^2 + y^4x^4}{x^2 - 2xy - 2x^5y^3}$$

Thus, we get $\boxed{\frac{dy}{dx} = \frac{-2xy + 1 + x^4y^2 + y^2 + y^4x^4}{x^2 - 2xy - 2x^5y^3}}$.

Answer 30E.

We know that $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Let $F(x, y) = e^y \sin x - x - xy$.

Find $F_x(x, y)$ and $F_y(x, y)$.

$$F_x(x, y) = \frac{\partial}{\partial x}\left(e^y \sin x - x - xy\right)$$
$$= e^y \cos x - 1 - y$$

$$F_y(x, y) = \frac{\partial}{\partial y}\left(e^y \sin x - x - xy\right)$$
$$= e^y \sin x - x$$

On substitute the known values in $\frac{dy}{dx} = -\frac{F_x}{F_y}$

we $\frac{dy}{dx} = -\frac{\left(e^y \cos x - 1 - y\right)}{\left(e^y \sin x - x\right)}$.

Thus, $\boxed{\frac{dy}{dx} = \frac{\left(e^y \cos x - 1 - y\right)}{\left(x - e^y \sin x\right)}}$.

Answer 31E.

We know that $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

Let $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 1$.

Find F_x, F_y and F_z.

$$F_x = \frac{\partial}{\partial x}\left(x^2 + 2y^2 + 3z^2 - 1\right)$$
$$= 2x$$

$$F_y = \frac{\partial}{\partial y}\left(x^2 + 2y^2 + 3z^2 - 1\right)$$
$$= 4y$$

$$F_z = \frac{\partial}{\partial z}\left(x^2 + 2y^2 + 3z^2 - 1\right)$$
$$= 6z$$

Substitute the known values in $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

$$\frac{\partial z}{\partial x} = -\frac{2x}{6z}$$
$$= -\frac{x}{3z}$$

$$\frac{\partial z}{\partial y} = -\frac{4y}{6z}$$
$$= -\frac{2y}{3z}$$

Thus, we get $\boxed{\frac{\partial z}{\partial x} = -\frac{x}{3z} \text{ and } \frac{\partial z}{\partial y} = -\frac{2y}{3z}}$.

Answer 32E.

We know that $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

Let $F(x, y, z) = x^2 - y^2 + z^2 - 2z - 4$.

Find F_x, F_y and F_z.

$$F_x = \frac{\partial}{\partial x}\left(x^2 - y^2 + z^2 - 2z - 4\right)$$
$$= 2x$$

$$F_y = \frac{\partial}{\partial y}\left(x^2 - y^2 + z^2 - 2z - 4\right)$$
$$= -2y$$

$$F_z = \frac{\partial}{\partial z}\left(x^2 + 2y^2 + 3z^2 - 1\right)$$
$$= 2z - 2$$

Substitute the known values in $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{2x}{2z-2}\\ &= -\frac{x}{z-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= -\frac{-2y}{2z-2}\\ &= \frac{y}{z-1}\end{aligned}$$

Thus, we get $\boxed{\frac{\partial z}{\partial x} = -\frac{x}{z-1} \text{ and } \frac{\partial z}{\partial y} = \frac{y}{z-1}}$.

Answer 33E.

We know that $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

Let $F(x, y, z) = e^z - xyz$.

Find F_x, F_y and F_z.

$$\begin{aligned}F_x &= \frac{\partial}{\partial x}\left(e^z - xyz\right)\\ &= -yz\end{aligned}$$

$$\begin{aligned}F_y &= \frac{\partial}{\partial y}\left(e^z - xyz\right)\\ &= -xz\end{aligned}$$

$$\begin{aligned}F_z &= \frac{\partial}{\partial z}\left(e^z - xyz\right)\\ &= e^z - xy\end{aligned}$$

Substitute the known values in $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

$$\frac{\partial z}{\partial x} = -\frac{-yz}{e^z - xy}$$
$$= \frac{yz}{e^z - xy}$$

$$\frac{\partial z}{\partial y} = -\frac{-xz}{e^z - xy}$$
$$= \frac{xz}{e^z - xy}$$

Thus, we get $\boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy} \text{ and } \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$.

Answer 34E.

We know that $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

Let $F(x, y, z) = yz + x\ln y - z^2$.

Find F_x, F_y, and F_z.

$$F_x = \frac{\partial}{\partial x}\left(yz + x\ln y - z^2\right)$$
$$= \ln y$$

$$F_y = \frac{\partial}{\partial y}\left(yz + x\ln y - z^2\right)$$
$$= z + \frac{x}{y}$$

$$F_z = \frac{\partial}{\partial z}\left(yz + x\ln y - z^2\right)$$
$$= y - 2z$$

Substitute the known values in $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

$$\frac{\partial z}{\partial x} = -\frac{\ln y}{y - 2z}$$

$$\frac{\partial z}{\partial y} = -\frac{z + \frac{x}{y}}{y - 2z}$$

$$= -\frac{zy + x}{y^2 - 2yz}$$

Thus, we get $\boxed{\frac{\partial z}{\partial x} = -\frac{\ln y}{y - 2z} \text{ and } \frac{\partial z}{\partial y} = -\frac{zy + x}{y^2 - 2yz}}$.

Answer 35E.

The temperature is given by T(x, y)

But $x = \sqrt{1+t}$, $y = 2 + \frac{t}{3}$

Then $T(x,y) = T\left(\sqrt{1+t}, 2+\frac{t}{3}\right)$

And therefore

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \qquad \text{(By chain rule)}$$

Now $\frac{dx}{dt} = \frac{1}{2\sqrt{1+t}}$

$\frac{dy}{dt} = \frac{1}{3}$

Then $\frac{dT}{dt} = T_x(x,y) \times \frac{1}{2\sqrt{1+t}} + T_y(x,y) \times \frac{1}{3}$

And $\left(\frac{dT}{dt}\right)_{t=3} = T_x\left(x(3), y(3)\right)\times\frac{1}{2\sqrt{1+3}} + T_y\left(x(3), y(3)\right)\times\frac{1}{3}$

$= T_x(2,3)\times\frac{1}{4} + T_y(2,3)\times\frac{1}{3}$

$= T_x(2,3)\times\frac{1}{4} + T_y(2,3)\times\frac{1}{3}$

$= 4\times\frac{1}{4} + 3\times\frac{1}{3}$

$= 2$

Hence after 3 seconds the temperature is rising at the rate3 of $\boxed{2^{\circ}}$c/s

Answer 36E.

(A)

Since $\frac{\partial W}{\partial T}$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels.

Since $\frac{\partial W}{\partial R}$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production

(B)

The wheat production function is W(T, R)

By chain rule $\frac{dW}{dt} = \frac{\partial W}{\partial T}\frac{dT}{dt} + \frac{\partial W}{\partial R}\frac{dR}{dt}$

$= (-2)(0.15) - (8)(0.1)$

$= -0.30 - 0.8$

$= -1.10$

That is the current rate of change of wheat production is $\boxed{-1.10}$

Answer 37E.

