Conic Section

12.01 Introduction :

Conic sections are the curves obtained by intersecting a double napped right circular tight cone by a plane, let O be the vertex of double napped right circular cone, AOA' be its axis and α be the semi vertical angle. If a plane P intersects the axis AOA' of double napped cone at an angle θ , then the classification of conic sections will be as follows.

Case I : When the plane P passes through the vertex O and cuts the cone, we get a pair of straight line. These lines are (fig. 12.01 (a))

(i) Real and distinct, if $\theta < \alpha$

(ii) coinciding, if $\theta = \alpha$

(iii) Imaginery, if $\theta > \alpha$

Case II : When the plane cuts the nappe (other than the vertex) of the cone, the section will be a curve. We have the following situations :

(i) When $\theta = 90^0$, the section is a circle i.e. plane P, cuts the axis AOA' of cone perpendicularly. The centre of circle lie on axis AOA' (fig. 12.01 (b).

(ii) When $\theta = \alpha$ the section is a parabola i.e. plane P is parallel to line OG. The axis of parabola will be parallel to this line.
(fig. 12.02).

(iii) When $\theta > \alpha$, then the section is an Ellipse i.e. plane P cuts both the generating lines of cone. (fig. 12.03)

(iv) When $\theta < \alpha$, then the section is Hyperbola i.e. plane P cuts both the nappes of the cone. (fig. 12.04)

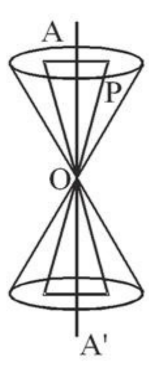


fig. 12.01 (a)
Pair of lines

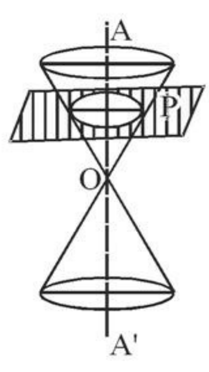


fig. 12.01 (b)
Circle

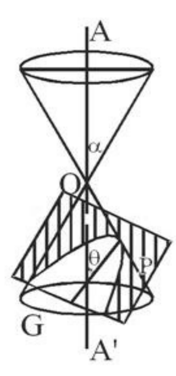


fig. 12.02
Parabola

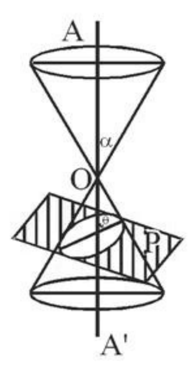


fig. 12.03
Ellipse

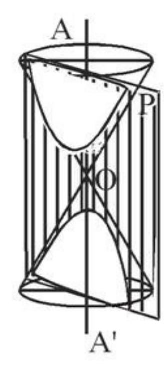


fig. 12.04
Hyperbola

12.02 Conic Sections

Definition : The locus of a point which moves in a plane in such a way that the ratio of the distances measured from fixed point and fixed line is always constant, is known as conic or conic section.

Fixed point S and fixed line M are called as focus and directrix of conic sections and the constant ratio is called Ecccentricity and is denoted by 'e'. A line passing through the vertex and perpendicular to the directrix is called as axis.

Conic section and its point of intersection with the axis is called as vertex. The line joining two points on the conic is known as chord every chord of conic section bisects at a point, that point is known as the centre of the conic section. This point is the centre point between the foci . The chord perpendicular to the axis and passing through the focus is known as latus rectum.

General equation of conic section

Let the coordinates of point S in a plane be (α, β), equation of a fixed line (directrix) is $ax + by + c = 0$ and eccentricity be e. Let $P(h, k)$ by any arbitrary point in a plane. Join PS and draw perpendicular PM on the directrix from point P. By the defination of conic,

$$\frac{SP}{PM} = e \qquad \text{or,} \qquad SP = e\,PM \qquad \Rightarrow \qquad SP^2 = e^2 PM^2$$

$$\Rightarrow \quad (h-\alpha)^2 + (k-\beta)^2 = e^2 \left\{ \frac{ah + bk + c}{\sqrt{a^2 + b^2}} \right\}^2$$

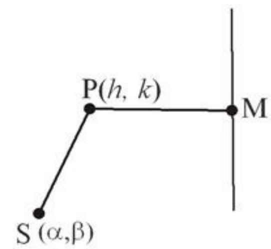


fig. 12.05

Thus, locus of point P or the equation of conic section is

$$(x-\alpha)^2 + (y-\beta)^2 = e^2 \frac{(ax + by + c)^2}{a^2 + b^2}$$

Thus, the general equation of conic can be written as :

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

12.03 Different forms of conic section

Conic section depends on the position of focus w.r.t. the derectrix and the value of eccentricity *'e'*

I. When focus is situated on the directrix.

Conic section will represent a pair of lines. These lines are

(i) real and distinct if, $e > 1$

(ii) real and coinciding if, $e = 1$

(iii) imaginary if, $e < 1$

(iv) parallel, if the directrix is at infinity

II. When focus is not on the directrix.

Conic sections for different values of eccentricity e :

(i) a circle, if $e = 0$

(ii) a parabola, if $e = 1$

(iii) an ellipse, if $e < 1$

(iv) a hyperbola, if $e > 1$

12.04 To prove that the general equation of second degree in *x* and *y* always represents a conic section

Let the general equation of conic be

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

To eliminate xy we place the origin at fixed point and we rotate the axes at angle θ . For this, we substitute

$x = (x\cos\theta - y\sin\theta)$ and $y = (x\sin\theta + y\cos\theta)$ in equation (1), we get

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2$$
$$+2g(x\cos\theta - y\sin\theta) + 2f(x\sin\theta + y\cos\theta) + c = 0 \quad (2)$$

On choosing θ in such a way that, the coefficient of xy becomes zero. That means,

$$2(b-a)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta) = 0 \quad \text{or} \quad \tan 2\theta = \frac{2h}{a-b}$$

$\therefore$ From eq (2),

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0, \quad (3)$$

where A, B, G, F and C are constants.

Case I : When $A \neq 0,\ B \neq 0$

From eq (3),

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C = K \text{ (let)}$$

On shifting the origin at $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, we get

$$Ax^2 + By^2 = K \quad (4)$$

(i) When $K = 0$ *and* $\Delta = 0$ i.e. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ then equation (4) denotes a pair of lines. If A and B are of opposite signs then they are real. If they are of same signs, then they are imaginary.

(ii) When $K \neq 0$, from eq (4), we have

$$\frac{x^2}{\frac{K}{A}} + \frac{y^2}{\frac{K}{B}} = 1$$

When K/A and K/B are of same signs, then it represents an ellipse and of opposite signs, it represents a hyperbola.

(iii) When $K \neq 0$, and $A = B$, then eq (4) represents a circle.

(iv) When $K \neq 0$, and $A = -B$, then eq (4) represents a rectangular Hyperbola.

Case II : When either A or B is zero. (let $B = 0$)

From eq (4),

$$A\left(x + \frac{G}{A}\right)^2 = -2Fy + \frac{G^2}{A} - C = -2F\left(y - \frac{G^2 - AC}{2F}\right) \quad (5)$$

(i) When F = 0, eq (5) will represent two parallel lines. If $(G^2/A) - C = 0$, then the lines will coincide.

(ii) When $F \neq 0$, shifting origin at $\left(-\frac{G}{A}, \frac{G^2 - AC}{2F}\right)$ in eq (5), we have

$$Ax^2 = -2Fy \quad \text{or} \quad x^2 = -\frac{2F}{A}y \quad (6)$$

which represent a parabola

$\therefore$ In all the conditions, the quadratic equation represents a conic section.

General quadratic equation in a tabular form (conditions for different conics are represented.

$$\phi(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

S.No.	Conditions	Nature of the sections
	$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$	
A	$\Delta = 0$ and	
	1. $h^2 \neq ab$	pair of line
	2. $h^2 = ab$	pair of parallel lines
	3. $a + b = 0$	pair of perpendicular lines
B	$\Delta \neq 0$ and	
	1. $a = b,\ h = 0$	Circle
	2. $h^2 = ab$	parabola
	3. $h^2 < ab$	Ellipse
	4. $h^2 > ab$	Hyperbola
	5. $h^2 > ab$ and $a + b = 0$	Rectangullar Hyperbola

In the following sections, we shall obtain the equations of each of these conic sections like Circle, Parabola, Ellipse and Hyperbola in standard form.

Circle

12.05 Equation of the circle in central form

Definition : A circle is the set of all points in a plane that are equidistant from a fixed point in the plane. The fixed point is called the centre of the circle and the distance from the centre to a point on the circle is called the radius of the circle.

Given C (h, k) be the centre and r the radius of circle. Let P(x, y) be any point on the circle, then

(by the fig.) $\quad CQ = ON - OM = x - h$

$$QP = NP - NQ = y - k$$

In a right angle triangle CQP,

$$CQ^2 + QP^2 = CP^2$$

or $\quad (x-h)^2 + (y-k)^2 = r^2$

It is called as central form of circle.

This is the required equation of the circle with centre at (h, k) and radius 'r'.

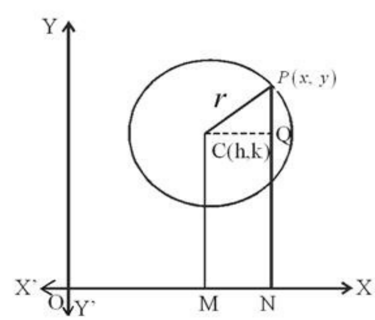
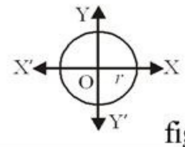
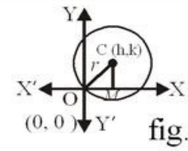
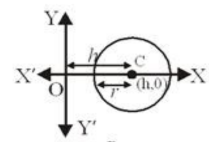
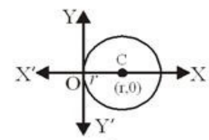
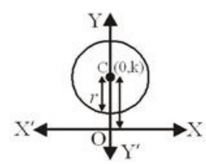
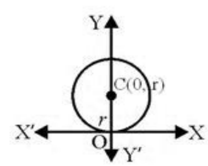
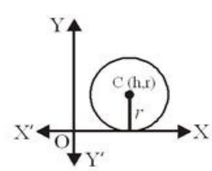
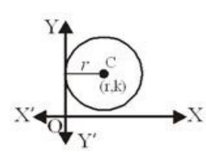
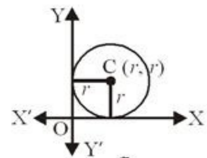


fig. 12.06

Equation of circle in different conditions :

S.No.	Various Conditions	Equation of Circle	Figure of Circle
1.	$h=0,\ k=0$	$x^2+y^2=r^2$ (Standard)	 fig. 12.07
2.	Circle passes through the origin and the centre is in the first quardrant	$x^2+y^2-2hx-2ky=0$	 fig. 12.08
3.	$k=0$ and $h>r$	$(x-h)^2+y^2=r^2$	 fig. 12.09
4.	$k=0$ and $h=r$	$x^2+y^2-2rx=0$	 fig. 12.10
5.	$h=0$ and $k>r$	$x^2+(y-k)^2=r^2$	 fig. 12.11
6.	$h=0$ and $k=r$	$x^2+y^2-2ry=0$	 fig. 12.12
7.	Circle touches the x-axis and $k=r$	$x^2+y^2-2hx-2ry+h^2=0$	 fig. 12.13
8.	Circle touches the y-axis and $h=r$	$x^2+y^2-2rx-2ky+k^2=0$	 fig. 12.14

9.	$h = k = r$	$x^2 + y^2 - 2rx - 2ry + r^2 = 0$	 fig. 12.15
10.	$r = 0$	$(x-h)^2 + (y-k)^2 = 0$	Circle is a point circle.

Illustrative Examples

Example 1 : Find an equation of the circle with centre at (2, -3) and radius is 4 units.

Solution : Here $h = 2,\ k = -3$ and $r = 4,$ thus required equation $(x-2)^2 + (y+3)^2 = 16$.

Example 2 : Find the centre and the radius of the circle $x^2 + y^2 + 8x + 10y - 8 = 0$

Solution : Given equation is

$$(x^2 + 8x) + (y^2 + 10y) = 8$$

Completing the square

$$(x^2 + 8x + 16) + (y^2 + 10y + 25) = 8 + 16 + 25$$

$$(x+4)^2 + (y+5)^2 = 49$$

$$\{x - (-4)\}^2 + \{y - (-5)\}^2 = 7^2$$

Thus the centre is (-4, -5) and radius is 7 units.

Example 3 : Find an equation of the circle which touches the x-axis and cuts an intercept of length 2ℓ on y-axis.

