11

Application of Integrals : Quadrature

11.01 Introduction

Quadrature means the process of finding out the area bounded by a given curve.

11.02 Area under a curve

Theorem : The area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis is expressed by definite integral $\int_a^b f(x)\,dx = \int_a^b y\,dx$

Proof : Let the equation of curve PQ be $y = f(x)$ where $f(x)$ is single valued real and continuous function of x in domain $[a, b]$. According to figure, we need to find the area of figure $PRSQP$.

Let $E(x, y)$ is any point on curve and $F(x + \delta x,\ y + \delta y)$ is a point in the neighbourhood. EA and FB are ordinates of E and F respectively.

Draw a perpendicular EC from E to FB and a perpendicualr FD from F to extended AE

$$AB = OB - OA = (x + \delta x) - x = \delta x$$

$$FC = FB - CB = (y + \delta y) - y = \delta y$$

Let area $RAEPR = A$

Now if the increment in x is δx and the increment in corresponding area is δA, then

$$\delta A = \text{area } ABFEA$$

$\therefore$ From figure, (area of rectangle $ABCE$) < area ($ABFEA$) < (area of rectangle $ABFD$)

$\Rightarrow$ $y\delta x < \delta A < (y + \delta y).\delta x$

$\Rightarrow$ $y < \dfrac{\delta A}{\delta x} < y + \delta y$

When, $F \to E$ then $\delta x \to o$ and $y + \delta y \to y$

$\Rightarrow$ $\lim\limits_{\delta x \to 0} y \le \lim\limits_{\delta x \to 0} \dfrac{\delta A}{\delta x} \le \lim\limits_{\delta x \to 0} (y + \delta y)$

$\Rightarrow$ $y \le \dfrac{dA}{dx} \le y$

$\Rightarrow$ $\dfrac{dA}{dx} = y \Rightarrow dA = ydx \Rightarrow dA = f(x)\,dx$

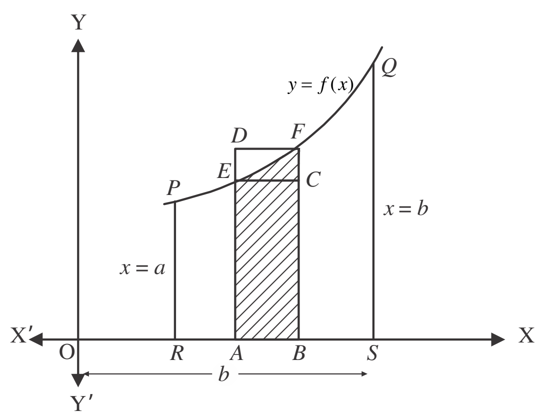


Fig. 11.01

Integrating both the sides with respect to x and within the limits $x = a$ and $x = b$.

$$\int_a^b dA = \int_a^b f(x)dx$$

or $$[A]_a^b = \int_a^b f(x)\,dx$$

or (area A when $x = b$) – (area A when $x = a$) $= \int_a^b f(x)\,dx$

or area $PRSQP - 0 = \int_a^b f(x)dx$

or area $PRSQP = \int_a^b f(x)dx$ or $\int_a^b y\,dx$

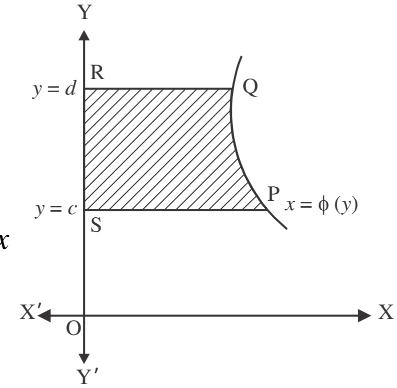


Fig. 11.02

The area of curve $y = f(x)$, under ordinates $x = a$ and $x = b$

and x-axis is $= \int_a^b f(x)dx$ or $\int_a^b y\,dx$

Similarly, the area between curve $x = \phi(y)$, y-axis and the abscissa $y = c$, $y = d$ is given by

$$= \int_c^d \phi(y)dy \quad \text{or} \quad = \int_c^d x\,dy$$

Remark : To find out the area of figure, a rough sketch should be made so that it is easy to determine the limits of curve and symmetry of curve with respect to axes.

11.03 Symmetrical Area

If the curve is symmetrical with respect to any axis or any straight line, then find the area of one symmetrical part and then by multiplying with number of symmetrical parts in order to get area.

For example : Find the area enclosed by circle $x^2 + y^2 = a^2$

Solution : Clearly the centre of circle is (0, 0) and radius is a and it is also symmetrical about both the axes.

Total area of circle = 4 × [area of OABO in first quandrant]

= 4 × [Area bounded by circle $y = \sqrt{a^2 - x^2}$, x-axis $x = 0$ and $x = a$]

$$= 4\int_a^b y\,dx = 4\int_a^b \sqrt{a^2 - x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(o + \frac{a^2}{2}.\frac{\pi}{2}\right) - (o + o)\right] = \pi a^2$$

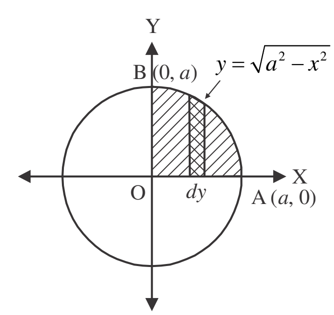


Fig. 11.03

11.04 Area of a curve around x-axis

Area is always considered as positive. It may happen that some is below the x-axis (which will be negative). Therefore the total area can be calcualted by adding up the numerical values of both the areas.

For Example : Find the area enclosed by the curve $y = \cos x$ and x-axis when $o \le x \le \pi$.

Solution : It is clear from the graph that the required area's portion is above x-axis and some portion is below x-axis.

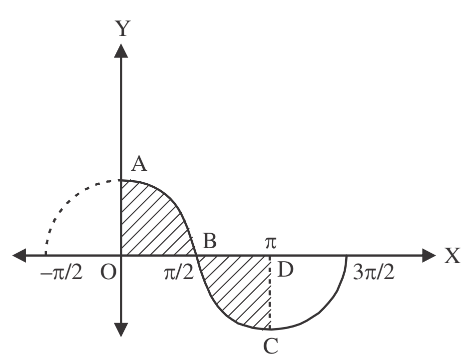


Fig. 11.04

So required area $= \int_0^{\pi/2} \cos x \, dx + \left| \int_{\pi/2}^{\pi} \cos x \, dx \right|$

$= [\sin x]_o^{\pi/2} + \left| [\sin x]_{\pi/2}^{\pi} \right| = (1-0) + |0-1|$

$= 1 + 1 = 2$ sq. units

Illustrative Examples

Example 1. Find the area bounded by the parabola $y^2 = 4x$ and line $x = 3$.

Solution : On tracing the given parabola and line

Required area = area AOBMA

= 2 × area AOMA (∵ Parabola is symmetrical about x-axis)

$= 2\int_0^3 y \, dx$

$= 2\int_0^3 \sqrt{4x} \, dx = 2 \times 2\int_0^3 \sqrt{x} \, dx$

$= 4 \times \left[\frac{2}{3} x^{3/2} \right]_0^3 = \frac{8}{3}\left[3^{3/2} - 0 \right]$

$= \frac{8}{3} \times 3\sqrt{3} = 8\sqrt{3}$ Sq. units.

