CHAPTER 02

Combinatorics

2.1 Introduction

Objects (or things) can be arranged in many ways. Suppose there are three objects marked *a, b, c* on a table from these, two objects can be selected at a line in three different ways as {*a, b*}, {*a, c*},{*b, c*}. In this way selection of two objects from three objects in three ways is called **Combinations.**

The above selection *ab, ac* can also be arranged as *ab, ba, ac, ca, bc, cb*. We can understand that two objects can be selected from three objects and arranged in six ways. These arrangements are called **Permutations.**

Fundamental Concepts

If A is a finite set, then the number of different elements in A is denoted by $n(A)$.e.g.,

If A = {2, 5, 7} then $n(A) = 3$

If $C = \phi$ then $n(C) = 0$

Let us assume that there are three routes say a_1, a_2, a_3 from Delhi to Noida and there are two routes, say b_1, b_2, from Noida to Agra. It may be written as:

$A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$

$n(A) = 3$; $n(B) = 2$

Now we can match the route a_1 from D to N with two routes b_1, b_2 from N, A

i.e., $(a_1, b_1), (a_1, b_2)$

Similarly the remaining routes can be written as

$(a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)$.

So to travel from D to A via N, there are 6 different routes

$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_2)$.

These 6 ways are nothing but the elements of the cartesian product of the two sets A and B.

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$

$n(A \times B) = 6 = 2 \times 3 = n(A) \times n(B)$.

Fundamental Multiplication Principle of Counting

"Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible out comes and if for each outcome of experiment 1 there are possible out comes of experiment 2, then together there are mn possible outcomes of the two experiments".

The Generalized basic Multiplication Principle of Counting

"If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment and if for each of the possible outcomes the first two experiments there are n_3 possible outcomes of the third experiment and so on, then there is a total of $n_1 \times n_2 \dots n_r$ possible outcomes of the r experiments".

Keywords to distinguish permutations from combinations:

Permutations : ordered, arrangement, sequence

Combinations : unordered, selection, set

Some useful properties from Number theory used in Combinatorics:

1. *Method for finding the number of positive divisors of a positive integer n* : If a positive integer n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \dots$ where p_1, p_2 etc. are distinct prime numbers, then the number of positive divisors of n is given by the formula $(n_1 + 1)(n_2 + 1)\dots$
   For example, the number 80 can be broken as $80 = 2^4 \times 5^1$. So the number of positive divisors of 80 is given by $(4 + 1)(1 + 1) = 10$.
2. *Method for finding the number of numbers from 1 to n, which are relatively prime to n* : The number of numbers from 1 to n, which are relatively prime to n i.e., $\gcd(m, n) = 1$, is given by the Euler Totient function $\phi(n)$. If n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \dots$ where p_1, p_2 etc. are distinct prime numbers, then $\phi(n) = \phi(p_1^{n_1})\,\phi(p_2^{n_2})\dots$ then by using the property

$$\phi(p^k) = p^k - p^{k-1}.$$

   we can find each of $\phi(p_1^{n_1}), \phi(p_2^{n_2})\dots$ etc.

   For *example*, the number of numbers from 1 to n, which are relatively prime to 80 can be found as follows: Since $80 = 2^4 \times 5^1$

The number of numbers from 1 to n, which are relatively prime to $80 = \phi(80) = \phi(2^4) \times \phi(5^1)$

Now $\phi(2^4) = 2^4 - 2^3 = 16 - 8 = 8$

Similarly, $\phi(5^1) = 5^1 - 5^0 = 5 - 1 = 4$

So, $\phi(80) = 8 \times 4 = 32$

2.2 Permutations

Permutations with no Repetitions

When we select objects from a set consisting of n distinct objects taking each object exactly once (no repetition), and then arrange them in a straight line, this situation is called permutations with no repetition.

The formula for counting this is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \dots (n-r+1)$$

Example-2.1 How many 2 letter passwords are there using the letters {a, b, c} if no letter is allowed to be used more then once?

Solution:

$$^3P_2 = \frac{3!}{(3-2)!} = 3 \times 2 = 6$$

The 6 permutations are ab, ba, ac, ca, bc cb.

Example-2.2 How many ways can four (distinct) dolls be arranged in a straight line?

Solution:

$$^4P_4 = \frac{4!}{(4-4)!} = 4! = 4 \times 3 \times 2 \times 1 = 24 \text{ ways.}$$

Alternately more problems can be solved by box method, which is more general and more powerful. For example arranging 4 dolls can be thought of as filling 4 boxes corresponding to position of the dolls.

The first box can be filled in 4 ways. Since repetition is not allowed, the second box can be filled only in 3 ways.

The third box in 2 ways and last box in only 1 way.

$\therefore$ Total arrangements = $4 \times 3 \times 2 \times 1 = 24$ ways.

Permutations with Unlimited Repetition

When we select an object, from a set of distinct objects, taking each object any number of times (unlimited repetition) and arrange them in a straight line, this situation is called permutation with unlimited repetition. The formula for counting this is n^r.

Example-2.3 How many 2 letter passwords can be made from {*a*, *b*, *c*}, if a letter can be used any number of times?

Solution:

$3^2 = 9$ passwords

The nine permutations are : *aa, ab, ac, ba, bb, bc, ca, cb, cc*.

NOTE

For objects such as passwords & number, if nothing is mentioned regarding repetition, the default assumption is that unlimited repetition is allowed.

Using box method we make the password by filling 2 boxes. Each can be filled in 3 ways (since repetition allowed) $3 \times 3 = 9$ passwords.

Example-2.4 If there are 10 multiple choice question with four choices for each question, How many answer sheets are possible?

Solution:

5^{10} answer sheets. We can think of each of the question as a box and each of the 10 boxes can be filled is 5 ways (Choice *a*, *b*, *c* or *d* or leave it blank):

$\therefore$ The number of ways of filling up all 10 boxes is $5 \times 5 \times 5 \times \ldots$ 10 times = 5^{10}

The box method is very powerful for use in all permutation problems (with repetition or without repetition)

Permutations with Limited Repetitions

There are situations where the objects can be used only a limited number of times, due to limited availability. In such cases if n_1 repetition of object 1, n_2 repetition of object 2, ..., n_r repetition of object r is allowed, and total number of objects = n, then the number of permutations is given by

$$P(n; n_1; n_2 ... n_r) = \frac{n!}{n_1! n_2! .... n_r!}$$

Example-2.5 How many ways are there to arrange 3 indentical blue flags, 2 indentical red flags and 3 indentical yellow flags in a straight line?

Solution:

$$\frac{8!}{3!2!3!} \text{ ways}$$

Example-2.6 How many ways are there to arrange the letters of the village "VALLIKKAVU". In other words, how many distinct anagrams of VALLIKKAVU are there? There are 2 V's, 2 A's, 2L's, 2K's, 1I, 1U

Solution:

$$\frac{10!}{2!2!2!2!1!1!} = \frac{10!}{(2!)^4}$$

Circular Permutations

The number of ways of arranging n objects in a circle is $(n-1)!$ This is less than $n!$, since in a circle many of the linear permutations become indistinguishable except for rotation.

