Exercise 7.5

Answer 1E.

Consider the following integral:

$$\int \cos x\left(1+\sin^2 x\right)dx$$

Evaluate the integral $\int \cos x\left(1+\sin^2 x\right)dx$.

$$\int \cos x\left(1+\sin^2 x\right)dx = \int\left(\cos x+\cos x\sin^2 x\right)dx$$
$$= \int \cos x dx + \int \cos x\sin^2 x dx$$
$$= \sin x + \int \cos x\sin^2 x dx$$

$$\int \cos x\left(1+\sin^2 x\right)dx = \sin x + \int \cos x\sin^2 x dx \quad \text{...... (1)}$$

Find the value of the integral, $\int \cos x\sin^2 x dx.$

Let $\sin x = t$.

Differentiate on both sides.

$$\sin x = t$$
$$\cos x dx = dt$$

The integral can be written as follows:

$\int \cos x\sin^2 x dx = \int t^2 dt$ Use $\sin x = t$ and $\cos x dx = dt$

$$= \frac{t^{2+1}}{2+1}+C$$
$$= \frac{t^3}{3}+C$$

$= \frac{\sin^3 x}{3}+C$ Use $\sin x = t$

The integral is $\int \cos x\sin^2 x dx = \frac{\sin^3 x}{3}+C$.

From equation (1), the following can be ascertained:

$\int \cos x\left(1+\sin^2 x\right)dx = \sin x + \frac{\sin^3 x}{3}+C$ Use $\int \cos x\sin^2 x dx = \frac{\sin^3 x}{3}+C$

Hence, the resulted integral is $\int \cos x\left(1+\sin^2 x\right)dx = \boxed{\sin x + \frac{\sin^3 x}{3}+C}$.

Answer 2E.

We have to evaluate the following integral

$$\int_{x=0}^{x=1} (3x+1)^{\sqrt{2}}\, dx$$

Let us begin by making the substitution:

$$u = 3x+1$$

$$du = 3dx$$

Plugging this substitution into our integral:

$$\int_{x=0}^{x=1} (3x+1)^{\sqrt{2}}\, dx$$

$$= \frac{1}{3}\int_{u=1}^{u=4} (u)^{\sqrt{2}}\, du$$

$$\left(\text{since } u = 3x+1, x: 0 \to 1 \Rightarrow u: 1 \to 4\right)$$

$$= \left.\frac{u^{1+\sqrt{2}}}{3\left(\sqrt{2}+1\right)}\right|_{u=1}^{u=4}$$

Therefore

$$\int_{x=0}^{x=1} (3x+1)^{\sqrt{2}}\, dx$$

$$= \frac{(4)^{1+\sqrt{2}}}{3\left(\sqrt{2}+1\right)} - \frac{1}{3\left(\sqrt{2}+1\right)}$$

$$= \boxed{\frac{(4)^{1+\sqrt{2}}-1}{3\left(\sqrt{2}+1\right)} \approx 3.8}$$

Answer 3E.

We have $\displaystyle\int \frac{\sin x + \sec x}{\tan x} dx = \int \frac{\sin x + \frac{1}{\cos x}}{\frac{\sin x}{\cos x}} dx$

$$= \int \frac{\sin x \cos x + 1}{\sin x} dx$$

$$= \int \cos x + \frac{1}{\sin x} dx$$

$$= \int \cos x\, dx + \int \csc x\, dx$$

$$= \boxed{\sin x + \ln\left|\csc x - \cot x\right| + C}$$

Answer 4E.

Consider the integral,

$$\int \frac{\sin^3 x}{\cos x} dx$$

Need to evaluate the integral.

$$\int \frac{\sin^3 x}{\cos x} dx = \int \frac{\sin^2 x \cdot \sin x}{\cos x} dx$$

$$= \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos x} dx \text{ Since } \sin^2 x = 1-\cos^2 x$$

$$= \int \frac{-\left(\cos^2 x - 1\right)\left(\sin x\right)}{\cos x} dx$$

$$= \int \frac{\left(\cos^2 x - 1\right)\left(-\sin x\right)}{\cos x} dx$$

Group the negative sign common from $1-\cos^2 x$ to $\sin x$

Let $u = \cos x$ then $du = -\sin x\, dx$

Substitute these values in $\int \frac{\left(\cos^2 x - 1\right)\left(-\sin x\right)}{\cos x} dx$, obtain

$$\int \frac{\left(\cos^2 x - 1\right)\left(-\sin x\right)}{\cos x} dx = \int \frac{\left(u^2 - 1\right)}{u} du$$

$$= \int \left(\frac{u^2}{u} - \frac{1}{u}\right) du$$

$$= \int \left(u - \frac{1}{u}\right) du$$

$$= \int u du - \int \frac{1}{u} du$$

$$= \frac{1}{2}u^2 - \ln|u| + C \text{ Use } \left\langle \begin{array}{l} \int x^n dx = \frac{x^{n+1}}{n+1} \text{ and} \\ \int \frac{1}{x} dx = \ln|x| \end{array} \right.$$

$$= \frac{1}{2}\cos^2 x - \ln|\cos x| + C \text{ Since } u = \cos x$$

Therefore,

$$\int \frac{\sin^3 x}{\cos x} dx = \boxed{\frac{1}{2}\cos^2 x - \ln|\cos x| + C}$$

Answer 5E.

Consider an integral $\int \frac{t}{t^4+2} dt$ ...... (1)\

Simplify the integral:

$$\frac{t}{t^4+2} = \frac{t}{\left(t^2\right)^2 + \left(\sqrt{2}\right)^2}$$

Assume the following:

$$t^2 = x$$

$$2t\, dt = dx$$

$$t\, dt = \frac{dx}{2}$$

Substitute these values in (1).

$$\int \frac{t}{t^4+2}dt = \int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\cdot\frac{dx}{2}$$

$$= \frac{1}{2}\int \frac{1}{x^2+\left(\sqrt{2}\right)^2}dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C \quad \text{Since } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right)+C$$

Hence, $\boxed{\int \frac{t}{t^4+2}dt = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right)+C}$

Answer 6E.

There are various techniques to evaluate the integral.

One of the techniques is the method of substitution.

In the method of substitution, an expression is substituted for one variable in order to simplify the integrand so that it is brought into a form that is easy to evaluate.

Consider the integral:

$$\int_0^1 \frac{x}{(2x+1)^3}dx$$

Use the method of substitution to evaluate the above integral.

Make the substitution as shown below:

$$\begin{aligned} 2x+1 &= t \\ 2x &= t-1 \\ x &= \frac{t-1}{2} \\ dx &= \frac{1}{2}dt \end{aligned}$$

Now change the limits of definite integral.

Take $x=0$ to find *t*.

$$\begin{aligned} t &= 2\times 0+1 \\ &= 1 \end{aligned}$$

Take $x=1$ to find *t*.

$$\begin{aligned} t &= 2\times 1+1 \\ &= 3 \end{aligned}$$

Evaluate the definite integral with new variable:

$$\int_0^1 \frac{x}{(2x+1)^3}dx = \int_1^3 \frac{1}{t^3}\times\frac{t-1}{2}\times\frac{1}{2}dt$$
$$= \frac{1}{4}\int_1^3 \frac{t-1}{t^3}dt$$
$$= \frac{1}{4}\int_1^3 \left[\frac{1}{t^2}-\frac{1}{t^3}\right]dt$$
$$= \frac{1}{4}\int_1^3 \frac{1}{t^2}dt - \frac{1}{4}\int_1^3 \frac{1}{t^3}dt$$

Evaluate the above integral further:

$$\int_0^1 \frac{x}{(2x+1)^3}dx = \frac{1}{4}\int_1^3 t^{-2} - \frac{1}{4}\int_1^3 t^{-3}$$
$$= \frac{1}{4}\left(\frac{t^{-2+1}}{-2+1}\right)_1^3 - \frac{1}{4}\left(\frac{t^{-3+1}}{-3+1}\right)_1^3$$
$$= \frac{1}{4}\left(\frac{t^{-1}}{-1}\right)_1^3 - \frac{1}{4}\left(\frac{t^{-2}}{-2}\right)_1^3$$
$$= -\frac{1}{4}\left(\frac{1}{t}\right)_1^3 + \frac{1}{8}\left(\frac{1}{t^2}\right)_1^3$$

Substitute the limits and simplify the expression:

$$\int_0^1 \frac{x}{(2x+1)^3}dx = -\frac{1}{4}\left(\frac{1}{3}-1\right) + \frac{1}{8}\left(\frac{1}{3^2}-\frac{1}{1^2}\right)$$
$$= -\frac{1}{4}\left(-\frac{2}{3}\right) + \frac{1}{8}\left(-\frac{8}{9}\right)$$
$$= \frac{1}{6}-\frac{1}{9}$$
$$= \frac{1}{18}$$

Hence, the final value of the integral is $\boxed{\int_0^1 \frac{x}{(2x+1)^3}dx = \frac{1}{18}}$.

Answer 7E.

We have to evaluate $\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}dy = \int_{-1}^{1} \frac{e^{\tan^{-1} y}}{1+y^2}dy$

Let $\tan^{-1} y = u \Rightarrow \frac{1}{1+y^2}dy = du$

When $y = 1,\ u = \tan^{-1}(1) = \frac{\pi}{4}$

And $y = -1,\ u = \tan^{-1}(-1) = -\frac{\pi}{4}$

Thus $\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}dy = \int_{-\pi/4}^{\pi/4} e^u du$
$$= \left[e^u\right]_{-\pi/4}^{\pi/4}$$
$$= \boxed{e^{\pi/4} - e^{-\pi/4}}$$

Thus $\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}dy = \int_{-\pi/4}^{\pi/4} e^u du$
$$= \left[e^u\right]_{-\pi/4}^{\pi/4}$$
$$= \boxed{e^{\pi/4} - e^{-\pi/4}}$$

Answer 8E.

Consider the integral $\int t\sin t\cos t\,dt$

$$\int t\sin t\cos t\,dt=\int t\frac{\sin 2t}{2}dt \quad \left[\begin{array}{l}\text{Since } \sin 2t=2\sin t\cos t\\ \sin t\cos t=\frac{\sin 2t}{2}\end{array}\right]$$

$$=\frac{1}{2}\int t\sin 2t\,dt$$

Let $f(x), g(x)$ are two differential functions, then

$$\int f(x)g'(x)dx=f(x)g(x)-\int g(x)f(x)dx$$

This is called the formula for the integration by parts.

In another way let $u=f(x), v=g(x)$ then the differentials are

$$du=f'(x)dx, dv=g'(x)dx$$

By the substitution rule formula for the integration by parts becomes

$$\int u\,dv=uv-\int v\,du \quad \text{...... (1)}$$

Consider $\int t\sin 2t\,dt$

Let

$$u=t \qquad dv=\sin 2t\,dt$$
$$du=dt \quad \text{and} \quad \int dv=\int \sin 2t\,dt$$
$$v=-\frac{\cos 2t}{2}$$

Plug in the required values to the formula (1).

$$\int t\sin 2t\,dt=t\left(-\frac{\cos 2t}{2}\right)-\int\left(-\frac{\cos 2t}{2}\right)dt$$
$$=-t\frac{\cos 2t}{2}+\int\left(\frac{\cos 2t}{2}\right)dt$$
$$=-t\frac{\cos 2t}{2}+\frac{1}{2}\frac{\sin 2t}{2}+C$$
$$=-\frac{t}{2}\cos 2t+\frac{1}{4}\sin 2t+C$$

Now $\frac{1}{2}\int t\sin 2t\,dt=\frac{1}{2}\left(-\frac{t}{2}\cos 2t+\frac{1}{4}\sin 2t+C\right)$

$$=-\frac{t}{4}\cos 2t+\frac{1}{8}\sin 2t+\frac{C}{2}$$
$$=-\frac{t}{4}\cos 2t+\frac{1}{8}\sin 2t+C_1 \qquad \left(C_1=\frac{C}{2}\right)$$

Therefore $\int t\sin t\cos t\,dt=\frac{1}{2}\int t\sin 2t\,dt$

$$=\boxed{-\frac{t}{4}\cos 2t+\frac{1}{8}\sin 2t+C_1}$$

Answer 9E.

We have to evaluate $\int_1^3 r^4\ln r\,dr$

Integrate by parts by taking

$$u=\ln r\Rightarrow du=\frac{1}{r}dr \text{ and } dv=r^4dr,\ v=\frac{r^5}{5}$$

Therefore

$$\int_1^3 r^4 \ln r\,dr = \left[\ln r.\frac{r^5}{5}\right]_1^3 - \int_1^3 \frac{1}{r}.\frac{r^5}{5}dr$$
$$= \left[\frac{r^5}{5}\ln r\right]_1^3 - \frac{1}{5}\int_1^3 r^4 dr$$
$$= \left[\frac{r^5}{5}\ln r\right]_1^3 - \left[\frac{1}{5}.\frac{r^5}{5}\right]_1^3$$
$$= \frac{1}{5}\left(3^5 \ln 3 - 1\ln 1\right) - \frac{1}{25}\left(3^5 - 1\right)$$
$$= \boxed{\frac{243}{5}\ln 3 - \frac{242}{25}}$$

Answer 10E.