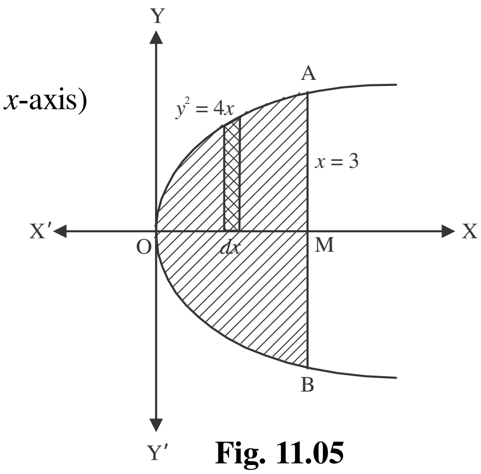


Fig. 11.05

Example 2. Find the area enclosed above x-axis by curve $y = 2\sqrt{1-x^2}$ and x-axis.

Solution : On simplifying $y = 2\sqrt{1-x^2}$

$$y^2 = 4(1-x^2) \quad \text{or} \quad \frac{x^2}{1} + \frac{y^2}{4} = 1 \qquad (1)$$

Clearly, curve $y = 2\sqrt{1-x^2}$ is upper part of ellipse (1) so according to figure, we have to find out the area of shaded region.

∴ required area = 2 × area OABO

$= 2\int_0^1 y \, dx = 2\int_0^1 2\sqrt{1-x^2} \, dx$

$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right]_0^1$

$= 4\left[\left(0 + \frac{1}{2} . \frac{\pi}{2} \right) - (o + o) \right] = \pi$ sq. unit.

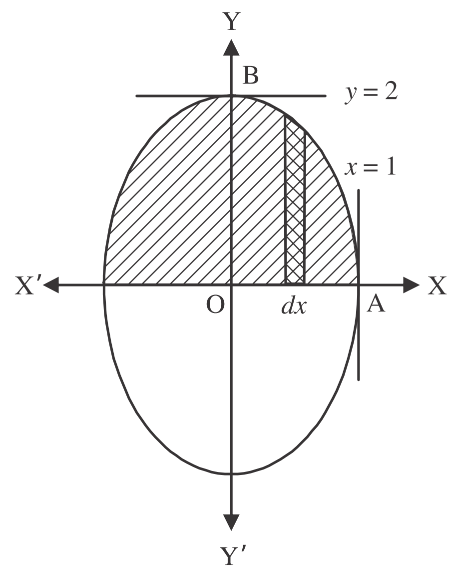


Fig. 11.06

Example 3. Find the area enclosed by $y^2 = 4ax$, x-axis, line $x = 2a$ and latus rectum.

Solution : We have know that the equation of latus rectum of parabola $y^2 = 4ax$ is $x = a$. This is presented by LSL' in figure and line PMQ is $x = 2a$.

So required area $= \text{area } SMPL$

$$= \int_a^{2a} y\,dx = \int_a^{2a} \sqrt{4ax}\,dx$$

$$= 2\sqrt{a}\int_a^{2a} \sqrt{x}\,dx = 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_a^{2a}$$

$$= 2\sqrt{a}\left[\frac{2}{3}\times(2a)^{3/2} - \frac{2}{3}a^{3/2}\right]$$

$$= 2\sqrt{a}\left[\frac{4\sqrt{2}}{3}a\sqrt{a} - \frac{2}{3}a\sqrt{a}\right]$$

$$= \frac{4a^2}{3}\left[2\sqrt{2} - 1\right].$$

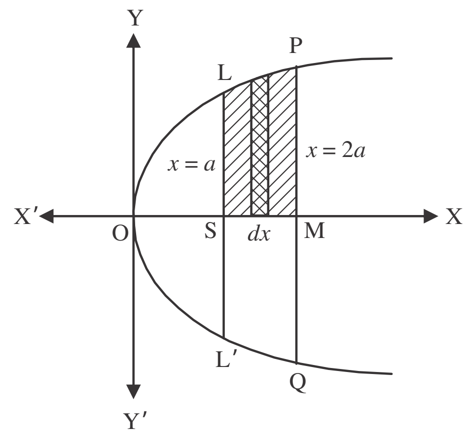


Fig. 11.07

Example 4. Find the area enclosed by parabola $y = 4x^2$ and lines, $y = 1$ and $y = 4$.

Solution : Parabola $y = 4x^2$ so $x^2 = \frac{1}{4}y$ and lines $y = 1$ and $y = 4$ will be traced as followed.

So, required area $= \text{area } PQRSP$

$= 2 \times \text{area } RQLM$

$$= 2\int_1^4 x\,dy$$

$$= 2\int_1^4 \frac{1}{2}\sqrt{y}\,dy = \int_1^4 \sqrt{y}\,dy$$

$$= \frac{2}{3}\left[(y)^{3/2}\right]_1^4 = \frac{2}{3}\left[4^{3/2} - 1^{3/2}\right]$$

$$= \frac{2}{3}[8-1] = \frac{14}{3} \text{ sq. units.}$$

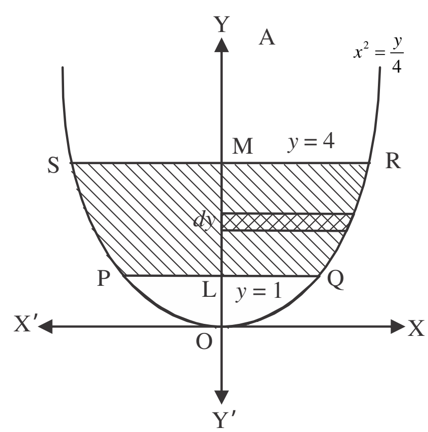


Fig. 11.08

Example 5. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$, where $b^2 = a^2(1-e^2), e < 1.$

Solution : The required area of the region $BPSQB'OB$ is enclosed by the ellipse and the lines $x = o$ and $x = ae$

The area is symmetrical about x-axis. So

$$\text{required area } BPSQB'OB = 2\int_o^{ae} y\,dx$$

So, by the equation of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \text{ or } \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

or $$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right) \text{ or } y=\frac{b}{a}\sqrt{a^2-x^2}$$

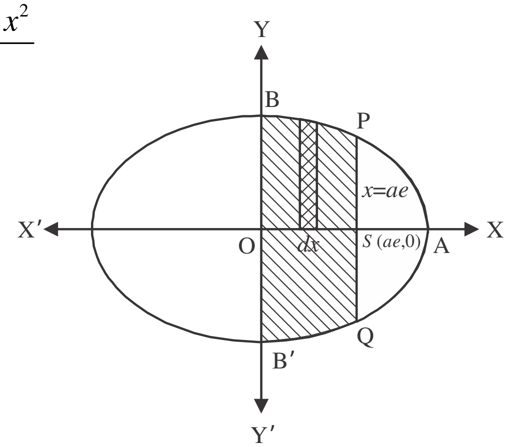


Fig. 11.09

So, required area $= 2\int_o^{ae}\frac{b}{a}\sqrt{a^2-x^2}\,dx$

$$=\frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_o^{ae}$$

$$=\frac{2b}{a}\left[\left(\frac{ae}{2}\sqrt{a^2-a^2e^2}+\frac{a^2}{2}\sin^{-1}\frac{ae}{a}\right)-(o+o)\right]$$

$$=\frac{2b}{a}\left[\frac{ae}{2}.a\sqrt{1-e^2}+\frac{a^2}{2}\sin^{-1}(e)\right]$$

$$=\frac{2a^2b}{2a}\left[e\sqrt{1-e^2}+\sin^{-1}e\right]$$

$$=ab\left[e\sqrt{1-e^2}+\sin^{-1}e\right] \text{ sq. units.}$$

Example 6. Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{2}y$ and the circle $x^2+y^2=9$.