Solution : Putting $k = r$ and $h = \sqrt{r^2 - \ell^2}$ in the standard equation of circle we have

$$\left\{x - \sqrt{r^2 - \ell^2}\right\} + (y - r)^2 = r^2$$

or $$x^2 + y^2 - 2x\sqrt{r^2 - \ell^2} - 2ry + r^2 - \ell^2 = 0$$

Example 4 : Find an equation of the circle which passes through the point (1, 2) and whose centre is the point of intersection of the lines $x + y + 3 = 0$ and $2x + 3y + 7 = 0$

Solution : Given equations $x + y + 3 = 0$ (1)

and $2x + 3y + 7 = 0$ (2)

on sloving we get $(-2, -1)$ which is the coordinate of the centre. Since the circle passes through the point $(1, 2)$ thus the radius will be

radius $= \sqrt{\left\{(1+2)^2 + (2+1)^2\right\}} = \sqrt{(9+9)} = 3\sqrt{2}$

Thus the required equation of the circle

$$(x+2)^2 + (y+1)^2 = (3\sqrt{2})^2$$

or $$x^2 + y^2 + 4x + 2y - 13 = 0$$

Exercise 12.1

1. In each of the following, find the equation of the circle with

   (i) centre $(-2,3)$ and radius 4 (ii) centre (a, b) and radius $a-b$.

2. In each of the following, find the centre and radius of the circles.

   (i) $x(x+y-6)=y(x-y+8)$ (ii) $\sqrt{1+k^2}\left(x^2+y^2\right)=2ax+2aky$ (iii) $4\left(x^2+y^2\right)=1$

3. Find the equation of the circle which touches the y-axis and cuts an intercept of length 2ℓ on x-axis.
4. Find the equation of the circle which touches the x-axis at a distance of +3 units from the origin and cuts an intercept of 6 units length on y-axis.
5. Find the centre and radius of the circle $x^2+y^2-8x+10y-12=0$.
6. Find the centre and radius of the circle $2x^2+2y^2-x=0$.
7. Find the equation of the circle passing through the points (2, 3) and (–1, 1) and whose centre is on the line $x-3y-11=0$.
8. Find the equation of the circle with radius 5 whose centre lies on x-axis passes through the point (2, 3).
9. Find the equation of the circle passing through the points (0, 0) and making intercept a and b on the coordinates axis.

12.06 Intersection of a circle and a line

Let a line $y=mx+c$ ..... (1) intersects a circle $x^2+y^2=a^2$ ..... (2)

On solving both the equations we have

$$x^2+(mx+c)^2=a^2$$

or $$\left(1+m^2\right)x^2+2mcx+\left(c^2-a^2\right)=0 \qquad (3)$$

This is a quadratic equation hence by solving this we will get two values of x which will be the points of intersection. Placing the values of x in equation (2) we get 2 coorresponding values or y that will be the ordinated of intersecting points. We can say that a straight line intersects the circle at two different points.

Nature of Points of intersection

Points of intersection will be real & distinct, coincident or imaginary according to the nature of the roots of quadritic equation (3) as real & district, coincident or imaginary.

Case I : When intersecting points are real and distinct :

For this, the equation will have real and distinct roots hence the Discriminants is

$$B^2-4AC>0$$

or $$4m^2c^2-4\left(1+m^2\right)\left(c^2-a^2\right)>0$$

or $$a^2\left(1+m^2\right)-c^2>0$$

or $$a^2>\frac{c^2}{1+m^2} \quad \text{thus} \quad a>\left|\frac{c}{\sqrt{1+m^2}}\right|$$

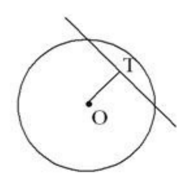


fig. 12.16

Clearly, a straight line will intersect the circle in two different points, if the perpendicular dropped from the centre of the circle to the line is less than the radius.

Case II: When intersecting points are coincide :

For this, the equation will have equal roots hence the Discriminant is

$$B^2-4AC=0$$

and $\quad 4m^2c^2 - 4(1+m^2)(c^2 - a^2) = 0$

and $\quad a^2 = \dfrac{c^2}{1+m^2}$ and $a = \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

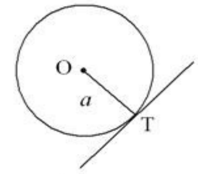


fig. 12.17

Clearly, s straight line will touch the circle if the perpendicular dropped from the centre of the circle to the line is equal to the radius.

Case III: When intersecting points are imaginary :

For this, the equation will have imaginary roots hence the discriminant is

$$B^2 - 4AC < 0$$

or $\quad 4m^2c^2 - 4(1+m^2)(c^2 - a^2) < 0$

or $\quad a^2 < \dfrac{c^2}{1+m^2}$ thus $a < \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

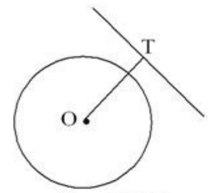


fig. 12.18

Clearly, a straight line is away from the circle if the perpendicular dropped from the centre of the circle to the line is greater than the radius.

12.07 Length of intercept cut off from a line by a circle

To determine the length of intercept by the circle $x^2 + y^2 = a^2$ on the line $y = mx + c$

Given circle $\quad x^2 + y^2 = a^2 \quad$ (1)

and straight line $\quad y = mx + c \quad$ (2)

let the coordinates of P and Q be (x_1, y_1) and (x_2, y_2) respectively. Now we have we have to find the length of the intercept PQ therefore

$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad$ (3)

Since P and Q lies on line (2) therefore

$$y_1 = mx_1 + c$$

and $\quad y_2 = mx_2 + c$

substracting $\quad y_1 - y_2 = m(x_1 - x_2)$

from (3) $\quad PQ = \sqrt{(x_1 - x_2)^2 + m^2(x_1 - x_2)^2}$

$$= (x_1 - x_2)\sqrt{1+m^2} \quad (4)$$

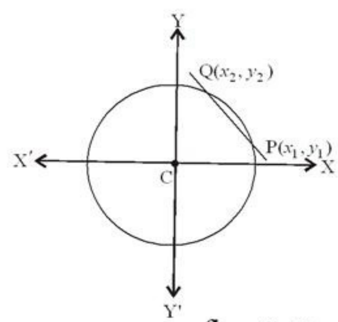


fig. 12.19

Solving (1) and (2) in terms of x,

$$x^2 + (mx + c)^2 = a^2$$

or $\quad (1+m^2)x^2 + 2mcx + (c^2 - a^2) = 0 \quad$ (5)

it is a quadratic equation in x. Therefore x will have two values, let it be x_1 and x_2.

$\therefore \quad x_1 + x_2 = -\dfrac{2mc}{1+m^2}$ and $x_1x_2 = \dfrac{c^2 - a^2}{1+m^2}$

but $\quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

putting the value $$(x_1 - x_2)^2 = \frac{4m^2c^2}{(1+m^2)^2} - 4\frac{(c^2 - a^2)}{1+m^2}$$

$\therefore$ $$x_1 - x_2 = \frac{2}{1+m^2}\sqrt{a^2(1+m^2) - c^2}$$

putting the value in (4)

$$PQ = \frac{2}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2) - c^2}$$

$$PQ = 2\sqrt{a^2 - \frac{c^2}{1+m^2}} \qquad (6)$$

which is the required length of the intercept PQ

Condition of tangency :

To find the condition of a line $y = mx + c$ touching the circle $x^2 + y^2 = a^2$

First Method : If the line $y = mx + c$, touches the circle $x^2 + y^2 = a^2$, then the length of the intercept cut on the line by the circle will be zero. Therefore $PQ = 0$

or $$\frac{2}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2) - c^2} = 0$$

or $$a^2(1+m^2) = c^2$$

or $$c = \pm a\sqrt{1+m^2}$$

which is the required condition of tangency

12.08 Tangent

Definition : According to fig. 12.20, if the secant PQ of a circle moves in such a manner that point Q coincides with point P, then secant PQ is known as tangent (PT) to a circle at point P and P is called point of contanct.

(i) Equation of a tangent $x^2 + y^2 = a^2$ at point (x_1, y_1)

Given circle $x^2 + y^2 = a^2$ (1)

Let $P(x_1, y_1)$ be any point on the circle, now we have to find the equation of tangent. Let $Q(x_2, y_2)$ be any point on the circle then equation of chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad (2)$$

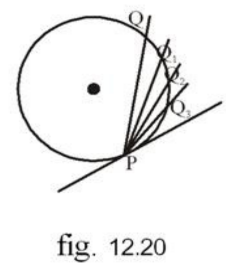


fig. 12.20

But P and Q lies on a circle.

$\therefore$ $$x_1^2 + y_1^2 = a^2 \qquad (3)$$

and $$x_2^2 + y_2^2 = a^2 \qquad (4)$$

substracting (3) from (4)

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) = 0$$

or $$(x_2 - x_1)(x_2 + x_1) = -(y_2 - y_1)(y_2 + y_1)$$

or $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1} \qquad (5)$$

putting the value in (2) $$y - y_1 = -\frac{x_2 + x_1}{y_2 + y_1}(x - x_1) \qquad (6)$$

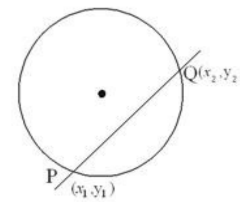


fig. 12.21

If Q moves and coincides with P, then $x_2 = x_1$ and $y_2 = y_1$ and chord PQ will be a tangent to a circle at point P.

$\therefore$ By (6), $$y - y_1 = -\frac{2x_1}{2y_1}(x - x_1)$$

or $$yy_1 - y_1^2 = -xx_1 + x_1^2$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2$$

$\therefore$ $$xx_1 + yy_1 = a^2 \qquad \text{[from (3)]}$$

This is the required equation of a tangent at point $P(x_1, y_1)$

(ii) To find an equation of tangent in terms of slope

The equation of a tangent is -

$$xx_1 + yy_1 = a^2 \qquad (1)$$

writing eq. (1) in terms of $y = mx + c$, we have

$$y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1}$$

slope of this line is

$$m = -x_1 / y_1 \quad \Rightarrow \quad x_1 = -my_1 \qquad (2)$$

But point (x_1, y_1) lies on a circle $x^2 + y^2 = a^2$

$\therefore$ $$x_1^2 + y_1^2 = a^2 \qquad (3)$$

from (2) and (3) $$m^2 y_1^2 + y_1^2 = a^2$$

or $$y_1 = \pm\frac{a}{\sqrt{1+m^2}} \qquad (4)$$

from (2) $$x_1 = \mp\frac{am}{\sqrt{1+m^2}} \qquad (5)$$

Thus, the coordinate of point of contact is $\left(\mp\frac{am}{\sqrt{1+m^2}}, \pm\frac{a}{\sqrt{1+m^2}}\right)$ and the slope form of the tangent will be $y = mx \pm a\sqrt{1+m^2}$

12.09 Normal

Definition : A line which is perpendicular to the tangent at the point of contact is known as normal to the curve.

Normal at any point on the circle passes through the centre of the circle. The equation of the normal will be the line passes through the point of contact and centre of the circle.

To find the equation of normal at any point of the circle.

(i) When equation of circle is of the form $x^2 + y^2 = a^2$

We know that, the equation of tangent of circle $x^2 + y^2 = a^2$ at (x_1, y_1) is

$$xx_1 + yy_1 = a^2 \qquad (1)$$

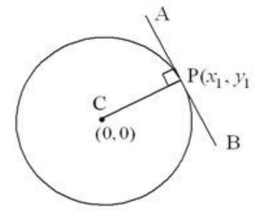


fig. 12.22

or $$y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1},$$

Its slope $= -x_1 / y_1$. Thus, slope of a normal will be $= y_1 / x_1$

From fig. 12.22 , equation of line passing through the point $P(x_1, y_1)$ and perpendicular to (1) is

$$y - y_1 = \frac{y_1}{x_1}(x - x_1) \text{ or } xy_1 - yx_1 = 0 .$$

Which is the required equation of the normal.

(ii) Slope from of a normal of the circle $x^2 + y^2 = a^2$:

In (i) , we have seen that, if slope of a tangent is m then slope of a normal will be $-1/m$. Thus equation of perpendicular line passing through origin (0, 0) will be

$$y - 0 = -\frac{1}{m}(x - 0)$$

or $$x + my = 0$$

Which is the required slope form of the normal.

Illustrative Examples

Example 5 : Find the coordinates of the point of intersection of the circle $x^2 + y^2 - 5x - y + 4 = 0$ and line $x - 2y + 1 = 0$, also find the length of the chord.