This formula is derived by putting the first object in any one of the n positions on a circle and then the remaining $(n-1)$ position can be filled by $(n-1)$ objects in $(n-1)!$ ways.

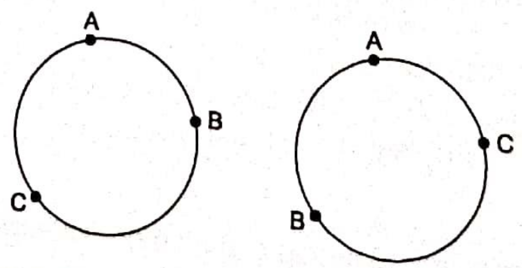
$\therefore 1 \times (n-1)! = (n-1)!$ ways for circular permutation.

Example-2.7 How many ways are there to arrange 3 children in a circle?

Solution:

$$(3-1)! = 2! = 2 \text{ ways}$$

Let the children A, B, C. The two arrangements on shown below.



If clockwise and anti-clockwise arrangements are also considered to be same, then no of circular permutations (disregarding clockwise and anti-clockwise arrangements) is $\frac{(n-1)!}{2}$

i.e. in the 3 child example; $\therefore$ disregarding clockwise and anticlockwise variations, $\frac{(3-1)!}{2} = 1$ ways only.

2.3 Combinations

Combinations with no Repetitions

Let n , r are integers. If $n \geq r$, for the set of n elements a sub set of r elements is called a "Combination". Therefore a combination is an unordered selection of r elements from a set of n elements.
The number of combination of r elements selected from n elements is denoted by ${}^nC_r$ or (n, r).
Here the same element cannot be selected more than once (no repetitions allowed)

Formula: $${}^nC_r = \frac{n!}{r!(n-r)!}$$

Results:
1. ${}^nC_r = {}^nC_{n-r}$
2. ${}^nC_r = 1$
3. ${}^nC_0 = 1$

Pascal's formula: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

Combinations with Unlimited Repetitions

The number of combinations of r elements selected from n elements, when same element can be selected any number of times is given by ${}^{n-1+r}C_r$ which can also be written as ${}^{n-1+r}C_{n-1}$.

Example-2.8 10 CD's are available for discount in a music shop. How many ways can a shopper select 3 CD's it if he can choose same CD any number of times? Here; $n = 10$, $r = 3$.

Solution:

$${}^{10-1+3}C_3 = {}^{12}C_3 = \frac{12 \times 11 \times 10}{1 \times 2 \times 3} = 220 \text{ ways}$$

This same formula can be used in distribution problems involving indentical objects:

Example-2.9 How many ways can we distribute 10 identical balls into 4 (distinct) boxes there:

r = Number of objects to be distributed = 10
n = Number of boxes = 4

Solution:
The same problem can also be used to find the number of non negative integral solutions to the equation

$$x_1 + x_2 + x_3 \dots + x_n = r$$

Where, $x_1 \geq 0, x_2 \geq 0, \dots x_n \geq 0$

Solution using ${}^{n-1+r}C_r$:
Notice that question can also be reduced to non negative integral solution problem as follows.
Let x_1, x_2, x_3, x_4 be the number of balls put into boxes 1, 2, 3 and 4 respectively. Since we have totally 10 balls.

$\therefore \quad x_1 + x_2 + x_3 + x_4 = 10$

Also we can only put 0 or more balls in each box.

$\therefore x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

$${}^{n-1+r}C_r = {}^{4-1+10}C_{10} = {}^{13}C_{10} = {}^{13}C_3 \text{ ways}$$

Distribution of Distinct Objects

When the problem involved distributing distinct objects to distinct people the formula to be used is $\frac{n!}{n_1!n_2!...n_r!}$. This is the same formula used in permutation with limited repetition, but the distribution problem here in different, although we are using the same formula.

Example-2.10 How many ways can we deal a deck of 52 cards to 4 people giving equal no of card to each?

Solution:

$$\frac{52!}{13!13!13!13!} = \frac{52!}{(13!)^4}$$

Example-2.11 How many ways can we divide a group of 10 boys into 3 teams such that team 1 has 2 boys, team 2 has 5 boys and team 3 has 3 boys?

Solution:

$$\frac{10!}{2!5!3!} \text{ ways}$$

Notice that in both the above problem, within the teams there is no particulars order of the boys, although the teams themselves than order (as in team 1, 2 and 3). Similarly also the card distribution problem. Although the 4 people are distinct, the 13 cards given to a person is in no particulars order.

∴ These problems are called ordered partitions.

The general formula for ordered partition of n objects of the type $(n_1, n_2, n_3, ..., n_r)$ is given by

$$P(n, n_1, n_2, n_3, ... n_r) = \frac{n!}{n_1!n_2!n_3!...n_r!}$$

(Which is really same as the permutation with limited repetition formula).

Now if the type of partition is unspecified we must take every possible type of partitioning into account.

Now the number of ways of distributing n distinct objects to r distinct people is

$$= \Sigma P(n, n_1, n_2, n_3 ... n_r) = n_r$$

$$n_1 + n_2 + ... n_r = n$$

(which is really the same as formula for permutations with unlimited repetitions).

Example-2.12 How many ways can 4 (distinct) dolls be distributed amongst 3 (distinct) children? Here $n = 4$, $r = 3$

Solution:

Alternatively we can think of each 4 dolls as boxes each of which can be issued 3 ways (3 children)

$$3 \times 3 \times 3 \times 3 = 3^4 \text{ ways} = 81 \text{ ways}$$

Notice that a child may receives 0 or more dolls.

If the above problem is changed follows: How many ways can 4 dolls be given to 3 children such that first child gets 2 dolls, second 1 doll and third 1 doll, then the answer would be $P(4; 2, 1, 1) = \frac{4!}{2!1!1!}$

(Since the method of distributing the dolls is specified in this case).

Unordered Partition

In the above problem the children were distinct.
But consider the problem of dividing 6 dolls into 3 groups of 2 dolls each.
Now this is a problem of unordered partition.

The number of ways of dividing 6 dolls into 3 groups of 2 dolls each are $\frac{6!}{(2!)^3 3!}$.

The general formula for dividing n objects into t groups of r objects is given by $\frac{n!}{(r!)^t t!}$.

We have to divide by t! the ordered the ordered partition formula since t!, permutations are indistinguishable as the groups or (cells as they are called) are indistinguishable in order.

2.4 Binomial Identities

Binomial Theorem

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots\binom{n}{k}x^k+\ldots\binom{n}{n}x^n \quad \ldots(2.1)$$

There are some basic properties of binomial coefficients:

1. **Symmetry identity:**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \quad \ldots(2.2)$$

This identity says that the number of ways to select a subset of k objects out of a set of n objects is equal to the number of ways to select a subset of $(n-k)$ of the objects to set aside, for rejection.