Solution : The centre of circle $x^2+y^2=9$ is (o, o) and radius is 3 unit. Straight line $x=\sqrt{2}y$ passes through origin and cuts the circle at P. On solving the equations of circle and line.

$$x^2+\frac{x^2}{2}=9 \Rightarrow x^2=6 \Rightarrow x=\pm\sqrt{6} \text{ then } y=\pm\sqrt{3}$$

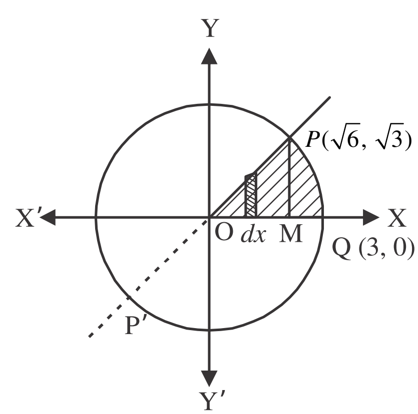
$\therefore$ Coordinates of P $(\sqrt{6}, \sqrt{3})$, Q $(3, 0)$ and M $(\sqrt{6}, 0)$.

required area = area OMPO + area PMQP

$$=\int_{0(y\text{ from line})}^{\sqrt{6}} y\,dx+\int_{\sqrt{6}(y\text{ from circle})}^{3} y\,dx$$

$$=\int_0^{\sqrt{6}}\frac{x}{\sqrt{2}}\,dx+\int_{\sqrt{6}}^{3}\sqrt{9-x^2}\,dx$$

$$=\left[\frac{x^2}{2\sqrt{2}}\right]_0^{\sqrt{6}}+\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{\sqrt{6}}^{3}$$



Fi.g 11.10

$$= \left(\frac{3}{\sqrt{2}} + 0\right) + \left[\left(0 + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{\sqrt{6}}{2}\sqrt{3} + \frac{9}{2}\sin^{-1}\frac{\sqrt{6}}{3}\right)\right]$$

$$= \frac{3}{\sqrt{2}} + \frac{9\pi}{4} - \frac{3}{\sqrt{2}} - \frac{9}{2}\sin^{-1}\frac{\sqrt{2}}{\sqrt{3}} = \frac{9}{4}\left(\pi - 2\sin^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right) \text{ sq. units.}$$

Example 7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut by the line $x = \frac{a}{\sqrt{2}}$.

Solution : On solving the equations of circle and line,

$$\frac{a^2}{2} + y^2 = a^2 \Rightarrow y^2 = \frac{a^2}{2} \Rightarrow y = \frac{a}{\sqrt{2}}$$

$\therefore$ Coordinates of P $\left(a/\sqrt{2},\ a/\sqrt{2}\right)$

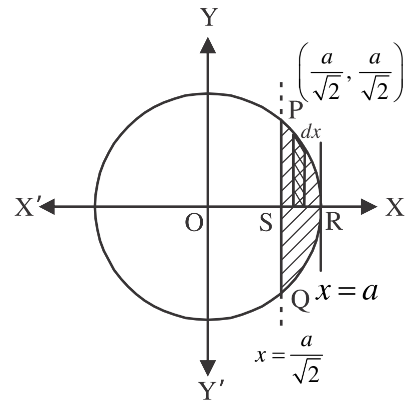


Fig. 11.11

required area = area $PSQRP$

= 2 × area PSRP

$$= 2\int_{a/\sqrt{2}}^{a} y\, dx = 2\int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2}\, dx$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a/\sqrt{2}}^{a}$$

$$= 2\left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - \left(\frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} + \frac{a^2}{2}\sin^{-1}\frac{a}{a\sqrt{2}}\right)\right]$$

$$= 2\left[o + \frac{a^2}{2}.\frac{\pi}{2} - \frac{a^2}{4} - \frac{a^2}{2}.\frac{\pi}{4}\right]$$

$$= 2\left(\frac{\pi a^2}{4} - \frac{a^2}{4} - \frac{\pi a^2}{8}\right) = 2\left(\frac{\pi a^2}{8} - \frac{a^2}{4}\right) = \frac{\pi a^2}{4} - \frac{a^2}{2}$$

$$= \frac{a^2}{4}(\pi - 2) \text{ sq. units}$$

Example 8. Find the area of the smaller part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, cut by line $y = c$, when $c < b$.

Solution : According to figure the area bounded between ellipse and line is shaded.

required area = area $BQPRB$

= 2 × area $BQPRB$

$$= 2\int_c^b x\,dy$$

$$= 2\int_c^b \frac{a}{b}\sqrt{b^2 - y^2}\,dy$$

$$= 2\frac{a}{b}\left[\frac{y}{2}\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)\right]_c^b$$

$$= \frac{2a}{b}\left[0 + \frac{b^2}{2}\sin^{-1}(1) - \frac{c}{2}\sqrt{a^2 - c^2} - \frac{b^2}{2}\sin^{-1}\left(\frac{c}{b}\right)\right] \text{ sq. units}$$

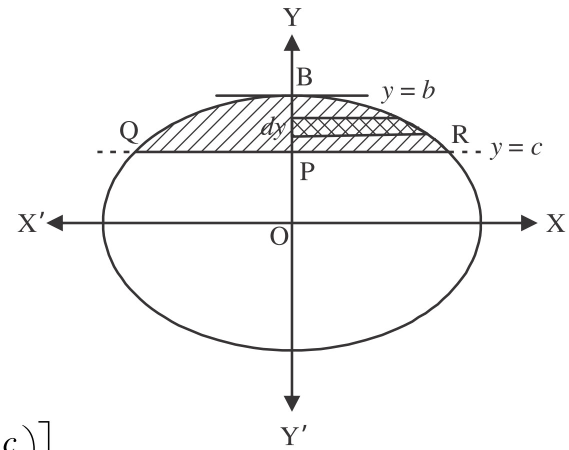


Fig. 11.12

Example 9. Find the area bounded by line $2x + y = 4$, x-axis and ordinates $x = 0$ and $x = 3$.

Solution : According to figure, line $2x + y = 4$, meets x-axis at $x = 2$ and y-axis at $y = 4$. When x is from 0 to 2, then the graph is above x-axis and when x is from 2 and 3, then graph is below x-axis.

So, required area = area OABO + area ALMA

$$= \int_0^2 y\,dx + \left|\int_2^3 y\,dx\right|$$

$$= \int_0^2 (4 - 2x)\,dx + \left|\int_2^3 (4 - 2x)\,dx\right|$$

$$= \left[4x - x^2\right]_0^2 + \left|\left[4x - x^2\right]_2^3\right|$$

$$= \left[(8-4) - (0-0)\right] + \left|(12-9) - (8-4)\right|$$

$$= 4 + |3 - 4| = 4 + 1 = 5$$

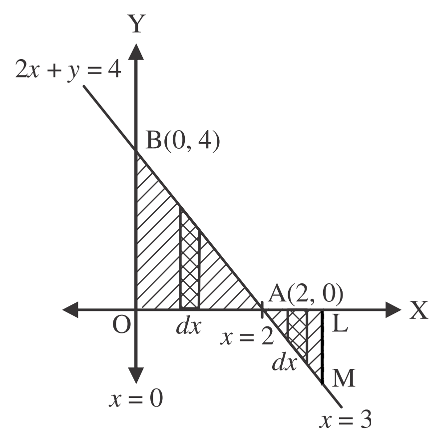


Fig. 11.13

Exercise 11.1

1. Find the area enclosed by parabola $y^2 = 4ax$ and its latus rectum.
2. Find the area bounded by circle $x^2 + y^2 = 4$, y-axis and $x = 1$.
3. Find the area enclosed by $y = \sin x$ and x-axis, when $0 \le x \le 2\pi$.
4. Find the area enclosed by $y = 2\sqrt{x}$ and between $x = 0, x = 1$.
5. Find the area enclosed by $y = |x|$, $x = -3$, $x = 1$ and x-axis.
6. Find the area enclosed by $x^2 = 4ay$, x-axis and line $x = 2$.
7. Find the area enclosed by ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ above x –axis.
8. Find the total area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
9. Find the area enclosed by line $\frac{x}{a} - \frac{y}{b} = 2$ and both axes.