We have to evaluate $\int_0^4 \frac{x-1}{x^2-4x-5}dx$

$$\frac{x-1}{x^2-4x-5} = \frac{x-1}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1} \qquad \text{(Let)}$$

So $\quad x-1 = A(x+1) + B(x-5)$

Put x = 5 $\quad 4 = 6A \Rightarrow A = \frac{2}{3}$

Put x = -1 $\quad -2 = -6B \Rightarrow B = \frac{1}{3}$

Then $\int_0^4 \frac{x-1}{x^2-4x-5}dx = \int_0^4 \left(\frac{2}{3(x-5)} + \frac{1}{3(x+1)}\right)dx$

$$= \left[\frac{2}{3}\ln|x-5| + \frac{1}{3}\ln|x+1|\right]_0^4$$
$$= \left[\frac{1}{3}\ln(x-5)^2 + \frac{1}{3}\ln|x+1|\right]_0^4$$
$$= \left[\frac{1}{3}\ln\left((x-5)^2.(x+1)\right)\right]_0^4$$
$$= \frac{1}{3}\ln\left((-1)^2.(5)\right) - \frac{1}{3}\ln\left((-5)^2.(1)\right)$$
$$= \frac{1}{3}\ln 5 - \frac{1}{3}\ln 25$$
$$= \frac{1}{3}\ln\frac{5}{25}$$
$$= \boxed{\frac{1}{3}\ln\frac{1}{5}}$$

Answer 11E.

We have $\int \frac{x-1}{x^2-4x+5}dx = \int \frac{x-1}{x^2-4x+4+1}dx$

$$= \int \frac{x-1}{(x-2)^2+1}dx$$

Let $\quad u = x-2$

$x = u+2$

$\Rightarrow dx = du$

Then $\int \frac{x-1}{x^2-4x+5}dx = \int \frac{(u+2)-1}{u^2+1}du$

$$= \int \frac{u+1}{u^2+1}du$$
$$= \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du \qquad \text{--- (1)}$$

Now substitute $u^2+1=t \Rightarrow 2udu=dt$ in first integral

$$\int \frac{u}{u^2+1}du = \int \frac{1}{t}\frac{dt}{2}$$
$$= \frac{1}{2}\ln|t|$$
$$= \frac{1}{2}\ln\left(u^2+1\right)$$

Therefore equation (1) becomes

$$\int \frac{x-1}{x^2-4x+5}dx = \frac{1}{2}\ln\left|u^2+1\right| + \tan^{-1}(u)+C$$
$$= \frac{1}{2}\ln\left|(x-2)^2+1\right| + \tan^{-1}(x-2)+C$$
$$= \boxed{\frac{1}{2}\ln\left|x^2-4x+5\right| + \tan^{-1}(x-2)+C}$$

Answer 12E.

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Consider the integral,

$$\int \frac{x}{x^4+x^2+1}dx.$$

The object is to evaluate the above integral.

Rewrite the integral as follows:

$$\int \frac{x}{x^4+x^2+1}dx$$
$$= \int \frac{x}{\left(x^2\right)^2+x^2+1}dx$$
$$= \int \frac{x}{\left(x^2\right)^2+x^2+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}dx \quad \text{Add and subtract } \left(\frac{1}{2}\right)^2.$$
$$= \int \frac{x}{\left(x^2+\frac{1}{2}\right)^2+\frac{3}{4}}dx \quad \text{Write } \left(x^2\right)^2+x^2+\left(\frac{1}{2}\right)^2=\left(x^2+\frac{1}{2}\right)^2$$
$$= \int \frac{x}{\left(x^2+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx \quad \text{Write } \frac{3}{4} \text{ as } \left(\frac{\sqrt{3}}{2}\right)^2$$

Use substitution method to evaluate the above integral as follows:

Let $x^2+\frac{1}{2}=u$

Differentiate on both sides of the function $x^2+\frac{1}{2}=u$.

$$2xdx=du$$
$$xdx=\frac{1}{2}du$$

So the integral reduced as,

$$\int \frac{x}{x^4+x^2+1}dx = \int \frac{x}{\left(x^2+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \int \frac{\frac{1}{2}du}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\int \frac{du}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\left(x^2+\frac{1}{2}\right)}{\sqrt{3}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

Therefore, $\int \frac{x}{x^4+x^2+1}dx = \boxed{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C}$

Answer 13E.

Given $\int \sin^5 t \cos^4 t\, dt$

$$= \int \cos^4 t \sin^4 t \sin t\, dt$$

$$= \int \cos^4 t\left(1-\cos^2 t\right)^2 \sin t\, dt$$

Put $\cos t = x \Rightarrow -\sin t\, dt = dx$

Therefore $\int \sin^5 t \cos^4 t\, dt = \int x^4\left(1-x^2\right)^2(-dx)$

$$= -\int x^4\left(1+x^4-2x^2\right)dx$$

$$= -\left[\frac{x^5}{5}+\frac{x^9}{9}-2\frac{x^7}{7}\right]+C$$

$$= -\left[\frac{\cos^5 t}{5}+\frac{\cos^9 t}{9}-2\frac{\cos^7 t}{7}\right]+C$$

Answer 14E.

Consider the following integral:

$$\int \frac{x^3}{\sqrt{1+x^2}}dx.$$

Recollect from the table of Trigonometric Substitution.

Expression	Substitution	Identity
$\sqrt{a^2+x^2}$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$

Let $x = \tan\theta$.

$$\begin{aligned}\sqrt{1+x^2} &= \sqrt{1+\tan^2\theta}\\ &= \sqrt{\sec^2\theta}\\ &= \sec\theta\end{aligned}$$

And $dx = \sec^2\theta \cdot d\theta$.

Integrate the function as shown below:

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}dx &= \int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta \cdot d\theta\\ &= \int \tan^3\theta\sec\theta \cdot d\theta\\ &= \int \tan^2\theta \cdot (\sec\theta\tan\theta)\cdot d\theta\end{aligned}$$

$= \int(\sec^2\theta - 1)\cdot(\sec\theta\tan\theta\, d\theta)$ Since, $\tan^2\theta = \sec^2\theta - 1$

Let $u = \sec\theta$ and $du = \sec\theta\tan\theta\, d\theta$ in the above integral.

Integrate as shown below:

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}dx &= \int (u^2-1)\cdot du\\ &= \frac{u^3}{3} - u + C\\ &= \frac{u}{3}(u^2-3) + C\end{aligned}$$

$= \frac{\sec\theta}{3}(\sec^2\theta - 3) + C$ Since, $u = \sec\theta$

$= \frac{\sqrt{1+x^2}}{3}(1+x^2-3) + C$ Since, $\sec\theta = \sqrt{1+x^2}$

$= \frac{\sqrt{1+x^2}}{3}(x^2-2) + C$ Simplifying

Therefore, the value of the integral $\int \frac{x^3}{\sqrt{1+x^2}}dx$ is $\boxed{\int \frac{x^3}{\sqrt{1+x^2}}dx = \frac{\sqrt{1+x^2}}{3}(x^2-2) + C}$.

Answer 15E.

We have to evaluate $\int \frac{dx}{(1-x^2)^{3/2}}$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Then $\cos\theta = \sqrt{1-x^2}$ and $\tan\theta = \frac{x}{\sqrt{1-x^2}}$

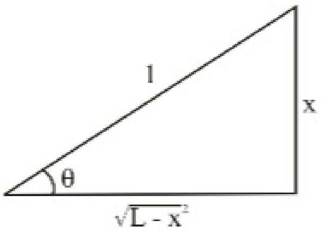

Fig. 1

$$\text{Therefore } \int \frac{dx}{\left(1-x^2\right)^{3/2}} = \int \frac{\cos\theta d\theta}{\left(1-\sin^2\theta\right)^{3/2}}$$

$$= \int \frac{\cos\theta d\theta}{\left(\cos^2\theta\right)^{3/2}}$$

$$= \int \frac{\cos\theta d\theta}{\cos^3\theta}$$

$$= \int \frac{1}{\cos^2\theta} d\theta$$

$$= \int \sec^2\theta d\theta$$

$$= \tan\theta + C$$

$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$

Answer 16E.

The original integral can be written as follows:

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos\theta} \cos\theta d\theta \qquad \text{Use } \sqrt{1-\sin^2\theta} = \cos\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2} d\theta \quad \text{Write } \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1-\cos 2\theta) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{\sin\left(\frac{\pi}{2}\right)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

Therefore, $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \boxed{\frac{\pi}{8} - \frac{1}{4}}$.

Answer 17E.

Consider the integral $\int_0^{\pi} t\cos^2 t\,dt$

$$\int_0^{\pi} t\cos^2 t\,dt = \int_0^{\pi} t\left(\frac{1+\cos 2t}{2}\right)dt \quad \left[\begin{aligned}\text{Since } \cos 2t &= \cos^2 t - \sin^2 t \\ &= \cos^2 t - \left(1-\cos^2 t\right) \\ &= 2\cos^2 t - 1 \\ \left(\frac{1+\cos 2t}{2}\right) &= \cos^2 t\end{aligned}\right]$$

$$= \frac{1}{2}\int_0^{\pi} t\left(1+\cos 2t\right)dt \quad \text{...... (1)}$$

Let $f(x), g(x)$ are two differential functions, then

$$\int_a^b f(x)g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b g(x)f'(x)\,dx$$

This is called the formula for the integration by parts.

In another way let $u = f(x), v = g(x)$ then the differentials are

$$du = f'(x)\,dx, dv = g'(x)\,dx$$

By the substitution rule formula for the integration by parts becomes

$$\int_a^b u\,dv = (uv)_a^b - \int_a^b v\,du \quad \text{...... (2)}$$

Consider $\int_0^{\pi} t\cos^2 t\,dt = \frac{1}{2}\int_0^{\pi} t\left(1+\cos 2t\right)dt$ From (1)

$$= \frac{1}{2}\int_0^{\pi} t\,dt + \frac{1}{2}\int_0^{\pi} t\cos 2t\,dt$$

$$= \frac{1}{2}\left(\frac{t^2}{2}\right)_0^{\pi} + \frac{1}{2}\int_0^{\pi} t\cos 2t\,dt$$

$$= \frac{1}{2}\left(\frac{\pi^2}{2}\right) + \frac{1}{2}\int_0^{\pi} t\cos 2t\,dt$$

$$= \frac{\pi^2}{4} + \frac{1}{2}\int_0^{\pi} t\cos 2t\,dt \quad \text{...... (3)}$$

Let $\frac{1}{2}\int_0^{\pi} t\cos 2t dt$

$$u = t \qquad \text{And} \qquad dv = \cos 2t dt$$
$$du = dt \qquad\qquad \int dv = \int \cos 2t dt$$
$$v = \frac{\sin 2t}{2}$$

Substitute in the required values to the formula (2).

$$\int_a^b u dv = (uv)_a^b - \int_a^b v du$$

$$\frac{1}{2}\int_0^{\pi} t\cos 2t dt = \frac{1}{2}\left[\left(t\frac{\sin 2t}{2}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin 2t}{2}dt\right]$$
$$= \frac{1}{2}\left[\left(\pi\frac{\sin 2\pi}{2} - 0\frac{\sin 2(0)}{2}\right) + \frac{1}{4}(\cos 2t)_0^{\pi}\right]$$
$$= \frac{1}{2}\left[0 + \frac{1}{4}(\cos 2\pi - \cos 0)\right]$$
$$= \frac{1}{8}(1-1)$$
$$= 0 \text{ ...... } (4)$$

Now from equation (3)

$$\int_0^{\pi} t\cos^2 t\, dt = \frac{\pi^2}{4} + \frac{1}{2}\int_0^{\pi} t\cos 2t dt$$
$$= \frac{\pi^2}{4} + 0 \qquad \text{(Since From (4))}$$
$$= \frac{\pi^2}{4}$$

Therefore $\int_0^{\pi} t\cos^2 t\, dt = \boxed{\frac{\pi^2}{4}}$

Answer 18E.

Given $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$

Put $\sqrt{t} = x \Rightarrow \frac{1}{2\sqrt{t}} dt = dx$

$$\Rightarrow \frac{1}{\sqrt{t}} dt = 2dx$$
$$t = 1 \Rightarrow x = 1$$
$$t = 4 \Rightarrow x = 2$$
$$\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} dt = \int_1^2 e^x 2dx$$
$$= 2\left[e^x\right]_1^2$$
$$= 2\left[e^2 - e\right]$$

Answer 19E.

Consider the following integral:

$$\int e^{x+e^x} dx$$

The objective is to evaluate the integral.

Rewrite the above integral as,

$$\int e^{x+e^x} dx = \int e^x \cdot e^{e^x} dx$$
$$= \int e^{e^x} \cdot e^x dx \qquad \text{......(1)}$$

Apply u substitution to have the following.

$u = e^x$

$du = e^x dx$

Substitute these values in the equation (1) and apply integration, we get

$$\begin{aligned} \int e^{x+e^x} dx &= \int e^{e^x} \cdot e^x dx \\ &= \int e^u du \\ &= e^u + C \\ &= e^{e^x} + C \quad \text{Back substitute } u = e^x \end{aligned}$$

Therefore, $\int e^{x+e^x} dx = \boxed{e^{e^x} + C}$.