We have a differentiable function $C = f(T, D)$ where T and D are each differentiable functions of time, t. We want to find $\partial C / \partial t$ when $t = 20$. Hence, we use the general version of the chain rule:

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial T}\frac{dT}{dt} + \frac{\partial C}{\partial D}\frac{dD}{dt} \quad \text{......(1)}$$

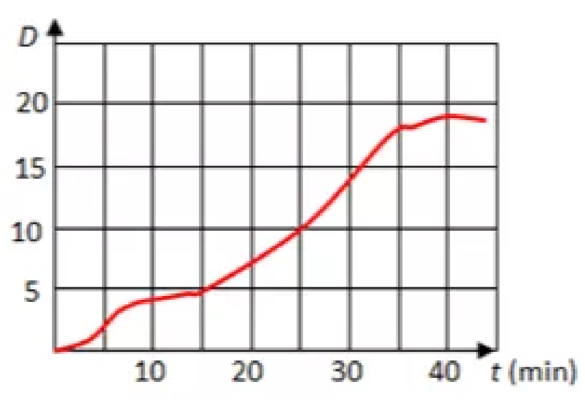
Consider the equation $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$

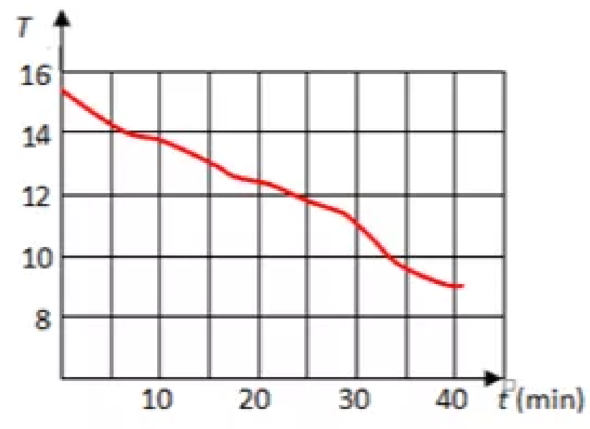
Now, calculate $\partial C / \partial T$:

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{\partial}{\partial T}\left(1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D\right) \\ &= 0 + 4.6 - 0.055(2T) + 0.00029\left(3T^2\right) + 0 \\ &= 4.6 - 0.11T + 0.00087T^2\end{aligned}$$

To evaluate this partial derivative when $t = 20$, we use the graph of T to estimate the temperature at 20 minutes.

The following graphs shows the divers depth and the surrounding water temperature over time are recorded in the following graphs.





From the graph of T, we estimate that when $t = 20$, we have $T \approx 12.5$. Therefore, we estimate $\partial C / \partial T$:

$$\begin{aligned}\frac{\partial C}{\partial T} &= 4.6 - 0.11T + 0.00087T^2 \\ &\approx 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \\ &\approx 4.6 - 1.375 + 0.136 \\ &= 3.361\end{aligned}$$

From the given equation, we calculate $\partial C / \partial D$:

$$\frac{\partial C}{\partial D} = \frac{\partial}{\partial D}\left(1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D\right)$$
$$= 0 + 0 - 0 + 0 + 0.016$$
$$= 0.016$$

To estimate dT / dt when $t = 20$, we use the graph of $T(t)$ and approximate the slope of the curve at the point where $t = 20$. The graph of *T* is decreasing, hence the derivative will be negative. By lining a straight edge along the curve at the point $(20,12.5)$ and estimating the slope, we approximate that $dT / dt \approx -1/10$.

Similarly, to estimate dD / dt when $t = 20$, we use the graph of $D(t)$ and approximate the slope of the curve at the point where $t = 20$. The graph of *D* is increasing, hence the derivative will be positive. By lining a straight edge along the curve at the point $(20,7)$ and estimating the slope, we approximate that $dD / dt \approx 1/2$.

Answer 38E.

If r is the radius and h is the height of the right circular cone, then the volume is

$$V(r,h) = \frac{1}{3}\pi r^2 h$$

Where $r = r(t)$, $h = h(t)$

Then $$\frac{dv}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt}$$
$$= \frac{2}{3}\pi rh \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

But $\frac{dr}{dt} = 1.8^{in}/_{s}$ and $\frac{dh}{dt} = -2.5^{in}/_{s}$

And $r = 120\, in$, $h = 140 in$

Then $$\frac{dv}{dt} = \frac{2}{3}\pi(120)(140)(1.8) - \frac{1}{3}\pi(120)^2(2.5)$$
$$= 63334.507 - 37694.11$$
$$= 25635.38^{in^3}/_{s}$$

Hence the volume of the cone is changing at a rule of $\boxed{25635.4}$ cubic inch/s

Answer 39E.

$$\frac{dl}{dt}=2\,\text{m}/\text{s},\ \frac{dw}{dt}=2\,\text{m}/\text{s},\ \frac{dh}{dt}=-3\,\text{m}/\text{s}$$

(A)

The volume is $v = lhw$

Then $$\frac{dv}{dtg}=v_l\frac{dl}{dt}+v_h\frac{dh}{dt}+v_w\frac{dw}{dt}$$

$$=(hw)\frac{dl}{dt}+lw\frac{dh}{dt}+lh\frac{dw}{dt}$$

$$=(2)(2)(2)+(1)(2)(2)+(1)(2)(-3)$$

$$=8+4-6$$

$$=6$$

Hence $$\frac{dv}{dt}=\boxed{6\,\text{m}^3/\text{s}}$$

(B)

The surface area $A=2(hw+lw+lh)$

Then $$\frac{dA}{dt}=A_l\frac{dl}{dt}+A_h\frac{dh}{dt}+A_w\frac{dw}{dt}$$

$$=2(h+w)\frac{dl}{dt}+2(w+l)\frac{dh}{dt}+2(l+h)\frac{dw}{dt}$$

$$=2(2+2)(2)+2(2+1)(2)+2(1+2)(-3)$$

$$=28-18=10$$

Hence $$\frac{dA}{dt}=\boxed{10\,\text{m}^2/\text{s}}$$

(C)

The length of the diagonal is

$$L=\sqrt{l^2+h^2+w^2}$$

Then $$\frac{dL}{dt}=L_l\frac{dl}{dt}+L_h\frac{dh}{dt}+L_w\frac{dw}{dt}$$

$$=\frac{l}{\sqrt{l^2+h^2+w^2}}\frac{dl}{dt}+\frac{h}{\sqrt{l^2+h^2+w^2}}\frac{dh}{dt}+\frac{w}{\sqrt{l^2+h^2+w^2}}\frac{dw}{dt}$$

$$=\frac{1}{\sqrt{1+4+4}}(2)+\frac{2}{\sqrt{1+4+4}}(2)+\frac{2}{\sqrt{1+4+4}}(-3)$$

$$=\frac{2}{3}+\frac{4}{3}-\frac{6}{3}=0$$

Hence $$\frac{dL}{dt}=\boxed{0}$$

Answer 40E.

It is given that

$$V = IR$$

Then $$\frac{dV}{dt} = V_I \frac{dI}{dt} + V_R \frac{dR}{dt}$$

$$= R\frac{dI}{dt} + I\frac{dR}{dt}$$

Now $R = 400\,\Omega,\ I = 0.08A$

$$\frac{dV}{dt} = -0.01 \,{}^{V}\!/_{s},\quad \frac{dR}{dt} = 0.03 \,{}^{\Omega}\!/_{s}$$

Then $$-0.01 = 400\frac{dI}{dt} + 0.08(0.03)$$

$$-0.01 = 400\frac{dI}{dt} + 0.0024$$

$$-0.01 - 0.0024 = 400\frac{dI}{dt}$$

$$-0.124\times10^{-3} = 400\frac{dI}{dt}$$

Then $$\frac{dI}{dt} = \frac{-0.124\times10^{-3}}{400}$$

$$= 3.1\times10^{-5}$$

Hence the current is decreasing at the rate of $\boxed{3.1\times10^{-5}\ {}^{A}\!/_{s}}$

Answer 41E.