2. **Pascal's identity:**

$$\binom{n}{k} = \binom{n-1}{k}+\binom{n-1}{k-1} \quad \ldots(2.3)$$

The left hand side counts the number of ways of selecting to k things from n things.
The r. h. s. counts the same as a sum of two types of selections.
One selection ${}^{n-1}C_k$ is the number of ways of selecting the k things so that a particular element say "x" is excluded and the other selection ${}^{n-1}C_{k-1}$ is the number of ways of selecting the k things so that same particular element "x" is always included.
In addition to these identities there are a few more important identities to remember:

3. **Newton's identity:**

$$\binom{n}{k}\cdot\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \quad \ldots(2.4)$$

The left hand side counts the ways to select a group of k people choosen from a set of n people and then to select a subset of m leaders from the group of k people.
Equivalently, as counted on the right side, we could first select the subset of m leaders from the set of n people and then select the remaining $k-m$ non leading members of the group from $(n-m)$ people.

4. **Row summation:**

$$\sum_{k=0}^{n} {}^nC_k = {}^nC_0 + {}^nC_1 + {}^nC_2 \ldots + {}^nC_n = C(n, 0) + C(n, 1) + C(n, 2) \ldots C(n, n) = 2^n$$

5. **Alternating sign row summation:**

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 \ldots (-1)^n\, {}^nC_n = 0$ or

$({}^nC_0 + {}^nC_2 + {}^nC_4 \ldots +) = ({}^nC_1 + {}^nC_3 + {}^nC_5 \ldots) = 2^{n-1}$

i.e. even summation = odd summation = 2^{n-1}

Example: Row summation: ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 = 16$

Even summation and Odd summation:

${}^4C_0 + {}^4C_2 + {}^4C_4 = {}^4C_1 + {}^4C_3 = 2^{4-1} = 8$

The row summation identity can be understood as RHS = 2^n = no of subsets of a set of n elements. Now, LHS is the same counted as no of 0 element subsets (= ${}^nC_0$) + No of 1 element subsets (= ${}^nC_1$) + no of 2 element subsets (= ${}^nC_2$) ... and so m + no of n element subsets (= ${}^nC_r$). Obviously these two counts should be equal. i.e. LHS ≡ RHS.

6. **Column summation:**

$$\sum_{k=0}^{n} {}^kC_r = \sum_{k=r}^{n} {}^kC_r = {}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r \ldots + {}^nC_r = {}^{n+1}C_{r+1}$$

Example: ${}^2C_2 + {}^3C_2 + {}^4C_2 = {}^5C_3$

7. **Vandermondes indentity:**

${}^{n+m}C_r = {}^nC_0\,{}^mC_r + {}^nC_1\,{}^mC_{r-1} + {}^nC_2\,{}^mC_{r-2} \ldots {}^nC_r\,{}^mC_0$

This indentity can be understood as selecting r objects from a group of $n + m$ objects as written in LHS.

RHS counts the same as, selecting 0 objects from group of n objects and r objects from group of m objects or selecting 1 object from group of n objects and $r - 1$ objects from group of m objects and so on ...

∴ LHS ≡ RHS

8. **Row square summation:**

Special case of vandermodes indentity is row square summation obtained by putting $n = m$ and $r = n$ in vandermondes identity.

$$\begin{aligned} {}^{2n}C_n &= {}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + \ldots {}^nC_n\,{}^nC_0 \\ &= {}^nC_0\,{}^nC_0 + {}^nC_1\,{}^nC_1 + \ldots {}^nC_n\,{}^nC_n \\ &= {}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 \ldots + {}^nC_n^2 \end{aligned}$$

∴ $$\sum_{k=0}^{n} {}^nC_k^2 = {}^{2n}C_n$$

9. Another special case of Vandermonde's identity is obtained by putting $r = n$

$${}^{n+m}C_n = {}^nC_0\,{}^mC_n + {}^nC_1\,{}^mC_{n-1} + \ldots {}^nC_n\,{}^mC_0$$

10. Yet another binomial indentity is written as follows:

∴ $$\sum_{r=1}^{n} r\,{}^nC_r = 1\,{}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 \ldots + n\,{}^nC_n = n\,2^{n-1}$$

2.4.1 Multinomial Coefficients

$$(x_1 + x_2 + ... + x_t)^n = \Sigma P(n, q_1, q_2, ... q_t) = x_1^{q_1} x_2^{q_2} x_3^{q_3} ... x_t^{q_t} \quad q_1 + q_2 ... + q_t = n$$

The coefficient of the term $x_1^3 x_2^4 x_3^2$ in the expansion of $(x_1+x_2+x_3)^9$, would be exactly $P(9; 3, 4, 2) = \frac{9!}{3!4!2!}$

$$\Sigma P(9, q_1, q_2, ... q_t) = 3^9 \quad q_1 + q_2 ... q_t = n$$

e.g. $\Sigma P(q, q_1, q_2, ... q_3) = 3^9 \quad q_1 + q_2 + q_3 + = 9$ (summation of all multinomial coefficients)

2.5 Generating Functions

Consider a sequence $(a_0, a_1, a_2 ..., a_r) = \{a_r\} = \{a_r\}_r^{\infty} = 0$ of real numbers.

We may write a power series of the form $A(X) = a_0X^0 + a_1X^1 + a_2X^2 + a_3X^3 ... + a_rX^r = \sum_{r=0}^{\infty} a_rX^r$

Now, A(X) is called as the generating function corresponding to the sequence a_r.

Generating functions can be successfully used to solve counting problems involving constraints such as in finding the number of ways to fill r balls into n boxes with upper constraints on the no of balls that can go into each box.

The following identities are useful:

$$\sum_{r=0}^{n} X^r = 1 + X + X^2 + ... X^n = \frac{1-X^{n+1}}{1-X} \quad ...(2.5)$$

$$\sum_{r=0}^{\infty} X^r = 1 + X + X^2 + ... = \frac{1}{1-X} \quad ...(2.6)$$

$$\sum_{r=0}^{n} {}^nC_rX^r = 1 + \binom{n}{1}X + \binom{n}{2}X^2 + ... \binom{n}{n}X^n = (1 + X)^n \quad ...(2.7)$$

$$\sum_{r=0}^{\infty} {}^{n-1+r}C_rX^r = \frac{1}{(1-X)^n} \quad ...(2.8)$$

In the equations (2.6) and (2.7) if we replace x by –x we can get.

$$\sum_{r=0}^{\infty} (-1)^r X^r = 1 - X + X^2 - X^3 ... = \frac{1}{1+X}$$

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_rX^r = 1 - \binom{n}{1}X + \binom{n}{2}X^2 - \binom{n}{3}X^3 ... (-1)^n \binom{n}{n}X^n = (1 - X)^n$$

Also : if in equation (2.6), we replace X by aX and $-aX$ we can get

$$\sum_{r=0}^{\infty} a^rX^r = 1 + aX + a^2X^2 + a^3X^3 = \frac{1}{1-aX}$$

$$\sum_{r=0}^{\infty} (-1)^r a^rX^r = 1 - aX + a^2X^2 - a^3X^3 ... = \frac{1}{1+aX}$$

NOTE

- When generating function is written as a series $1 + X + X^2 + ...$ it said to be in open form and when it written instead as $\frac{1}{1-X}$, it is said to be in closed form.
- Notice that the sequence corresponding to (2.1) $\sum_{r=0}^{n} X^r$ is $[a_r]_{r=0}^{n} = 1$.