Answer 20E.

Consider the following integral:

$$\int e^2 \, dx$$

Since, e^2 is a constant, the integral can be written as shown below:

$$\int e^2 \, dx = e^2 \int 1 dx$$

$= e^2(x) + C$ Since, $\int 1 dx = \int x^0 dx = \frac{x^{0+1}}{0+1} = x$

$= e^2 x + C$

Therefore, the value of the integral $\int e^2 \, dx$ is $\boxed{\int e^2 \, dx = e^2 x + C}$.

Answer 21E.

We have to evaluate the following integral

$$\int \tan^{-1}\left(\sqrt{x}\right) dx$$

Let us begin by making the substitution:

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

Our integral now becomes:

$$2\int u \tan^{-1}(u) du$$

We can now apply integration by parts, given by:

$$\int f(u) g'(u) du = f(u) g(u) - \int f'(u) g(u) du$$

By taking:

$f(u) = \tan^{-1}(u)$

$f'(u) = \frac{1}{u^2+1}$

And

$g(u) = \frac{u^2}{2}$

$g'(u) = u$

Plugging into our formula:

$$2\int u \tan^{-1}(u) du = u^2 \tan^{-1}(u) - \int \frac{u^2}{u^2+1} du$$

We can now do long division on the second integral, to get a function that we can integrate, which leaves us with:

$$u^2\tan^{-1}(u)-\int\frac{u^2}{u^2+1}du=u^2\tan^{-1}(u)-\int 1-\frac{1}{u^2+1}du$$
$$=u^2\tan^{-1}(u)-\int 1du+\int\frac{1}{u^2+1}du$$
$$=u^2\tan^{-1}(u)-u+\tan^{-1}(u)+C$$

Finally, we can plug x back:

$$\int\tan^{-1}\left(\sqrt{x}\right)dx=u^2\tan^{-1}(u)-u-\tan^{-1}(u)+C \quad \text{Where } u=\sqrt{x}$$
$$=x\tan^{-1}\left(\sqrt{x}\right)+\tan^{-1}\left(\sqrt{x}\right)-\sqrt{x}+C$$
$$=\boxed{(x+1)\tan^{-1}\left(\sqrt{x}\right)-\sqrt{x}+C}$$

Answer 22E.

Consider the following integral:

$$\int\frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx$$

Make the substitution.

$$u=\ln x$$
$$du=\frac{1}{x}dx$$

Therefore, the given integral can be written as follows:

$$\int\frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx=\int\frac{u}{\sqrt{1+u^2}}du \quad \text{......}(1)$$

Again, put $1+u^2=t$

Then $2u\,du=dt$

That is, $u\,du=\frac{dt}{2}$

Therefore, equation (1) can be written as follows:

$$\int\frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx=\int\frac{1}{\sqrt{t}}\cdot\frac{dt}{2}$$
$$=\frac{1}{2}\int t^{-1/2}dt$$
$$=\frac{1}{2}\cdot\frac{t^{1/2}}{\left(1/2\right)}+C$$
$$=\frac{1}{\cancel{2}}\cdot\cancel{2}t^{1/2}+C$$

$$=\sqrt{t}+C$$
$$=\sqrt{1+u^2}+C \qquad \text{Replace } t=1+u^2$$
$$=\sqrt{1+(\ln x)^2}+C \qquad \text{Replace } u=\ln x$$
$$\int\frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx=\boxed{\sqrt{1+(\ln x)^2}+C}.$$

Answer 23E.

Consider the following integral:

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx$$

Let us consider, $1+\sqrt{x}=t$.

Differentiate on both sides.

$$1+\sqrt{x}=t$$

$$\frac{1}{2\sqrt{x}}dx=dt$$

$$dx=2\sqrt{x}dt$$

$$=2(t-1)dt$$

Limits:

For $x=0;$

$$1+\sqrt{x}=t$$

$$1+\sqrt{0}=t$$

$$1=t$$

For $x=0;$

$$1+\sqrt{x}=t$$

$$1+\sqrt{1}=t$$

$$1+1=t$$

$$2=t$$

The integral $\int_0^1 \left(1+\sqrt{x}\right)^8 dx$ can be written as follows:

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx=\int_1^2 t^8 2(t-1)dt$$ Use $1+\sqrt{x}=t$ and $dx=2(t-1)dt$

$$=2\int_1^2 \left(t^9-t^8\right)dt$$

$$=2\int_1^2 t^9 dt-2\int_1^2 t^8 dt$$

$$=2\left[\frac{t^{9+1}}{9+1}\right]_1^2-2\left[\frac{t^{8+1}}{8+1}\right]_1^2$$ Use $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$=\frac{2}{10}\left[t^{10}\right]_1^2-\frac{2}{9}\left[t^9\right]_1^2$$

$$=\frac{1}{5}\left[2^{10}-1^{10}\right]-\frac{2}{9}\left[2^9-1^9\right]$$ Apply the limits

Simplify further as shown below:

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \frac{2^{10}}{5} - \frac{1}{5} - \frac{2^{10}}{9} - \frac{2}{9}$$
$$= 2^{10}\left(\frac{1}{5} - \frac{1}{9}\right) - \frac{1}{5} + \frac{2}{9}$$
$$= 2^{10}\left(\frac{9-5}{45}\right) - \left(\frac{9-8}{45}\right)$$
$$= 2^{10}\left(\frac{4}{45}\right) - \left(\frac{-1}{45}\right)$$
$$= 1024\left(\frac{4}{45}\right) + \frac{1}{45}$$
$$= \frac{4096}{45} + \frac{1}{45}$$
$$= \frac{4096+1}{45}$$
$$= \frac{4097}{45}$$

Hence, the integral value is $\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \boxed{\frac{4097}{45}}$.

Answer 24E.

Consider the integration $\int_0^4 \frac{6z+5}{2z+1} dz$

Here integrand is rational function.

Integrand is improper function, so dividing numerator by the denominator by using long division until the degree of the remainder is less that the denominator of the rational function.

That is

$$\begin{array}{r} 3 \\ 2z+1 \overline{)6z+5} \\ -(6z+3) \end{array}$$

2

By division algorithm, we can write as below

$$6z+5 = 3(2z+1)+2$$

$$\frac{6z+5}{2z+1} = \frac{3(2z+1)}{2z+1} + \frac{2}{2z+1}$$ Divided by $2z+1$

$$\frac{6z+5}{2z+1} = 3 + \frac{2}{2z+1} \quad \text{...... (1)}$$

Now

$$\int_0^4 \frac{6z+5}{2z+1}dz = \int_0^4 \left(3+\frac{2}{2z+1}\right)dz$$

$$= \int_0^4 3dz + \int_0^4 \frac{2}{2z+1}dz \qquad \text{From (1)}$$

$$= 3(z)_0^4 + \left(\frac{2\ln(2z+1)}{2}\right)_0^4 + C$$

$$= 3(4-0) + \left(\ln(2(4)+1) - \ln(2(0)+1)\right) + C$$

$$= 12 + (\ln 9 - \ln 1)$$

$$= 12 + \ln 9 \qquad (\text{Since } \ln 1 = 0)$$

Therefore $\int_0^4 \frac{6z+5}{2z+1}dz = \boxed{12+\ln 9}$

Answer 25E.

We have to evaluate $\int \frac{3x^2-2}{x^2-2x-8}dx$

$$\begin{array}{r|l} & 3 \\ x^2-2x-8 & 3x^2-2 \\ & 3x^2-24-6x \\ \hline & 22+6x \end{array}$$

By long division, we get

$$\int \frac{3x^2-2}{x^2-2x-8}dx = \int\left(3+\frac{6x+22}{x^2-2x-8}\right)dx$$

$$= \int 3dx + \int \frac{6x+22}{(x-4)(x+2)}dx \qquad \text{--- (1)}$$

Let $\frac{6x+22}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$

So $6x+22 = A(x+2) + B(x-4)$

Put $x = 4$ $\quad 46 = 6A \Rightarrow A = \frac{23}{3}$

Put $x = -2$ $\quad 10 = -6B \Rightarrow B = -\frac{5}{3}$

Then $\frac{6x+22}{(x-4)(x+2)} = \frac{23}{3(x-4)} - \frac{5}{3(x+2)}$

Place this value in equation (1)

$$\int \frac{3x^2-2}{x^2-2x-8}dx = \int 3dx + \int \frac{23}{3(x-4)}dx - \int \frac{5}{3(x+2)}dx$$

$$= \boxed{3x + \frac{23}{3}\ln|x-4| - \frac{5}{3}\ln|x+2| + C}$$

Answer 26E.

We have to evaluate $\int \frac{3x^2-2}{x^3-2x-8}dx$

Let $x^3 - 2x - 8 = t$

$(3x^2-2)dx = dt$

Then $\int \frac{3x^2-2}{x^3-2x-8}dx = \int \frac{1}{t}dt$

$$= \ln|t| + C$$

$$= \boxed{\ln|x^3-2x-8| + C}$$

Answer 27E.

Consider the following integral:

$$\int \frac{dx}{1+e^x}.$$

Let $1+e^x = u.$

Then, $e^x = u-1,\ x = \ln(u-1)$ and $dx = \frac{du}{u-1}.$

Integrate as shown below:

$$\int \frac{1}{1+e^x}dx = \int \frac{1}{u}\left(\frac{1}{u-1}\right)\cdot du$$

$$= \int \frac{1}{u(u-1)}\cdot du$$

Since, the degree of the numerator is less than the degree of the denominator, and the denominator has two linear factors, the partial fraction decomposition of the integrand has the following form:

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$= \frac{A(u-1)+Bu}{u(u-1)}$$

$$= \frac{(A+B)u-A}{u(u-1)}$$

Compare both sides of the equation, to get the following system of equations for A and B.

$$A+B=0$$

$$-A \quad =1$$

From the system of equations, $A=-1$, and $-1+B=0 \Rightarrow B=1$

So, the integrand can be written as follows:

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$= \frac{-1}{u} + \frac{1}{u-1}$$

Integrate furthers as shown below:

$$\int \frac{1}{1+e^x}dx = \int \frac{1}{u(u-1)}\cdot du$$

$$= \int\left(\frac{-1}{u} + \frac{1}{u-1}\right)\cdot du$$

$$= \int\left(\frac{-1}{u}\right)\cdot du + \int\left(\frac{1}{u-1}\right)\cdot du$$

$$= -\ln u + \ln(u-1) + C$$

$= -\ln(1+e^x) + \ln(1+e^x-1) + C$ Since, $u = 1+e^x$

$$= -\ln(1+e^x) + \ln(e^x) + C$$

Therefore, the value of the integral $\int \frac{dx}{1+e^x}$ is $\boxed{\int \frac{dx}{1+e^x} = -\ln(1+e^x) + \ln(e^x) + C}$.

Answer 28E.

We have to evaluate $\int \sin\sqrt{at}\,dt$

Let $\quad \sqrt{at} = u \Rightarrow at = u^2$

$$\Rightarrow a\,dt = 2u\,du$$

$$\Rightarrow dt = \frac{2}{a}u\,du$$

Then $\int \sin\sqrt{at}dt = \frac{2}{a}\int \sin u.udu$

Integrate by parts with u as first function

$$\int \sin\sqrt{at}dt = \frac{2}{a}\left[u(-\cos u) - \int 1.(-\cos u)du\right]$$
$$= -\frac{2}{a}u\cos u + \frac{2}{a}\int \cos udu$$
$$= -\frac{2}{a}u\cos u + \frac{2}{a}\sin u + C$$
$$= \boxed{-\frac{2}{a}\sqrt{at}\ \cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C}$$

Answer 29E.

Consider the integration $\int \ln\left(x+\sqrt{x^2-1}\right)dx$

Rewrite the above integration as below

$\int \ln\left(x+\sqrt{x^2-1}\right)\cdot 1\ dx$ ...... (1)

Let $f(x), g(x)$ are two differential functions, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f(x)dx$$

This is called the formula for the integration by parts.

In another way let $u = f(x), v = g(x)$ then the differentials are

$$du = f'(x)dx, dv = g'(x)dx$$

By the substitution rule formula for the integration by parts becomes

$\int udv = uv - \int vdu$ ...... (2)

Now

$$\int \ln\left(x+\sqrt{x^2-1}\right)\cdot 1\ dx$$

$u = \ln\left(x+\sqrt{x^2-1}\right)$ $\qquad$ $dv = 1$

$du = \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right)dx$ $\qquad$ And $\int dv = \int 1dx$

$v = x$

Substitute the above values in equation (2)

$$\int \ln\left(x+\sqrt{x^2-1}\right)\cdot 1\ dx = \ln\left(x+\sqrt{x^2-1}\right)x - \int x\left(\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right)\right)dx$$

$$= x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{2\left(x+\sqrt{x^2-1}\right)}{\left(x+\sqrt{x^2-1}\right)2\sqrt{x^2-1}}\cdot x\,dx$$

$= x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{x}{\sqrt{x^2-1}}dx$ ..... (3)

Consider $\int \frac{x}{\sqrt{x^2-1}}dx$

$$x^2-1=t$$
$$2xdx=dt$$
$$xdx=\frac{dt}{2}$$

$$\int \frac{x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$=\frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$=\frac{1}{2}\left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) \qquad \left(\text{Since } \int x^n dx = \frac{x^{n+1}}{n+1}\right)$$
$$=\sqrt{t}$$
$$=\sqrt{x^2-1}$$

Substitute the value of $\int \frac{x}{\sqrt{x^2-1}}dx$ in the equation (3)

$$\int \ln\left(x+\sqrt{x^2-1}\right)\cdot 1\,dx=\boxed{x\ln\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}+C}$$

Answer 30E.