Consider the data,

Rate of change of pressure of 1 mole of ideal gas, $\frac{dP}{dt} = 0.05\,\text{kPa/s}$

Rate of change of temperature, $\frac{dT}{dt} = 0.15\,\text{K/s}$

Pressure is $20\ \text{KPa}$ and the temperature is $320\ \text{K}$

And the equation relating pressure, volume and temperature is given by

$$P = 8.31\frac{T}{V} \quad \text{......}\ (1)$$

The chain rule gives

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}$$

Find $\frac{\partial P}{\partial T}, \frac{\partial P}{\partial V}$ from (1).

$$\frac{\partial P}{\partial T} = \frac{8.31}{V}$$

$$\frac{\partial P}{\partial V} = 8.31T\left(-\frac{1}{V^2}\right) \quad \text{Use power rule}$$

$$= -\frac{8.31T}{V^2}$$

So,

$$\frac{dP}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt} \quad \text{......}\ (2)$$

Substitute $\frac{dP}{dt}, \frac{dT}{dt}$ in (2).

Then,

$$0.05=\frac{8.31}{V}(0.15)-\frac{8.31T}{V^2}\frac{dV}{dt} \quad \text{...... (3)}$$

From equation (1)

$$V=\frac{8.31T}{P}$$

$$=\frac{8.31\times 320}{20}$$

$$=132.96 \text{ litres}$$

Using this value of V in equation (3)

$$0.05=\frac{8.31}{132.96}(0.15)-\frac{8.31}{(132.96)^2}(320)\frac{dV}{dt}$$

$$0.05=0.00937-0.1504\frac{dV}{dt}$$

Therefore,

$$\frac{dV}{dt}=\frac{0.00937-0.05}{0.1504}$$

$$=-0.27\,\text{L/s}$$

Hence, the volume is decreasing at a rate of $\boxed{0.27\,\text{L/s}}$.

Answer 42E.

If $z=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable functions of t, then z is a differentiable function of t and $\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$.

Consider the function:

$$P(L,K)=1.47L^{0.65}K^{0.35}$$

Rate of change of production is:

$$\frac{dP}{dt}=\frac{\partial P}{\partial L}\frac{dL}{dt}+\frac{\partial P}{\partial K}\frac{dK}{dt}$$

Consider the values shown below:

$$L=30$$
$$K=8$$
$$\frac{dL}{dt}=-2$$
$$\frac{dK}{dt}=0.5$$

So, the equation of derivatives becomes:

$$\frac{dP}{dt}=(-2)\frac{\partial P}{\partial L}+(0.5)\frac{\partial P}{\partial K}$$

Determine the expression for $\frac{\partial P}{\partial L}$ and $\frac{\partial P}{\partial K}$:

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial L}\left(1.47L^{0.65}K^{0.35}\right)$$
$$= \frac{0.9555K^{0.35}}{L^{0.35}}$$
$$\frac{\partial P}{\partial K} = \frac{\partial}{\partial K}\left(1.47L^{0.65}K^{0.35}\right)$$
$$= \frac{0.5145L^{0.65}}{K^{0.65}}$$

Consider the equation of derivatives shown below:

$$\frac{dP}{dt} = (-2)\left(\frac{0.9555K^{0.35}}{L^{0.35}}\right) + (0.5)\left(\frac{0.5145L^{0.65}}{K^{0.65}}\right)$$

Replace *L* with 30 and *K* with 8 and simplify.

$$\frac{dP}{dt} = (-2)\left[\frac{0.9555(8)^{0.35}}{(30)^{0.35}}\right] + (0.5)\left[\frac{0.5145(30)^{0.65}}{(8)^{0.65}}\right]$$
$$= (-2)[0.6016] + (0.5)[1.2148]$$
$$= -0.5958$$

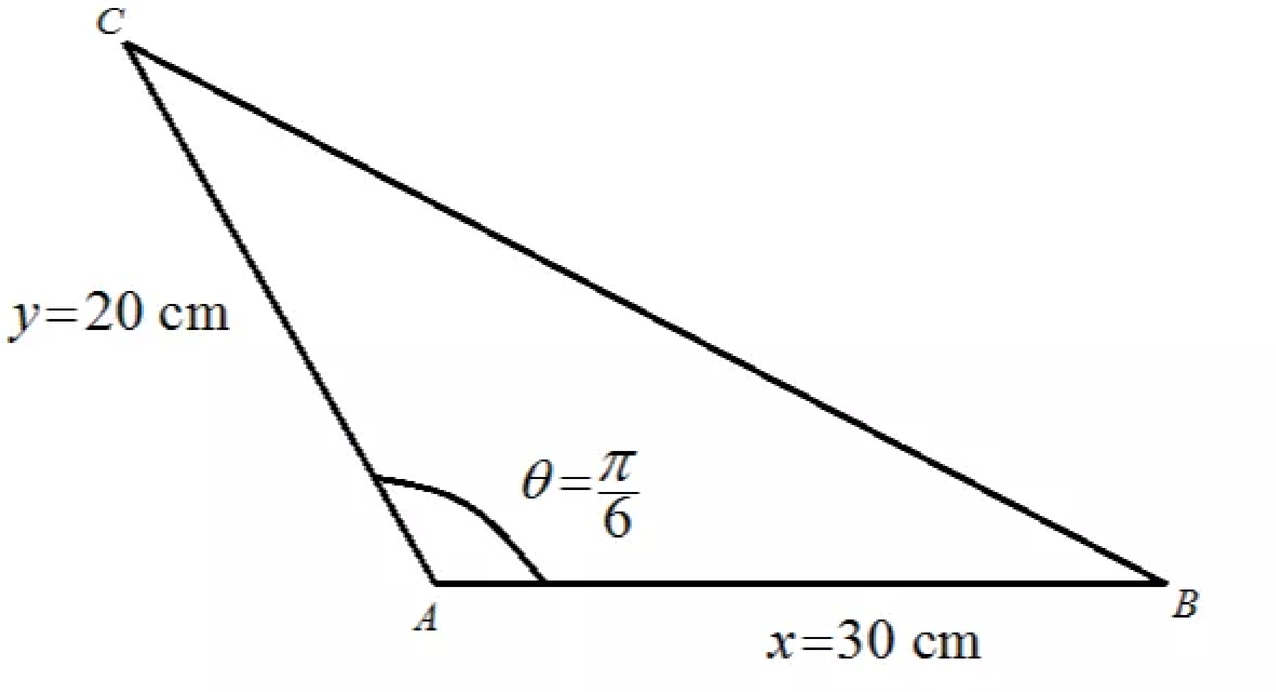
Determine the units for $\frac{dP}{dt}$.

So, the units are in million dollars per year.

Hence, the final value of the rate of change of production is a **decrease of** $\boxed{\$595{,}800 \text{per year}}$.

Answer 43E.

Consider the following figure:



One side of a triangle is increasing at a rate $\frac{dy}{dt} = 3 \text{ cm/s}$.

Second side is decreasing at a rate $\frac{dx}{dt} = -2 \text{ cm/s}$.

First side is $y = 20 \text{ cm}$.

Second side is $x = 30 \text{ cm}$.

The area of the triangle remains constant.

The equation for finding the area of the triangle is $xy\sin(\pi - \theta) = A$.

According to the trigonometric identity,

$$xy\ \sin\theta = A$$

Differentiate with respect to time ' t '.