The sequence corresponding to (2.6) is $[a_r]_{r=0}^{\infty} = 1$.

The sequence corresponding to (2.7) is $\left[{}^{n}C_r\right]_{r=0}^{n}$ and the sequence corresponding (2.8) is $\left\{\left({}^{n-1+r}C_r\right)\right\}_{r=0}^{\infty}$.

Example - 2.13 Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + ...)^5$. What is the coefficient of x^r?

Solution:

To simplify the expression, we extract x^2 from each polynomial factor and then apply identity (2.6).

$$(x^2 + x^3 + x^4 + x^5)^5 = [x^2(1 + x + x^2 + ...)]^5 = x^{10}(1 + x + x^2 + ...)^5 = x^{10}\frac{1}{(1-x)^5}.$$

Thus the coefficient of x^{16} in $(x^2 + x^3 + x^4 + ...)^5$ is the coefficient of x^{16} in $x^{10}(1-x)^{-5}$. But the coefficient of x^{16} in this latter expression will be the coefficient of x^6 in $(1-x)^{-5}$.

Now, we need to find coefficient of X^6 in $\frac{1}{(1-X)^5}$.

We expand $\frac{1}{(1-X)^5}$ using identity (4)

$$\frac{1}{(1-X)^n} = \sum_{r=0}^{\infty} {}^{n-1+r}C_r X^r$$

$$\therefore \quad \frac{1}{(1-X)^5} = \sum_{r=0}^{\infty} {}^{5-1+r}C_r X^r = \sum_{r=0}^{\infty} {}^{r+4}C_r X^r$$

Now the coefficient of X^r in this expansion is ${}^{r+4}C_4$. Therefore, coefficient of X^6 in this expansion would be ${}^{6+4}C_4 = {}^{10}C_4$. Which is the required answer for this problem.

NOTE: Ball in the box problems, when balls are indistinguishable, along with upper unstraints on the no of balls that can be put into a box, can be effectively converted into a problem of finding coefficient of some power of X in the expansion of a corresponding generating function.

2.6 Summation

Basic Results

(i) $1 + 2 + 3 .... + n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

(*iii*) $1^3+2^3+3^3+ \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

In general sum to n terms is denoted as S_n where the summation is of the sequence u_n.

i.e. $$S_n = t_1 + t_2 + \ldots + t_n$$

Arithmetic Progression

$a, a + d, a + 2d, \ldots a + (n-1)d + \ldots$ If the sequence is an A. P. (arithmetic progression) then, $t_n = a + (n-1)d$ (n^{th} term of sequence).

$$S_n = a + (a + d) \ldots = \frac{n}{2}(2a + (n-1)d) \text{ (sum up to } n \text{ terms)}$$

$$= \frac{n}{2}(a + l) \quad \text{(where } l \text{ is the last term)}$$

Geometric Progression

$a, ar, ar^2, \ldots ar^{n-1} + \ldots t_n = ar^{n-1}$

$$S_n = a + ar + ar^2 \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

$$S_\infty = \frac{a}{1-r} \quad [\text{if } |r| < 1]$$

Example: A. P : 1, 3, 5, 7 ...

Here,

$$a = 1$$

$$d = 3 - 1 = 5 - 3 = 2$$

$$u_{20} = 1 + (20 - 1)\,2 = 39$$

$$S_{20} = \frac{20}{2}(2\times1+(20-1)\times2) = 10\,(40) = 400$$

Example: G. P. : 1, 2, 4, 8, 16, ...

Here,

$$a = 1,\ r = \frac{2}{1} = \frac{4}{2} = 2$$

$$u_{10} = 1 \times 2^{(10-1)} = 2^9 = 51$$

$$S_{10} = \frac{a(r^n - 1)}{r - 1} = \frac{1\times(2^{10} - 1)}{2-1} = 2^{10} - 1 = 1023$$

Example: Infinite G. P. : $1, \frac{1}{2}, \frac{1}{4}, \ldots$

Here,

$$a = 1,\ r = \frac{1}{2},\ \text{since } |r| < 1,$$

$\therefore$ $$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

Method of Summation when u_n is a product of r successive terms of an A. P.

Example: 1.4 + 4.7 + 7.10 + ...

The first factor is in A.P. So is the second factor.

n^{th} term of 1^{st} factor $= 1 + (n-1)3 = 3n-2$

n^{th} term of 2^{nd} factor $= 4 + (n-1)3 = 3n+1$

$$\therefore \quad u_n = (3n-2)(3n+1) = 9n^2 - 3n - 2$$

$$\sum u_n = 9\sum n^2 - 3\sum n - 2\sum 1 = \frac{9n(n+1)(2n+1)}{6} - 3\frac{(n)(n+1)}{2} - 2n$$

$$= n(3n^2 + 3n - 2)$$

Method of Summation when u_n is the reciprocal of the product of r successive terms of an A.P.

Example - 2.14 Find the sum to n terms and if possible to infinity the series $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots$

Solution:

Here, $$u_n = \frac{1}{(3n-2)(3n+1)}$$

Let $$V_n = \frac{1}{3n+1}, \text{ then } V_{n-1} = \frac{1}{3n-2}$$

$$V_n - V_{n-1} = \frac{1}{3n+1} - \frac{1}{3n-2} = \frac{-3}{(3n+1)(3n-2)} = -3u_n$$

$$\therefore \quad u_n = -\frac{1}{3}(v_n - v_n - 1)$$

Now, $$u_1 = -\frac{1}{3}(v_1 - v_0)$$

$$u_2 = -\frac{1}{3}(v_2 - v_1)$$

$$u_3 = -\frac{1}{3}(v_3 - v_2)$$

$$u_n = -\frac{1}{3}(v_n - v_{n-1})$$

Adding, we get $$S_n = \sum_{n=1}^{n} u_n = -\frac{1}{3}(v_n - v_0) = -\frac{1}{3}\left(\frac{1}{3n+1} - \frac{1}{1}\right)$$

$$\therefore \quad S_n = \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$$

$$\lim_{n\to\infty} S_n = \frac{1}{3}\left(1 - \frac{1}{\infty}\right) = \frac{1}{3}$$

Arithmetic-Geometric Series

When numerator of u_n is in arithmetic series and denominator is in geometric series, we say, it is an Arithmetic- Geometric series.

Sum to infinity, the series below:

Example - 2.15 $1+\frac{5}{3}+\frac{9}{3^2}+\frac{13}{3^3}+\frac{17}{3^4}+...\infty$, find sum.