10. Find the area bounded by lines $x + 2y = 8$, $x = 2$, $x = 4$ and x-axis.

11. Find the area bounded by $y = x^2$, x-axis and ordinates $x = 1$, $x = 2$.

12. Find the area bounded by $y = 4x^2$ (in first quadrant), $x = 0$, $y = 1$ and $y = 4$.

11.05 Area between two Curves

Theorem : The area between two curves $y = f(x)$ and $y = g(x)$ and between the ordinates $x = a$ and $x = b$ is $= \int_a^b [f(x) - g(x)]dx$

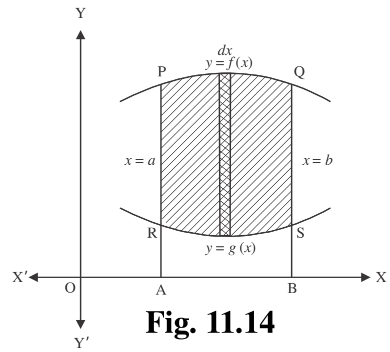


Fig. 11.14

Proof : In the figure the shaded region represents the area between the curves $y = f(x)$ and $y = g(x)$ and two lines $x = a$ and $x = b$.

The area of this region = area $PQBAP$ − area $RSBAR$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

or
$$\int_{a\,(y=f(x))}^{b} y\,dx - \int_{a\,(y=g(x))}^{b} y\,dx$$

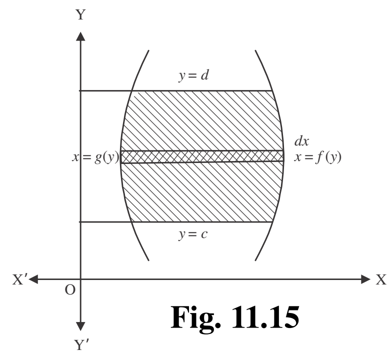


Fig. 11.15

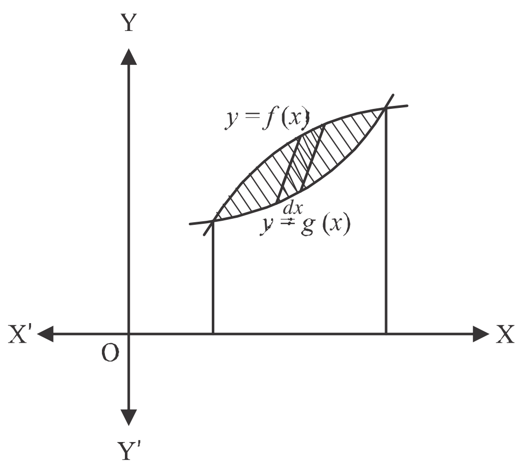
Remark : The area between two curves $x = f(y)$ and $x = g(y)$ and lines $y = c$ and $y = d$ is

$$= \int_a^b [f(y) - g(y)]\,dy$$

Special Cases :

Case-I : If two curves intersect each other at two points then area fo common region is

$$= \int_a^b [f(x) - g(x)]\,dx$$



Fig, 11.16

Case-II: If two curves intersect at one point and the area between them is bounded by x–axis then,

required area $= \int_a^c f(x)\,dx + \int_c^b g(x)\,dx$

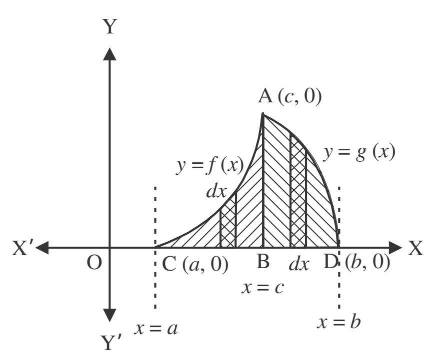


Fig. 11.17

[where both curves intersect each other at $A(C, O)$]

Case-III: If two curves, intersect each other at more than two points.

In the interval $[a, b]$, two curves $y = f(x)$ and $y = g(x)$, intersect each other at A, B and C. Clearly in $[a, c]$ $f(x) \geq g(x)$ and $g(x) \geq f(x)$ in $[c, d]$.

required area = area $APBQA$ + area $BECDB$

$$= \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$$

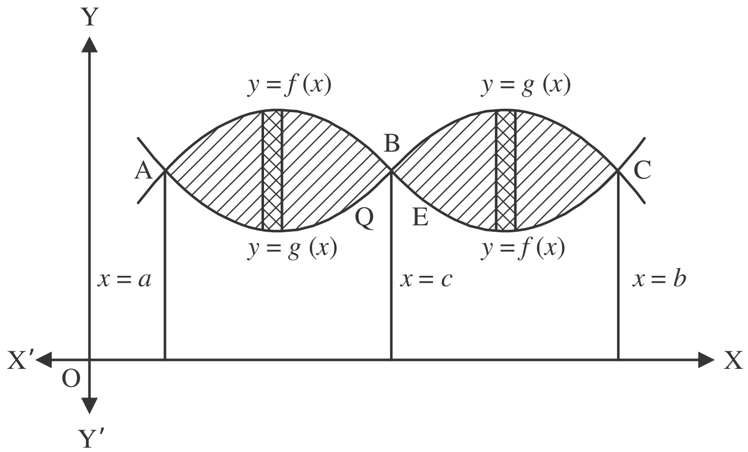


Fig. 11.18

Illustrative Examples

Example 10. Find the area bounded by parabola $y^2 = 4ax$ and line $y = x$ in first qudrant.

Solution : On solving the equations of parabola and line

$$y^2 = 4ax \quad \text{or} \quad x(x - 4a) = 0 \Rightarrow x = 0, 4a \quad \therefore y = 0, 4a$$

So, the line cuts the parabola at $0\ (0, 0)$ and $A(4a, 4a)$ so the area between parabola and lines is

$$= \int_{0(y \text{ from parabola})}^{4a} y\,dx - \int_{0(y \text{ from line})}^{4a} y\,dx$$

$$= \int_0^{4a} \sqrt{4ax}\,dx - \int_0^{4a} x\,dx = 2\sqrt{a}\int_0^{4a} \sqrt{x}\,dx - \int_0^{4a} x\,dx$$

$$= 2\sqrt{a} \times \frac{2}{3}\left[x^{3/2}\right]_0^{4a} - \left[\frac{x^2}{2}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left[(4a)^{3/2} - 0\right] - \left[\frac{(4a)^2}{2} - 0\right]$$

$$= \frac{32a^2}{3} - 8a^2 = \frac{8a^2}{3} \text{ Sq. units.}$$

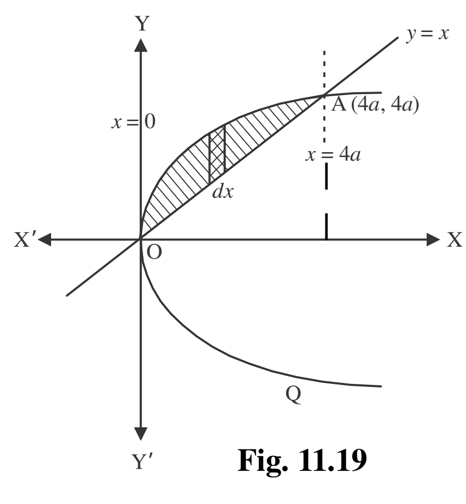


Fig. 11.19

Example 11. Find the area bounded by circle $x^2+y^2=a^2$ and curve $y=|x|$.