Apply the formula: $\frac{d}{dx}(uvw) = \left(\frac{d}{dx}u\right)vw + u\left(\frac{d}{dx}v\right)w + uv\left(\frac{d}{dx}w\right)$

$$\left(\frac{d}{dt}x\right)y\sin\theta + x\left(\frac{d}{dt}y\right)\sin\theta + xy\left(\frac{d}{dt}\sin\theta\right) = 0$$

$$\left(\frac{dx}{dt}\right)y\sin\theta + x\left(\frac{dy}{dt}\right)\sin\theta + xy\left(\cos\theta\frac{d}{dt}\theta\right) = 0$$

$$(-2)(20)\sin\left(\frac{\pi}{6}\right) + (30)(3)\sin\left(\frac{\pi}{6}\right) + (30)(20)\cos\left(\frac{\pi}{6}\right)\frac{d\theta}{dt} = 0$$

$(-2)(20)\left(\frac{1}{2}\right) + (30)(3)\left(\frac{1}{2}\right) + (30)(20)\left(\frac{\sqrt{3}}{2}\right)\frac{d\theta}{dt} = 0$ Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$-20 + 45 + 300\sqrt{3}\frac{d\theta}{dt} = 0$$

$$300\sqrt{3}\frac{d\theta}{dt} = -25$$

$$\frac{d\theta}{dt} = \frac{-25}{300\sqrt{3}}$$

$$= -\frac{1}{12\sqrt{3}}\ \text{rad/s}$$

Therefore, the rates of angle between the sides change is $\boxed{\frac{d\theta}{dt} = -\frac{1}{12\sqrt{3}}\ \text{rad/s}}$.

Answer 44E.

To determine the perceived frequency, substitute the known values into the given equation. The speed of sound $c \approx 332$ m/s. The observer's train is traveling at 34 m/s, hence $v_o = 34$. The approaching train from whence the sound emanates is traveling at 40 m/s, hence $v_s = 40$. The frequency of the whistle is 460 Hz, hence $f_s = 460$. Therefore, write the equation as follows:

$$f_o = \left(\frac{c + v_o}{c - v_s}\right)f_s$$

$$= \left(\frac{332 + 34}{332 - 40}\right)(460)$$

$$= \frac{366}{292}(460)$$

$$\approx 576.6$$

So, the perceived frequency that you hear from the train whistle is approximately $\boxed{577\ \text{Hz}}$.

Consider the differentiable function $f_o = f(v_o, v_s)$ where v_o and v_s are each differentiable functions of time, *t*. The objective is to find the perceived frequency's rate of change, $\partial f_o / \partial t$. Hence, use the general version of the chain rule:

$$\frac{\partial f_o}{\partial t} = \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt} \quad \text{......}(1)$$

From formula (1) calculate $\partial f_o / \partial v_o$:

$$\begin{aligned}\frac{\partial f_o}{\partial v_o} &= \frac{\partial}{\partial v_o}\left(\left(\frac{c+v_o}{c-v_s}\right)f_s\right)\\ &= \left(\frac{f_s}{c-v_s}\right)\frac{\partial}{\partial v_o}(c+v_o)\\ &= \left(\frac{f_s}{c-v_s}\right)(0+1)\\ &= \frac{f_s}{c-v_s}\end{aligned}$$

Evaluate this partial derivative for $f_s = 460$, $c = 332$ and $v_s = 40$:

$$\begin{aligned}\frac{f_s}{c-v_s} &= \frac{460}{332-40}\\ &= \frac{460}{292}\\ &\approx 1.575\end{aligned}$$

Therefore,

$$\frac{\partial f_o}{\partial v_o} \approx 1.575.$$

Similarly, calculate $\partial f_o / \partial v_s$:

$$\frac{\partial f_o}{\partial v_s} = \frac{\partial}{\partial v_s}\left(\left(\frac{c+v_o}{c-v_s}\right)f_s\right)$$

$$= \left(f_s\left(c+v_o\right)\right)\frac{\partial}{\partial v_s}\left(c-v_s\right)^{-1}$$

$$= \left(f_s\left(c+v_o\right)\right)\left[-\left(c-v_s\right)^{-2}\right]\cdot\frac{\partial}{\partial v_s}\left(c-v_s\right)$$

$$= \frac{-f_s\left(c+v_o\right)}{\left(c-v_s\right)^2}\cdot(-1)$$

$$= \frac{f_s\left(c+v_o\right)}{\left(c-v_s\right)^2}$$

Evaluate this partial derivative for $f_s = 460$, $c = 332$, $v_o = 34$ and $v_s = 40$:

$$\frac{f_s\left(c+v_o\right)}{\left(c-v_s\right)^2} = \frac{460\left(332+34\right)}{\left(332-40\right)^2}$$

$$= \frac{460(366)}{292^2}$$

$$\approx 1.975$$

Therefore, obtain the value as shown below:

$$\frac{\partial f_o}{\partial v_s} \approx 1.975$$

Acceleration, *a*, is the derivative of velocity with respect to time.

Hence $a_o = dv_o / dt$ and $a_s = dv_s / dt$.

The acceleration of the observer's train $1.2\ \text{m/s}^2$, hence $dv_o / dt = 1.2$.

The acceleration of the source's train $1.4\ \text{m/s}^2$, hence $dv_s / dt = 1.4$.

Substituting all results into formula (1) to get the following:

$$\frac{\partial f_o}{\partial t} = \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt}$$

$$\approx [1.575][1.2] + [1.975][1.4]$$

$$= 1.89 + 2.765$$

$$= 4.655$$

Note that f_o is measured in Hz and *t* is measured in seconds. Therefore, the units of $\partial f_o / \partial t$ are Hz per second. Since the result is positive, it can be said that the frequency is increasing at the rate $\boxed{4.655\ \text{Hz/s}}$.

Answer 45E.

Consider the function $z = f(x, y)$.

where $x = r\cos\theta$, $y = r\sin\theta$

(a)

The objective is to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

The Chain rule:

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(\theta)$ and $y = h(\theta)$ are both differentiable functions of θ. Then z is a differentiable function of θ and

$$\frac{dz}{d\theta} = \frac{\partial f}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial f}{\partial y} \cdot \frac{dy}{d\theta}$$

Consider $x = r\cos\theta$

Take partial differentiate with respect to r,

$$\frac{\partial x}{\partial r} = \cos\theta$$

Take partial differentiate with respect to θ,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

And $y = r\sin\theta$

Take partial differentiate with respect to r,

$$\frac{\partial y}{\partial r} = \sin\theta$$

Take partial differentiate with respect to θ,

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

Use Chain rule,

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta$$

And,

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot(-r\sin\theta)+\frac{\partial z}{\partial y}\cdot r\cos\theta$$

$$\frac{\partial z}{\partial \theta}=-r\sin\theta\cdot\frac{\partial z}{\partial x}+r\cos\theta\cdot\frac{\partial z}{\partial y}$$

Therefore, the partial derivatives are,

$$\frac{\partial z}{\partial r}=\boxed{\cos\theta\cdot\frac{\partial z}{\partial x}+\sin\theta\cdot\frac{\partial z}{\partial y}}$$

$$\frac{\partial z}{\partial \theta}=\boxed{-r\sin\theta\cdot\frac{\partial z}{\partial x}+r\cos\theta\cdot\frac{\partial z}{\partial y}}.$$

(b)

The objective is to show that,

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Consider the right hand side of the above equation,

$$\begin{aligned}
&\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\\
&=\left(\cos\theta\cdot\frac{\partial z}{\partial x}+\sin\theta\cdot\frac{\partial z}{\partial y}\right)^2+\frac{1}{r^2}\left(-r\sin\theta\cdot\frac{\partial z}{\partial x}+r\cos\theta\cdot\frac{\partial z}{\partial y}\right)^2\\
&=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta\cos\theta\\
&\qquad+\frac{1}{r^2}\left[\left(\frac{\partial z}{\partial x}\right)^2\cdot r^2\sin^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\cdot r^2\cos^2\theta-2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot r^2\sin\theta\cos\theta\right]
\end{aligned}$$

Continuation of the above step:

$$=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$$+\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta-2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial z}{\partial y}\right)^2\left(\sin^2\theta+\cos^2\theta\right)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2(1)+\left(\frac{\partial z}{\partial y}\right)^2(1)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Thus, $\boxed{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}$.