Solution:

Let,

$$S_n = 1+\frac{5}{3}+\frac{9}{3^2}+\frac{13}{3^3}+\frac{17}{3^4}+...\infty$$

$$\frac{S_n}{3} = \frac{1}{3}+\frac{5}{3^2}+\frac{9}{3^3}+\frac{13}{3^4}+\frac{17}{3^5}+...\infty$$

$$S_n - \frac{S_n}{3} = 1+\left(\frac{5}{3}-\frac{1}{3}\right)+\left(\frac{9}{3^2}-\frac{5}{3^2}\right)+\left(\frac{13}{3^3}-\frac{9}{3^3}\right)...\infty$$

$$= 1+\left(\frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}+...\infty\right) = 1+4\left(\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+...\infty\right)$$

$$\frac{2S_n}{3} = 1+\frac{4\left(\frac{1}{3}\right)}{1-\frac{1}{3}} = 1+2 = 3$$

$$\therefore \quad S_n = \frac{3\times 3}{2} = 4.5$$

2.7 Recurrence Relations

Sometimes it may be difficult to define the n^{th} term of a sequence explicitly in terms of n. However, it may be fairly easy to define the n^{th} term in terms of either the previous term, or in terms of a collection of previous terms. e.g.,

(i) The sequence of powers of 2 is given by $a_n = 2a_{n-1}$ and $a_0 = 1$ for $n \geq 1$. Here $a_n = 2a_{n-1}$ is called a recurrence relation and $a_0 = 1$ is called initial condition.

(ii) The recursive definition of binomial coefficients $\binom{n}{k}$ or ${}^nC_k$ or $C(n, k)$ where $n \geq 0$, $k \geq 0$ and $n \geq k$ is given by $C(n, k) = C(n-1, k) + C(n-1, k-1)$ with initial conditions, ${}^1C_0 = 1$, ${}^1C_1 = 1$.

Recurrence Relations

Let the number of bacteria in a colony double every hour. If a colony begins with five bacteria, how many will be present in n hours?

To solve this problem let a_n be the number of bacteria at the end of n hours. Since the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This relationship, together with initial condition $a_0 = 5$, uniquely determines a_n for all non negative integer n.

Definition: A recurrence relation for the sequence $\{a_n\}$ is a formula that expresses a_n in terms of one or more of the previous term of sequence, namely $a_0, a_1 ... a_{n-1}$, for all integers n with $n \geq n_0$ where n_0 is a non-negative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example-2.16 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3 \ldots$ and suppose that $a_0 = 3$, $a_1 = 5$. What are a_2 and a_3?

Solution:

Given the recurrence relation $a_n = a_{n-1} - a_{n-2}$.

Recurrence relation for $n = 2$ is $a_2 = a_1 - a_0 = 5 - 3 = 2$ and for $n = 3$ is $a_3 = a_2 - a_1 = -3$.

Example: The recurrence relation for the Fibonacci sequence is:

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ together with initial conditions $f_1 = 1$ and $f_2 = 1$.

It is possible to find any term of a recurrence relation by starting from initial condition and repeatedly applying the recurrence formula.

For example: Find the 5th term of the Fibonacci sequence

$f_n = f_{n-1} + f_{n-2} \quad n \geq 3, \quad f_1 = 1, \quad f_2 = 1$

Now,

$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$

However this brute force process is tedious and we wish to explicitly find the solution for n^{th} term, as a function of n, so that the n^{th} term may be obtained easily for even large values of n.

Recursively Defined Functions

To define a function with the set of non negative integers as its domain.

(*i*) Specify the value of the function at zero.

(*ii*) Give a rule for finding its value as an integer from its values at smaller integers such a definition is called a recursive or inductive definition.

Example-2.17 Give an inductive definition of factorial function $f(n) = n!$

Solution:

The initial value of the function is defined as $F(0) = 1$, then giving a rule for $F(n + 1)$ from $F(n)$.

$$F(n + 1) = (n + 1)\, F(n)$$

For $n = 5$, $F(6)$

$$= 6F(5) = 6.5F(4) = 6.5.4F(3)$$
$$= 6.5.4.3.2.F(0)$$

Now, since $F(0) = 1$

$$F(6) = 6.5.4.3.2.1 = 720$$
$$F(6) = 6! = 720$$

2.8 Solving Recurrence Relations

Types of Recurrence Relations

1. Linear recurrence relations with constant coefficient
2. Non-linear recurrence relations
3. Indeterminate order recurrence relations

1. Linear recurrence relations with constant coefficients:

A k^{th} order linear recurrence relation on S with constant coefficients can be written of the form,

$S(n) + C_1 S(n-1) + C_2 S(n-2) \ldots + C_k S(n-k) = f(n) \; [k \geq n]$

Where, $C_1, C_2, \ldots C_k$ are constant and f is a numeric function defined for $k \leq n$

When $f(n) = 0$, such a recurrence is called homogenous.

Examples:

(a) $f_n = f_{n-1} + f_{n-2}$ (Fibonacci sequence)

(b) $f_n - f_{n-1} - f_{n-2} = 0$

(c) $T_n - 3T_{n-1} + 4T_{n-2} = 2^n$

(d) $S_n = 2S_{n-1} + 5 + n^2$ i.e. $S_n - 2S_{n-1} = 5 + n^2$

(*i*) is a 2nd order homogenous linear recurrence

(*ii*) is a 2nd order inhomogenous linear recurrence

(*iii*) is a 1st order inhomogenous linear recurrence

All linear homogenous equations can be solved by standard methods. Linear inhomogenous equations also can be solved by standard methods, provided the right hand side function $f(n)$ is in some specific allowed forms.

2. **Non-linear recurrence relations:**

If the powers of the recurrence variable is more than 1 or if product of recurrence variables is found in the equation, then the recurrence becomes non-linear.

Example:

$$S_n^2 - 2S_{n-1} = n + 2 \text{ is a non linear 1st order relation}$$

and $S_n S_{n-1} + S_{n-2} = 5n$ is a non linear 2nd order relation

Some non-linear recurrences can be solved after converting them to linear homogenous or inhomogenous type, by appropriate substitutions.

Example: $S_n^2 - 2S_{n-1}^2 = n$

Let, $T_n = S_n^2$ than $T_{n-1} = S_{n-1}^2$

∴ The above non linear recurrence can be converted to a linear recurrence.

$T_n - T_{n-1} = n$ and solved.

3. **Indeterminate order recurrence relations:**

If the difference between the largest subscript of the recurrence and smallest subscript in not an integer but a function of n, we say that such a recurrence relation, is of indeterminate order.

Example: $S_n = 2S_{n/2} + 1$

Now $n - n/2 = n/2$ is not an integer, but a function of n.

∴ This is an indeterminate order recurrence relation.

These can be solved in some cases by converting into linear recurrence relation, by appropriate substitutions.

In some cases, the Master's theorem may be applied, to find the complexity order of the solution as a function of n.

Example: $S_n = 2S_{n/2} + 1 \quad S(1) = 2$

$S_n - 2S_{n/2} = 1$ Let $n = 2^k$

$S_n = S_{2^k} = B(k)$ and $S_{n/2} = S_{2^{k-1}} = B(k-1)$

∴ The recurrence becomes converted to the inhomogenous linear recurrence, $B(k) - 2B(k-1) = 1$ and can be solved by standard methods.