Solution : The lines represented by curve $y=|x|$ are $y=x$ and $y=-x$. They intersect the circle at points A and B whose coordinates are $(a/\sqrt{2}, a/\sqrt{2})$ and $(-a/\sqrt{2},\ a/\sqrt{2})$. Required area is shaded in Fig.

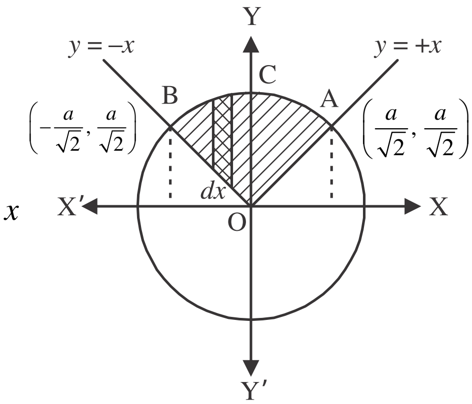


Fig. 11.20

$\therefore$ required area $=$ area $AOBCA$

$= 2 \times$ area $AOCA$

$$= 2\int_0^{a/\sqrt{2}} (\sqrt{a^2-x^2} - x)\,dx$$

where $f(x)$ in taken from circle and $g(x)$ is taken from line $y=x$

$$= 2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x^2}{2}\right]_0^{a/\sqrt{2}}$$

$$= 2\left[\frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}} + \frac{a^2}{2}\sin^{-1}\frac{a}{a\sqrt{2}} - \frac{a^2}{2\times 2}\right] - 2[0+0-0]$$

$$= 2\left[\frac{a}{2\sqrt{2}}\times\frac{a}{\sqrt{2}} + \frac{a^2}{2}\times\frac{\pi}{4} - \frac{a^2}{4}\right] = 2\left[\frac{a^2}{4} + \frac{\pi a^2}{8} - \frac{a^2}{4}\right]$$

$$= \frac{\pi a^2}{4} \text{ sq. units.}$$

Example 12. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution : The equations of given paraboals are

$$y^2 = 4ax \quad \text{and} \quad x^2 = 4by$$

On solving both the equations

$$\left(x^2/4b\right)^2 = 4ax \quad \text{or} \quad x^4 = 64ab^2x$$

or $\quad x(x^3 - 64ab^2) = 0 \Rightarrow x = 0,\ 4(ab^2)^{1/3}$

So both the curves will intersect x-axis at $x=0$ and $x = 4(ab^2)^{1/3}$

On tracing the curves we get the fig. 11.21

Hence the area between the curves is $OCABO$

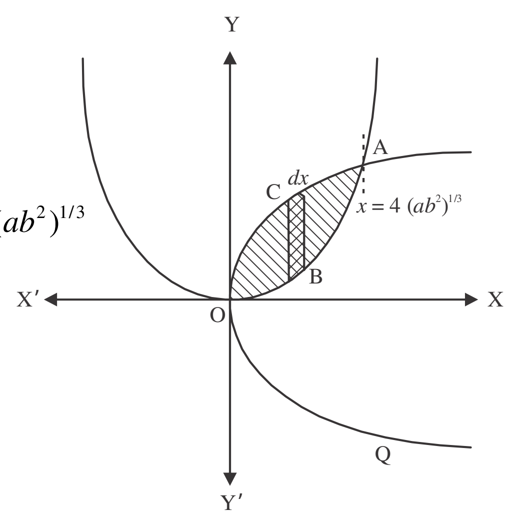


Fig. 11.21

$$= \int_{y\,(\text{From};\ y^2=4ax)}^{4(ab^2)^{1/3}} y\,dx - \int_{y\,(\text{From};\ x^2=4by)}^{4(ab^2)^{1/3}} y\,dx$$

$$= \int_0^{4(ab^2)^{1/3}} \sqrt{4ax}\,dx - \int_0^{4(ab^2)^{1/3}} \frac{x^2}{4b}\,dx$$

$$= 2\sqrt{a}.\frac{2}{3}[x^{3/2}]_0^{4(ab^2)^{1/3}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4(ab^2)^{1/3}}$$

$$= \frac{4}{3}\sqrt{a}\left[\left[4(ab^2)^{1/3}\right]^{3/2} - 0\right] - \frac{1}{12b}\left[\left\{4(ab^2)^{1/3}\right\}^3 - 0\right]$$

$$= \frac{4}{3}\sqrt{a}\left[8(ab^2)^{1/2}\right] - \frac{1}{12b}[64\, ab^2]$$

$$= \frac{32\sqrt{a}}{3}\sqrt{a}\, b - \frac{1}{12b} \times 64\, ab^2$$

$$= \frac{32}{3}ab - \frac{16ab}{3} = \frac{16ab}{3} \text{ Sq. units.}$$

Example 13. Find the area of smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution : As per diagram, the smaller region between the ellipse and line is represented by shaded region. Clearly the line cuts the ellipse at $A(a, 0)$ and $B(0, \text{b})$. So required area $ACBDA$

$$= \int_{0\,(y \text{ from ellipse})}^{a} y\, dx - \int_{0\,(y \text{ from line})}^{a} y\, dx$$

$$= \int_o^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx - \int_o^a \frac{b}{a}(a - x)\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_o^a - \frac{b}{a}\left(ax - \frac{x^2}{2}\right)_o^a$$

$$= \frac{b}{a}\left[\left(o + \frac{a^2}{2} \times \frac{\pi}{2}\right) - (o + o)\right] - \frac{b}{a}\left[\left(a^2 - \frac{a^2}{2}\right) - (o - o)\right]$$

$$= \frac{\pi ab}{4} - \frac{ab}{2} = \frac{ab}{4}(\pi - 2) \text{ sq. units.}$$

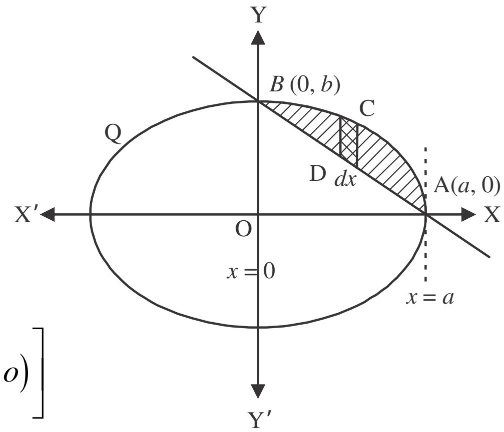


Fig. 11.22

Example 14. Find the area between the parabola $x^2 = 4y$ and line $x = 4y - 2$.

Solution : On solving the equations of parabola and straight line

$$x = x^2 - 2 \text{ or } x - x^2 - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2, -1$$

Clearly, the line cuts the parabola at $x = 2$ and $x = -1$.