Answer 46E.

$$u=f(x,y)$$

Where $x=e^s\cos t$, $y=e^s\sin t$

Then $\frac{\partial x}{\partial s}=e^s\cos t$, $\frac{\partial x}{\partial t}=-e^s\sin t$

$\frac{\partial y}{\partial s}=e^s\sin t$, $\frac{\partial y}{\partial t}=e^s\cos t$

Now the chain rule gives

$$\frac{\partial u}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

i.e. $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t$ --------------- (1)

And $\frac{\partial u}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

i.e. $\frac{\partial u}{\partial t}=-\frac{\partial u}{\partial x}e^s\sin t+\frac{\partial u}{\partial y}e^s\cos t$ --------------- (2)

Now $\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t+\left(\frac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$

$$+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t+\left(\frac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t$$

$$+\left(\frac{\partial u}{\partial y}\right)^2 e^{2s}\cos^2 t-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t$$

$$=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]e^{2s}$$

Hence $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]$

Answer 47E.

$$z=f(x-y)$$

Let $x-y=u$

Then $\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=-1$

By chain rule

$$\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}$$

$$=f'(u)(1)$$

i.e. $\frac{\partial z}{\partial x}=f'(x-y)$

And $\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}$

$$=f'(u)(-1)$$

$$=-f'(x-y)$$

Then $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=f'(x-y)-f'(x-y)=0$

Hence $\boxed{\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0}$

Answer 48E.

$$z = f(x, y)$$

Where $x = s + t$ and $y = s - t$

Then $z = f\left(x(s,t), y(s,t)\right)$

By chain rule,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

And $$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

But $$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}, \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

$$\frac{\partial x}{\partial s} = 1, \frac{\partial x}{\partial t} = 1, \frac{\partial y}{\partial s} = 1, \frac{\partial y}{\partial t} = -1$$

Therefore $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

And $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$$

Then $$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

i.e. $$\boxed{\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2}$$

Hence proved

Answer 49E.

$$z = f(x+at) + g(x-at)$$

Let $x+at = u$, $x-at = v$

Then $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial t} = a$

$\frac{\partial v}{\partial x} = 1$, $\frac{\partial v}{\partial t} = -a$

By chain rule

$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x}$$

And $\frac{\partial z}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t}$

i.e. $\frac{\partial z}{\partial x} = f'(u)(1) + g'(v)(1)$

$= f'(u) + g'(v)$

And $\frac{\partial z}{\partial t} = af'(u) - ag'(v)$

Again differentiating partially

$$\frac{\partial^2 z}{\partial x^2} = f''(u)\frac{\partial u}{\partial x} + g''(v)\frac{\partial v}{\partial x}$$

$= f''(u) + g''(v)$

And $\frac{\partial^2 z}{\partial t^2} = a f''(u)\frac{\partial u}{\partial t} - a g''(v)\frac{\partial v}{\partial t}$

$= a^2 f''(u) + a^2 g''(v)$

i.e. $\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at)$

And $\frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at)$

$= a^2\left[f''(x+at) + g''(x-at)\right]$

$= a^2 \frac{\partial^2 z}{\partial x^2}$

i.e. $\boxed{\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}}$

Hence proved

Answer 50E.

$$u = f(x,y)$$

Where $x = e^s \cos t$, $y = e^s \sin t$

Then $u = f(x(s,t), y(s,t))$

By chain rule,

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

And $$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

But $$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}, \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

$$\frac{\partial x}{\partial s} = e^s \cos t, \frac{\partial x}{\partial t} = -e^s \sin t$$

$$\frac{\partial y}{\partial s} = e^s \sin t, \frac{\partial y}{\partial t} = e^s \cos t$$

Therefore $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t$$

And $$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} e^s \sin t + \frac{\partial u}{\partial y} e^s \sin t$$

Now $$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left[\frac{\partial u}{\partial x} e^s \cos t\right] + \frac{\partial}{\partial s}\left[\frac{\partial u}{\partial y} e^s \sin t\right]$$

$$= \frac{\partial^2 u}{\partial x^2} e^s \cos t \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial x \partial y} e^s \cos t \frac{\partial y}{\partial s} + \frac{\partial u}{\partial x} e^s \cos t$$

$$\frac{\partial^2 u}{\partial x \partial y} e^s \sin t \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y^2} e^s \sin t \frac{\partial y}{\partial s} + \frac{\partial u}{\partial y} e^s \sin t$$

$$= \frac{\partial^2 u}{\partial x^2} e^{2s} \cos^2 t + \frac{\partial^2 u}{\partial x \partial y} e^{2s} \cos t \sin t + \frac{\partial^2 u}{\partial x \partial y} e^{2s} \sin t \cos t$$

$$+ \frac{\partial^2 u}{\partial y^2} e^{2s} \sin^2 t + \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t \qquad \text{-------------- (1)}$$

And $$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2} e^s \sin t \frac{\partial x}{\partial t} - \frac{\partial^2 u}{\partial y \partial x} e^s \sin t \frac{\partial y}{\partial t} - \frac{\partial u}{\partial x} e^s \cos t$$

$$\frac{\partial^2 u}{\partial y \partial x} e^s \cos t \frac{\partial x}{\partial t} + \frac{\partial^2 u}{\partial y^2} e^s \cos t \frac{\partial y}{\partial t} - \frac{\partial u}{\partial y} e^s \sin t$$

$$= \frac{\partial^2 u}{\partial x^2} e^{2s} \sin^2 t - \frac{\partial^2 u}{\partial x \partial y} e^{2s} \sin t \cos t - \frac{\partial u}{\partial x} e^s \cos t$$

$$-\frac{\partial^2 u}{\partial x \partial y} e^{2s} \sin t \cos t + \frac{\partial^2 u}{\partial y^2} e^{2s} \cos^2 t - \frac{\partial u}{\partial y} e^s \sin t \qquad \text{------------ (2)}$$

On adding (1) and (2)

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=e^{2s}\frac{\partial^2 u}{\partial x^2}\left[\cos^2 t+\sin^2 t\right]+e^{2s}\frac{\partial^2 u}{\partial y^2}\left[\cos^2 t+\sin^2 t\right]$$

$$+2\frac{\partial^2 u}{\partial x\,\partial y}e^{2s}\cos t\sin t-2\frac{\partial^2 u}{\partial x\,\partial y}e^{2s}\cos t\sin t$$

$$=e^{2s}\frac{\partial^2 u}{\partial x^2}+e^{2s}\frac{\partial^2 u}{\partial y^2}$$

i.e. $$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=e^{2s}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

Then $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{-2s}\left[\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right]$$

Hence proved

Answer 51E.