Note: The initial condition S(1) = 2 must also be changed, using the same transformation $n = 2^k$

Here, $S(1) = S(2^0) = B(0) = 2$

$\therefore$ The initial condition, S(1) = 2 is changed to B (0) = 2

Solving (Linear Homogenous Recurrence Relations with Constant Coefficients) by Characteristic Roots Method

Example-2.18 Solve the following recurrence relation by using characteristic root method:

$$a_n - 7a_{n-1} + 12a_{n-2} = 0, \quad n \geq 2, a_0 = 1, a_1 = 2$$

Solution:

Characteristic equation:

$$C(t) = t^2 - 7t + 12 = 0$$

i.e. $(t-3)(t-4) = 0$

Root are 3, 4

$\therefore$ The general solution is $a_n = C_1 3^n + C_2.4^n$

C_1, C_2 can be found from initial conditions.

$$a_0 = C_1 3^0 + C_2 4^0 = 1$$

$\Rightarrow$ $C_1 + C_2 = 1$ ...(1)

$$a_1 = C_1 3^1 + C_2 4^1 = 2$$

$\Rightarrow$ $3C_1 + 4C_2 = 2$ ... (2)

From equation (1) and (2), $C_2 = -1$, $C_1 = 2$

$\therefore$ $a_n = 2.\, 3^n - 4^n$

If the roots of characteristic equations are not distinct, then one of the terms of the solution has to be multiplied by *n*.

Example: $a_n - 4a_{n-1} + 4a_{n-2} = 0$

$C(t) = t^2 - 4t + 4 = 0 = (t-2)(t-2) = 0$ root are 2, 2

Now solution to recurrence is $a_n = C_1 2^n + C_2.n.2^n$

C_1, C_2 can now be solved from initial conditions.

If the number of repeated roots is 3, then we multiply one term by *n*, another term by n^2, to keep all terms distinct in form.

Solving Linear Inhomogenous Equations by Characteristic Roots Method

To solve linear inhomogenous equation, we must

Step-1: first put the R.H.S. *f*(*n*) to Zero and solve the homogenous case. Let us call the homogenous solution as a_n^h. Do not solve for constants C_1, C_2 etc at this stage.

Step-2: Then we choose a trial particular solution a_n^p based on the form of the RHS function according to the table:

RHS	Form of trial particular solution (a_n^p)	
Constant	C	d
Linear	$C_0 + C_1 n$	$d_0 + d_1 n$
Quadratic	$C_0 + C_1 n + C_2 n^2$	$d_0 + d_1 n + d_2 n^2$
Power fn	$C.a^n$	da^n
Power fn * Poly	$a^n (C_0 + C_1 n ...)$	$a^n (d_0 + d_1 n ...)$

If any of the terms of the trial solution is same as a term of the homogenous solution (in form), then the entire particular solution is multiplied by *n*.

If this does not make the particulars solution terms to be distinct from homogenous solution, then multiply by n^2 instead and so on until all terms of particulars solution are distinct in form from those of the homogenous solution.

Step-3: Now, substitute the trial solution into the recurrence and solve for constants d_0, d_1,.... etc.

Step-4: Now get general solution $a_n = a^h_n + a^p_n$

Step-5: Lastly, solve for constants C_1, C_2 etc in homogenous solution by substituting the initial conditions.

Example-2.19 Solve the following linear inhomogenous recurrence relation by using characteristic root method, $S(k) + 5S(k-1) = 9$, $S(0) = 6$.

Solution:

Step-1:

$$S(k) + 5\,S(k-1) = 0$$

$$C(t) = t + 5 = 0 \Rightarrow t = -5$$

$$S^h_k = C_1(-5)^k$$

Step-2:

Let,

$$S^p_k = d$$

Step-3:

$$d + 5d = 9 \Rightarrow d = 9/6 = 1.5$$

Step-4:

∴

$$S_k = S^h_k + S^p_k = C_1(-5)^k = 1.5$$

Step-5:

Substituting the initial condition,

$$S(0) = C_1(-5)^0 + 1.5 = 6 \Rightarrow C_1 = 4.5$$

∴

$$S_k = 4.5(-5)^k + 1.5$$

Example-2.20 Solve the following linear inhomogenous recurrence relation by using characteristic root method, $T(k) - 7T(k-1) + 10T(k-2) = 6 + 8k$.

$$T(0) = 1 \text{ and } T(1) = 2$$

Solution:

Step-1:

Here, the characteristic equation,

$$t^2 - 7t + 10 = 0 \Rightarrow t = 2.5$$

∴

$$T^h(k) = C_1 2^k + C_2 5^k$$

Step-2:

$$T^p(k) = d_0 + d_1 k$$

Step-3:

Substituting in recurrence we get,

$$(d_0 + d_1 k) - 7(d_0 + d_1(k-1)) + 10(d_0 + d_1(k-2)) = b + 8k$$

⇒

$$(4d_0 - 13d_1) + 4d_1 k = 6 + 8k$$

Equating ooefficients, $4d_0 - 13d_1 = 6$ and $4d_1 = 8$

⇒

$$d_0 = 8,\ d_1 = 2$$

$$T^p(k) = 8 + 2k$$

Step-4:

∴

$$T(k) = T^h(k) + T^p(k) = C_1 2^k + C_2 5^k + 8 + 2k$$

Step-5:

Now, C_1, C_2 can be solved using initial conditions:

$$T(0) = 1 \Rightarrow C_1 + C_2 = -7$$
$$T(1) = 2 \Rightarrow 2C_1 + 5C_2 = -8$$
$$\therefore \quad C_1 = -9,\ C_2 = 2$$
$$T(k) = -9.2^k + 2.5^K + 8 + 2K$$

Summary

- The number of ways of arranging n objects in a circle is $(n-1)!$ This is less than $n!$, since in a circle many of the linear permutations become indistinguishable except for rotation.
- Combinations with no Repetitions $\boxed{{}^nC_r = \dfrac{n!}{r!(n-r)!}}$
- Combinations with Unlimited Repetitions ${}^{n-1+r}C_r$ which can also be written as ${}^{n-1+r}C_{n-1}$.
- Distribution of Distinct Objects: $\dfrac{n!}{n_1!n_2!\ldots n_r!}$
- **Vandermondes indentity:** ${}^{n+m}C_r = {}^nC_0{}^mC_r + {}^nC_1{}^mC_{r-1} + {}^nC_2{}^mC_{r-2} \ldots {}^nC_r{}^mC_0$
- Special case of Vandermonde's identity is obtained by putting $r = n$

  $${}^{n+m}C_n = {}^nC_0{}^mC_n + {}^nC_1{}^mC_{n-1} + \ldots {}^nC_n{}^mC_0$$
- When generating function is written as a series $1 + X + X^2 + \ldots$ it said to be in open form and when it written instead as $\dfrac{1}{1-X}$, it is said to be in closed form.
- Notice that the sequence corresponding to (4.1) $\sum_{r=0}^{n} X^r$ is $[a_r]_{r=0}^{n} = 1$.
- Ball in the box problems, when balls are indistinguishable, along with upper unstraints on the no of balls that can be put into a box, can be effectively converted into a problem of finding coefficient of some power of X in the expansion of a corresponding generating function.
- Solving Linear Inhomogenous Equations by Characteristic Roots Method

RHS	Form of trial particular solution (a^P_n)	
Constant	**C**	**d**
Linear	$C_0 + C_1n$	$d_0 + d_1n$
Quadratic	$C_0 + C_1n + C_2n^2$	$d_0 + d_1n + d_2n^2$
Power fn	$C.a^n$	da^n
Power fn * Poly	$a^n(C_0 + C_1n\ldots)$	$a^n(d_0 + d_1n\ldots)$

Student's Assignment

Q.1 How many strings are there of lowercase letters of length four or less?