Solution : Equation of the circle $x^2 + y^2 - 5x - y + 4 = 0$ (1)

and line $x - 2y + 1 = 0$ (2)

solving equation (1) and (2) for x and y by putting the values $x = 2y - 1$, in equation (1) we have

$$(2y - 1)^2 + y^2 - 5(2y - 1) - y + 4 = 0$$

or $$y^2 - 3y + 2 = 0$$

$\therefore$ $y = 1$ and 2

$\therefore$ when $y = 1$ then $x = 1$ and when $y = 2$ then $x = 3$

Thus the coordinates of P and Q will be (1, 1) and (3, 2)

$\therefore$ length of the intersecting chord is $= \sqrt{(3-1)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$

Example 6 : Find the equation of tangent to a circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which is parallel to the line $4x - 3y + 6 = 0$

Solution : Equation of given circle $\quad x^2 + y^2 - 6x + 4y - 12 = 0 \qquad (1)$

centre (3, -2) and radius 5

let there be a line parallel to the line $4x - 3y + 6 = 0$ which is

$$4x - 3y + k = 0 \qquad (2)$$

it will touch the circle

Radius of the circle = length of perpendicular drown from the centre to the circle to the tangent

or $$5 = \pm \frac{4 \times 3 - 3 \times (-2) + k}{\sqrt{16 + 9}}$$

or $$25 = \pm(18 + k)$$

[taking + sign, $k = 7$] , [taking - sign, $k = -43$]

Thus from (2) the required equations are $4x - 3y + 7 = 0$ and $4x - 3y - 43 = 0$

Exercise 12.2

1. find the coordinates of the point of intersection of the circle $x^2 + y^2 = 25$ and line $4x + 3y = 12$, also find the length of the intersecting chord.
2. If the circle $x^2 + y^2 = a^2$ cuts an intercept of length $2l$ on a line $y = mx + c$ then prove that $c^2 = (1 + m^2)(a^2 - l^2)$.
3. Find the length of intercept cut by the circle $x^2 + y^2 = c^2$ on the line $\frac{x}{a} + \frac{y}{b} = 1$.
4. For what value of *k* line $3x + 4y = k$ touches the circle $x^2 + y^2 = 10x$.
5. Find the condition if

   (i) Line $y = mx + c$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$.

   (ii) Line $lx + my + n = 0$ touches the circle $x^2 + y^2 = a^2$.
6. (i) Find the equation of tangent to a circle $x^2 + y^2 = 64$ which passes through the point (4, 7)

   (ii) Find the equation of tangent to a circle $x^2 + y^2 = 4$ which makes an angle of 60^0 with *x*-axis.
7. Find the value of c if line $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at a point $(1,1)$.
8. Find the equation of tangent to a circle $x^2 + y^2 = 169$ at point (5, 12) and (12, -5). Prove that they are matually perpendicular also find the coordinates of the point of intersection.

Parabola

12.10 Definition

A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line) in the plane. The fixed line is called the directrix of the parabola and the fixed point F is called the focus. A line through the focus and perpendicular to the directrix is called the axis of the parabola. The point of intersection of parabola with the axis is called the vertex of the parabola.

12.11 Standard equation of the Parabola

Let ZZ' be the directrix and S be the Focus of parabola. Drop a perpendicular SK from focus S to ZZ'. Let A be the mid-point of SK, then by definition point A will lie on parabola. $[\because AS = AK]$

Let AS be the x-axis, A is the origin and AY be the y-axis . If SK = 2 a then AS = AK = a . Thus the coordinates of focus S will be $(a, 0)$ and equation of directrix will be $x = -a$

Let $P(x, y)$ be any point on the parabola. Join SP and drop perpendicular from P ons PN and PM (on x-axis and directrix ZZ, then according to the definition of parabola

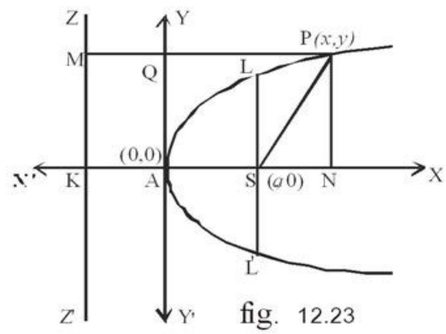


fig. 12.23

$$SP = PM \text{ or } SP^2 = PM^2$$

or $$(x-a)^2 + y^2 = (x+a)^2$$

or $$x^2 + y^2 - 2ax + a^2 = x^2 + 2ax + a^2$$

or $$y^2 = 4ax$$

which is the required equation of parabola.

12.12 Important definitions

(i) Focal chord : Chord which passes through the focus of the parabola is called focal chord.

(ii) Focal distance : The distance of focus from any point on the parabola is known as focal distance.

(iii) Focal Property : In the fig. 12.23

$$SP = PM = PQ + QM = a + x$$

(iv) Latus rectum : Chord perpendicular to axis of parabola and passes through the focus. Is known is Latus rectum.

(v) Double ordinate : The chord of parabola which is perpendicular to the axis of parabola is known as double ordinate.

(vi) Semi latus rectum : Half of length of latus rectum is known semi latus retum

(vii) Length of latus rectum : Let LSL' be the latus rectum of parabola $y^2 = 4ax$, $SL = SL'$ or $LL' = 2SL$

let $SL = \ell$, thus the coordinates of L will be (a, ℓ) and it lies on parabola

$\therefore \quad \ell^2 = 4a \cdot a \qquad$ or $\qquad \ell = \pm 2a$

thus length of latus rectum $= 2 \times 2a = 4a$

coordinates of L will be $(a, 2a)$ and of L' will be $(a, -2a)$

12.13 Tracing of the standard equation of the Parabola

From the standard equation of the parabola $y^2 = 4ax$ or $y = \pm 2\sqrt{ax}$ we have the following observations :

(i) If value of x is negative then the value of y is imaginary. There for no part of curve lies in the left side of y-axis.

(ii) For every positive value of x there are two equal and opposite values of y. Therefore curve is symmetric about x-axis.

(iii) If x increases in positive direction then y is also increses i.e. as $x \to \infty$ then $y \to \infty$. Therefore the curve extends to infinity to right of axis of y.

(iv) If $x = 0$ then $y = 0$. Curve passes through origin (0, 0).

(v) If we find y- coordinate with the half of x- coordinate lies on parabola $y^2 = 4ax$ and join then we get the curve as shown in fig. 12.24 .

12.14 Four different forms of Parabola

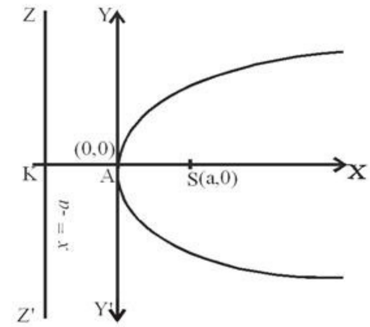


fig. 12.24

1. Parabola $y^2 = 4ax$
   - Vertex A(0, 0)
   - Focus S(a, 0)
   - Axis $y = 0$
   - Directrix $x = -a$
   - length of latus rectum $= 4a$
   - Equation of latus rectum $x = a$

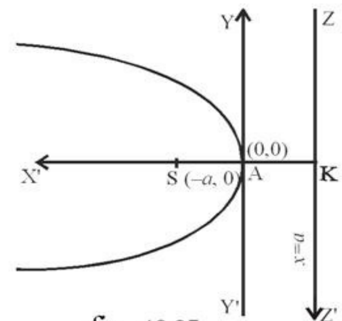


fig. 12.25

2. Parabola $y^2 = -4ax$
   - Vertex A(0, 0)
   - Focus S($-a$, 0)
   - Axis $y = 0$
   - Directrix $x = a$
   - length of latus rectum $= 4a$
   - Equation of latus rectum $x = -a$

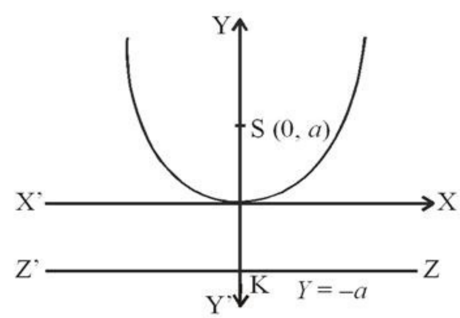


fig. 12.26

3. Parabola $x^2 = 4ay$
   - Vertex A(0, 0)
   - Focus S(0, a)
   - Axis $x = 0$
   - Directrix $y = -a$
   - length of latus rectum $= 4a$
   - Equation of latus rectum $y = a$

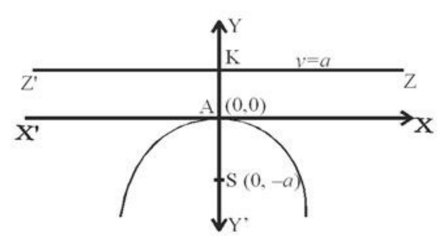


fig. 12.27

4. Parabola $x^2 = -4ay$
   - Vertex A(0, 0)
   - Focus S(0, $-a$)
   - Axis $x = 0$
   - Directrix $y = a$
   - length of latus rectum $= 4a$
   - Equation of latus rectum $y = -a$

Illustrative Examples

Example 7 : Find the equation of parabola with focus $(-1, -2)$ and equation of directrix $x - 2y + 3 = 0$

Solution : Let $P(h, k)$ be any on the parabola, then according to the difinition of parabola

$$SP = PM \quad \text{and} \quad SP^2 = PM^2$$

or $$(h+1)^2+(k+2)^2=\left(\frac{h-2k+3}{\sqrt{1+4}}\right)^2$$

or $$5\left\{(h+1)^2+(k+2)^2\right\}=(h-2k+3)^2$$

or $$5\left(h^2+k^2+2h+4k+5\right)=h^2+4k^2+9-4hk+6h-12k$$

or $$4h^2+k^2+4hk+4h+32k+16=0$$

Thus the locus of point $P(h, k)$ is $4x^2+y^2+4xy+4x+32y+16=0$ which is the required equation of parabola.

Example 8 : Find the vertex, axis, focus, directrix and latus rectum of the parabola $y^2=2y-2x$

Solution : Equation of parabola $y^2=2y-2x$

$\therefore$ $$y^2-2y=-2x$$

or $$y^2-2y+1=-2x+1$$

or $$(y-1)^2=-2(x-1/2) \qquad (1)$$

putting $y-1=Y$ and $x-(1/2)=X$ in eq. (1) we have

$$Y^2=-2X \qquad (2)$$

which is of the form $y^2=4ax$ hence comparing with (2)

$$4a=-2 \Rightarrow a=-1/2$$

For Parabola $Y^2=-2X$

(a) Vertex (0, 0) i.e. $X=0; Y=0$

(b) Focus (–1/2, 0) i.e. $X=-1/2; Y=0$

(c) Axis $Y=0$

(d) Directrix $X=1/2$

(e) latus rectum $=4a=4(-1/2)=2$ (positive)

putting $X=x-(1/2)$ and $Y=y-1$ in the above terms for parabola

(a) Vertex $x-(1/2)=0$ or $x=1/2$, $y-1=0$ or $y=1$, thus the coordinates are $(1/2, 1)$

(b) Focus $x-(1/2)=-1/2$ or $x=0$ and $y-1=0$ or $y=1$ thus the coordinates are $(0, 1)$

(c) Axis $y-1=0$ or $y=1$

(d) Directrix $x-1/2=1/2 \Rightarrow x=1$

(e) latus rectum $=2$

Example 9 : Find the vertex, axis, focus, directrix and latus rectum (equation and length) of the parabola $4y^2-6x-4y=5$

Solution : Given Equation of parabola $4y^2-6x-4y=5$

or $$\left[y-(1/2)\right]^2=\frac{6}{4}(x+1)$$

or $$\left[y-(1/2)\right]^2=\frac{3}{2}(x+1) \qquad (1)$$

shifting the origin to $(-1,\ 1/2)$ the changed form of (1)

$$Y^2 = \frac{3}{2}X \tag{2}$$

here $Y = y-(1/2)$ and $X = x+1$ length of latus rectum $= 3/2$

$\therefore$ $a = 3/8$ latus rectum $= 3/2$, thus coordinates of focus and vertex are $(3/8,\ 0)$ and $(0, 0)$ respectively from parabola (2) equation of Latus rectum $X = 3/8$ and equation of directrix is $X = -3/8$ of (2)

$\therefore$ coordinates of focus and vertex of parabola (1) are $(-5/8,\ 1/2)$ and $(-1,\ 1/2)$ equation of latus rectum $x+1 = 3/8$ or $x = -5/8$ and equation of directrix is $x+1 = -3/8$ or $x = -11/8$.