Consider the function:

$$z=f(x,y)$$

Here x and y are each differentiable functions of r and s.

$$x=r^2+s^2,\ y=2rs$$

The objective is to compute $\partial^2 z/\partial r\partial s$:

To calculate the derivative $\partial^2 z/\partial r\partial s$ we first need to differentiate z with respect to r.

To find the derivative $\partial z/\partial r$ we use the general version of the chain rule:

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{......(1)}$$

Differentiate x and y with respect to r and s.

$$\frac{\partial x}{\partial r}=\frac{\partial}{\partial r}\left(r^2+s^2\right) = 2r \quad \text{And} \quad \frac{\partial x}{\partial s}=\frac{\partial}{\partial s}\left(r^2+s^2\right) = 2s$$

$$\frac{\partial y}{\partial r}=\frac{\partial}{\partial r}(2rs) = 2s \quad \text{And} \quad \frac{\partial y}{\partial s}=\frac{\partial}{\partial s}(2rs) = 2r$$

Substitute the above partial derivatives into formula (1) we get:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}[2r] + \frac{\partial z}{\partial y}[2s] \\ &= 2\left[r\frac{\partial z}{\partial x} + s\frac{\partial z}{\partial y}\right]\end{aligned}$$

Now to find $\partial^2 z / \partial r \partial s$ we differentiate $\partial z / \partial r$ with respect to *s*:

$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial s} &= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial r}\right) \\ &= \frac{\partial}{\partial s}\left(2\left[r\frac{\partial z}{\partial x} + s\frac{\partial z}{\partial y}\right]\right) \\ &= 2\left[\frac{\partial}{\partial s}\left(r\frac{\partial z}{\partial x} + s\frac{\partial z}{\partial y}\right)\right]\end{aligned}$$

We note that both $\partial z / \partial x$ and $\partial z / \partial y$ are functions of *x* and *y*. Hence to differentiate $\partial z / \partial x$ and $\partial z / \partial y$ with respect to *s* we will need to use the chain rule again. This is necessary since *x* and *y* are each functions of *s*.

Therefore,

$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial s} &= 2\left[\frac{\partial}{\partial s}\left(r\frac{\partial z}{\partial x} + s\frac{\partial z}{\partial y}\right)\right] \\ &= 2\left[r\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial s}\left(s\frac{\partial z}{\partial y}\right)\right] \\ &= 2\left[r\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial s}\right) + \frac{\partial}{\partial s}\left(s\frac{\partial z}{\partial y}\right)\right] \\ &= 2\left[r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\right) + \frac{\partial}{\partial s}\left(s\frac{\partial z}{\partial y}\right)\right]\end{aligned}$$

To continue, we need to use the product rule along with the chain rule. Therefore,

$$\frac{\partial^2 z}{\partial r \partial s} = 2\left[r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\right) + \frac{\partial}{\partial s}\left(s\frac{\partial z}{\partial y}\right)\right]$$

$$= 2\left[r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\right) + s\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial s}{\partial s}\right)\right]$$

$$= 2\left[r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\right) + s\left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}(1)\right]$$

$$= 2\left[r\left(\frac{\partial^2 z}{\partial x^2}(2s) + \frac{\partial^2 z}{\partial x \partial y}(2r)\right) + s\left(\frac{\partial^2 z}{\partial y^2}(2r) + \frac{\partial^2 z}{\partial y \partial x}(2s)\right) + \frac{\partial z}{\partial y}\right]$$

$$= 4rs\frac{\partial^2 z}{\partial x^2} + 4r^2\frac{\partial^2 z}{\partial x \partial y} + 4rs\frac{\partial^2 z}{\partial y^2} + 4s^2\frac{\partial^2 z}{\partial y \partial x} + 2\frac{\partial z}{\partial y}$$

$$= 4rs\frac{\partial^2 z}{\partial x^2} + \left(4r^2 + 4s^2\right)\frac{\partial^2 z}{\partial x \partial y} + 4rs\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial y}$$

Therefore, $\partial^2 z / \partial r \partial s = \boxed{4rs\frac{\partial^2 z}{\partial x^2} + \left(4r^2 + 4s^2\right)\frac{\partial^2 z}{\partial x \partial y} + 4rs\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial y}}$.

Answer 52E.

$$z = f(x, y)$$

Where $x = r\cos\theta$, $y = r\sin\theta$

Then $z = f\left(x(r,\theta), y(r,\theta)\right)$

By chain rule

(A)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

But $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

Then $\boxed{\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta}$ ------------ (1)

(B)

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

Using the values of $\frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial \theta}$

$$\boxed{\frac{\partial z}{\partial \theta}=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}} \qquad \text{-------- (2)}$$

(C)

Now differentiating equation (2) partially with respect to r

$$\frac{\partial^2 z}{\partial r\,\partial \theta}=-r\sin\theta\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\right]-\sin\theta\frac{\partial z}{\partial x}$$

$$+r\cos\theta\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\right]+\cos\theta\frac{\partial z}{\partial y}$$

$$=-r\sin\theta\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}-r\sin\theta\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial r}-\sin\theta\frac{\partial z}{\partial x}$$

$$+r\cos\theta\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial r}+r\cos\theta\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}+\cos\theta\frac{\partial z}{\partial y}$$

$$=-r\sin\theta\cos\theta\frac{\partial^2 z}{\partial x^2}-r\sin^2\theta\frac{\partial^2 z}{\partial x\,\partial y}-\sin\theta\frac{\partial z}{\partial x}$$

$$+r\cos\theta\frac{\partial^2 z}{\partial x\,\partial y}+r\cos\theta\sin\theta\frac{\partial^2 z}{\partial y^2}+\cos\theta\frac{\partial z}{\partial y}$$

Hence

$$\boxed{\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial \theta}&=-r\sin\theta\cos\theta\frac{\partial^2 z}{\partial x^2}+r\cos\theta\sin\theta\frac{\partial^2 z}{\partial y^2}\\&\quad+r\left(\cos^2\theta-\sin^2\theta\right)\frac{\partial^2 z}{\partial x\,\partial y}-\sin\theta\frac{\partial z}{\partial x}+\cos\theta\frac{\partial z}{\partial y}\end{aligned}}$$

Answer 53E.

Consider the function

$z=f(x,y)$

Where $x=r\cos\theta$, $y=r\sin\theta$

The Chain rule:

Suppose that $z=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now

$$x = r\cos\theta$$

Take partial differentiate with respect to r

$$\frac{\partial x}{\partial r} = \cos\theta$$

Take partial differentiate with respect to θ

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

And $y = r\sin\theta$

Take partial differentiate with respect to r

$$\frac{\partial y}{\partial r} = \sin\theta$$

Take partial differentiate with respect to θ

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

By chain rule

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \quad \text{......(1)}$$

Then

$$\frac{1}{r}\frac{\partial z}{\partial r} = \frac{\cos\theta}{r}\frac{\partial z}{\partial x} + \frac{\sin\theta}{r}\frac{\partial z}{\partial y} \quad \text{......(2)}$$