So, required area $ABOA$ $= \int_{-1(y,\text{ from line})}^{2} y\, dx - \int_{-1(y,\text{ from parabola})}^{2} y\, dx$

$$= \int_{-1}^{2} \frac{x + 2}{4}\, dx - \int_{-1}^{2} \frac{x^2}{4}\, dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2} + 2x\right)_{-1}^{2} - \left[\frac{x^3}{12}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \left[\frac{8}{12} - \left(\frac{-1}{12}\right)\right]$$

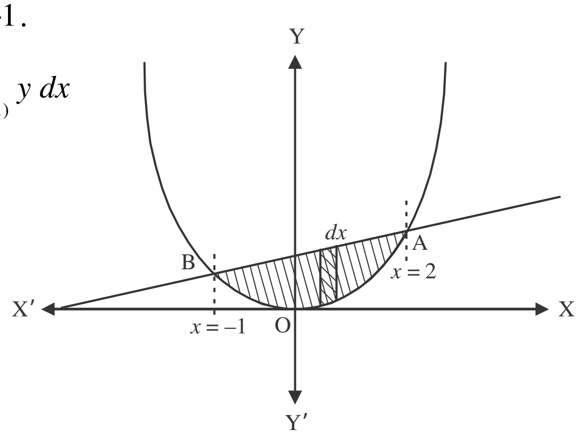


Fig. 11.23

$$= \frac{1}{4}\left[6+\frac{3}{2}\right]-\frac{9}{12} = \frac{1}{4}\times\frac{15}{2}-\frac{9}{12} = \frac{15}{8}-\frac{3}{4} = \frac{9}{8} \text{ sq. units.}$$

Example 15. Find the area of smaller region between $x^2 + y^2 = 2$ and $x = y^2$.

Solution : The area of smaller region between circle $x^2 + y^2 = 2$ and parabola $x = y^2$ is presented by shaded region, to find out the points of intersection. On solving the equation.

$$x^2 + x = 2 \Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = -2, 1 \text{ when } x = 1 \text{ then } y = \pm 1$$

So both the curves intersect each other at $A(1, 1)$ and $B(1, -1)$.

So required area = area $AOBCO$ = 2 × area $AODCA$

$$= 2[AODA + ADCA]$$

$$= 2\left[\int_{0(y\text{ from parabola})}^{1} y\,dx + \int_{1(y\text{ from circle})}^{\sqrt{2}} y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{x}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx\right]$$

$$= 2\left[\frac{2}{3}\left\{x^{3/2}\right\}_0^1 + \left\{\frac{x}{2}\sqrt{2-x^2} + \frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right\}_1^{\sqrt{2}}\right]$$

$$= 2\left[\frac{2}{3}\times(1-0) + \left(0+\sin^{-1}1\right) - \left(\frac{1}{2}+\sin^{-1}\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\left[\frac{2}{3}+\frac{\pi}{2}-\frac{1}{2}-\frac{\pi}{4}\right] = 2\left[\frac{1}{6}+\frac{\pi}{4}\right] = \left[\frac{1}{3}+\frac{\pi}{2}\right] \text{ sq. units}$$

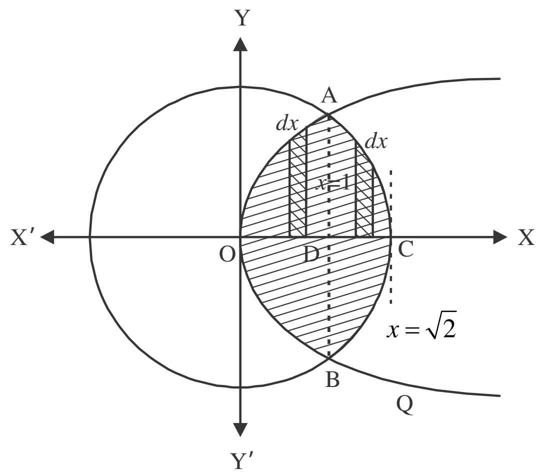


Fig. 11.24

Example 16. Using integration, find the area of region bounded by the triangle whose vertices are (–1, 1), (0, 5) and (3, 2).

Soluton : Let $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$ are vertices of triangle.

Equation of line AB

$$y-1 = \frac{5-1}{0+1}(x+1)$$

or $\quad y - 1 = 4x + 4$

or $\quad 4x - y + 5 = 0 \qquad (1)$

equation of line BC

$$y-5 = \frac{2-5}{3-0}(x-0)$$

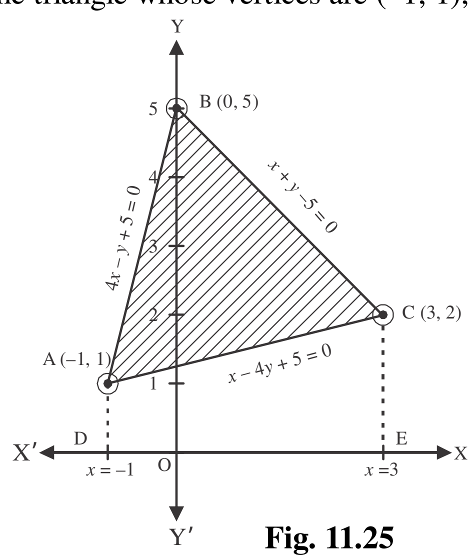


Fig. 11.25

or $\quad 3y-15=-3x$

or $\quad x+y-5=0 \qquad (2)$

equation of line CA

$$y-1=\frac{2-1}{3+1}(x+1)$$

or $\quad 4y-4=x+1$

or $\quad x-4y+5=0 \qquad (3)$

So, area of ΔABC = area of trapezium $ABOD$ + area of trapezium $BOEC$ – area of trapezium $ACED$

$$=\int_{-1(\text{from line AB})}^{0} y\,dx+\int_{0(\text{from line BC})}^{3} y\,dx-\int_{-1(\text{from line CA})}^{3} y\,dx$$

$$=\int_{-1}^{0}(4x+5)\,dx+\int_{0}^{3}(5-x)\,dx-\int_{-1}^{3}\frac{x+5}{4}\,dx$$

$$=\left[2x^2+5x\right]_{-1}^{0}+\left[5x-\frac{x^2}{2}\right]_{0}^{3}-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$=\left[(0+0)-(2-5)\right]+\left[(15-9/2)-(0-0)\right]-\frac{1}{4}\left[(9/2+15)-(1/2-5)\right]$$

$$=[3]+[21/2]-\frac{1}{4}(39/2+9/2)$$

$$=3+\frac{21}{2}-6=\frac{21}{2}-3=\frac{15}{2} \text{ sq. units.}$$

Exercise 11.3

1. Find the area between parabola $y^2=2x$ and circle $x^2+y^2=8$.
2. Find the area between parabola $4y=3x^2$ and line $3x-2y+12=0$.
3. Find the area between curves $y=\sqrt{4-x^2}$, $x=\sqrt{3}y$ and x-axis.
4. Find the the area between circle $x^2+y^2=16$ and line $y=x$ in first quadrant.
5. Find the common area between parabolas $y^2=4x$ and $x^2=4y$.
6. Find the area between $x^2+y^2=1$ and $x+y=1$ in first quadrant.
7. Find the area between $y^2=4ax$, line $y=2a$ and y-axs.
8. Find the area of circle $x^2+y^2=16$ whcih is exterior to the parabola $y^2=6x$.
9. Using integration, find the area of region bounded by triangle whose vertices are $A(2, 0)$, $B(4, 5)$ and $C(6, 3)$.
10. Using integration, find the area of triangular region whose sides have the equations $3x-2y+3=0$, $x+2y-7=0$ and $x-2y+1=0$.