Q.2 A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

(a) How many socks must he take out to be sure that he has at least two socks of the same color?

(b) How many socks must he take out to be sure that he has at least two black socks?

Q.3 A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

(a) How many balls must she select to be sure of having at least three balls of the same color?

(b) How many balls must she select to be sure of having at least three blue balls?

Q.4 What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Q.5 Atleast how many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

Q.6 How many subsets with an odd number of elements does a set with 10 elements have?

Q.7 A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

(a) are there in total?

(b) contain exactly three heads?

(c) contain at least three heads?

(d) contain the same number of heads and tails?

Q.8 What is the coefficient of $x^{12} y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Q.9 Find the coefficient of $x^5 y^8$ in $(x + y)^{13}$.

Q.10 How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?

Q.11 What is the coefficient of $x^{101} y^{99}$ in the expansion of $(2x - 2y)^{200}$?

Q.12 In how many ways can 5 numbers be selected and arranged in ascending order from the set {1,2,3...10}?

Q.13 In how many different ways can 5 ones and 20 twos be permuted so that each one is followed by atleast 2 twos?

Q.14 How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

Q.15 A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

(a) six bagels?

(b) a dozen bagels?

(c) a dozen bagels with at least one of each kind?

(d) a dozen bagels without least three egg bagels and no more than two salty bagels?

Q.16 How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Q.17 How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, i=1, 2, 3, 4, 5, is a non-negative integer such that

(a) $x_1 \geq 1$

(b) $x_i \geq 2$ for i = 1, 2, 3, 4, 5

(c) $0 \leq x_1 \leq 10$

Q.18 How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of twelve 0 bits, and must have at least two 0 bits following each 1 bit?

Q.19 An agency has 10 available foster families $F_1...F_{20}$ and 6 children $C_1...C_6$ to place. In how many ways can they do this if

(a) No family can get more than one child.

(b) A family can get more than one child.

Q.20 How many distinguishable permutation can be generated from word "BANANA"?

(a) 720 (b) 60

(c) 240 (d) 120

Q.21 How many integers in S = {1, 2, 3, ..., 1000} are divisible by 3 or 5?

(a) 599 (b) 467

(c) 333 (d) 66

Q.22 How many numbers from a set {1, 2, 3, ..., 20} should be chosen in order to be sure to have one number multiple of another.

(a) 10 (b) 11

(c) 3 (d) 9

Q.23 How many numbers must be chosen from set {1, 2, 3, ..., 8}, such that atleast 2 of them must have Sum = 9

(a) 28 (b) 9

(c) 5 (d) 10

Q.24 In a string of length n, with distinct letters, how many substrings can be generated (other than null string)?

(a) 2^n (b) n^2

(c) $n(n+1)/2$ (d) $n(n-1)/2$

Answer Key:

20. (b) **21.** (b) **22.** (b) **23.** (c) **24.** (c)

Student's Assignments Explanations

1. Here repetition is allowed by default (nothing is mentioned about repetition).

Number of possible strings of 0 length (empty string) : 1

Number of possible strings of length 1 : 26

Number of possible strings of length 2 : 26 × 26

Number of possible strings of length 3 : 26 × 26 × 26

Number of possible strings of length 3 : 26 × 26 × 26 × 26

Therefore the total number of strings of length 4 or less = $1+26+26^2+26^3+26^4$

2. (a)

He must take out three socks to make sure that he gets a pair. Because the first socks could be of one colour and second socks could be of another colour.

Hence the third socks that he draws from the drawer is of one of the colour of the two socks that he had drawn earlier, which is sufficient to make a pair.

(b)

He must take out 14 socks to be sure that he has at least 2 black socks. Because there are 12 black and 12 brown socks in the drawer. So when he draws first 12 socks they could be all brown and hence 2 more socks need to be drawn to make sure a pair of black socks.

3. (a)

She must select five balls to be sure of having at least 3 balls of the same colour. This is because first 4 attempts have the following possibilities which does not ensure three balls of same colour.

Red, Blue, Red, Blue or

Blue, Red, Red, Blue ...and so on

Hence we need fifth ball to ensure three balls of same colour.

(b)

Thirteen balls must be selected to be sure of having atleast three blue balls.

The first ten balls that she draws could be all ten red balls. Hence she need to draw three more balls to ensure three blue balls.

4. Here the number of students are similar to number of pigeons and number of states are similar to pigeon holes.

Number of pigeons = n

Number of pigeons hole = 50

Therefore $\left\lfloor \frac{n-1}{50} \right\rfloor + 1 = 100$

The minimum value of n is obtained after removing the floor function and solving for n.
$\Rightarrow n = 4951$ pigeons (students).

5. The number of pairs that add upto 7 are the pigeon holes.
{1, 6}, {2, 5}, {3, 4} are the holes.
Let 'N' be the number of such numbers. If two numbers sit in any one of these pigeon holes, we have a pair whose total is 7.

Therefore $\left\lfloor \frac{N-1}{3} \right\rfloor + 1 = 2$

$\Rightarrow$ Atleast N = 4 numbers must be selected.

6. The number of subsets with 1 element = $^{10}C_1$
The number of subsets with 3 elements = $^{10}C_3$
The number of subsets with 5 elements = $^{10}C_5$
The number of subsets with 7 elements = $^{10}C_7$
The number of subsets with 9 elements = $^{10}C_9$
Hence the total number of subsets with an odd number of elements are:
$^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9$
$= 2^{10-1} = 2^9 = 512$

7. When a coin is tossed there are only 2 possibilities (H and T).
(a) Total number of possibilities are:
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$
(b) The total number of possibilities with exactly three heads are: $^8C_3$ (any of the 3 coins can turn up heads).
(c) The total number of possibilities with atleast three heads are: $2^8 - \{^8C_0 + {}^8C_1 + {}^8C_2\}$.
(d) The total number of possibilities with same number of heads and tails are: $^8C_4$ (any of the 4 coins could be head and remaining 4 coins could be tails).

8. First, note that this expression equals $(2x+(-3y))^{25}$. By the Binomial Theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

Consequently, the coefficient of $x^{12} y^{13}$ in the expression is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25}{13!\,12!} 2^{12} 3^{13}$$

9. The coefficient of $x^{n_1} y^{n_2}$ is $\frac{n}{n_1!\,n_2!}$. Therefore the coefficient is $\frac{13!}{5!\,8!}$ which is $= 13 \times 11 \times 9 =$ 1287.

10. We will get like terms when we expand $(x+y)^{100}$ as $(x + y)(x + y)(x + y)(x + y)$ ...100 times. The binomial expansion collects all like terms. In the expansion the terms are in the form $x^{n_1} y^{n_2}$ with the condition that $n_1 + n_2 = 100$ always. Thus, the number of solution of this equation gives the number of terms in the binomial expansion.
$^{2-1+100}C_{100} = 101$
$\therefore$ The number of solutions of this equation gives the number of terms are 101.