Consider the integration $\int_{-1}^{2}\left|e^x-1\right|dx$ ...... (1)

For any real number *a* the absolute value of *a* is denoted by $|a|$ and it is defined by

$$|a|=a \quad \text{if } a\geq 0$$
$$=-a \text{ if } a<0$$

From the above definition, write the equation (1) as given below.

$$\int_{-1}^{2}\left|e^x-1\right|dx=\int_{-1}^{0}-\left(e^x-1\right)dx+\int_{0}^{2}\left(e^x-1\right)dx$$
$$=\int_{-1}^{0}\left(1-e^x\right)dx+\int_{0}^{2}\left(e^x-1\right)dx$$
$$=\left[x-e^x\right]_{-1}^{0}+\left[e^x-x\right]_{0}^{2}$$
$$=\left[\left(0-e^0\right)-\left(-1-e^{-1}\right)\right]+\left[\left(e^2-2\right)-\left(e^0-0\right)\right]$$
$$=\left[\left(0-1\right)-\left(-1-e^{-1}\right)\right]+\left[\left(e^2-2\right)-\left(1-0\right)\right]$$
$$=-1+1+e^{-1}+e^2-2-1$$
$$=e^2+\frac{1}{e}-3$$

Therefore $\int_{-1}^{2}\left|e^x-1\right|dx=\boxed{e^2+\frac{1}{e}-3}$

Answer 31E.

We have $\int \sqrt{\frac{1+x}{1-x}}dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}dx$

$$= \int \frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= \sin^{-1} x - \frac{1}{2}\int -2x.\left(1-x^2\right)^{-1/2} dx$$

$$= \sin^{-1} x - \frac{1}{2}\left(\frac{\left(1-x^2\right)^{-1/2+1}}{-1/2+1}\right) + C$$

$$= \sin^{-1} x - \frac{1}{2}.\frac{\left(1-x^2\right)^{1/2}}{1/2} + C$$

$$= \boxed{\sin^{-1} x - \sqrt{1-x^2} + C}$$

Answer 32E.

Substitute the value of $dx = tdt$ in the integral $\int \frac{\sqrt{2x-1}}{2x+3}dx$.

$$\int \frac{\sqrt{2x-1}}{2x+3}dx = \int \frac{t\cdot tdt}{\left(t^2+1\right)+3}$$

$$= \int \frac{t^2}{t^2+4}dt$$

$$= \int \frac{t^2+4-4}{t^2+4}dt$$

$$= \int \left(\frac{t^2+4}{t^2+4} - \frac{4}{t^2+4}\right)dt$$

$$= \int \left(1 - \frac{4}{t^2+4}\right)dt$$

$$= \int dt - \int \frac{4}{t^2+2^2}dt$$

$$= t - 4.\frac{1}{2}\tan^{-1}\frac{t}{2} + C \qquad \left(\text{Since } \int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right)$$

$$= t - 2\tan^{-1}\frac{t}{2} + C$$

$$= \sqrt{2x-1} - 2\tan^{-1}\left(\frac{\sqrt{2x-1}}{2}\right) + C \qquad \left(\text{Since } t = \sqrt{2x-1}\right)$$

Therefore, $\int \frac{\sqrt{2x-1}}{2x+3}dx = \boxed{\sqrt{2x-1} - 2\tan^{-1}\left(\frac{\sqrt{2x-1}}{2}\right) + C}$.

Answer 33E.

Consider the integral $\int \frac{\sqrt{2x-1}}{2x+3}dx$.

Let $\sqrt{2x-1} = t$.

$$2x - 1 = t^2$$

$$2x = t^2 + 1$$

$$2dx = 2tdt$$

$$dx = tdt$$

We have $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-1-2x-x^2}\,dx$

$$= \int \sqrt{4-(1+x)^2}\,dx$$

Let $1+x = 2\sin\theta$ then $dx = 2\cos\theta\, d\theta$

And so $\theta = \sin^{-1}\left(\frac{1+x}{2}\right)$

And $\cos\theta = \sqrt{1-\left(\frac{1+x}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{4-(1+x)^2}$$

$$= \frac{1}{2}\sqrt{3-2x-x^2}$$

Then $\int \sqrt{4-(1+x)^2}\,dx = \int \sqrt{4-4\sin^2\theta}.2\cos\theta\, d\theta$

$$= 4\int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$$

$$= 4\int \cos^2\theta\, d\theta$$

$$= 4\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2\int (1+\cos 2\theta)\, d\theta$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2.\left(\sin^{-1}\left(\frac{x+1}{2}\right)\right) + 2.\left(\frac{1+x}{2}\right).\frac{1}{2}\sqrt{3-2x-x^2} + C$$

$$= \boxed{2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{1+x}{2}.\sqrt{3-2x-x^2} + C}$$

Answer 34E.

We have $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \frac{1+4\frac{\cos x}{\sin x}}{4-\frac{\cos x}{\sin x}}\,dx$

$$= \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,dx$$

Put $4\sin x - \cos x = t$

Then $(4\cos x + \sin x)\,dx = dt$

When $x = \frac{\pi}{2}, t = 4\sin\frac{\pi}{2} = 4$

And $x = \frac{\pi}{4}, t = 4\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{4}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

Then $\int_{\pi/2}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{3/\sqrt{2}}^{4} \frac{dt}{t}$

$$= \left[\ln|t|\right]_{3/\sqrt{2}}^{4}$$

$$= \ln 4 - \ln\frac{3}{\sqrt{2}}$$

$$= \boxed{\ln\left(\frac{4}{3}\sqrt{2}\right)}$$

Answer 35E.

Given $\int \cos 2x \cos 6x\, dx$

We know $\cos A \cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$

$$\int \cos 2x \cos 6x\, dx = \int \frac{1}{2}\left(\cos(-4x)+\cos 8x\right) dx$$
$$= \frac{1}{2}\int\left(\cos 4x + \cos 8x\right) dx$$
$$= \frac{1}{2}\left[\frac{\sin 4x}{4}+\frac{\sin 8x}{8}\right]+C$$
$$= \frac{1}{8}\sin 4x + \frac{1}{16}\sin 8x + C$$

Answer 36E.

We have to evaluate the following integral

$$\int_{-\pi/4}^{\pi/4} \frac{x^2 \tan(x)}{(1+\cos^4 x)} dx$$

Although this integral seems baffling at first sight, we can notice a simple fact that will allow us to figure out the answer for the definite integral, without actually having to integrate it all. We know that functions can be even, odd, or neither based on:

If $f(-x) = f(x)$ then f is even

If $f(-x) = -f(x)$ then f is odd

Or, if it does not satisfy any of these then f is neither.

Let us examine our integral now:

$$f(x) = \frac{x^2 \tan(x)}{(1+\cos^4(x))}$$
$$f(-x) = \frac{(-x)^2 \tan(-x)}{(1+\cos^4(-x))}$$
$$= \frac{-x^2 \tan(x)}{(1+\cos^4(x))}$$

Using the fact that $\tan(-x) = -\tan(x)$ and that $\cos(-x) = \cos(x)$

This satisfies our condition for an odd function. We also know that generally, for any continuous smooth function:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Furthermore, if $a = -b$, and taking for a fact that the integral of an odd function is even, we can say:

$$\int_{-b}^{b} f(x)\,dx = F(b) - F(-b)$$
$$= F(b) - F(b)$$
$$= 0$$

Thus, we can deduce that

$$\int_{-\pi/4}^{\pi/4} \frac{x^2 \tan(x)}{(1+\cos^4 x)} dx = \boxed{0}$$

Answer 37E.

Given $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\, d\theta$

Put $\tan\theta = t$

$\Rightarrow \sec^2\theta\, d\theta = dt$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi/4 \Rightarrow t = 1$

$$\int_0^{\pi/4} \tan^3\theta \sec^2\theta\, d\theta = \int_0^1 t^3\, dt$$
$$= \left[\frac{t^4}{4}\right]_0^1$$
$$= \frac{1}{4}$$

Answer 38E.

Given $\int_{\pi/6}^{\pi/3} \frac{\sin\theta\cot\theta}{\sec\theta}\, d\theta$

$$= \int_{\pi/6}^{\pi/3} \sin\theta\cos\theta \cdot \frac{\cos\theta}{\sin\theta}\, d\theta$$
$$= \int_{\pi/6}^{\pi/3} \cos^2\theta\, d\theta$$
$$= \int_{\pi/6}^{\pi/3} \left[\frac{1+\cos 2\theta}{2}\right] d\theta$$
$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/3}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{3}-\frac{\pi}{6}\right) + \frac{1}{2}\left(\sin\frac{2\pi}{3} - \sin\frac{2\pi}{6}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$$
$$= \frac{\pi}{12}$$

Answer 39E.

Consider the following integral:

$$\int \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta}\, d\theta.$$

Let $u = \sec\theta$.

Differentiate both sides of the above equation with respect to *x*.

$$du = \sec\theta\tan\theta \cdot d\theta$$
$$\int \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta}\, d\theta = \int \frac{1}{u^2 - u}\, du$$
$$= \int \frac{1}{u(u-1)} \cdot du$$

Since, the degree of the numerator is less than the degree of the denominator, and the denominator has two linear factors, the partial fraction decomposition of the integrand has the following form:

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$
$$= \frac{A(u-1)+Bu}{u(u-1)}$$
$$= \frac{(A+B)u - A}{u(u-1)}$$

Compare both sides of the equation to get the following system of equations for A and B.

$$A+B=0$$
$$-A \quad =1$$

From the system of equations, $A=-1$, and $-1+B=0 \Rightarrow B=1$.

So, the integrand can be written as follows:

$$\frac{1}{u(u-1)}=\frac{A}{u}+\frac{B}{u-1}$$
$$=\frac{-1}{u}+\frac{1}{u-1}$$

Integrate further as follows:

$$\int \frac{\sec\theta\tan\theta}{\sec^2\theta-\sec\theta}d\theta=\int\frac{1}{u(u-1)}\cdot du$$
$$=\int\left(\frac{-1}{u}+\frac{1}{u-1}\right)\cdot du$$
$$=\int\left(\frac{-1}{u}\right)\cdot du+\int\left(\frac{1}{u-1}\right)\cdot du$$
$$=-\ln u+\ln(u-1)+C$$

$=-\ln(\sec\theta)+\ln(\sec\theta-1)+C$ Since, $u=\sec\theta$

Therefore, the value of the integral $\int \frac{\sec\theta\tan\theta}{\sec^2\theta-\sec\theta}d\theta$ is,

$$\boxed{\int \frac{\sec\theta\tan\theta}{\sec^2\theta-\sec\theta}d\theta=-\ln(\sec\theta)+\ln(\sec\theta-1)+C}.$$

Answer 40E.

Consider the following integral:

$$\int\frac{1}{\sqrt{4y^2-4y-3}}dy.$$

Evaluate the given integral.

$$\int\frac{1}{\sqrt{4y^2-4y-3}}dy=\int\frac{1}{\sqrt{4y^2-4y+1-4}}dx$$
$$=\int\frac{1}{\sqrt{(2y-1)^2-4}}dx$$
$$=\frac{1}{2}\int\frac{1}{\sqrt{t^2-2^2}}dt \qquad [\text{Let } 2y-1=t \Rightarrow 2dy=dt]$$
$$=\frac{1}{2}\ln\left|t+\sqrt{t^2-2^2}\right|+C \qquad \left[\text{Since } \int\frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|\right]$$
$$=\frac{1}{2}\ln\left|2y-1+\sqrt{(2y-1)^2-2^2}\right|+C \qquad [\text{Replace } t=2y-1]$$
$$=\frac{1}{2}\ln\left|2y-1+\sqrt{4y^2-4y-3}\right|+C$$

Therefore, $\int\frac{1}{\sqrt{4y^2-4y-3}}dx=\boxed{\frac{1}{2}\ln\left|2y-1+\sqrt{4y^2-4y-3}\right|+C}$.

Answer 41E.