Miscellaneous Examples

Example 17. Find the area in first quadrant bounded by curves $x^2 + y^2 = \pi^2$ and $y = \sin x$.

Solution : The area bounded by $x^2 + y^2 = \pi^2$ and $y = \sin x$ in first qudrant is shaded in figure.

Required area = $OCABO$

$$= \int_{0(y \text{ from circle})}^{\pi} y\,dx - \int_{0(y \text{ from } y=\sin x)}^{\pi} y\,dx$$

$$= \int_0^{\pi} \sqrt{\pi^2 - x^2}\,dx - \int_0^{\pi} \sin x\,dx$$

$$= \left[\frac{x}{2}\sqrt{\pi^2 - x^2} + \frac{\pi^2}{2}\sin^{-1}\frac{x}{\pi}\right]_0^{\pi} - \left[-\cos x\right]_0^{\pi}$$

$$= \left[\left\{0 + \frac{\pi^2}{2}\sin^{-1}(1)\right\} - \{0+0\}\right] + \left[\cos\pi - \cos 0\right]$$

$$= \frac{\pi^2}{2} \times \frac{\pi}{2} + (-1) - 1 = \frac{\pi^3}{4} - 2 = \frac{\pi^3 - 8}{4} \text{ sq. units}$$

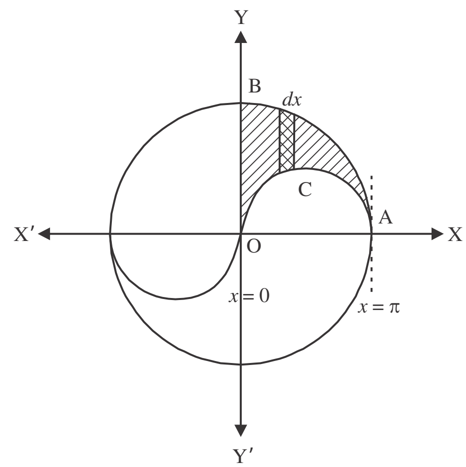


Fig. 11.26

Example 18. Find the area between the circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

Solution : Given circles are

$$x^2 + y^2 = 1 \qquad (1)$$

$$(x-1)^2 + y^2 = 1 \qquad (2)$$

Centres of circles (1) and (2) are (0, 0) and (1, 0) respectively and the radii of both circles are 1. On solving the equations of circles (1) and (2).

$$x^2 - (x-1)^2 = 0$$

or $\quad x^2 - x^2 + 2x - 1 = 0$

$\Rightarrow \quad x = 1/2 \quad \Rightarrow \quad y = \pm\sqrt{3}/2$

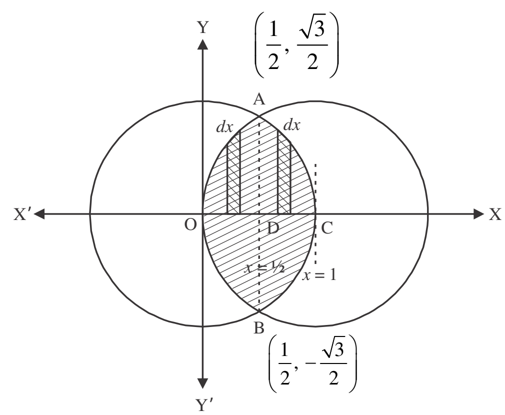


Fig. 11.27

$\therefore$ Coordinates of $A = (1/2, \sqrt{3}/2)$ and coordinates of B $(1/2, -\sqrt{3}/2)$

where A and B are point of intersection of both the circles

So required area = area $OACBO$

= 2 × area $OACDO$

= 2 [area $OADO$ + area $ADCA$]

$$= 2\left[\int_{0\,(y,\text{from circle (2)})}^{1/2} y\,dx + \int_{1/2\,(y,\text{ from circle (1)})}^{1} y\,dx\right]$$

$$= 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1} \sqrt{1-x^2}\,dx\right]$$

$$= 2\left[\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{1/2} + 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/2}^{1}$$

$$= 2\left[\left\{-\frac{1}{4}\sqrt{1-\frac{1}{4}} + \frac{1}{2}\sin^{-1}\left(\frac{-1}{2}\right)\right\} - \left\{\frac{-1}{2}\times 0 + \frac{1}{2}\sin^{-1}(-1)\right\}\right]$$

$$+2\left[\left\{0 + \frac{1}{2}\sin^{-1}(1)\right\} - \left\{\frac{1}{4}\sqrt{1-\frac{1}{4}} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$= 2\left[-\frac{\sqrt{3}}{8} + \frac{1}{2}\times\left(-\frac{\pi}{6}\right) - \frac{1}{2}\times\left(-\frac{\pi}{2}\right)\right] + 2\left[\frac{1}{2}\times\frac{\pi}{2} - \frac{\sqrt{3}}{8} - \frac{1}{2}\times\left(\frac{\pi}{6}\right)\right]$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

Example 19. Find the area between the curves $y = \sin x$, $y = \cos x$, y-axis and $o \le x \le \pi/2$.

Solution : On solving $y = \sin x$ and $y = \cos x$, $\sin x = \cos x \Rightarrow \tan x = 1$

$\Rightarrow \qquad x = \pi/4$

Hence both intersect at $x = \pi/4$

So at B $x = \pi/4$ hence

required area = area of $AOBA$

= area $ABEO$ – area $OBEO$

$$= \int_{0\,(y,\text{ From } y=\cos x)}^{\pi/4} y\cdot dx - \int_{0\,(y,\text{ from } y=\sin x)}^{\pi/4} y\cdot dx$$

$$= \int_0^{\pi/4}\cos x\, dx - \int_0^{\pi/4}\sin x\, dx$$

$$= [\sin x]_0^{\pi/4} - [\cos x]_0^{\pi/4}$$

$$= \sin\frac{\pi}{4} - 0 + \left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \frac{2}{\sqrt{2}} - 1 = (\sqrt{2}-1) \text{ sq. units.}$$

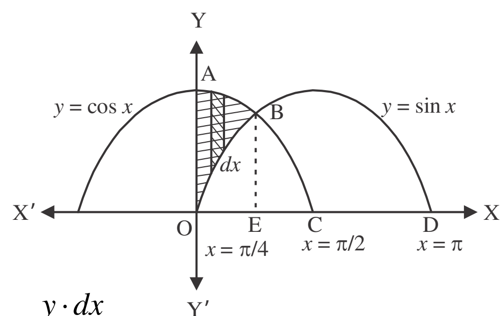


Fig. 11.28

Example 20. Find the area of region $\{(x, y) \mid x^2 \le y \le x\}$.