11. $\frac{200!}{100!\,99!}(2x)^{101}(-2y)^{99}$

$= -\frac{200!}{100!\,99!} \times 2^{200} \times x^{101} \times y^{99}$

12. Since, after selection there is only one way to put 5 numbers in ascending order, the answer is $^{10}C_5 \times 1 = {}^{10}C_5 = 252$.

13. First itself, attach 2 twos to each 1 to form the compound symbols 122,122,122,122,122 and permute these 5 symbols along with the remaining 10 twos in $\frac{15!}{5!\,10!}$ ways.

14. First job can be given to all the 5 employees.
Second job can be given to all the 5 employees.
Third job can be given to all the 5 employees.
$\therefore$ The total number of ways of distributing three jobs among five employees are $5 \times 5 \times 5 = 125$.

15. Onion bagels (x_1), Poppy seed bagels (x_2), Egg bagels (x_3), Salty bagels (x_4), Pumpernickel bagels (x_5), Sesame seed bagels (x_6), Raisin bagels (x_7), and Plain bagels (x_8).

(a) This problem is same as distributing six chocolates among eight children.

i.e. $x_1 + x_2 + x_3 + ... + x_8 = 6$

$(n = 8,\ r = 6)$

${}^{8-1+6}C_6 = 1716$

(b) $r = 12, n = 8$

${}^{8-1+12}C_{12} = 50388$

(c) The problem is same as

$$x_1 + x_2 + x_3 + \dots + x_8 = 12$$

with the constraints $x_1 \geq 1, x_2 \geq 1, \dots x_8 \geq 1$.

Now select one bagel of each type to satisfy the constraints.

Now the number of ways to choose $12 - 8 = 4$ bagels from among the 8 type of bagels is same as the number of solutions to the equation

$$x_1 + x_2 + x_3 + \dots + x_8 = 4$$

which is ${}^{8-1+4}C_4 = {}^{11}C_4$

(d) The problem is same as

$$x_1 + x_2 + x_3 + ... + x_8 = 12$$

with the constraints $x_3 \geq 3$ and $x_4 \leq 2$.

Now to meet the constraint on the egg bagel, choose 3 egg bagels. We now have choice in choosing the remaining 9 bagels.

Now we are left with the problem of finding the number of solutions to the equation

$$x_1 + x_2 + x_3 + ... + x_8 = 9$$

with the constraint $x_4 \leq 2$

We can solve this problem by complementary counting method.

Solve the complementary problem

$$x_1 + x_2 + x_3 + \dots + x_8 = 9$$

with the constraint $x_4 \geq 3$

which has a solution same as

$$x_1 + x_2 + x_3 + \dots + x_8 = 6$$

with the constraint $x_4 \geq 0$

The solution of this is ${}^{8-1+6}C_6 = {}^{13}C_6$

Now subtract from the universal set which is ${}^{8-1+9}C_9 = {}^{16}C_9$ and the final answer is ${}^{16}C_9 - {}^{13}C_6$.

16. There are only two varieties hence $n = 2$.

We need to choose 8 coins.

This problem is same as selecting 8 chocolates from 2 types of chocolates.

$$x_1 + x_2 = 8$$

$${}^{2-1+8}C_8 = {}^9C_8 = {}^9C_1 = 9 \text{ ways}$$

17. (a) Treat the right side as 21 balls. Put one ball in First box, to satisfy the constraint $x_1 \geq 1$. The remaining 20 balls can be distributed into 5 boxes in

$${}^{5-1+20}C_{20} = {}^{24}C_{20} = {}^{24}C_4 \text{ ways}$$

(b) Treat the right side as 21 balls. Put 2 balls in each of the 5 boxes, to satisfy the constraints $x_i \geq 2$.

The remaining 11 balls can be distributed into 5 boxes in ${}^{5-1+11}C_{11} = {}^{15}C_{11} = {}^{15}C_4$ ways.

(c) This problem can be solved by complementary counting method.

The given problem is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $0 \leq x_1 \leq 10$.

The universal set for this problem is the unconstrained solution to this problem which is

$${}^{5-1+21}C_{21} = {}^{25}C_{21} = {}^{25}C_4$$

The complementary problem is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $x_1 \geq 11$.

The solution to this is to put 11 balls into first box and distribute the remaining 10 balls into 5 boxes in

$${}^{5-1+10}C_{10} = {}^{14}C_{10} = {}^{14}C_4 \text{ ways.}$$

So the final answer to the given problem is obtained by subtracting ${}^{14}C_4$ from the universal set which is ${}^{25}C_4$ to given the final answer of ${}^{25}C_4 - {}^{14}C_4$.

18. 100 □ 100 □ 100 □ 100 □

The remaining four 0's can be placed in four boxes as shown above.

This is same as distributing four identical balls to four boxes.

$$x_1 + x_2 + x_3 + x_4 = 4$$

$${}^{4-1+4}C_4 = {}^7C_4 = 35 \text{ ways}$$

19. (a) Since all children must be placed, consider the children as boxes and assign one family to each box (permutation with no repetition).
$10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151200$ ways.

(b) Since each family can be assigned more than one child, each of the boxes (children) can be assigned to any one of the ten families (permutation with unlimited repetition).
$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$ ways.

20. (b)
Total letters = 6
Repetition frequency per word = N : 2, A : 3, B : 1

Number of permutation $\frac{6!}{3!\,2!\,1!} = 60$

21. (b)

$$D_3 = \{n \mid n\%3 = 0\}$$

$$\Rightarrow \quad |D_3| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$$

$$D_5 = \{n \mid n\%5 = 0\}$$

$$\Rightarrow \quad |D_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$D_3 \cap D_5 = \{n \mid n\%3 = 0 \text{ and } n\%5 = 0\}$
Since 15 is LCM of 3 and 5, $D_3 \cap D_5$ certains exactly those numbers which are divisible by 15.

$$|D_3 \cap D_5| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$$

Therefore,

$$|D_3 \cup D_5| = |D_3| + |D_5| - |D_3 \cap D_5|$$
$$= 333 + 200 - 66 = 467$$

22. (b)
{11, 12, 13, ... 20} is the largest set with no multiples. Adding one more number will make it sure that there is at least one number which is multiple of another in this set.

23. (c)
{1, 8}, {2, 7}, {3, 6} and {4, 5} are the 4 pairs which give sum = 9. Maximum 4 numbers can belong to different sets. The 5[Th] number will pair up and form 9 sum, for sure.

24. (c)
suppose, S = abcd, |S| = 4. The substrings are a, b, c, d, ab, bc, cd, abc, bcd, abcd. Total substrings are 10. Here there is a trend, with strings of length n, we have

substring of length 1 = n
substring of length 2 = $n - 1$
substring of length 3 = $n - 3$
⋮
substring of length n = 1

Total substring = $1 + 2 + \ldots + (n-1) + n$
$= n(n+1)/2$

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