We require to evaluate $\int\theta\tan^2\theta d\theta$

We have $\int\theta\tan^2\theta d\theta=\int\theta(\sec^2\theta-1)d\theta \qquad (\text{using trigonometry } \tan^2\theta=\sec^2\theta-1)$

$$=\int\theta\sec^2\theta d\theta-\int\theta d\theta$$
$$=\int\theta\sec^2\theta d\theta-\frac{\theta^2}{2} \qquad \left(\text{using } \int x^n dx=\frac{x^{n+1}}{n+1}\right)$$

$$= \theta . \tan\theta - \int 1.\tan\theta d\theta - \frac{\theta^2}{2} + C$$

$$\left(\text{using integration by parts, } \int udv = uv - \int vdu\right)$$

$$= \theta\tan\theta - \ln|\sec\theta| - \frac{\theta^2}{2} + C \quad \left(\text{using } \int \tan x dx = \ln|\sec x|\right)$$

Therefore $\boxed{\int \theta\tan^2\theta d\theta = \theta\tan\theta - \ln|\sec\theta| - \frac{\theta^2}{2} + C}$

Answer 42E.

Consider the following integral:

$$\int \frac{\tan^{-1} x}{x^2} dx$$

Evaluate the integral by using integration by parts.

Write the formula for Integration by parts.

$$\int udv = uv - \int vdu$$

Take $u = \tan^{-1} x$ and $dv = \frac{1}{x^2} dx.$

$$du = \frac{1}{x^2+1} \text{ and } v = -\frac{1}{x}$$

By the formula, calculate $\int \frac{\tan^{-1} x}{x^2} dx$ as shown below:

$$\int \frac{\tan^{-1} x}{x^2} dx = \tan^{-1} x\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\cdot\frac{1}{x^2+1}$$

$$= -\frac{\tan^{-1} x}{x} + \int \frac{1}{x(x^2+1)} \quad \text{...... (1)}$$

The partial fraction decomposition of $\frac{1}{x(x^2+1)}$ is as follows:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{1}{x(x^2+1)} = \frac{Ax^2+A+Bx^2+C}{x(x^2+1)}$$

$$1 = Ax^2 + A + Bx^2 + C$$

$$1 = A + x^2(A+B) + C$$

Compare both sides.

$$A = 1 \text{ and } (A+B) = 0, C = 0$$

$$A + B = 0$$

$$B = -A$$

$$B = -1$$

Substitute these values in $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$.

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{(-1)x+0}{x^2+1}$$
$$= \frac{1}{x} - \frac{x}{x^2+1}$$
$$= -\frac{x}{x^2+1} + \frac{1}{x}$$

Thus, equation (1) is solved as shown below:

$$-\frac{\tan^{-1}x}{x} + \int \frac{1}{x(x^2+1)}$$
$$= -\frac{\tan^{-1}x}{x} + \int\left(-\frac{x}{x^2+1} + \frac{1}{x}\right)dx$$
$$= -\frac{\tan^{-1}x}{x} - \int \frac{x}{x^2+1}dx + \int \frac{1}{x}dx$$
$$= -\frac{\tan^{-1}x}{x} - \frac{1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x}dx$$ Multiply and divide by x
$$= -\frac{\tan^{-1}x}{x} - \frac{1}{2}\ln(x^2+1) + \ln x + C$$ Use $\int \frac{2u}{u^2+1}du = \ln(u^2+1) + C$
$$= -\frac{\tan^{-1}x}{x} + \ln x - \frac{1}{2}\ln(x^2+1) + C$$

Therefore, $\int \frac{\tan^{-1}x}{x^2}dx = \boxed{-\frac{\tan^{-1}x}{x} + \ln x - \frac{1}{2}\ln(x^2+1) + C}$.

Answer 43E.

Consider the equation $\int \frac{\sqrt{x}}{1+x^3}dx$ ...... (1)

This is a nonrational function; it can be changed into a rational function by appropriate substitution.

Let $x^{3/2} = t$

$$\frac{3}{2}x^{1/2}dx = dt$$

$$\sqrt{x}\,dx = \frac{2}{3}dt$$ ...... (2)

Now $x^{3/2} = t$

$$\left(x^{3/2}\right)^{\frac{2}{3}} = (t)^{\frac{2}{3}}$$
$$x = t^{\frac{2}{3}}$$

Cubing on both sides

$$(x)^3 = \left(t^{\frac{2}{3}}\right)^3$$

$$x^3 = t^2$$ ...... (3)

Plug the values of equations (2), (3) in the equation (1)

$$\int \frac{\sqrt{x}}{1+x^3} dx = \int \frac{1}{1+t^2} \frac{2}{3} dt$$

$$= \frac{2}{3} \int \frac{1}{1+t^2} dt$$

$$= \frac{2}{3} \tan^{-1} t + C \qquad \left(\text{Since } \int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C \right)$$

$$= \frac{2}{3} \tan^{-1}\left(x^{3/2}\right) + C \quad \left(\text{Since } t = x^{3/2}\right)$$

Therefore $\int \frac{\sqrt{x}}{1+x^3} dx = \boxed{\frac{2}{3} \tan^{-1}\left(x^{3/2}\right) + C}$

Answer 44E.

Consider the integral

$$\int \sqrt{1+e^x}\, dx$$

As a first step take substitution

$$1+e^x = u^2$$

Differentiating either side

$$e^x dx = 2u du$$

$$dx = \frac{2u}{e^x} du$$

$$= \frac{2u}{\sqrt{u^2-1}} du \quad \left(e^x = \sqrt{u^2-1}\right)$$

Above integral transforms as

$$\int \sqrt{1+e^x}\, dx = \int \sqrt{u^2} \left(\frac{2u}{u^2-1}\right) du$$

$$= 2\int \frac{u^2}{u^2-1} du$$

Consider the rational function

$$\frac{u^2}{u^2-1}$$

Observe that degree of the numerator is not less than that of the denominator.

On dividing

$$\begin{array}{r|l} u^2-1 \,)\; u^2 & (1 \\ u^2-1 & \\ - \quad + & \\ \hline 1 & \end{array}$$

On substituting

$$\int \sqrt{1+e^x}\, dx = 2\int \frac{u^2}{u^2-1} du$$

$$= 2\int \left(1 + \frac{1}{u^2-1}\right) du$$

$$= 2\int du + 2\int \frac{1}{u^2-1} du$$

$$= 2u + 2\int \frac{1}{u^2-1} du$$

Integral formula;

$$\int \frac{dt}{t^2-a^2} = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + C$$

So that

$$\int \sqrt{1+e^x}\,dx = 2u + 2\int \frac{1}{u^2-1}du$$

$$= 2u + 2\left(\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right) + C$$

$$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C$$

$$= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C \qquad \left(u=\sqrt{1+e^x}\right)$$

Therefore

$$\int \sqrt{1+e^x}\,dx = \boxed{2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C}$$

Answer 45E.

We have $\int x^5.e^{-x^3}dx = \int x^3.x^2.e^{-x^3}dx$

Let $x^3 = t \Rightarrow 3x^2dx = dt$

Then $\int x^5.e^{-x^3} = \int t.e^{-t}.\left(\frac{dt}{3}\right)$

$$= \frac{1}{3}\int t.e^{-t}.dt$$

Integrate by parts

$$\int x^5.e^{-x^3}dx = \frac{1}{3}\left[t.\frac{e^{-t}}{-1} - \int 1.\frac{e^{-t}}{-1}dt\right]$$

$$= -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}dt$$

$$= -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C$$

$$= -\frac{1}{3}(t+1)e^{-t} + C$$

$$= \boxed{-\frac{1}{3}e^{-x^3}\left(x^3+1\right) + C}$$

Answer 46E.

Consider the integral $\int \frac{(x-1)e^x}{x^2}dx$

$$\int \frac{(x-1)e^x}{x^2}dx = \int \frac{x}{x^2}e^x dx - \int \frac{1}{x^2}e^x dx$$

$$= \int \frac{1}{x}e^x dx - \int \frac{1}{x^2}e^x dx \quad \text{...... (1)}$$

Let $f(x), g(x)$ are two differential functions, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f(x)dx$$

This is called the formula for the integration by parts.

In another way let $u = f(x), v = g(x)$ then the differentials are

$$du = f'(x)dx, dv = g'(x)dx$$

By the substitution rule formula for the integration by parts becomes

$$\int udv = uv - \int vdu \quad \text{...... (2)}$$

Consider $\int \frac{1}{x} e^x dx$ From equation (1)

$$u = \frac{1}{x} \qquad dv = e^x dx$$

$$du = -\frac{1}{x^2} dx \quad \text{and} \quad \int dv = \int e^x dx$$

$$v = e^x$$

Substitute the above values in the equation (2)

$$\int \frac{1}{x} e^x dx = \frac{1}{x} e^x - \int e^x \left(-\frac{1}{x^2}\right) dx$$

$$= \frac{1}{x} e^x + \int e^x \left(\frac{1}{x^2}\right) dx \quad \text{...... (3)}$$

Plug the value of integration $\int \frac{1}{x} e^x dx$ in equation (1).

$$\int \frac{(x-1)e^x}{x^2} dx = \int \frac{1}{x} e^x dx - \int \frac{1}{x^2} e^x dx$$

$$= \frac{1}{x} e^x + \int e^x \frac{1}{x^2} dx - \int \frac{1}{x^2} e^x dx$$

$$= \frac{1}{x} e^x + C$$

Therefore $\int \frac{(x-1)e^x}{x^2} dx = \boxed{\frac{1}{x} e^x + C}$

Answer 47E.

Consider the following integral:

$$\int x^3 (x-1)^{-4} dx \quad \text{...... (1)}$$

Rewrite equation (1) as follows:

$$\int x^3 (x-1)^{-4} dx = \int \frac{x^3}{(x-1)^4} dx \quad \text{...... (2)}$$

Let $x - 1 = t \Rightarrow x = t + 1$

Plug the values in equation (2).

$$\int \frac{x^3}{(x-1)^4} dx = \int \frac{(t+1)^3}{(t)^4} dt$$

$$= \int \frac{t^3 + 3t^2 + 3t + 1}{t^4} dt$$

$$= \int \left(\frac{t^3}{t^4} + \frac{3t^2}{t^4} + \frac{3t}{t^4} + \frac{1}{t^4}\right) dt$$

$$= \int \left(\frac{1}{t} + 3t^{-2} + 3t^{-3} + t^{-4}\right) dt$$

$$= \int \frac{1}{t} dt + \int 3t^{-2} dt + \int 3t^{-3} dt + \int t^{-4} dt$$

$$= \log t + 3\left(\frac{t^{-2+1}}{-2+1}\right) + 3\left(\frac{t^{-3+1}}{-3+1}\right) + \left(\frac{t^{-4+1}}{-4+1}\right) + C$$

$$= \log t - 3\left(t^{-1}\right) - \frac{3}{2}\left(t^{-2}\right) - \frac{1}{3}\left(t^{-3}\right) + C$$

$$\int \frac{x^3}{(x-1)^4} dx = \log t - 3\left(t^{-1}\right) - \frac{3}{2}\left(t^{-2}\right) - \frac{1}{3}\left(t^{-3}\right) + C$$

Substitute, $x - 1 = t$ in the above equation.

Therefore, $\int \frac{x^3}{(x-1)^4} dx = \boxed{\log(x-1) - 3(x-1)^{-1} - \frac{3}{2}(x-1)^{-2} - \frac{1}{3}(x-1)^{-3} + C}$

Answer 48E.

Given integral is $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

Assume $1-x^2=t^2 \quad \Rightarrow \quad -2x\,dx = 2t\,dt$

If $\quad x \to 0$ then $t \to 1 \quad \Rightarrow \quad x\,dx = -t\,dt$

If $\quad x \to 1$ then $t \to 0$

Therefore $\int_1^0 -t\sqrt{2-t}\,dt = \int_0^1 t\sqrt{2-t}\,dt$

Assume $2-t=p^2 \quad \Rightarrow \quad -dt = 2p\,dp$

If $\quad t \to 0$ then $p \to \sqrt{2}$

If $\quad t \to 1$ then $p \to 1$

Therefore $\int_{\sqrt{2}}^1 \left(2-p^2\right)p\left(-2p\,dp\right)$

$$= \int_{\sqrt{2}}^1 \left(2p-p^3\right)\left(-2p\,dp\right)$$

$$= \int_{\sqrt{2}}^1 \left(2p^4-4p^2\right)dp$$

$$= \left[2\frac{p^5}{5}-4\frac{p^3}{3}\right]_{\sqrt{2}}^1$$

$$= \frac{2}{5}-\frac{4}{3}-\left[\frac{8\sqrt{2}}{5}-\frac{8\sqrt{2}}{3}\right]$$

$$= \frac{2}{5}-\frac{4}{3}-\frac{8\sqrt{2}}{5}+\frac{8\sqrt{2}}{3}$$

$$\boxed{= 0.57516}$$

Answer 49E.