Solution : Given that:

$$y = x^2 \qquad (1)$$

and $$y = x \qquad (2)$$

Curve (1) is upward parabola and line $y = x$ passes through origin. The region between parabola and lines has been shaded. On solving equ. (1) and (2).

$$x^2 = x \Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

$$\therefore \quad y = 0, 1$$

Hence parabola and line intersect each other at (0, 0) and (1, 1)

$\therefore$ Required area = area $OCABO$

$$= \int_{0(y \text{ from line})}^{1} y\,dx - \int_{0(y \text{ from parabola})}^{1} y\,dx$$

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left[x^2/2\right]_0^1 - \left[x^3/3\right]_0^1$$

$$= (1/2 - 0) - (1/3 - 0) = 1/6 \text{ sq. units}$$

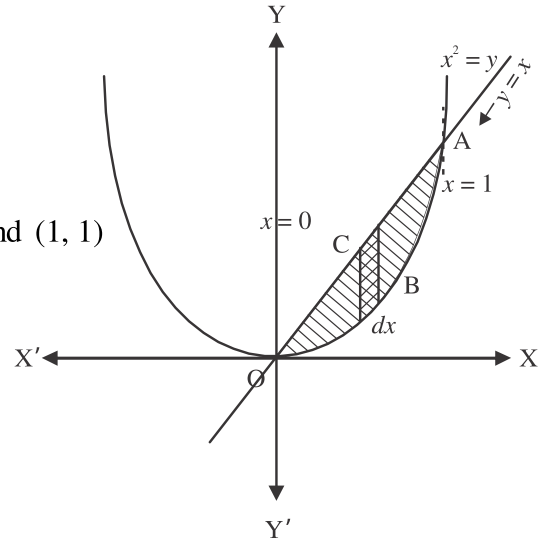


Fig. 11.29

Example 21. Find the area bounded by the $y = x^2 + 2$, lines $y = x, x = 0$ and $x = 3$.

Solution : Curve $y = x^2 + 2$ is a parabola whose vertex (0, 2) is on y–axis. $y = x$ is a line passes through origin. The required area bounded by curve $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$ is shaded in figure. In the figure the coordinates of point Q are (3, 11) which is a point of intersection of $x = 3$ and $y = x^2 + 2$.

Required area = area $OPQRO$

$$= \int_{0(y \text{ from parabola})}^{3} y\,dx - \int_{0(y \text{ from line})}^{3} y\,dx$$

$$= \int_0^3 (x^2 + 2)dx - \int_0^3 x\,dx$$

$$= \left[(x^3/3) + 2x\right]_0^3 - \left[x^2/2\right]_0^3$$

$$= \{(27/3 + 6)\} - (0 + 0) - [(9/2) - 0]$$

$$= 9 + 6 - (9/2) = 21/2 \text{ sq. units.}$$

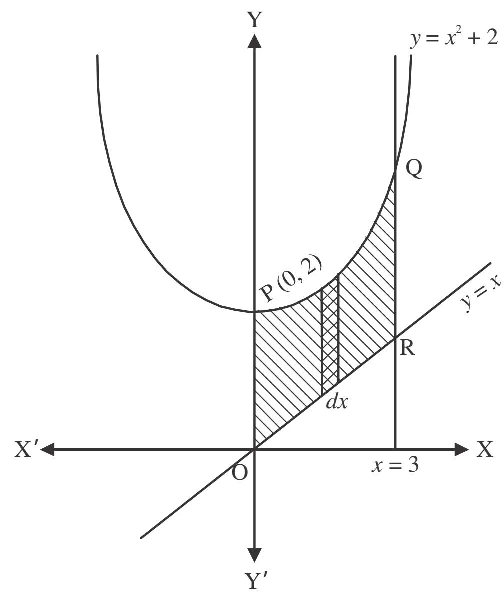


Fig. 11.30

Miscellaneous Exercise – 11

1. The area bounded by curve $y = \sqrt{x}$ and $y = x$ is (in sq. units)

   (a) 1 (b) 1 / 9 (c) 1 / 6 (d) 2 / 3

2. The area (in sq. units) bounded by curves $y^2 = x$ and $x^2 = y$ is

   (a) 1 / 3 (b) 1 (c) 1 / 2 (d) 2

3. The area (in sq. units) bounded by parabola $x^2 = 4y$ and its latus rectum is

(a) 5 / 3 (b) 2 / 3 (c) 4 / 3 (d) 8 / 3

4. The area (in sq. units) bounded by $y = \sin x, \frac{\pi}{2} \le x \le \frac{3\pi}{2}$ and x-axis is

(a) 1 (b) 2 (c) 1 / 2 (d) 4

5. The area (in sq. units) bounded by $y^2 = 2x$ and circle $x^2 + y^2 = 8$ is

(a) $(2\pi + 4/3)$ (b) $(\pi + 2/3)$ (c) $(4\pi + 4/3)$ (d) $(\pi + 4/3)$

6. Find the area between parabola $y^2 = x$ and line $x + y = 2$.

7. Find the area between $y^2 = 2ax - x^2$ and $y^2 = ax$ in first quadrant.

8. Find the area between parabola $y = x^2$ and $y = |x|$.

9. Find the common area between circle $x^2 + y^2 = 16$ and parabola $y^2 = 6x$.

10. Find the area bounded by $x^2 + y^2 = 1$ and $x + y \ge 1$.

11. Using integration find the area of a triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

12. Find the area bounded by line $y = 3x + 2$, x-axis and ordinates $x = -1$ and $x = 1$.

13. Find the area between $y^2 = 2x$, $y = 4x - 1$ and $y \ge o$.

14. Find the area between $y^2 = 4x$, y-axis and line $y = 3$.

15. Find the area between the two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

IMPORTANT POINTS

1. The area bounded by curve $y = f(x)$, x-axis and ordinates $x = a$ and $x = b$ is given by definite integral $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$ i.e. area $= \int_a^b f(x)\,dx = \int_a^b y\,dx$.

2. The area of the region bounded by the curve $x = \phi(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula : Area $= \int_c^d \phi(y)\,dy = \int_c^d x\,dy$.

3. If the curve is symmetrical about any principal axis or any straight line, then the total area may be calculated by multiplying the area of one symmetrical part by number of symmetrical parts.

4. Quadrature is always considered as positive. So if some portion of area is above x-axis and some portion is below x-axis then calculate the required area as a sum of individual parts of both areas.

5. The area of the region enclosed between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by the formula.

Area $= \int_a^b [f(x) - g(x)]\,dx$, where $f(x) \ge g(x)$ in [a, b]

6. The area of the region enclosed between two curves $x = \phi(y)$ and $x = \psi(y)$ and $y = c$ and $y = d$ is given by the formula $= \int_c^d [\phi(y) - \psi(y)]dy$

ANSWERS

Exercise 11.1

1. $8/3\ a^2$ sq. units
2. $(\sqrt{3} + 2\pi/3)$ sq. units
3. 4 sq. units
4. 4 / 3 sq. units
5. 5 sq. units
6. 2 / $3a$ sq. units
7. 3p sq. units
8. πab sq. units
9. 2ab sq. units
10. 5 sq. units
11. 7 / 3 sq. units
12. 7 / 3 sq. units

Exercise 11.2

1. $(2\pi + 4/3)$ sq. units
2. 27 sq. units
3. π / 3 sq. units
4. 2π sq. units
5. 16 / 3 sq. units
6. $\pi - 2/4$ sq. units
7. $2\ a^2/3$ sq. units
8. 9 / 2 sq. units
9. 7 sq. units
10. 4 sq. units

Miscellaneous Exercise – 11

1. (c)
2. (a)
3. (d)
4. (b)
5 (a)
6. 9 / 2 sq. units
7. $a^2(\pi/4 - 2/3)$ sq. units
8. 1 / 3 sq. units
9. $4/3(\sqrt{3} + 4\pi)$ sq. units
10. $\pi - 2/4$ sq. units
11. 4 sq. units
12. 13 / 3 sq. units
13. 1 / 3 sq. units
14. 9 / 4 sq. units
15. $(8\pi/3 - 2\sqrt{3})$ sq. units