Example 10 : Find the vertex, axis, focus and latus rectum of the parabola $9x^2 - 6x + 36y + 19 = 0$

Solution : Given equation of parabola $9x^2 - 6x + 36y + 19 = 0$ है।

or $$9x^2 - 6x = -36y - 19$$

or $$x^2 - (2/3)x = -4y - (19/9)$$

or $$x^2 - (2x/3) + (1/3)^2 = -4y - (19/9) + (1/9)$$

or $$\left[x-(1/3)\right]^2 = -4y - 2$$

or $$\left[x-(1/3)\right]^2 = -4\left[y+(1/2)\right] \tag{1}$$

shifting the origin to $(1/3,\ -1/2)$ from (1)

$$x-(1/3) = X \quad \text{and} \quad y+(1/2) = Y$$

(1) will take the form

$$X^2 = -4Y \tag{2}$$

(of the type $X^2 = 4aY$ where $a = -1$)

axis at $X = 0$

coordinates of vertex $= (0,\ 0)$ i.e. $X = 0$ and $Y = 0$

coordinates of focus $= (0,\ -1)$ i.e. $X = 0$ and $Y = -1$

length of latus rectum $= 4\times1 = 4$ $(\because\ a = -1)$

Substituting the value of X and Y in (1) equation of axis

$$X = 0 \Rightarrow x - 1/3 = 0 \Rightarrow x = 1/3$$

coordinates of vertex $X = 0 \Rightarrow x = 1/3;\ Y = 0 \Rightarrow y+(1/2) = 0 \Rightarrow y = -1/2$

$\Rightarrow$ So, coordinates of vertex $= (1/3,\ -1/2)$

coordinates of focus $X = 0 \Rightarrow x-(1/3) = 0 \Rightarrow x = 1/3;\ Y = -1 \Rightarrow y+(1/2) = -1 \Rightarrow y = -3/2$

$\Rightarrow$ So, coordinates of focus $= (1/3,\ -3/2)$, length of latus rectum $= 4\times1 = 4$

Exercise 12.3

1. Find the equation of parabola when

(i) focus (2, 3) and directrix $x-4y+3=0$ (ii) focus (-3, 0) and directrix $x+5=0$

2. Find the vertex, axis, focus and length of latus rectum of the parabola.

(i) $y^2=8x+8y$ (ii) $x^2+2y=8x-7$

3. If the length of double ordinate of parabola $y^2=4ax$ is $8a$ then prove that lines meeting the double ordinate from the origin are perpendicular.

4. If the distance of vertex and focus from origin to the x-axis are a and or a', then prove that equation of parabola is $y^2=4(a'-a)(x-a)$

5. PQ is a double ordinate of a parabola. Find the focus of points trisecting it.

6. Prove that the locus of all points passing through the midpoint of the chord and vertex of the parabola $y^2=4ax$ is also a parabola of the form $y^2=2ax$

12.15 Intersection of a Parabola and a line

Let the equation of parabola be $y^2=4ax$ (1)

and equation of line be $y=mx+c$ (2)

putting the value of y from (1) to (2), we get

$$(mx+c)^2=4ax$$

or $$m^2x^2+2(mc-2a)x+c^2=0 \tag{3}$$

which is a quadratic equation in x, let the roots be x_1 and x_2, then

$$x_1+x_2=\frac{-2(mc-2a)}{m^2} \tag{4}$$

and $$x_1x_2=\frac{c^2}{m^2} \tag{5}$$

$\therefore$ $$x_1-x_2=\sqrt{(x_1+x_2)^2-4x_1x_2}=\sqrt{\left(\frac{4(cm-2a)^2}{m^4}-\frac{4c^2}{m^2}\right)}$$

$$=\frac{1}{m^2}\sqrt{4m^2c^2-16amc+16a^2-4m^2c^2}$$

$\therefore$ $$x_1-x_2=\frac{4}{m^2}\sqrt{a(a-mc)} \tag{6}$$

By solving equations (4) and (6) we can get the values for x_1 and x_2 followed by y_1 and y_2. Hence the line cuts the parabola at two points.

12.16 Nature of the points of intersection

The intersecting points can be real, coinciding or imaginary depending upon the nature of the roots of equation (3) as be real, equal or imaginary.

$$\{2(mc-2a)\}^2-4m^2c^2 >=< 0$$

or $$a(a-mc) >=< 0$$

or $$a >=< mc$$

12.17 Condition for straight line $y=mx+c$ to be a tangent to the parabola $y^2=4ax$

If the intersecting points of a line and parabola coincide, then line will touch the parabola. Thus line (2) will touch parabola (1), if the roots of equation (3) are equal i.e.

$$4(mc-2a)^2 = 4m^2c^2 \qquad (B^2 = 4AC)$$

or $$a(a-mc) = 0$$

or $$a - mc = 0$$

$\therefore$ $$c = a/m$$

This is the required condition of line touching the parabola, puitting the value of c in equation (2) we get $y = mx + (a/m)$. Thus line $y = mx + (a/m)$ (for each value of m) touches the parabola $y^2 = 4ax$

12.18 Co-ordinates of the point of contact

In equation (3) of article 12.15 putting, $c = a/m$ we have

$$m^2x^2 + 2x\left(\frac{ma}{m} - 2a\right) + \frac{a^2}{m^2} = 0$$

or $$\left(mx - \frac{a}{m}\right)^2 = 0$$

i.e. putting the value $x = \frac{a}{m^2}, x$ in (2) we have $y = m\left(\frac{a}{m^2}\right) + \frac{a}{m} = \frac{2a}{m}$

Thus co-ordinates of point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

12.19 Length of intercept

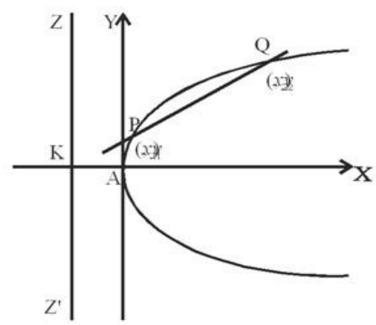


fig. 12.28

Let the equation of straight line be

$$y = mx + c \qquad (1)$$

and equation of parabola

$$y^2 = 4ax \qquad (2)$$

The intersecting points of (1) and (2) are

$$P(x_1, y_1) \text{ and } Q(x_2, y_2)$$

Since P and Q lie on a line,

$\therefore$ $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

$\therefore$ $$y_1 - y_2 = m(x_1 - x_2) \qquad (3)$$

Now $$PQ = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = \sqrt{(x_1-x_2)^2 + m^2(x_1-x_2)^2}$$

$$= (x_1 - x_2)\sqrt{1+m^2}$$

but $$(x_1 - x_2) = \frac{4}{m^2}\sqrt{a(a-mc)}$$ [article 12-15, equation (6)]

$\therefore$ $$PQ = \frac{4}{m^2}\sqrt{\left[a(a-mc)(1+m^2)\right]}$$

12.20 Tangent

(i) Equation of tangent at any point on the parabola :

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the Parabola $y^2 = 4ax$

$\therefore \qquad y_1^2 = 4ax_1 \qquad (1)$

and $\qquad y_2^2 = 4ax_2 \qquad (2)$

Equation of line PQ, $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad (3)$

form equation (1) and (2),

$$y_2^2 - y_1^2 = 4a(x_2 - x_1)$$

or $$(y_2 - y_1)(y_2 + y_1) = 4a(x_2 - x_1)$$

or $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_1 + y_2} \qquad (4)$$

form equation (3) and (4) of PQ

$$y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1) \qquad (5)$$

When Q coincides with P, then chord PQ is a tangent at point P, therefore putting : $y_2 = y_1$ in (5)

$$y - y_1 = \frac{4a}{2y_1}(x - x_1)$$

or $$yy_1 - y_1^2 = 2a(x - x_1)$$

or $$yy_1 = 2ax - 2ax_1 + y_1^2$$

or $$yy_1 = 2ax - 2ax_1 + 4ax_1$$

$yy_1 = 2a(x + x_1)$, which is the required equation of tangent.

(ii) Equation of tangent in terms of slope :

Let the equation of parabola be $\qquad y^2 = 4ax \qquad (1)$

equation of tangent at point $P(x_1, y_1)$

$$yy_1 = 2a(x + x_1) \qquad (2)$$

let $$m = \frac{2a}{y_1}$$

or $$y_1 = \frac{2a}{m}$$

$\because$ the point $P(x_1, y_1)$ lies on the parabola

$$\therefore \qquad y_1 = 4ax_1$$

or, $$\frac{4a^2}{m^2} = 4ax_1$$

or, $$x_1 = \frac{a}{m^2}$$

substituting the value of x_1 and y_1 in equation (2), we get the required equation of tangent

$$y\left(\frac{2a}{m}\right) = 2a\left(x + \frac{a}{m^2}\right),$$

or, $y = mx + \frac{a}{m}$ and coordinates of point of the contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Illustrative Examples

Example 11 : Prove that line $y = mx + c$ touches to the parabola $y^2 = 4a(x+a)$ if $c = am + \frac{a}{m}$.

Solution : If the line $y = mx + c$ touches the parabola $y^2 = 4a(x+a)$ then it will cut at two concide points now elimininting y from the equation of line we have

$$(mx+c)^2 = 4a(x+a)$$

or $$m^2x^2 + 2(mc - 2a)x + (c^2 - 4a^2) = 0$$

The roots of the equation will be same if $B^2 - 4AC = 0$

$$4(mc-2a)^2 - 4m^2(c^2 - 4a^2) = 0$$

or $$m^2c^2 + 4a^2 - 4amc - m^2c^2 + 4a^2m^2 = 0$$

or $$4a(a - mc + am^2) = 0$$

or $$a - mc + am^2 = 0$$

$\therefore$ $$c = am + \frac{a}{m}$$

Example 12 : Find the equation of line touches the parabola $y^2 = 4x$ and $x^2 = 32y$

Solution : We know that the line

$$y = mx + \frac{1}{m} \qquad (1)$$

touches the parabola $y^2 = 4x$ for every value of m. If (1) touches the parabola $x^2 = 32y$ then their intersecting points must be concide $x^2 = 32y$

this equation

$$x^2 = 32\left(mx + \frac{1}{m}\right)$$

or $$mx^2 - 32m^2x - 32 = 0$$

must have equal roots thus using $B^2 - 4AC = 0$

$$\left(-32m^2\right)^2 - 4m\left(-32\right) = 0$$

$\Rightarrow$ $$8m^3 = -1 \Rightarrow m = -1/2$$

thus the equation of line is

$$y = -\frac{x}{2} + \frac{1}{(-1/2)}$$

or $$x + 2y + 4 = 0$$

Example 13 : Find the equation of line touches the parabola $y^2 = 8x$ and perpendicular to the line $2y - x + 1 = 0$

Solution : Equation of line perpendicular to the line $2y - x + 1 = 0$

$$y + 2x + k = 0 \quad (1)$$

Line (1) will touch the parabola $y^2 = 8x$ if equation is $\left(2x + k\right)^2 = 8x$

or $$4x^2 + 4\left(k - 2\right)x + k^2 = 0$$

and roots are equal i.e. $(B^2 = 4AC)$

$$16\left(k - 2\right)^2 = 4 \times 4 \times k^2$$

$\Rightarrow$ $k^2 - 4k + 4 = k^2$ $\Rightarrow$ $k = 1$

Thus the required equation is

$$y + 2x + 1 = 0$$

12.21 Equation of normal

(i) To find the equation of normal to parabola $y^2 = 4ax$ at point $\left(x_1, y_1\right)$

Equation of tangent to parabola $y^2 = 4ax$ at point $\left(x_1, y_1\right)$ is

$$yy_1 = 2a\left(x + x_1\right) \quad (1)$$

The slope of (1) is $m = \frac{2a}{y_1}$. Then, at point $P\left(x_1, y_1\right)$, the slope of normal $= \frac{-y_1}{2a}$.

$\therefore$ The equation of normal drawn on parabola $y^2 = 4ax$ at point $\left(x_1, y_1\right)$ will be

$$y - y_1 = \frac{-y_1}{2a}\left(x - x_1\right)$$

(ii) Equation of normal in slope form

Equation of normal of parabola $y^2 = 4ax$ at point $P\left(x_1, y_1\right)$ is

$$y - y_1 = \frac{-y_1}{2a}\left(x - x_1\right) \quad (1)$$

If the slope of normal is m then

$$\frac{-y_1}{2a} = m \quad \text{or} \quad y_1 = -2am \quad (2)$$

Since point $P(x_1, y_1)$ lies on a parabola

$$y_1^2 = 4ax_1 \qquad \text{or,} \qquad (-2am)^2 = 4ax_1 \qquad \text{or} \qquad x_1 = am^2$$

putting the values of x_1, y_1 in equation (1), the required equation of normal will be

$$y = mx - 2am - am^3 \qquad (3)$$

and the coordinates of P will be $(am^2, -2am)$

12.22 Three normals can be drawn from any point to the parabola

Let the equation of parabola be $y^2 = 4ax$ and equation of normal be $y = mx - 2am - am^3$. If the normal passes through the point (x_1, y_1), then $y_1 = mx_1 - 2am - am^3$ or $am^3 + (2a - x_1)m + y_1 = 0$

This is a cubic equation in m, and will have three values of m, therefore, three normals can be drawn from point (x_1, y_1).