And

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \quad \text{......(3)}$$

Again differentiating (3) partially with respect to θ

$$\frac{\partial^2 z}{\partial\theta^2} = -r\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) - r\cos\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right) - r\sin\theta\frac{\partial z}{\partial y}$$

$$= -r\sin\theta\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta} - r\sin\theta\frac{\partial^2 z}{\partial x\partial y}\frac{\partial y}{\partial\theta} - r\cos\theta\frac{\partial z}{\partial x}$$

$$+ r\cos\theta\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial\theta} + r\cos\theta\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta} - r\sin\theta\frac{\partial z}{\partial y}$$

$$= r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2} - r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} - r\cos\theta\frac{\partial z}{\partial x}$$

$$- r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2} - r\sin\theta\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial\theta^2} = r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2} - r\cos\theta\frac{\partial z}{\partial x} - r\sin\theta\frac{\partial z}{\partial y} \quad \text{......(4)}$$

Then

$$\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} = \sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2} - \frac{\cos\theta}{r}\frac{\partial z}{\partial x} - \frac{\sin\theta}{r}\frac{\partial z}{\partial y} \quad \text{......(5)}$$

Now differentiating (1) partially with respect to *r*

$$\frac{\partial^2 z}{\partial r^2} = \cos\theta\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \cos\theta\frac{\partial^2 z}{\partial x\partial y}\frac{\partial y}{\partial r}$$

$$+ \sin\theta\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \sin\theta\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}$$

$$= \cos^2\theta\frac{\partial^2 z}{\partial x^2} + \cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + \cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial r^2} = \cos^2\theta\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2} \quad \text{......(6)}$$

Add equations (2), (5), and (6)

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}$$

$$=\cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\,\partial y}+\sin^2\theta\frac{\partial^2 z}{\partial y^2}$$

$$+\sin^2\theta\frac{\partial^2 z}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\,\partial y}+\cos^2\theta\frac{\partial^2 z}{\partial y^2}$$

$$-\frac{\cos\theta}{r}\frac{\partial z}{\partial x}-\frac{\sin\theta}{r}\frac{\partial z}{\partial y}+\frac{\cos\theta}{r}\frac{\partial z}{\partial x}+\frac{\sin\theta}{r}\frac{\partial z}{\partial y}$$

$$=\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 z}{\partial x^2}+\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}$$

$$=(1)\frac{\partial^2 z}{\partial x^2}+(1)\frac{\partial^2 z}{\partial y^2}\quad\left(\text{Since }\ \sin^2\theta+\cos^2\theta=1\right)$$

Therefore

$$\boxed{\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}}$$

Answer 54E.

$$z=f(x,y)$$

Where $x=g(s,t)$ and $y=h(s,t)$

(A)

Then $z=f\left(x(s,t),y(s,t)\right)$

By chain rule

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\qquad\text{----------------- (1)}$$

Then $$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t}$$

$$+ \frac{\partial z}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)$$

$$= \left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial t}\right]\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2}$$

$$+ \left[\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial t}\right]\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2}$$

$$+ \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

i.e. $$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2$$

$$+ \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

(B)

Differentiating (1) partially with respect to s

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial t}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial}{\partial s}\left(\frac{\partial y}{\partial t}\right)$$

$$= \left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\right]\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t}$$

$$+ \left[\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\right]\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}$$

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t}$$

$$+ \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial x}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}$$

Hence $$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\left[\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial s}\frac{\partial y}{\partial t}\right]$$

$$+ \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}$$

Answer 55E.

(a)

Consider the function

$$f(x,y) = x^2y + 2xy^2 + 5y^3$$

Check whether the function is homogenous or not.

By the definition of the homogenous function,

A function $f(x,y)$ is homogenous if $f(tx,ty) = t^n f(x,y)$

Consider

$$f(tx,ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3$$

$$= t^3x^2y + 2t^3xy^2 + 5t^3y^3$$

$$= t^3\left(x^2y + 2xy^2 + 5y^3\right) \text{ Plug out } t^3$$

$$= t^3 f(x,y)$$

Clearly, $f(tx,ty) = t^3 f(x,y)$

Thus, *f* is a homogenous function of degree 3.

(B)

Suppose that *f* is a homogeneous function of degree *n*

Then

$$f(tx,ty) = t^n f(x,y)$$

Let $tx = u$ and $ty = v$

Then $f(tx,ty) = f(u,v)$

Differentiate this with respect to *x* and *y*.

Then

$$\frac{\partial u}{\partial x} = t,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = t$$

Differentiate $f(u,v)$ with respect to *x*.

$$\frac{\partial f(u,v)}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \text{ Use chain rule}$$

$$= \frac{\partial f}{\partial u}t \quad \text{...... (1)}$$

$$\frac{\partial f(u,v)}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \text{ Use chain rule}$$

$$= \frac{\partial f}{\partial v}t \quad \text{...... (2)}$$

Multiply (1) by *x* and (2) by *y* and add them as follows.

$$x\frac{\partial f(u,v)}{\partial x}+y\frac{\partial f(u,v)}{\partial y}=x\frac{\partial f}{\partial u}t+y\frac{\partial f}{\partial v}t$$ Add on both sides

$$x\frac{\partial f(u,v)}{\partial x}+y\frac{\partial f(u,v)}{\partial y}=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$$

$$x\frac{\partial}{\partial x}\left(f(tx,ty)\right)+y\frac{\partial}{\partial y}\left(f(tx,ty)\right)=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$$ Replace *u* by tx and *v* by ty

$$x\frac{\partial}{\partial x}\left(t^n f(x,y)\right)+y\frac{\partial}{\partial y}\left(t^n f(x,y)\right)=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$$ As *f* is homogeneous

$$xt^n\frac{\partial f}{\partial x}+t^n y\frac{\partial f}{\partial y}=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$$

$$t^n\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t \quad \ldots\ldots (3)$$

To simplify the expression on the right hand side of (3) use the assumption to simplify.

From the assumption,

$$f(u,v)=f(tx,ty)=t^n f(x,y)$$

$$\frac{\partial f}{\partial t}=nt^{n-1}f(x,y)$$ Differentiate with respect to *t*

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}=nt^{n-1}f(x,y)$$ Use chain rule as *u* and *v* are functions of *t*

$$\frac{\partial f}{\partial u}x+\frac{\partial f}{\partial v}y=nt^{n-1}f(x,y) \quad \ldots\ldots (4)$$

From (3) and (4)

$$t^n\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=nt^{n-1}f(x,y)t$$

$$\boxed{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)}$$

Answer 56E.

Consider that *f* is a homogeneous equation

Recall that,

A function *f* is homogenous of degree *n* if it satisfies the condition

$f(tx,ty)=t^n f(x,y)$ For all *t*, *n* is a positive integer

Need to prove that $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f(x,y)$

Use the condition of homogeneity to prove this.

Assume that, $u=tx$ and $v=ty$

Differentiate these equation partially with respect to *x*, *t* and *y*.

Differentiate *u*, *v* with *x*.

$\frac{du}{dx}=t$, $\frac{dv}{dx}=0$

Differentiate with *t*.

$\frac{du}{dt}=x$, $\frac{dv}{dt}=y$

Differentiate with *y*.

$\frac{du}{dy}=0$, and $\frac{dv}{dy}=t$

As per the assumption

$f(u,v)=f(tx,ty)$

Use chain rule to differentiate $f(u,v)$.