We require to evaluate $\int \frac{1}{x\sqrt{4x+1}}dx$

Substitute $4x+1=t^2 \Rightarrow x = \frac{\left(t^2-1\right)}{4}$

$$\Rightarrow dx = \frac{1}{4}(2t)\,dt$$

$$\Rightarrow dx = \frac{t}{2}dt$$

Thus $\int \frac{1}{x\sqrt{4x+1}}dx = \int \frac{1}{\left(\frac{t^2-1}{4}\right)\sqrt{t^2}}\left(\frac{t}{2}\right)dt$

$\left(\text{substituting } x = \frac{t^2-1}{4}, 4x+1=t^2 \text{ and } dx = \frac{dt}{2}\right)$

$$= \int \frac{4}{\left(t^2-1\right)t}\left(\frac{t}{2}\right)dt$$

$$= \frac{4}{2}\int \frac{1}{t^2-1}dt$$

$$= 2\int \frac{1}{t^2-1^2}dt$$

$$= 2\left[\frac{1}{2(1)}\ln\left|\frac{t-1}{t+1}\right|\right]+C \quad \left(\text{using } \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|\right)$$

$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right|+C \qquad \left(4x+1=t^2 \Rightarrow t=\sqrt{4x+1}\right)$$

Therefore $\boxed{\int \frac{1}{x\sqrt{4x+1}}dx = \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right|+C}$

Answer 50E.

We have to evaluate $\int \frac{1}{x^2\sqrt{4x+1}}dx$

Substitute $\sqrt{4x+1}=t$

Then $4x+1=t^2$

$\Rightarrow 4dx=2tdt$

$\Rightarrow dx=tdt/2$

Thus $$\int \frac{1}{x^2\sqrt{4x+1}}dx=\int \frac{\frac{1}{2}tdt}{\left(\frac{t^2-1}{4}\right)^2 t}$$

$$=8\int \frac{dt}{\left(t^2-1\right)^2}$$

$$=8\int \frac{dt}{(t-1)^2(t+1)^2}$$

Let $$\frac{1}{(t-1)^2(t+1)^2}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2}$$

Then $1=A(t-1)(t+1)^2+B(t+1)^2+C(t+1)(t-1)^2+D(t-1)^2$

Now we put $t=1$ then we get $B=1/4$

$t=-1$ Then we get $D=1/4$

Now $1=A(t-1)(t+1)^2+B(t+1)^2+C(t+1)(t-1)^2+D(t-1)^2$

Or $1=A\left(t^3+t^2-t-1\right)+B\left(t^2+1+2t\right)+C\left(t^3-t^2-t+1\right)+D\left(t^2-2t+1\right)$

Or $1=(A+C)t^3+(A+B-C+D)t^2+(-A+2B-C-2D)t+(-A+B+C+D)$

Equating the coefficients of t^3 and 1,

We have $A+C=0$ .... (1)

And $-A+B+C+D=1$

Or $-A+\frac{1}{4}+C+\frac{1}{4}=1$ [Putting the values of B and D]

Or $-A+C+=1/2$ .... (2)

Adding equations (1) and (2) we get $C=1/4$

Subtracting equation (2) from (1) we get $A=-1/4$

Therefore

$$\int \frac{1}{x^2\sqrt{4x+1}}dx=\frac{8}{4}\int\left[-\frac{1}{t-1}+\frac{1}{(t-1)^2}+\frac{1}{t+1}+\frac{1}{(t+1)^2}\right]dt$$

$$=2\int\left[-\frac{1}{t-1}+(t-1)^{-2}+\frac{1}{t+1}+(t+1)^{-2}\right]dt$$

$$=2\left[-\ln|t-1|-(t-1)^{-1}+\ln|t+1|-(t+1)^{-1}\right]+C$$

$$=\boxed{-2\ln\left|\sqrt{4x+1}-1\right|-\frac{2}{\left(\sqrt{4x+1}-1\right)}+2\ln\left(\sqrt{4x+1}+1\right)-\frac{2}{\left(\sqrt{4x+1}+1\right)}+C}$$

Answer 51E.

We have to evaluate $\int \frac{1}{x\sqrt{4x^2+1}}dx$

Substitute $x=\frac{1}{2}\tan\theta$

Then $dx=\frac{1}{2}\sec^2\theta d\theta$

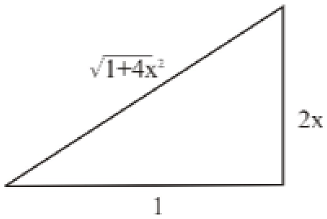

Fig. 1

From figure we have $\csc\theta = \frac{\sqrt{1+4x^2}}{2x}$ and $\cot\theta = \frac{1}{2x}$

Therefore $\int \frac{1}{x\sqrt{4x^2+1}}dx = \int \frac{1}{\frac{\tan\theta}{2}\sqrt{\tan^2\theta+1}}\cdot\frac{\sec^2\theta}{2}d\theta$

$$= \int \frac{1}{\tan\theta.\sec\theta}.\sec^2\theta d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta}d\theta$$

$$= \int \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}d\theta$$

$$= \int \frac{1}{\sin\theta}d\theta$$

$$= \int \csc\theta d\theta$$

$$= \ln\left|\csc\theta - \cot\theta\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{1+4x^2}}{2x} - \frac{1}{2x}\right| + C}$$

Answer 52E.

We have $\int \frac{dx}{x(x^4+1)} = \int \frac{x^3}{x^4(x^4+1)}dx$

Put $x^4 = t \Rightarrow 4x^3dx = dt$

Then $\int \frac{dx}{x(x^4+1)} = \int \frac{\frac{1}{4}dt}{t(t+1)}$

$$= \frac{1}{4}\int \frac{1}{t(t+1)}$$

$$= \frac{1}{4}\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{4}\int \left(\ln|t| - \ln|t+1|\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{t}{t+1}\right| + C$$

$$= \frac{1}{4}\ln\left|\frac{x^4}{x^4+1}\right| + C$$

$$= \boxed{\frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C} \qquad \text{since } \frac{x^4}{x^4+1} \geq 0$$

Answer 53E.

We require to evaluate $\int x^2 \sinh(mx)dx$

Using integration by parts, we have $\int udv = uv - \int vdu$ ...(1)

Take $u = x^2$, $dv = \sinh(mx)dx$

Then $du = 2xdx$, $v = -\frac{\cosh(mx)}{m}$

Then using (1),

$$\int x^2 \sinh(mx)\,dx = x^2\left(-\frac{\cosh(mx)}{m}\right) - \int\left(-\frac{\cosh(mx)}{m}\right)(2x\,dx)$$

$$= -\frac{1}{m}x^2\cosh(mx) + \frac{2}{m}\int x\cosh(mx)\,dx$$

Again using (1), Take $u = x,\ dv = \cosh(mx)\,dx$

Then $du = dx,\ \ v = \frac{\sinh(mx)}{m}dx$

$$\int x^2 \sinh(mx)\,dx = -\frac{1}{m}x^2\cosh(mx) + \frac{2}{m}\left[x\left(\frac{\sinh(mx)}{m}\right) - \int\frac{\sinh(mx)}{m}dx\right]$$

$$= -\frac{1}{m}x^2\cosh(mx) + \frac{2}{m^2}x\sinh(mx) - \frac{2}{m^2}\int\sinh(mx)dx$$

$$= -\frac{1}{m}x^2\cosh(mx) + \frac{2}{m^2}x\sinh(mx) - \frac{2}{m^2}\left(-\frac{\cosh(mx)}{m}\right) + C$$

$$= -\frac{1}{m}x^2\cosh(mx) + \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$$

Therefore $\boxed{\int x^2\sinh(mx)\,dx = -\frac{1}{m}x^2\cosh(mx) + \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C}$

Answer 54E.

We require to evaluate $\int (x+\sin x)^2\,dx$

We have $\int(x+\sin x)^2\,dx = \int\left(x^2+\sin^2 x+2x\sin x\right)dx$

$$= \int x^2dx + \int\sin^2 x\,dx + 2\int x\sin x\,dx$$

$$= \frac{x^3}{3} + \int\sin^2 x\,dx + 2\int x\sin x\,dx \qquad \left(\text{using } \int x^n dx = \frac{x^{n+1}}{n+1}\right)$$

$$= \frac{x^3}{3} + \int\left(\frac{1-co2x}{2}\right)dx + 2\int x\sin x\,dx$$

$$\left(\text{using trignometry, } \cos 2x = 1-2\sin^2 x\right)$$

$$\int(x+\sin x)^2\,dx = \frac{x^3}{3} + \frac{1}{2}\int dx - \frac{1}{2}\int\cos 2xdxdx + 2\int x\sin x\,dx$$

$$= \frac{x^3}{3} + \frac{1}{2}(x) - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + 2\int x\sin x\,dx$$

$$\left(\text{using } \int dx = x, \int\cos 2x\,dx = \frac{\sin 2x}{2}\right)$$

$$= \frac{x^3}{3} + \frac{x}{2} - \frac{\sin 2x}{4} + 2\left[x(-\cos x) - \int(-\cos x)(1)\,dx\right]$$

$$\left(\text{using integrati on by parts, } \int u\,dv = uv - \int v\,du\right)$$

$$= \frac{x^3}{3} + \frac{x}{2} - \frac{\sin 2x}{4} - 2x\cos x + 2\int\cos x\,dx$$

$$= \frac{x^3}{3} + \frac{x}{2} - \frac{\sin 2x}{4} - 2x\cos x + 2\sin x + C \quad \left(\because \int\cos x\,dx = \sin x\right)$$

Therefore $\boxed{\int(x+\sin x)^2\,dx = \frac{x^3}{3} + \frac{x}{2} - \frac{\sin 2x}{4} - 2x\cos x + 2\sin x + C}$

Answer 55E.

Consider the following integral:

$$\int \frac{dx}{x+x\sqrt{x}} \quad \text{......} (1)$$

Let, $\sqrt{x}=t \Rightarrow x=t^2$.

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

Rewrite equation (1) as shown below:

$$\begin{aligned}
\int \frac{dx}{x+x\sqrt{x}} &= \int \frac{dx}{\sqrt{x}\left(x+\sqrt{x}\right)} \\
&= \int \frac{2dt}{\left(t^2+t\right)} \qquad \left(\text{Since } \sqrt{x}=t, \frac{1}{\sqrt{x}}dx = 2dt\right) \\
&= \int \frac{2dt}{t\cdot(t+1)} \\
&= 2\int \left(\frac{1}{t} - \frac{1}{(t+1)}\right)dt
\end{aligned}$$

$$\begin{aligned}
&= 2\int \frac{1}{t}dt - 2\int \frac{1}{(t+1)}dt \\
&= 2\ln(t) - 2\ln(t+1) + C \\
&= 2\left(\frac{\ln(t)}{\ln(t+1)}\right) + C \\
&= 2\left(\frac{\ln\left(\sqrt{x}\right)}{\ln\left(\sqrt{x}+1\right)}\right) + C
\end{aligned}$$

Therefore, $\int \frac{dx}{x+x\sqrt{x}} = \boxed{2\left(\frac{\ln\left(\sqrt{x}\right)}{\ln\left(\sqrt{x}+1\right)}\right) + C}$.

Answer 56E.

Consider the following integration:

$$\int \frac{dx}{\sqrt{x}+x\sqrt{x}} \quad \text{......} (1)$$

Let, $\sqrt{x}=t \Rightarrow x=t^2$.

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

Rewrite equation (1) as shown below:

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}+x\sqrt{x}} &= \int \frac{dx}{\sqrt{x}(x+1)} \\
&= \int \frac{2dt}{\left(t^2+1\right)} \qquad \left(\text{Since } \sqrt{x}=t, \frac{1}{\sqrt{x}}dx = 2dt\right) \\
&= 2\tan^{-1}\left(\frac{t}{1}\right) \\
&= 2\tan^{-1}(t) \\
&= 2\tan^{-1}\left(\sqrt{x}\right) + C
\end{aligned}$$

Therefore, $\int \frac{dx}{\sqrt{x}+x\sqrt{x}} = \boxed{2\tan^{-1}\left(\sqrt{x}\right) + C}$.

Answer 57E.

We have $\int x\sqrt[3]{x+c}\ dx = \int x(x+c)^{1/3}\,dx$

Let $(x+c)^{1/3} = t$

$\Rightarrow x+c = t^3$

$\Rightarrow x = t^3 - c$

$\Rightarrow dx = 3t^2 dt$

Then $\int x\sqrt[3]{x+c}\ dx = \int (t^3 - c)\, t.3t^2 dt$

$= 3\int (t^6 - t^3 c)\ dt$

$= 3\frac{t^7}{7} - 3c\,\frac{t^4}{4} + K$

$= \boxed{\frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + K}$

Answer 58E.

Consider the integral,

$$\int \frac{x\ln x}{\sqrt{x^2-1}}dx$$

The objective is to evaluate the integral.