Property : (i) If three normals are drawn from a point to the parabola, then the sum of their slopes is zero.

(ii) The algebraic sum of foot of co-ordinates (y-coordinates) is always zero.

Proof: Equation of normal passing through (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$am^3 + (2a - x_1)m + y_1 = 0 \qquad (1)$$

This is cubic equation in m, let the roots be m_1, m_2 and m_3

(i) Sum of the roots $m_1 + m_2 + m_3 = \dfrac{-\text{coefficient of } m^2}{\text{coefficient of } m^3} = \dfrac{0}{a} = 0$

Thus, sum of the slopes is zero.

(ii) Sum of ordinates (y-coordinate) of foot of normal

$$= -2am_1 - 2am_2 - 2am_3$$

$$= -2a(m_1 + m_2 + m_3) = -2a \times 0 = 0$$

Illustrative Examples

Example 14 : Prove that the chord of parabola $y^2 = 4ax$ whose equation is $y - x\sqrt{2} + 4a\sqrt{2} = 0$ is normal to the parabola and its length is $6a\sqrt{3}$.

Solution : equation of the normal at point m is given by

$$y = mx - 2am - am^3 \qquad (1)$$

Equation of the chord is

$$y = x\sqrt{2} - 4a\sqrt{2} \qquad (2)$$

equation (2) is normal to the parabola if it is of the form of equation (1). Therefore comparing (1) and (2) we have

$$m = \sqrt{2} \ ; \ -4a\sqrt{2} = -2am - am^3$$

Thus line (2) is normal to the parabola as it satisfy itself. Now putting the value of y from (2) in equation of parabola we have

$$\left(x\sqrt{2} - 4a\sqrt{2}\right)^2 = 4ax$$

or $$2x^2+32a^2-16ax-4ax=0$$

$$x^2-10ax+16a^2=0$$

or $$(x-2a)(x-8a)=0$$

Therefore $x=2a,\ 8a$ similarly $y=-2a\sqrt{2},\ 4a\sqrt{2}$

Therefore length of chord of normal $=\sqrt{\left[(8a-2a)^2+\left(4a\sqrt{2}+2a\sqrt{2}\right)^2\right]}=\sqrt{36a^2+72a^2}=6a\sqrt{3}$.

Exercise 12.4

1. Find the co-ordinates of the point of intersection when a line $4y+3x+6=0$ and parabola $2y^2=9x$.
2. Find the length of chord cut by the line $4y-3x=8$ on the parabola $y^2=8x$.
3. Prove that the line $x+y=1$ touches the parabola $y=x-x^2$.
4. Find the condition when a parabola $y^2=4ax$ touches the line $\ell x+my+n=0$.
5. Prove that the length of the focal chord which makes an angle "α" with x-axis to the parabola $y^2=4ax$ is $4a\operatorname{cosec}^2\alpha$.
6. Find the condition when the line $x\cos\alpha+y\sin\alpha=p$ touches the parabola $y^2=4ax$.
7. Determine the equation of tangent to the following given parabolas –

   (i) $y^2=6x$, which is parallel to the line $2x-3y=4$

   (ii) $y^2=8x$, which is perpendicular to the line $2x-y+1=0$
8. For what value of k does the line $2x-3y=k$ touches the parabola $y^2=6x$?
9. Find the equation of tangent drawn from the point (4, 10) to the parabola $y^2=8x$.
10. Find the equation of normal on the following given parabolas -

    (i) $y^2=8x$ at point $(2,4)$ (ii) $y^2+12x=0$ on the upper end of latus rectum.
11. Find the equation of normal to the following given parabolas -

    (i) $y^2=4x$ which is perpendicular to the line $y-2x+5=0$.

    (ii) $y^2=4x$ which is parallel to the line $x+3y-1=0$.
12. Prove that the line $2x+y-12a=0$ normal to the parabola $y^2=4ax$ and its length is $5\sqrt{5}a$ units.

Ellipse

12.23 Definition

The locus of a point which moves in such a way that the ration of the distances measured from the fixed line is always less than one, is called an ellipse. The fixed point is called focus, the fixed line is called directrix and ratio is called the eccentricity of the ellipse.

12.24 Standard equation of the Ellipse

Let S be the focus of Ellipse, line ZK be the directrix, e eccentricity and $P(x,\ y)$ be any arbitarary point. Since $e<1$, therefore the ellipse will divide SK internally and externally in the ratio $e:1$ at points A and A' respectively.

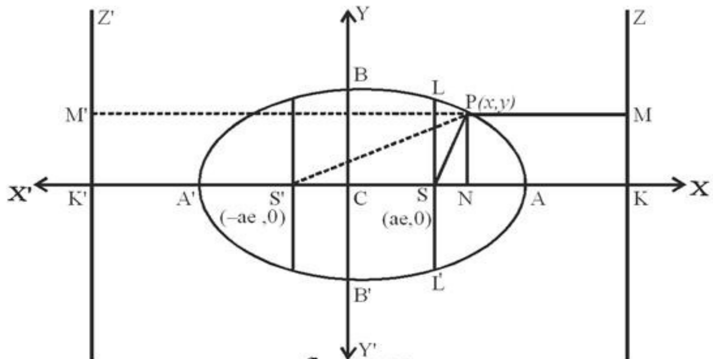


fig. 12.29

$$\frac{AS}{AK} = \frac{e}{1} \qquad \text{or} \qquad AS = e \cdot AK \tag{1}$$

Also A' lies on the ellipse. Therefore,

$$\frac{A'S}{A'K} = \frac{e}{1} \qquad \text{or} \qquad A'S = e \cdot A'K \tag{2}$$

Let $AA' = 2a$. C is the mid-point.

$\therefore$ $CA = CA' = a$ On adding (1) and (2), we get

$$AS + A'S = e\left(AK + A'K\right) \qquad \text{or} \qquad AA' = e\left\{CK - CA + CA' + CK\right\}$$

or $\quad AA' = e\left(2CK\right),$ $\qquad [\because\ CA = CA']$

or $\quad 2a = 2e \cdot CK,$ $\qquad [\because\ AA' = 2a]$

$\therefore \quad CK = a/e$ $\qquad$ (3)

Substracting (1) from (2), $\quad A'S - AS = e\left(A'K - AK\right)$

or $\quad \left(CA' + CS\right) - \left(CA - CS\right) = e\left(AA'\right)$

$$2CS = eAA', \qquad [\because CA = CA']$$

$$2CS = e \cdot 2a, \qquad [\because AA' = 2a]$$

$$\therefore \quad CS = ae \tag{4}$$

Considering C as the origin, CAX as x-axis and perpendicular line CY as y-axis, we will derive the equation of ellipse. The co-ordinates of focus is $S\left(ae,\ 0\right)$ from equation (4) and equation of directrix is $X = a/e$ from equation (3). Draw perpendicular from point P on PN and PM respectively i.e. on x-axis and directrix ZK. Join P and S. Then, by defination of ellipse, we have

$$SP = ePM$$

or $$SP^2 = e^2 PM^2$$

or $$SN^2 + NP^2 = e^2 NK^2$$

or $$\left(CN - CS\right)^2 + NP^2 = e^2\left(CK - CN\right)^2$$

or $$\left(x - ae\right)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2 \qquad \text{(from (3) and (4))}$$

or $$x^2\left(1 - e^2\right) + y^2 = a^2\left(1 - e^2\right) \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2\left(1 - e^2\right)} = 1 \tag{5}$$

substituting $a^2(1-e^2)=b^2$ in (5), the equation of ellipse can be written as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \text{where} \quad b^2=a^2(1-e^2) \qquad (6)$$

Note : (1) putting $x=0$ in (5), we get the co-ordinates of B and B' as $\left(O,\ a\sqrt{1-e^2}\right)$ and $\left(O,\ -a\sqrt{1-e^2}\right)$ i.e. $(0,\ b)$ and $(0,\ -b)$,

(2) In equation (6) , if $a=b$, then ellipse will be of the form $x^2+y^2=a^2$, which is the standard equation of a circle with centre at origin and radius a .

12.25 Some Definitions

1. **Major axis :** The line joining A and A' is known as major axis and its length is $2a$ (fig. 12,29)
2. **Minor axis :** The line BB' , passing through C(0, 0) is known as minor axis and its length is $2b$. (fig. 12.29)
3. **Principal axes :** Major and Minor axis both are termed as Principal axes.
4. **Latus rectum :** A line passing through the focus and perpendicular to major axis is called the latus rectum.

12.26 Tracing of standard equation of the ellipse

Equation of ellipse is
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

$$y=\pm\frac{b}{a}\sqrt{(a^2-x^2)} \qquad (2)$$

and
$$x=\pm\frac{a}{b}\sqrt{(b^2-y^2)} \qquad (3)$$

substituting $-x$ in place of x in eq. (2), we see that there is no change in the value of y. Therefore, Ellipse is symmetric about y-axis. Similar putting $-y$ in place of y in (3), we see that there is no change in the value of x. Therefore, the Ellipse is symmetric about x-axis.

From equation (2), if $x>a$, then y is imaginary . Hence, no part of Ellipse will be right of A and to the left of A'.

Similarly from equation (3), if y > b , then x will be imaginary. Hence no part of Ellipse will be above B and below B'. Thus, Ellipse is a closed figure.

From (2), we see that as x increases then y decreases and from (3) as y increases then x decreases. Therefore an Ellipse looks like a stretched circle.

According to above facts and informations, drawn curve is according to fig. 12.29.

12.27 Another form of the ellipse

If an Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and $b>a$, then $a^2=b^2(1-e^2)$. In this case, x-axis will be the minor axis and y-axis will be the major axis. The length of major axis will be $2b$ and length of minor axis will be $2a$,

The co-ordinates of foci S and S' will be $(0,\ be)$ and $(0,\ -be)$ and the equation of directrices be $y=\pm b/e$ and length of latus rectum will be $2a^2/b$.

12.28 Second focus and second directrix of the ellipse

From fig. 12.29, points S' and K' are taken on to the left of x-axis in such a way that

$$CS'=CS=ae \qquad \text{and} \qquad CK'=CK=a/e$$

$\therefore$ Equation of line K' M' is

$$x=-a/e \qquad (1)$$

From point $P(x,\ y)$ lying on the Ellipse, draw perpendicular PM' from the line K' Z'. Now, if S' be the focus and line K'M' be the directrix, then by the definition of Ellipse

$$PS' = ePM' \qquad \text{or} \qquad (PS')^2 = e^2 (PM')^2 \qquad (2)$$

Since the co-ordinates of S' are $(-ae, 0)$. Therefore, from (2)

$$(x+ae)^2 + y^2 = e^2 (x + a/e)^2$$

or $$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

or $$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1-e^2)$$

This is the same equation with focus S and directrix KM. Hence, Ellipse has two foci and two directrices.

12.29 Focal Property

By the definition of ellipse,

$$SP = e \cdot PM = eNK = e(CK - CN) \qquad (1)$$

and $$S'P = ePM' = eNK' = e(CN + CK') \qquad (2)$$

$$\Rightarrow \qquad SP + S'P = e(CK + CK') = e\left(\frac{a}{e} + \frac{a}{e}\right) = 2a = \text{length of major axis.}$$

Thus, the distance between the foci from any point lying on the ellipse is constant and is equal to the length of the major axis. On the basis of above important property, the definition of ellipse can be given as follows :

"Ellipse is the locus of that point whose sum of distances of two fixed points (foci) is constant."

12.30 Length of the latus-rectum of the ellipse

In fig. 12.29, LSL' is a latus rectum. L and L' lies on the ellipse. Also $x = ae$ and let $SL = y'$. Now co-ordinates of L will be (ae, y'). Since point L lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$\therefore \qquad \frac{a^2e^2}{a^2} + \frac{y'^2}{b^2} = 1$$

or $$y'^2 = \frac{b^4}{a^2} \qquad \left[\because b^2 = a^2(1-e^2)\right]$$

or $$y' = \pm \frac{b^2}{a}$$

$\therefore$ length of the latus rectum $= LL' = 2SL = \dfrac{2b^2}{a}$

Co-ordinates of L and L' will be $(ae, b^2/a)$ and $(ae, -b^2/a)$

Similarly, the co-ordinates of the ends of latus rectum passing through S' will be $(-ae, -b^2/a)$ and $(-ae, b^2/a)$.