Recall that,

Suppose that $z=f(x,y)$ is a differentiable function of *x* and *y* where $x=g(t)$ and $y=h(t)$ are differentiable functions of *t*. then

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

Differentiate $f(u,v)$ with respect to x.

$$\frac{\partial f(u,v)}{\partial x}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}$$

$=\frac{\partial f}{\partial u}t$ Use $\frac{du}{dx}=t$, $\frac{dv}{dx}=0$

And $\frac{\partial f(u,v)}{\partial y}=\frac{\partial f}{\partial u}\frac{du}{dy}+\frac{\partial f}{\partial v}\frac{dv}{dy}$

$=\frac{\partial f}{\partial v}t$ Use $\frac{du}{dy}=0$, $\frac{dv}{dy}=t$

As, $f(u,v)=f(tx,ty)=t^n f(x,y)$

Then

$$\frac{\partial f}{\partial t}=nt^{n-1}f(x,y)$$

$\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}=nt^{n-1}f(x,y)$ Use chain rule for $f(u,v)$

$\frac{\partial f}{\partial u}x+\frac{\partial f}{\partial v}y=nt^{n-1}f(x,y)$ Use $\frac{du}{dt}=x$ and $\frac{dv}{dt}=y$ ...... (1)

To use $\frac{\partial f}{\partial u}x+\frac{\partial f}{\partial v}y$ value find $x\frac{\partial f(u,v)}{\partial x}+y\frac{\partial f(u,v)}{\partial y}$.

$$x\frac{\partial f(u,v)}{\partial x}+y\frac{\partial f(u,v)}{\partial y}=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$$

$$x\frac{\partial}{\partial x}(f(tx,ty))+y\frac{\partial}{\partial y}(f(tx,ty))=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$$

$xt^n\frac{\partial f}{\partial x}+t^n y\frac{\partial f}{\partial y}=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$ Use $f(tx,ty)=t^n f(x,y)$

$t^n\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)t$ ...... (2)

From (1) and (2)

$$t^n\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=n\,t^{n-1}f\left(x,y\right)t$$

$$t^n\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=n\,t^{n}f\left(x,y\right)$$

$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=n\,f\left(x,y\right)$ Cancel out t^n on each side

...... (3)

Differentiate (3) partially with respect to *x*

$$\frac{\partial f}{\partial x}+x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\,\partial y}=n\frac{\partial f}{\partial x} \quad ...... (4)$$

Differentiating (3) partially with respect to *y*

$$x\frac{\partial^2 f}{\partial y\,\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}=n\frac{\partial f}{\partial y} \quad ...... (5)$$

Multiply (4) by *x* and (5) by *y*.

$$x\left(\frac{\partial f}{\partial x}+x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\,\partial y}=n\frac{\partial f}{\partial x}\right)$$

$$x\frac{\partial f}{\partial x}+x^2\frac{\partial^2 f}{\partial x^2}+xy\frac{\partial^2 f}{\partial x\,\partial y}=nx\frac{\partial f}{\partial x}$$

$$y\left(x\frac{\partial^2 f}{\partial y\,\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}\right)=ny\frac{\partial f}{\partial y}$$

$$xy\frac{\partial^2 f}{\partial y\,\partial x}+y\frac{\partial f}{\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=ny\frac{\partial f}{\partial y}$$

Add them as follows. And also use the result $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$$x\frac{\partial f}{\partial x} + x^2\frac{\partial^2 f}{\partial x^2} + xy\frac{\partial^2 f}{\partial x \partial y} + \left(xy\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial f}{\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) = nx\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right]$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} + nf(x,y) = n \cdot nf(x,y)$$ Use (3)

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n^2 - n)f(x,y)$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x,y)$$

Thus, $\boxed{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x,y)}$.

Answer 57E.

Show that

$$f_x(tx, ty) = t^{n-1}f_x(x, y)$$

We can use the Chain Rule to differentiate the equation $f(tx, ty) = t^n f(x, y)$

$$\frac{\partial}{\partial x}f(tx, ty) = \frac{\partial}{\partial x}\left[t^n f(x, y)\right]$$

$$\frac{\partial}{\partial (tx)}f(tx, ty) \cdot \frac{\partial (tx)}{\partial x} + \frac{\partial}{\partial (ty)}f(tx, ty) \cdot \frac{\partial (ty)}{\partial x} = t^n \frac{\partial}{\partial x}f(x, y)$$

$$tf_x(tx, ty) = t^n f_x(x, y)$$

$$f_x(tx, ty) = \frac{t^n f_x(x, y)}{t}$$

$$f_x(tx, ty) = t^{n-1}f_x(x, y)$$

Answer 58E.

Consider the equation $F(x,y,z)=0$ implicitly defines each of the three variables *x*, *y*, and *z* as a function of other two.

We find each derivative with implicit differentiation. We have an equation of the form $F(x,y,z)=0$ which defines *z* implicitly as a function of *x* and *y*, i.e. that $z=f(x,y)$ for all (x,y) in the domain of *f*. Then, since $F_z \neq 0$, we solve for $\partial z/\partial x$ by the formula:

$$\frac{\partial z}{\partial x}=\frac{-\partial F/\partial x}{\partial F/\partial z}$$

Similarly, since *F* defines *x* as a function of *y* and *z* and $F_x \neq 0$, we solve for $\partial x/\partial y$ by the formula:

$$\frac{\partial x}{\partial y}=\frac{-\partial F/\partial y}{\partial F/\partial x}$$

Finally, since *F* defines *y* as a function of *x* and *z* and $F_y \neq 0$, we solve for $\partial y/\partial z$ by the formula:

$$\frac{\partial y}{\partial z}=\frac{-\partial F/\partial z}{\partial F/\partial y}$$

Hence by substitution, the product is:

$$\begin{aligned}\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}&=\left(\frac{-\partial F/\partial x}{\partial F/\partial z}\right)\left(\frac{-\partial F/\partial y}{\partial F/\partial x}\right)\left(\frac{-\partial F/\partial z}{\partial F/\partial y}\right)\\&=\frac{(-\partial F/\partial x)(-\partial F/\partial y)(-\partial F/\partial z)}{(\partial F/\partial z)(\partial F/\partial x)(\partial F/\partial y)}\\&=\left(\frac{-\partial F/\partial x}{\partial F/\partial x}\right)\left(\frac{-\partial F/\partial y}{\partial F/\partial y}\right)\left(\frac{-\partial F/\partial z}{\partial F/\partial z}\right)\\&=(-1)(-1)(-1)\end{aligned}$$

$$=\boxed{-1}$$

Answer 59E.

Given that $F(x,y)=0$.

Then, $\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy = 0$ or $\frac{dy}{dx} = -\frac{F_x}{F_y}$, $F_y \neq 0$.

We have to find $\frac{d^2y}{dx^2}$ given by $\frac{d}{dx}\left[-\frac{F_x}{F_y}\right]$.

$$\frac{d^2y}{dx^2} = \frac{F_y\left[-F_{xx}-F_{xy}y'\right]-(-F_x)\left(F_{yx}+F_{yy}y'\right)}{F_y^2}$$

$$= \frac{\left[-F_{xx}F_y - F_{xy}F_y\left(-\frac{F_x}{F_y}\right)\right]+\left[F_xF_{yx}+F_xF_{yy}\left(-\frac{F_x}{F_y}\right)\right]}{F_y^2}$$

$$= \frac{-F_{xx}F_y^2+F_xF_yF_{xy}+F_xF_yF_{xy}-F_x^2F_{yy}}{F_y^3}$$

$$= \frac{-F_{xx}F_y^2+2F_xF_yF_{xy}-F_x^2F_{yy}}{F_y^3}$$

Thus, we have proved that $\boxed{\frac{d^2y}{dx^2} = \frac{-F_{xx}F_y^2+2F_xF_yF_{xy}-F_x^2F_{yy}}{F_y^2}}$.