Put $x = \sec t$

$dx = \sec t \tan t dt$

Substitute *x* and *dx* values, we get,

$$\int \frac{x\ln x}{\sqrt{x^2-1}}dx = \int \frac{\sec t \ln(\sec t)}{\sqrt{\sec^2 t - 1}} \cdot \sec t \tan t dt$$

$$= \int \frac{\sec t \ln(\sec t)}{\tan t} \cdot \sec t \tan t\, dt$$

$$= \int \sec^2 t \cdot \ln(\sec t) dt$$

$$= \ln(\sec t)\int \sec^2 t dt - \int\left(\frac{d}{dt}(\ln(\sec t))\int \sec^2 t dt\right)dt$$

On preceding the next step as follows:

$$\int \frac{x\ln x}{\sqrt{x^2-1}}dx = \ln(\sec t)\tan t - \int\left(\frac{1}{\sec t}\cdot \sec t \tan t \cdot \tan t\right)dt$$

$$= \tan t \ln(\sec t) - \int \tan^2 t dt$$

$$= \tan t \ln(\sec t) - \int(\sec^2 t - 1)dt$$

$$= \tan t \ln(\sec t) - \int \sec^2 t dt + \int 1 \cdot dt$$

$$= \tan t \ln(\sec t) - \tan t + t + C$$

$$= \sqrt{\sec^2 t - 1}\ln(\sec t) - \sqrt{\sec^2 t - 1} + t + C$$

$$= \sqrt{x^2-1}\ln x - \sqrt{x^2-1} + \sec^{-1}(x) + C \qquad [\text{Since, } \sec t = x]$$

$$= \sqrt{x^2-1}(\ln x - 1) + \sec^{-1}(x) + C$$

Therefore, the evaluated integral is,

$$\int \frac{x\ln x}{\sqrt{x^2-1}}dx = \boxed{\sqrt{x^2-1}(\ln x - 1) + \sec^{-1}(x) + C}.$$

Answer 59E.

Consider the following integral:

$$\int \cos x \cos^3(\sin x)\,dx \quad ...... (1)$$

Let, $\sin x = t \Rightarrow \cos x dx = dt$.

$$\int \cos x \cos^3(\sin x)\,dx = \int \cos^3(t)\,dt \quad (\text{Since } \sin x = t, \cos x dx = dt)$$

Apply the reduction formula for $\int \cos^n(x)\,dx$.

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$$\int \cos^3(t)\,dt = \frac{1}{3}\cos^{3-1} t \sin t + \frac{3-1}{3}\int \cos^{3-2} t dt$$

$$= \frac{1}{3}\cos^{3-1} t \sin t + \frac{2}{3}\int \cos t dt$$

$$= \frac{1}{3}\cos^2 t \sin t + \frac{2}{3}\int \cos t dt$$

$$= \frac{1}{3}\cos^2 t \sin t + \frac{2}{3}\sin t$$

$$= \frac{1}{3}\cos^2(\sin x)\sin(\sin x) + \frac{2}{3}\sin(\sin x) \quad (\text{Since } \sin x = t)$$

Therefore, $\int \cos x \cos^3(\sin x)\,dx = \boxed{\frac{1}{3}\cos^2(\sin x)\sin(\sin x) + \frac{2}{3}\sin(\sin x)}$.

Answer 60E.

Consider the following integral:

$$\int \frac{dx}{x^2\sqrt{4x^2-1}} \quad ...... (1)$$

Let, $\frac{\sqrt{4x^2-1}}{x} = t$.

Simplify as shown below:

$$\frac{x\frac{d}{dx}\left(\sqrt{4x^2-1}\right) - \left(\sqrt{4x^2-1}\right)\frac{d}{dx}(x)}{x^2}dx = dt$$

$$\frac{x\left(\frac{8x}{2\sqrt{4x^2-1}}\right) - \left(\sqrt{4x^2-1}\right)}{x^2}dx = dt$$

$$\frac{\left(\frac{4x^2}{\sqrt{4x^2-1}}\right) - \left(\sqrt{4x^2-1}\right)}{x^2}dx = dt$$

$$\frac{4x^2 - (4x^2-1)}{x^2\sqrt{4x^2-1}}dx = dt$$

$$\frac{1}{x^2\sqrt{4x^2-1}}dx = dt$$

Consider, $\int \frac{dx}{x^2\sqrt{4x^2-1}} = \int dt$.

$$= t + C$$

$$= \frac{\sqrt{4x^2-1}}{x} + C$$

Therefore, $\int \frac{dx}{x^2\sqrt{4x^2-1}} = \boxed{\frac{\sqrt{4x^2-1}}{x} + C}$

Answer 6E.

Consider the integration $\int \frac{d\theta}{1+\cos\theta}$

Rationalize the numerator and denominator with $(1-\cos\theta)$

$$\int \frac{d\theta}{1+\cos\theta} = \int \frac{(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} d\theta$$

$$= \int \frac{(1-\cos\theta)}{(1-\cos^2\theta)} d\theta \qquad \left(\text{Since } 1-\cos^2\theta = \sin^2\theta\right)$$

$$= \int \frac{(1-\cos\theta)}{\sin^2\theta} d\theta$$

$$= \int \frac{1}{\sin^2\theta} d\theta - \int \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int \csc^2\theta\, d\theta - \int \csc\theta \cot\theta\, d\theta \qquad \left(\begin{array}{l}\text{Since } \frac{1}{\sin^2\theta} = \csc^2\theta \\ \frac{\cos\theta}{\sin\theta} = \cot\theta \\ \frac{1}{\sin\theta} = \csc\theta\end{array}\right)$$

$$= -\cot\theta + \csc\theta + C$$

$$= \csc\theta - \cot\theta + C$$

Therefore $\int \frac{d\theta}{1+\cos\theta} = \boxed{\csc\theta - \cot\theta + C}$

Answer 62E.

Consider the integration $\int \frac{d\theta}{1+\cos^2\theta}$

Multiply the numerator and denominator by $\sec^2\theta$.

$$\int \frac{1}{1+\cos^2\theta} d\theta = \int \frac{\sec^2\theta}{\sec^2\theta\left(1+\cos^2\theta\right)} d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^2\theta + \sec^2\theta\cos^2\theta} d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^2\theta + \sec^2\theta \frac{1}{\sec^2\theta}} d\theta \qquad \left(\text{Since } \sec^2\theta = \frac{1}{\cos^2\theta}\right)$$

$$= \int \frac{\sec^2\theta}{\sec^2\theta + 1} d\theta$$

$$= \int \frac{\sec^2\theta}{1+\tan^2\theta + 1} d\theta \qquad \left(\text{Since } \sec^2\theta - \tan^2\theta = 1\right)$$

$$= \int \frac{\sec^2\theta}{2+\tan^2\theta} d\theta$$

Now $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$$\int \frac{d\theta}{1+\cos^2\theta} d\theta = \int \frac{\sec^2\theta}{2+\tan^2\theta} d\theta$$

$$= \int \frac{1}{t^2 + \left(\sqrt{2}\right)^2} dt \qquad \left(\text{Since } \left(\sqrt{2}\right)^2 = 2\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C \qquad \left(\text{Since } \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}} \tan\theta\right) + C$$

Therefore $\int \frac{d\theta}{1+\cos^2\theta} = \boxed{\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\tan\theta\right) + C}$

Answer 63E.

Consider the integral,

$$\int \sqrt{x} e^{\sqrt{x}} dx$$

The objective is to evaluate the integral.

Put $\sqrt{x} = t$

$$x = t^2$$

$$dx = 2t dt$$

Substitute x and dx to obtain,

$$\begin{aligned}\int \sqrt{x} e^{\sqrt{x}} dx &= \int t \cdot e^t \cdot 2t dt \\ &= 2\int t^2 e^t dt \\ &= 2\left[t^2 \int e^t dt - \int\left(\frac{d}{dt}(t^2)\int e^t dt\right)dt\right] \\ &= 2\left[t^2 e^t - \int 2te^t dt\right]\end{aligned}$$

On proceeding the next step as follows:

$$\begin{aligned}&= 2t^2 e^t - 2\cdot 2\int te^t dt \\ &= 2t^2 e^t - 4\left[t\int e^t dt - \int\left(\frac{d}{dt}(t)\int e^t dt\right)dt\right] \\ &= 2t^2 e^t - 4\left[te^t - \int 1\cdot e^t dt\right] \\ &= 2t^2 e^t - 4te^t + 4\int e^t dt \\ &= 2t^2 e^t - 4te^t + 4e^t + C \\ &= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C \quad \left[\text{Since, } t = \sqrt{x}\right] \\ &= 2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C\end{aligned}$$

Therefore, the evaluated integral is,

$$\int \sqrt{x} e^{\sqrt{x}} dx = \boxed{2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C}.$$

Answer 64E.

Given $\int \frac{1}{\sqrt{\sqrt{x}+1}} dx$

Put $\sqrt{\sqrt{x}+1} = t$

$\Rightarrow \sqrt{x}+1 = t^2$

$\Rightarrow \sqrt{x} = t^2 - 1$

$\Rightarrow \frac{1}{2\sqrt{x}} dx = 2t\,dt$

$\Rightarrow dx = 4\sqrt{x}\,t\,dt$

$\Rightarrow dx = 4\left(t^2 - 1\right)t\,dt$

$$\begin{aligned}\int \frac{1}{\sqrt{\sqrt{x}+1}} dx &= \int \frac{1}{t} 4\left(t^2 - 1\right)t\,dt \\ &= 4\int\left(t^2 - 1\right)dt \\ &= 4\left[\frac{t^3}{3} - t\right] + C \\ &= \frac{4}{3}\left(\sqrt{\sqrt{x}+1}\right)^3 - 4\sqrt{\sqrt{x}+1} + C\end{aligned}$$

Answer 65E.

Consider the following integration:

$$\int \frac{\sin 2x}{1+\cos^4 x}dx \text{ .... (1)}$$

Let, $\cos^2 x = t.$

Then, $-2\cos x \sin x dx = dt$

$-\sin 2x = dt.$

Rewrite equation (1) as shown below:

$$\int \frac{\sin 2x}{1+\cos^4 x}dx = -\int \frac{-\sin 2x}{1+\left(\cos^2 x\right)^2}dx$$

$$= -\int \frac{dt}{1+(t)^2}$$

$$= -\tan^{-1}(t)+c \qquad \left(\text{Since } \int \frac{dx}{1+(x)^2}\right)$$

$$= -\tan^{-1}\left(\cos^2 x\right)+c \qquad \left(t = \cos^2 x\right)$$

Therefore, $\int \frac{\sin 2x}{1+\cos^4 x}dx = \boxed{-\tan^{-1}\left(\cos^2 x\right)+c}.$

Answer 66E.

Consider the integration $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x}dx$. .... (1)

Rewrite the (1) as:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x}dx = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{2\sin x \cos x}dx$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin 2x}dx \qquad (\text{Since } \sin 2x = 2\sin x \cos x)$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc(2x)\ln(\tan x)dx \qquad \left(\csc(2x) = \frac{1}{\sin(2x)}\right)$$

Let $\ln(\tan x) = t$

$$\frac{1}{\tan x}\frac{d}{dx}(\tan x) = dt$$

$$\frac{1}{\tan x}\left(\sec^2 x\right) = dt$$

$$\frac{\cos x}{\sin x}\left(\frac{1}{\cos^2 x}\right) = dt$$

$$\frac{2}{2\sin x \cos x} = dt$$

$$\frac{2}{\sin 2x} = dt$$

$$2\csc(2x) = dt$$

Limits: if $x = \frac{\pi}{4}$, then $t = \ln\left(\tan\left(\frac{\pi}{4}\right)\right) \Rightarrow t = \ln(1) \Rightarrow t = 0$

If $x = \frac{\pi}{3}$, then $t = \ln\left(\tan\left(\frac{\pi}{3}\right)\right) \Rightarrow t = \ln\left(\sqrt{3}\right)$

Now

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\csc(2x)\ln(\tan x)\,dx$$

$$= \int_0^{\ln(\sqrt{3})} t\,dt$$

$$= \left(\frac{t^2}{2}\right)_0^{\ln(\sqrt{3})}$$

$$= \frac{\left(\ln\sqrt{3}\right)^2}{2} - 0$$

$$= \frac{\left(\ln\sqrt{3}\right)^2}{2} + C$$

Therefore $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x \cos x} dx = \boxed{\frac{\left(\ln\sqrt{3}\right)^2}{2} + C}$

Answer 67E.

We have $\int \frac{1}{\sqrt{x+1}+\sqrt{x}} dx = \int \frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} dx$ [Rationalizing]

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x} dx$$

$$= \int \sqrt{x+1} - \sqrt{x}\; dx$$

$$= \frac{(x+1)^{3/2}}{3/2} - \frac{x^{3/2}}{3/2} + C$$

$$= \boxed{\frac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C}$$

Answer 68E.

$$\frac{x^2}{x^6+3x^3+2} = \frac{x^2}{x^6+2x^3+x^3+2}$$

$$= \frac{x^2}{x^3\left(x^3+2\right)+1\left(x^3+2\right)}$$

$$= \frac{x^2}{\left(x^3+1\right)\left(x^3+2\right)}$$

$$= \frac{x^2}{x^3+1} - \frac{x^2}{x^3+2}$$

$$\int \frac{x^2}{x^6+3x^3+2} dx = \int \frac{x^2}{x^3+1} dx - \int \frac{x^2}{x^3+2} dx$$

Assume $x^3+1=t$

$3x^2\,dx = dt, 3x^2 dx = dp$

$\Rightarrow \quad x^2 dx = \frac{dt}{3}, x^2 dx = \frac{dp}{3}$

$$= \int \frac{\frac{dt}{3}}{t} - \frac{\frac{dp}{3}}{p}$$

$$= \frac{1}{3}\log t - \frac{1}{3}\log p + C$$

$$= \frac{1}{3}\log\left(x^3+1\right) - \frac{1}{3}\log\left(x^3+2\right) + C$$

$\boxed{\text{Therefore} \quad \frac{1}{3}\log\left(\frac{x^3+1}{x^3+2}\right) + C}$

Answer 69E.