Illustrative Examples

Example 15 : Find the equation of Ellipse whose focus is (–1, 1), directrix $x - y + 3 = 0$ and eccentricity $e = 1/2$

Solution : Let P(h, k) be any point on the ellipse by definition

distance of P from focus = e (distance of P from the directix)

or $$PS = e(PM)$$

or $$(PS)^2 = e^2 (PM)^2$$

or $$(h+1)^2 + (k-1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

or $$8\left(h^2 + k^2 + 2h - 2k + 2\right) = (h-k+3)^2$$

or $$7h^2 + 2hk + 7k^2 + 10h - 10k + 7 = 0$$

which is the required equation of ellipse $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$ as $P(h, k)$ lies on the ellipse.

Example 16 : Find the eccentricity, length of latus rectum and coordinates of foci of the Ellipse :

(i) $3x^2 + 4y^2 = 12$ (ii) $9x^2 + 5y^2 - 30y = 0$

Solution : (i) Equation of ellipse can be written as

$$\frac{x^2}{4} + \frac{y^2}{3} = 1, \qquad (1)$$

where $a^2 = 4$ and $b^2 = 3$

(i) eccentricity since $b^2 = a^2\left(1 - e^2\right)$ $\therefore$ $3 = 4\left(1 - e^2\right) \Rightarrow e = 1/2$

(ii) latus rectum $= \dfrac{2b^2}{a} = \dfrac{2 \times 3}{2} = 3$

(iii) coordinates of foci (ae, 0) and (–ae, 0) i.e. (1, 0) and (–1, 0)

(ii) Equation of ellipse

$$9x^2 + 5y^2 - 30y = 0$$

or $$9x^2 + 5(y-3)^2 = 45$$

or $$\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1 \qquad (1)$$

shifting the origin at point (0, 3)

$$\frac{X^2}{5} + \frac{Y^2}{9} = 1, \text{ here } a^2 = 5 \text{ and } b^2 = 9 \text{ or } b > a \qquad (2)$$

(i) **Eccentricity :** $a^2 = b^2\left(1 - e^2\right)$ is $5 = 9\left(1 - e^2\right) \Rightarrow e = \dfrac{2}{3}$

(ii) **Length of latus rectum :** $= \frac{2a^2}{b} = \frac{2\cdot 5}{3} = \frac{10}{3}$

(iii) **Coordinates of foci :** Co-ordinates of foci of (2) $(0, \pm be)$ since focus lie on Y-axis hence the coordinates of focus will be (0, +2).

Co-ordinates of foci will be $(0, 3\pm 2)$ i.e. (0, 1) and (0, 5)

Example 17 : Find the equation of ellipse whose coordinates of vertices and foci are $(\pm 5, 0)$ and $(\pm 4, 0)$

Solution : Equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

coordinates of vertices and foci are $(\pm a, 0)$ and $(\pm ae, 0)$ respectively On comparing we have $a = 5$ and $ae = 4$

$$a = 5 \text{ and } e = 4/5$$

again, $e = \sqrt{(1 - b^2/a^2)}$ $\therefore$ $4/5 = \sqrt{(1 - b^2/25)}$ $\Rightarrow$ $b = 3$

by putting the value of a and b in (1) $= \frac{x^2}{25} + \frac{y^2}{9} = 1$

This is the required equation of ellipse.

Example 18 : Find the equation of ellipse whose eccentricity is $3/5$ and length of latus rectum is 6.

Solution : let the equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (a > b) \qquad (1)$$

given $= \frac{2b^2}{a} = 6$

or $2\frac{a^2(1-e^2)}{a} = 6$

or $a\left(1 - \frac{9}{25}\right) = 3$ $\therefore$ $a = \frac{75}{16}$

from eq. (2) $2b^2 = 6 \times \frac{75}{16}$

or $b = \frac{15}{4}$

putting the value of a and b in (2) we get the required equation of ellipse, $256x^2 + 400y^2 = 5625$.

Exercise 12.5

1. Find the equation of Ellipse whose

(i) Focus (−1, 1), directrix $x - y + 4 = 0$ and eccentricity $e = 1/\sqrt{5}$.

(ii) Focus (–2, 3), directrix $3x+4y=1$ and eccentricity $e=1/3$.

2. Find the eccentricity, length of latus rectum and coordinates of focus of the ellipse.

(i) $4x^2+9y^2=1$ (ii) $25x^2+4y^2=100$ (iii) $3x^2+4y^2-12x-8y+4=0$

3. Find the equation of ellipse whose major axis is along the x-axis and passing through the points (6, 2) and (4, 3)

4. Find the eccentricity of ellipse whose latus rectum is half of its minor axis.

5. Find the locus of a point so that its sum of the distance from the points(1, 0) and (–1, 0) is always 3.

12.31 Intersection of an ellipse and a straight line

Let the equation of Ellipse

$$y=mx+c \tag{1}$$

and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{2}$$

putting the value of y in (2) from (1)

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

or

$$x^2\left(a^2m^2+b^2\right)+2a^2mcx+a^2\left(c^2-b^2\right)=0 \tag{3}$$

It is a quadratic equation with two roots. Therefore it shows that a line will intersect the ellipse at two points But if the line touches the ellipse, then (3) has equal roots i.e.

$$\left(2a^2mc\right)^2-4\left(a^2m^2+b^2\right)a^2\left(c^2-b^2\right)=0, \qquad \left(B^2=4AC\right)$$

or $$4a^4m^2c^2-4a^2\left(a^2m^2c^2-a^2b^2m^2+b^2c^2-b^4\right)=0$$

or $$a^2b^2m^2-b^2c^2+b^4=0$$

or $$b^2\left(a^2m^2-c^2+b^2\right)=0$$

or $$a^2m^2-c^2+b^2=0$$

or $$c=\pm\sqrt{a^2m^2+b^2}$$

$\Rightarrow$ $$c=\pm\sqrt{a^2m^2+b^2} \tag{4}$$

Putting the value of c in (1) the equation of tangent line is $y=mx\pm\sqrt{\left(a^2m^2+b^2\right)}$. Therefore, the line $y=mx\pm\sqrt{a^2m^2+b^2}$ touches the ellipse $x^2/a^2+y^2/b^2=1$ and the coordinates of the point of contact are

$$\left(\mp\frac{a^2m}{\sqrt{\left(a^2m^2+b^2\right)}},\ \pm\frac{b^2}{\sqrt{\left(a^2m^2+b^2\right)}}\right)$$

Illustrative Examples

Example 19 : Find the coordinates of the point of intersection of the parabola $x^2+4y^2=8$ and the line $y=2x-3$

Solution : Given equation of ellipse and a line

$$x^2+4y^2=8 \qquad (1)$$

and $\quad y=2x-3 \qquad (2)$

solving (1) and (2) we have

$$x^2+4(2x-3)^2=8 \qquad \text{or} \qquad 17x^2-48x+28=0$$

$$\Rightarrow \quad x=2,\ 14/17 \ \text{ from (2) } \ y=1,\ -23/17$$

Thus the coordinates are $(2, 1)$ and $\left(\frac{14}{17}, \frac{-23}{17}\right)$

Example 20 : If the line $y=x+c$ touches the parabola $2x^2+3y^2=6$ then find the value of c.

Solution : Given equation of ellipse and a line

$$y=x+c \qquad (1)$$

and $\quad 2x^2+3y^2=6$. $\qquad (2)$

solving (1) and (2) we have

$$2x^2+3(x+c)^2=6 \qquad \text{or} \qquad 5x^2+6cx+\left(3c^2-6\right)=0 \qquad (3)$$

Since the line (1) touches the Ellipse (2) therefore :

$$(6c)^2-4.5\left(3c^2-6\right)=0 \quad \text{or} \qquad 24c^2=120 \qquad \Rightarrow \qquad c=\pm\sqrt{5}$$

Example 21 : Find the condition of a line $x\cos\alpha+y\sin\alpha=p$ to be a tengent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Substituting the value of y from the line $x\cos\alpha+y\sin\alpha=p$ in the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have $\qquad \frac{x^2}{a^2}+\left(\frac{p-x\cos\alpha}{\sin\alpha}\right)\frac{1}{b^2}=1$

or $\quad x^2\left(a^2\cos^2\alpha+b^2\sin^2\alpha\right)-2a^2px\cos\alpha+\left(a^2p^2-a^2b^2\sin^2\alpha\right)=0 \qquad (1)$

the line will touch the ellipse if the roots of (1) are equal

$$\left(-2a^2p\cos\alpha\right)^2-4\left(a^2\cos^2\alpha+b^2\sin^2\alpha\right)\left(a^2p^2-a^2b^2\sin^2\alpha\right)=0$$

or $\qquad 4a^2b^2\sin^2\alpha\left\{a^2\cos^2\alpha-p^2+b^2\sin^2\alpha\right\}=0$

therefore $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$

Exercise 12.6

1. Prove that the line $y=x+\sqrt{(5/6)}$ touches the ellipse $2x^2+3y^2=1$. Also find the co-ordinates of the point of contact.
2. Show that the line $x-3y-4=0$ touches the ellipse $3x^2+4y^2=20$
3. For what value of k does the line $3x-4y=k$ touches the ellipse $5x^2+4y^2=20$
4. Prove that the line $x+y=\sqrt{\left(a^2+b^2\right)}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find the co-ordinates of the point of contact.

5. Find the condition when the line $\ell x + my = n$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

6. find the equation of tangent to the ellipse $4x^2+3y^2=5$ which makes an angle 60^0 with the x-axis. Also find the coordinates of the point of contact.

Hyperbola

12.32 Defination :

A locus of a point which moves in a plane in such a way that the ratio of the distances measured from the fixed point and the fixed line always greater than one, is called hyperbola.

12.33 Standard equation of Hyperbola :

Let there be a fixed point S and a fixed line ZK. Let P(x, y) be any arbitrary point. Drop a perpendicular from S on the directrix ZK, let it be SK.

Let e be eccentricity. Divide SK internally and externally in the ratio e : 1 at points A an A'. Let C be the origin and $AA' = 2a$

Since $e > 1$

$\therefore \qquad \frac{SA}{AK}=\frac{e}{1}$

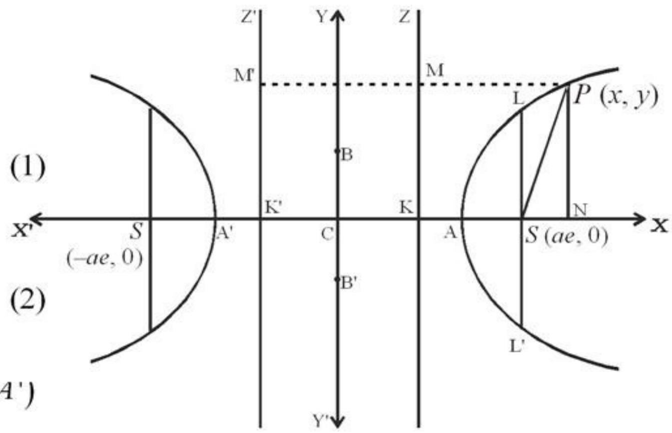
or $\qquad SA = eAK \qquad (1)$

and $\qquad \frac{SA'}{KA'}=\frac{e}{1}$

or $\qquad SA' = eKA' \qquad (2)$

On adding (1) and (2), we get

$$SA + SA' = e\left(AK + KA'\right)$$



or $\qquad \left(CS - CA\right)+\left(CS + CA'\right) = e\left\{\left(CA - CK\right)+\left(CA' + CK\right)\right\}$ fig. 12.30

or $\qquad 2CS = 2eCA \qquad [\because CA = CA']$

or $\qquad CS = ae \qquad (3)$

On subtracting (1) from (2), we get

$$SA' - SA = e\left(KA' - AK\right)$$

$$AA' = e\left\{\left(CA' + CK\right)-\left(CA - CK\right)\right\}$$

or $\qquad 2a = e \cdot 2CK$

or $\qquad CK = a/e \qquad (4)$

$\therefore$ coordinates of focus are $(ae, 0)$ and equation of directrix is $x = a/e$. Now by the definition of Hyperbola, we have

Distance of P from the focus S = e (length of perpendicular on the directrix from point P)

or $\qquad SP = ePM$

$\Rightarrow \qquad SP^2 = e^2 PM^2$

or $\qquad (x-ae)^2 + (y-0)^2 = e^2\left(CN - CK\right)^2$

or $$\left(x-ae\right)^2+y^2=e^2\left(x-\frac{a}{e}\right)^2$$

or $$x^2+a^2e^2-2aex+y^2=e^2\left(x^2+\frac{a^2}{e^2}-\frac{2ax}{e}\right)$$

or $$x^2\left(e^2-1\right)-y^2=a^2\left(e^2-1\right)$$

or $$\frac{x^2}{a^2}-\frac{y^2}{a^2\left(e^2-1\right)}=1$$

$\therefore$ $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{where} \quad b^2=a^2\left(e^2-1\right)$$

$\therefore$ Required equation of Hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. $(e>1)$.

12.34 Important definitions :

1. **Transverse axis :** Line joining the vertex A and A' is known as Transverse axis ACA'. Its length is $AA'=2a$
2. **Conjugate axis :** A line perpendicular to the transverse axis and passing through the point C is known as Conjugate axis. Let the points B and B' be such that $CB=CB'=b,$ then $BB'=2b,$ will be the length of conjugate.
3. **Principal axes :** Transverse axis and conjugate axis are called as principal axes of the hyperbola.
4. **Focus :** There are two foci of Hyperbola $S(ae,\ 0)$ and $S'(-ae,\ 0)$
5. **Directrix :** There are two directrices equidistant from the centre 'c' and their equations are $x=a/e$ and $x=-a/e$
6. **Vertex :** Points A(a, 0) and A' ($-a$, 0) are called the vertices of hyperbola.
7. **Latus rectum :** The line perpendicular to the transverse axis and passing through the focus is called the latus rectum . In fig. 12.30 LSL' is a latus rectum. Let the coordinated of L be $(ae,\ \ell)$. Since the point L lies on a Hyperbola, therefore,

$$\frac{a^2e^2}{a^2}-\frac{\ell^2}{b^2}=1 \quad \Rightarrow \quad \frac{\ell^2}{b^2}=e^2-1 \quad \text{or} \quad \ell^2=\frac{b^2}{a^2} \qquad \left[\because\ b^2=a^2\left(e^2-1\right)\right]$$

$\therefore$ $\ell=\pm b^2/a$

$\therefore$ length of latus rectum $=LSL'=2\ell=2b^2/a$.

8. **Focal property :** Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (1)$$

Coordinates of foci are $(\pm ae, 0)$

$\therefore$ $SP=ePM=eNK=e(CN-CK)$ $\qquad SP=e\left(x-\frac{a}{e}\right)=ex-a \qquad (2)$

Similarly $S'P=ePM'=eNK'=e(CN+CK')$

or, $S'P = e\left(\frac{a}{e} + x\right) = a + ex$

$\therefore$ $S'P - SP = 2a$ (constant) = length of transverse axis

Thus, focal length of a point lying on the hyperbola is equal to the length of transverse axis.

12.35 Special types of hyperbola :

(1) Rectangular Hyperbola :

If the transverse and conjugate axis of a hyperbola are equal i.e. a = b, then it is known as rectangular hyperbola, its equation will be

$$x^2 - y^2 = a^2$$

Also, $$b^2 = a^2\left(e^2 - 1\right)$$

when $a = b$, then $$a^2 = a^2\left(e^2 - 1\right) \Rightarrow e = \sqrt{2}$$

Thus eccentricity of a rectangular Hyperbola is $\sqrt{2}$.

(2) Conjugate Hyperbola :

If two hyperbolas are such that, transverse and conjugate axes of one is the conjugate and transverse axes of other, then together they are called conjugate cyperbola.

If the equation of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of conjugate hyperbola will be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

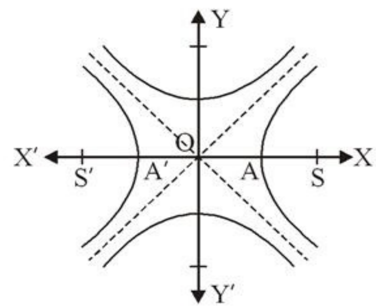


fig. 12.31

12.36 Intersection of hyperbola and a straight line :

Let the equation of a line and a hyperbola are

$$y = mx + c \quad (1)$$

and $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (2)$$

Solving (1) and (2), we get

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

or $$x^2\left(a^2m^2 - b^2\right) + 2a^2mcx + a^2\left(c^2 + b^2\right) = 0 \quad (3)$$

This is a quadratic equation in x and will have two roots. Line (1) will touch the Hyperbola (2), if the roots of (3) are equal

i.e. $$\left(2a^2mc\right)^2 - 4\left(a^2m^2 - b^2\right)\left\{\left(b^2 + c^2\right)a^2\right\} = 0 \quad \left[B^2 - 4AC = 0\right]$$

or, $$b^2\left(a^2m^2 - b^2\right) - b^2c^2 = 0$$

$\Rightarrow$ $$c = \pm\sqrt{\left(a^2m^2 - b^2\right)} \quad (4)$$

which is required condition of (1) touching (2). Putting the value of c in (1), we have

$$y = mx \pm \sqrt{\left(a^2m^2 - b^2\right)}$$

and coordinates of points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

Illustrative Examples

Example 22 : Find the axis, foci, eccentricity and length of latus rectum of the hyperbola

$$(x-1)^2-2(y-2)^2=-6$$

Solution : The equation of hyperbola can be written as

$$\frac{(x-1)^2}{(\sqrt{6})^2}-\frac{(y-2)^2}{(\sqrt{3})^2}=-1$$

comparing (1) with the standard equation we have centre (1, 2), transverse axis $=2\sqrt{3}$,

conjugate axis $=2\sqrt{6}$, eccentricity $=\sqrt{\left(\frac{6+3}{3}\right)}=\sqrt{3}$ (1)

foci $=\left[1,\left(2+\sqrt{3}\right)\times\sqrt{3}\right]$ and $\left[1,\left(2-\sqrt{3}\right)\times\sqrt{3}\right]$ or (1, 5) and (1, –1)

Example 23 : Find the equation of Hyperbola whose focus is (0, 0) directrix $x\cos\alpha+y\sin\alpha=\ell$ and eccentricity is $5/4$

Solution : Let $P(h, k)$ be any point on the Hyperbola

$\therefore$ distance of P from the focus = e (perpendicular directrix from P)

$$\sqrt{\left\{(h-0)^2+(k-0)^2\right\}}=\frac{5}{4}\left[\frac{h\cos\alpha+k\sin\alpha-\ell}{\sqrt{(\cos^2\alpha+\sin^2\alpha)}}\right]$$

$$\Rightarrow \quad 4\sqrt{(h^2+k^2)}=5(h\cos\alpha+k\sin\alpha-\ell)$$

squaring

$$16h^2+16k^2=25(h\cos\alpha+k\sin\alpha-\ell)^2.$$

required equation of Hyperbola is

$$\text{or} \quad 16(x^2+y^2)=25(x\cos\alpha+y\sin\alpha-\ell)^2$$

Example 24 : Find the equation of Hyperbola whose foci are $(4, 0)$ and $(-4, 0)$ whose eccentricity is 8.

Solution : Distance between the foci $=\sqrt{\left[(4-4)^2+(0-0)^2\right]}=8$

$\therefore \quad 2ae=8$ or $2a\times 8=8 \quad \therefore \quad a=1/2$

again $b^2=a^2(e^2-1)=\frac{1}{4}(64-1)=\frac{63}{4} \Rightarrow b=\frac{\sqrt{63}}{2}$

thus required equation of Hyperbola is $\frac{x^2}{(1/2)^2}-\frac{y^2}{(\sqrt{63}/2)^2}=1$

Example 25 : If the eccentricities of hyparabola and its conjugate hyperbola are e and e' then prove that $\frac{1}{e^2}+\frac{1}{e'^2}=1$.

Solution : Let the equation of hyparabola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (1)

then equation of its conjugate hyparabola is

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (2)$$

eccentricity of (1) $e^2=1+\frac{b^2}{a^2}=\frac{a^2+b^2}{a^2}$ (3)

eccentricity of (1) $e'^2=1+\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2}$ (4)

from equation (3) and (4)

$$\frac{1}{e^2}+\frac{1}{e'^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1$$

Example 26 : Find the condition for which the line $x\cos\alpha+y\sin\alpha=p$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solution : Putting the value of y from $x\cos\alpha+y\sin\alpha=p$, in the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $\frac{x^2}{a^2}-\frac{1}{b^2}\left[\frac{p-x\cos\alpha}{\sin\alpha}\right]^2=1$

or $x^2\left(b^2\sin^2\alpha-a^2\cos^2\alpha\right)+2a^2px\cos\alpha-\left(a^2b^2\sin^2\alpha+a^2p^2\right)=0$ (1)

the line will touch the hyperbola if (1) has equal roots $\left(B^2=4AC\right)$

or $\left(2a^2p\cos\alpha\right)^2=-4\left(b^2\sin^2\alpha-a^2\cos^2 a\right)\left(a^2b^2\sin^2\alpha+a^2p^2\right)$

or $4a^4p^2\cos^2\alpha=-4\left[a^2b^4\sin^4\alpha+a^2b^2p^2\sin^2\alpha-a^4b^2\sin^2\alpha\cos^2\alpha-p^2a^4\cos^5\alpha\right]$

$\therefore$ $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$ is the required condition.

Exercise 12.7

1. Find the length of axis, foci, eccentricity and length of latus rectum and equation of directrices of the hyperbola $9x^2-16y^2=144$
2. Find the equation of hyperbola whose
   (i) Focus is (2, 1) directrix is $x+2y-1=0$ and eccentricity.
   (ii) Focus is (1, 2) directrix is $2x+y=1$ and eccentricity $\sqrt{3}$.
3. Find the vertices, foci, eccentricity and length of latus rectum of the hyperbola $x^2-6x-4y^2-16y-11=0$

4. Find the equation of hyperbola whose

(i) Length of latus rectum is 8 and conjugate axis = 1/2 (distance between the foci)

(ii) Distance between the foci is 16 and conjugate axis is $\sqrt{2}$

(iii) Length of conjugate axis is 7 and passing through point (3, -2).

5. Prove that the locus of point of intersection of the lines $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$ is a hyperbola.

6. Find the common points of the hyperbola $5x^2-9y^2=45$ and $y=x+2$.

7. Prove that the line $\ell x+my=1$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2\ell^2-b^2m^2=1$.

8. Find the equation of tangent to the hyperbola $4x^2-9y^2=1$ which is parallel to the line $4y=5x+7$

9. Prove that the locus of foot of perpendicular dropped from the focus to the tangent to the hyperbola is a circle.

Miscellaneous Exercise 12

1. The radius of the circle $9x^2+y^2+8x=4(x^2-y^2)$ is :

(A) 1 (B) 2 (C) 4 / 5 (D) 5 / 4

2. The equation of circle whose centre is (α,β) and lies in the first quadrant and touches the x-axis is :

(A) $x^2+y^2-2\alpha x-2\beta y+\alpha^2=0$ (B) $x^2+y^2+2\alpha x-2\beta y+\alpha^2=0$

(C) $x^2+y^2-2\alpha x+2\beta y+\alpha^2=0$ (D) $x^2+y^2+2\alpha x+2\beta y+\alpha^2=0$

3. The value of c when the line $y=mx+c$ touches the circle $x^2+y^2=4y$ is :

(A) $2+\sqrt{1+m^2}$ (B) $2-\sqrt{1+m^2}$ (C) $2+2\sqrt{1+m^2}$ (D) $1+\sqrt{1+m^2}$

4. The point where the line $3x+4y=25$ touches the circle $x^2+y^2=25$ is :

(A) $(4,3)$ (B) $(3,4)$ (C) $(-3,-4)$ (D) $(3,-4)$

5. A parabola sections from the cone if :

(A) $e=0$ (B) $e<1$ (C) $e>1$ (D) $e=1$

6. Equation of directrix of parabola $x^2=-8y$ is :

(A) $y=-2$ (B) $y=2$ (C) $x=2$ (D) $x=-2$

7. The vertex of the parabola $x^2+4x+2y=0$ is :

(A) $(0,\ 0)$ (B) $(2,\ -2)$ (C) $(-2,\ -2)$ (D) $(-2,\ 2)$

8. The equation of parabola with focus $(-3,\ 0)$ and directrix $x+5=0$ is :

(A) $y^2=4(x+4)$ (B) $y^2+4x+16=0$ (C) $y^2+4x=16$ (D) $x^2=4(y+4)$

9. The equation of parabola with vertex and focus are $(2,\ 0)$ and $(5,\ 0)$ is :

(A) $y^2=12x+24$ (B) $y^2=12x-24$ (C) $y^2=-12x-24$ (D) $y^2=-12x+24$

10. The focus of parabola $x^2=-8y$ is :

(A) $(2, 0)$ (B) $(0, 2)$ (C) $(-2, 0)$ (D) $(0, -2)$

11. Equation of tangent to the parabola $y^2 = x$ is :

(A) $y = mx + 1/m$ (B) $y = mx + 1/4m$ (C) $y = mx + 4/m$ (D) $y = mx + 4m$

12. The point of contact when a line $2y - x = 2$ touches the parabola $y^2 = 2x$ is :

(A) $(4, 3)$ (B) $(-4, 1)$ (C) $(2, 2)$ (D) $(1, 4)$

13. The equation of a line parallel to a line $x + 2y + 1 = 0$ to the parabola $x^2 = 8y$

(A) $x + 2y + 1 = 0$ (B) $x - 2y + 1 = 0$ (C) $x + 2y - 1 = 0$ (D) $x - 2y - 1 = 0$

14. The equation of normal to the parabola $y^2 = 4x$ is :

(A) $y = x + 4$ (B) $y + x = 3$ (C) $y + x = 2$ (D) $y + x = 1$

15. The length of half latus rectum of the parabola $3x^2 + 4y^2 = 12$ is :

(A) $\frac{3}{2}$ (B) 3 (C) $\frac{8}{\sqrt{3}}$ (D) $\frac{4}{\sqrt{3}}$

16. The eccentricity of parabola $3x^2 + 4y^2 = 12$ is :

(A) –2 (B) ½ (C) 1 (D) 2

17. The value of c when a line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is :

(A) $c = \frac{a}{m}$ (B) $c = \pm\sqrt{a^2m^2 - b^2}$ (C) $c = \pm\sqrt{a^2m^2 + b^2}$ (D) $c = a\sqrt{1 + m^2}$

18. The co-ordinates of the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (b > a)$ is :

(A) $(\pm ae, 0)$ (B) $(\pm be, 0)$ (C) $(0, \pm ae)$ (D) $(0, \pm be)$

19. The eccentricity of the rectangular hyperbola is :

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) 2

20. The eccentricity of hyperbola $9x^2 - 16y^2 = 144$ is :

(A) 1 (B) 0 (C) 5 / 16 (D) 5 / 4

21. Find the equation of a circle whose centre is $(a\cos\alpha, a\sin\alpha)$ and radius is a.

22. If the tangents are perpendicular to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at points (x_1, y_1) and (x_2, y_2) then prove that

$$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + g^2 + f^2 = 0$$

23. Find the equation of circle with radius r and whose centre lies in the first quadrant and touches the y-axis from origin. Also find the equation of another tangent passing through the origin.

24. A tangent drawn at a point (α, β) on the circle $x^2 + y^2 = a^2$ meets the axes at point A and B respectively.

Prove that the area of triangle OAB is $\frac{a^4}{2\alpha\beta}$, where O is the origin.

25. Find the equation of tangent to a circle $x^2 + y^2 = a^2$ which makes a triangle of a^2 area with the axes.

26. Write the co-ordinates of the foci of the parabola $x^2 - 4x - 8y = 4$,

27. Write the eccentricity of the parabola $x^2 - 4x - 4y + 4 = 0$.

28. Write a condition that a line $\ell x + my + n = 0$ touches the parabola $y^2 = 4ax$.

29. Find the equation of parabola whose vertex is $(0, 0)$ and focus is $(0, -a)$.

30. Write the equation of the axis of the parabola $9y^2 - 16x - 12y - 57 = 0$.

31. Find the co-ordinates of the centre of the ellipse $\frac{x^2 - ax}{a^2} + \frac{y^2 - by}{b^2} = 0$.

32. Write the condition of a line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

33. Find the equation of hyperbola whose transverse axis and conjugate axis are 4 and 5.

34. Find the co-ordinates of the centre of the hyperbola $\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$

Important Points

1. When the plane P (whose semi vertical any v is α) passes through the vertex O of the cone, we get point of straight line. These lines are

   (A) Passes through vertex of cone then intersecting curve pair line will be

   (i) Real and distinct if $\theta < \alpha$

   (ii) Coinciding if $\theta = \alpha$

   (iii) Imaginaery if $\theta > \alpha$

   (B) If the plane do not passes through the vertex then the section will be -

   (i) Circle if $\theta = 90^o$ (ii) parabola if $\theta = \alpha$

   (iii) Ellipse if $\theta > \alpha$ (iv) Hyperbola if $\theta < \alpha$

2. Standard form of equation of circle

   $(x-h)^2 + (y-k)^2 = a^2$; centre (h,k), radius $= a$

3. Equation of circle when centre lies at the origin $x^2 + y^2 = a^2$

4. General equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$;

   centre $(-g, -f)$, radius $= \sqrt{g^2 + f^2 - c}$

5. Line $y = mx + c$, touches the circle $x^2 + y^2 = a^2$ if $c = \pm a\sqrt{1+m^2}$

6. Tangent to a circle : $x^2 + y^2 = a^2$

   (i) At point (x_1, y_1) equation is $xx_1 + yy_1 - a^2 = 0$

   (ii) Slope form (m), $y = mx \pm a\sqrt{1+m^2}$ co-ordinates of point of contact $\left(\mp \frac{am}{\sqrt{1+m^2}}, \pm \frac{a}{\sqrt{1+m^2}} \right)$

   (iii) At point (x_1, y_1) to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

7. Normal to a circle $x^2 + y^2 = a^2$

   (i) At point (x_1, y_1) equation is $xy_1 - yx_1 = 0$

   (ii) Slope (m) $x + my = 0$, where m is the slope of tangent to a circle

8. Standard equation of parabola is $y^2 = 4ax$

9. For parabola $y^2 = 4ax$, co-ordinates of vertex is (0, 0), co-ordinates of focus is (a, 0), length of latus rectum is $4a$, equation of directrix is $x + a = 0$

10. (i) Condition of line $y = mx + c$ to touches the parabola $y^2 = 4ax$ is $c = a/m$.

    (ii) Co-ordinates of point of contact $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

11. Standard equation of parabola $y^2 = 4ax$

    (i) Tangent to a point (x_1, y_1) is $yy_1 = 2a(x + x_1)$

    (ii) Equation of tangent in a slope form $y = mx + \frac{a}{m}$

    (iii) Equation of normal at point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

    (iv) Slope form of normal $y = mx - 2am - am^3$

12. Standard equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

13. In ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

    (i) Foci $(\pm ae, 0)$ (ii) Directrices $x = \pm a/e$

    (iii) Length of latus rectum $= \frac{2b^2}{a}$ (iv) Eccentricity $e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$

14. (i) Condition of line $y = mx + c$ to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c = \pm\sqrt{a^2m^2 + b^2}$.

    (ii) Co-ordinates of point of contact $\left[\mp\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 + b^2}}\right]$

15. Equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$

16. In hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$

    (i) Foci $(\pm ae, 0)$ (ii) Directrices $x = \pm a/e$

(iii) Length of latus rectum $=\dfrac{2b^2}{a}$ (iv) Eccentricity $e=\left(1+\dfrac{b^2}{a^2}\right)^{1/2}$

17. (i) Condition of line $y=mx+c$ to touch the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is $c=\pm\sqrt{\left(a^2m^2-b^2\right)}$

(ii) Co-ordinates of point of contact $\left[\pm\dfrac{a^2m}{\sqrt{a^2m^2-b^2}}, \mp\dfrac{b^2}{\sqrt{a^2m^2-b^2}}\right]$

18. Eccentricity of rectangular hyperbola $x^2-y^2=a^2$ is $e=\sqrt{2}$

Answers

Exercise 12.1

1. (i) $x^2+y^2+4x-6y-3=0$; (ii) $x^2+y^2-29x-2by+2ab=0$

2. (i) or (3, 4); or 5 ; (ii) or $\left(\dfrac{a}{\sqrt{1+k^2}}, \dfrac{ak}{\sqrt{1+k^2}}\right)$, or a ; (iii) or (0, 0) ;or

3. $x^2+y^2-2rx-2y\sqrt{r^2-\ell^2}+\left(r^2-\ell^2\right)=0$

4. $x^2+y^2-6x\pm 6\sqrt{2}y+9=0$

5. or (4, –5), or $\sqrt{53}$

6. or $\left(\dfrac{1}{4}, 0\right)$ or $\dfrac{1}{4}$

7. $x^2+y^2-7x+5y-14=0$

8. $x^2+y^2+4x-21=0$ and $x^2+y^2-12x+11=0$

9. $x^2+y^2-ax-by=0$

Exercise 12.2

1. $\left[\dfrac{48\pm 3\sqrt{481}}{25}, \dfrac{36\mp 4\sqrt{481}}{25}\right]$; $\dfrac{2}{5}\sqrt{481}$

3. $2\sqrt{c^2-\dfrac{a^2b^2}{a^2+b^2}}$ 4. $k=40$ or -10

5. (i) $m^2\left(a^2-r^2\right)+2ma(c-b)+(c-b)^2=r^2$ (ii) $n^2=a^2\left(l^2+m^2\right)$

6. (i) $4x+3y-40=0$ and $5x+12y-104=0$ (ii) $y=\sqrt{3}\pm 4$

7. $c=1$ 8. $5x+12y-169=0$

Exercise 12.3

1. (i) $16x^2+8xy+y^2-74x-78y+212=0$; (ii) $y^2=4x+16$

2.

	vertex	axis	latus rectum	focus
(i)	$(-2, 4)$	$y=4$	8	$(0, 4)$
(ii)	$(4, 9/2)$	$x=4$	2	$(4, 4)$

5. $9y^2=4ax$

Exercise 12.4

1. $(2, -3)$ 2. $\frac{80}{9}$ 4. $am^2=\ell n$ 6. $a\sin^2\alpha=-p\cos\alpha$

7. (i) $8x-2y+27=0$; (ii) $y+2x-4=0$; $y+2x+\frac{4}{9}=0$

8. $-\frac{27}{4}$ 9. $x^2-5xy+2y^2+42x-20y+16=0$

10. (i) $x+y-6=0$; (ii) $x+y+9=0$

11. (i) $2x-y-12=0$; (ii) $3x-y-33=0$

Exercise 12.5

1. (i) $3x^2+2xy+3y^2=8$

(ii) $216x^2+209y^2-24xy+906x+1358y+2924=0$

2.

(i)	$\frac{\sqrt{5}}{3}$	$\frac{4}{9}$	$\left(\pm\frac{\sqrt{5}}{9}, 0\right)$
(ii)	$\frac{\sqrt{21}}{5}$	$\frac{8}{5}$	$\left(0, \pm\sqrt{21}\right)$
(iii)	$\frac{1}{2}$	3	$(3, 1), (1, 1)$

3. $\frac{x^2}{52}+\frac{y^2}{13}=1$ 4. $\frac{\sqrt{3}}{2}$

5. $\frac{x^2}{9/4}+\frac{y^2}{5/4}=1$; or

Exercise 12.6

1. $\left[-\sqrt{(3/10)}, \sqrt{(2/15)}\right]$ 3. $\mp 2\sqrt{29}$

4. $\left[\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right]$ 5. $a^2\ell^2+b^2m^2=n^2$

6. $y=\sqrt{3}x\pm\frac{1}{2}\sqrt{\frac{65}{3}}$; $\left[\pm\frac{15}{2\sqrt{65}}, \mp\frac{10}{3}\sqrt{\frac{15}{65}}\right]$

Exercise 12.7

1. or = 8   or = 6

or = 5/4   or $(\pm 5,\ 0)$

or = 9 / 4   or $= \pm \frac{16}{5}$

2. (i) $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$

(ii) $7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$

3. $(3, -2)$; $(5, -2)$, $(1, -2)$; $(3 \pm \sqrt{5}, -2)$; 1; $\frac{\sqrt{5}}{2}$

4. (i) $x^2 - 3y^2 = 144$; (ii) $x^2 - y^2 = 32$; (iii) $65x^2 - 36y^2 = 441$

7. $(-9/2, -5/2)$ 9. $4y = 5x \pm 3/2$

Miscellaneous Exercise

1. (C)	2. (A)	3. (C)	4. (B)	5. (C)	6. (B)	7. (D)
8. (A)	9. (B)	10. (D)	11. (B)	12. (C)	13. (A)	14. (B)
15. (A)	16. (B)	17. (C)	18. (D)	19. (C)	20. (D)	

21. $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$ 23. $(x-r)^2 + (y-h)^2 = r^2$; $(r^2 - h^2)x + 2rhy = 0$

25. $y = \pm x \pm a\sqrt{2}$ 26. (2, 1)   27. 1   28. $\ell n = am^2$ 29. $x^2 = -4ay$

30. $3y = 2$ 31. $\left(\frac{a}{2}, \frac{b}{2}\right)$ 32. $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

33. $25x^2 - 16y^2 = 100$ 34. $(1, -2)$