Consider the integration $\int_{1}^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx$. ...... (1)

Let $x = \tan t$

$dx = \sec^2 t dt$

If $x = 1$ then $1 = \tan t \Rightarrow t = \frac{\pi}{4}$

If $x = \sqrt{3}$ then $\sqrt{3} = \tan t \Rightarrow t = \frac{\pi}{3}$.

$$\int_{1}^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{1+\tan^2 t}}{\tan^2 t} \sec^2 t dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\sec^2 x}}{\tan^2 t} \sec^2 t dt \qquad \left(\text{Since } 1+\tan^2 t = \sec^2 x\right)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec x}{\frac{\sin^2 t}{\cos^2 t}} \frac{1}{\cos^2 t} t dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2 t \sec t dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\cot^2 t + 1\right) \sec t dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\cot^2 t \sec t + \sec t\right) dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\cot t \csc t + \sec t\right) dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\cos t}{\sin t} \frac{1}{\sin t} + \sec t\right) dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\cos t}{\sin^2 t} + \sec t\right) dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\cos t}{\sin^2 t}\right) dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\sec t\right) dt \quad \text{...... (1)}$$

Now consider $\int\left(\frac{\cos t}{\sin^2 t}\right)dt$

Let $\sin t = s$

$\cos t dt = ds$

$$\int\left(\frac{\cos t}{\sin^2 t}\right)dt = \int\frac{ds}{s^2}$$
$$= -\frac{1}{s}$$
$$= -\frac{1}{\sin t}$$

Therefore $\int\left(\frac{\cos t}{\sin^2 t}\right)dt = -\frac{1}{\sin t}$ ...... (2)

Substitute (2) in (1).

$$\int_1^{\sqrt{3}}\frac{\sqrt{1+x^2}}{x^2}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{\cos t}{\sin^2 t}\right)dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\sec t)dt$$
$$= \left(-\frac{1}{\sin t}\right)_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \left(\ln|\sec t + \tan t|\right)_{\frac{\pi}{4}}^{\frac{\pi}{3}} + c$$
$$= \left(-\frac{1}{\sin\left(\frac{\pi}{3}\right)} + \frac{1}{\sin\left(\frac{\pi}{4}\right)}\right)$$
$$+\left(\ln\left|\sec\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right| - \ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right|\right) + c$$
$$= \left(-\frac{2}{\sqrt{3}} + \sqrt{2}\right) + \ln\left(2+\sqrt{3}\right) - \ln\left(\sqrt{2}+1\right) + C$$
$$= \left(\sqrt{2} - \frac{2}{\sqrt{3}}\right) + \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right) + C \quad \left(\text{Since } \ln\left(\frac{a}{b}\right) = \ln a - \ln b\right)$$

Answer 70E.

We have to evaluate $\int\frac{1}{1+2e^x - e^{-x}}dx$

Multiply numerator and denominator by e^x, we get $\int\frac{e^x}{e^x + 2e^{2x} - 1}dx$

Substitute $e^x = t$
$e^x dx = dt$

We have $$\int\frac{e^x}{e^x + 2e^{2x} - 1}dx = \int\frac{dt}{t + 2t^2 - 1}$$
$$= \int\frac{dt}{2t^2 + t - 1}$$
$$= \int\frac{dt}{(2t-1)(t+1)}$$

Let $\frac{1}{(2t-1)(t+1)} = \frac{A}{(2t-1)} + \frac{B}{(t+1)}$

Then $1 = A(t+1) + B(2t-1)$

Put $t = -1$ $\quad 1 = B.(-3) \Rightarrow B = -\frac{1}{3}$

Put $t = \frac{1}{2}$ $\quad 1 = A\left(\frac{3}{2}\right) \Rightarrow A = \frac{2}{3}$

Then

$$\int \frac{e^x}{e^x+2e^{2x}-1}dx = \int\left[\frac{2}{3(2t-1)} - \frac{1}{3(t+1)}\right]dt$$

$$= \frac{1}{3}.\ln|2t-1| - \frac{1}{3}\ln|t+1| + C$$

$$= \frac{1}{3}.\ln\left|\frac{2t-1}{t+1}\right| + C$$

$$= \boxed{\frac{1}{3}\ln\left|\frac{2e^x-1}{e^x+1}\right| + C}$$

Answer 71E.

We have to evaluate $\int \frac{e^{2x}}{1+e^x}dx$

Substitute $e^x = t$

Then $e^x dx = dt$

Therefore $\int \frac{e^{2x}}{1+e^x}dx = \int \frac{t\,dt}{1+t}$

$$= \int\left(1 - \frac{1}{1+t}\right)dt$$

$$= t - \ln|1+t| + C$$

$$= \boxed{e^x - \ln\left(1+e^x\right) + C}$$

Answer 72E.

We have $\int \frac{\ln(x+1)}{x^2}dx = \int \ln(x+1).\frac{1}{x^2}dx$

Integrating by parts

$$\int \ln(x+1).\frac{1}{x^2}dx = \ln(x+1)\left(-\frac{1}{x}\right) - \int \frac{1}{x+1}\left(-\frac{1}{x}\right)dx$$

$$= -\frac{\ln(x+1)}{x} + \int \frac{1}{x(x+1)}dx$$

$$= -\frac{\ln(x+1)}{x} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= -\frac{\ln(x+1)}{x} + \ln|x| - \ln|(x+1)| + C$$

$$= \boxed{-\left(1+\frac{1}{x}\right)\ln(x+1) + \ln|x| + C}$$

Answer 73E.

Consider the integration $\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx$. ...... (1)

Rewrite the (1) as shown below:

$$\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx \quad \text{...... (2)}$$

Now consider $\int \frac{x}{\sqrt{1-x^2}}dx$

Let $1-x^2 = t$

$$-2xdx = dt$$

$$xdx = -\frac{dt}{2}$$

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{\left(-\frac{dt}{2}\right)}{\sqrt{t}}dx$$

$$= -\frac{1}{2}\int \frac{(dt)}{\sqrt{t}}dx$$

$$= -\frac{1}{2}\int (t)^{-\frac{1}{2}}dt$$

$$= -\frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= -\frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{\frac{1}{2}}$$

$$= -\sqrt{t}$$

$$= -\sqrt{1-x^2}$$

Therefore $\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}$ ...... (3)

Now consider $\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx$

Let $\sin^{-1}(x) = t$

$$\frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx = \int t\,dt$$

$$= \frac{t^2}{2}$$

$$= \frac{\left(\sin^{-1}x\right)^2}{2}$$

Therefore $\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx = \frac{\left(\sin^{-1}x\right)^2}{2}$ ...... (4)

Plug the values (3) and (4) in (2).

$$\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx$$

$$= -\sqrt{1-x^2} + \frac{\left(\sin^{-1}x\right)^2}{2} + C$$

Therefore $\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx = \boxed{-\sqrt{1-x^2} + \frac{\left(\sin^{-1}x\right)^2}{2} + C}$

Answer 74E.

Consider the following integration:

$$\int \frac{4^x+10^x}{2^x}dx$$

Simplify as shown below:

$$\int \frac{4^x+10^x}{2^x}dx = \int\left(\frac{4^x}{2^x}+\frac{10^x}{2^x}\right)dx$$

$$= \int\left(\left(\frac{4}{2}\right)^x+\left(\frac{10}{2}\right)^x\right)dx \quad \left(\text{Since } \frac{a^m}{b^m}=\left(\frac{a}{b}\right)^m\right)$$

$$= \int\left((2)^x+(5)^x\right)dx$$

$$= \int (2)^x\,dx+(5)^x\,dx$$

$$= \frac{(2)^x}{\ln 2}+\frac{(5)^x}{\ln 5}+C$$

Therefore, $\int \frac{4^x+10^x}{2^x}dx = \boxed{\frac{(2)^x}{\ln 2}+\frac{(5)^x}{\ln 5}+C}$.

Answer 75E.

We have to evaluate $\int \frac{1}{(x-2)(x^2+4)}dx$

Let $\quad \frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2}+\frac{Bx+C}{x^2+4}$

Then $\quad 1 = A(x^2+4)+(Bx+C)(x-2)$

Or $\quad 1 = x^2(A+B)+x(-2B+C)+(4A-2C)$

Equating the coefficients

x^2: $\quad 0 = A + B \quad$ --- (1)

x : $\quad 0 = -2B + C \quad$ --- (2)

Constant $\quad 1 = 4A - 2C \quad$ --- (3)

From equation (1) and (2)

$$0 = 2A+C \Rightarrow C = -2A$$

From equation (3)

$$1 = 4A-2(-2A) \Rightarrow 1 = 8A \Rightarrow A = \frac{1}{8}$$

$$\Rightarrow C = -\frac{1}{4} \text{ and } B = -A = -\frac{1}{8}$$

Therefore

$$\int \frac{1}{(x-2)(x^2+4)}dx = \int\left[\frac{1}{8(x-2)}+\frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\right]dx$$

$$= \frac{1}{8}\int\frac{1}{x-2}dx-\left(\frac{1}{2}\right)\frac{1}{8}\int\frac{2x}{x^2+4}dx-\frac{1}{4}\int\frac{dx}{x^2+4}$$

$$= \frac{1}{8}\ln|x-2|-\frac{1}{16}\ln|x^2+4|-\frac{1}{4}\times\frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

$$= \boxed{\frac{1}{8}\ln|x-2|-\frac{1}{16}\ln(x^2+4)-\frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right)+C}$$

Answer 76E.

Consider the following integral:

$$\int \frac{dx}{\sqrt{x}\left(2+\sqrt{x}\right)^4}$$

Let, $\sqrt{x}=t.$

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

Simplify as shown below:

$$\int \frac{dx}{\sqrt{x}\left(2+\sqrt{x}\right)^4} = \int \frac{2dt}{(2+t)^4}$$

$$= 2\int (2+t)^{-4}dt$$

$$= 2\frac{(2+t)^{-4+1}}{-4+1}$$

$$= 2\frac{(2+t)^{-3}}{-3}$$

$$= -\frac{2}{3(2+t)^3}$$

Therefore, $\int \frac{dx}{\sqrt{x}\left(2+\sqrt{x}\right)^4} = \boxed{-\frac{2}{3(2+t)^3}}.$

Answer 77E.

Consider the integration $\int \frac{xe^x}{\sqrt{1+e^x}}dx$ . ... (1)

Let $\sqrt{1+e^x}=t$

$$1+e^x = t^2$$

$$e^x = t^2 - 1$$

$$x = \ln\left(t^2-1\right)$$

Differentiate, $\sqrt{1+e^x}=t$

$$\frac{e^x}{2\sqrt{1+e^x}}dx = dt$$

$$\frac{e^x}{\sqrt{1+e^x}}dx = 2dt$$

Substitute the above values in (1).

$$\int \frac{xe^x}{\sqrt{1+e^x}}dx = \int 2\ln\left(t^2-1\right)dt \quad ...... (2)$$

Solve the equation (2), using by parts.

$$\int u\,dv = uv - \int v\,du$$

Let $u = \ln\left(t^2-1\right)$ $\quad dv = dt$

$$du = \frac{2t}{t^2-1} \qquad v = t$$

$$\int 2\ln\left(t^2-1\right)dt = 2\left[t\ln\left(t^2-1\right)\right] - \int t\left(\frac{2t}{t^2-1}\right)dt$$

$$= 2\left[t\ln\left(t^2-1\right)\right] - \int\left(\frac{2t^2}{t^2-1}\right)dt \quad \text{...... (3)}$$

Consider $\int\left(\frac{2t^2}{t^2-1}\right)dt = 2\int\frac{t^2-1+1}{t^2-1}dt$

$$= 2\int\frac{t^2-1}{t^2-1}dt + 2\int\frac{1}{t^2-1}dt$$

$$= 2\int dt + 2\int\frac{1}{t^2-1}dt$$

$$= 2t + 2\frac{1}{2(1)}\ln\left|\frac{t-1}{t+1}\right|$$

$$= 2t + \ln\left|\frac{t-1}{t+1}\right| \quad \text{...... (4)}$$

Plug the equation (4) in (3).

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$$\int 2\ln\left(t^2-1\right)dt = 2\left[t\ln\left(t^2-1\right)\right] - \int\left(\frac{2t^2}{t^2-1}\right)dt$$

$$= 2\left[t\ln\left(t^2-1\right)\right] - 2t - \ln\left|\frac{t-1}{t+1}\right| \quad \text{...... (5)}$$

Plug the equation (5) in (2).

$$\int\frac{xe^x}{\sqrt{1+e^x}}dx = 2\left[t\ln\left(t^2-1\right)\right] - 2t - \ln\left|\frac{t-1}{t+1}\right|$$

$$= 2t\left(\ln\left(t^2-1\right)-1\right) - \ln\left|\frac{t-1}{t+1}\right|$$

$$= 2\left(\sqrt{1+e^x}\right)\left[\ln\left(\left(\sqrt{1+e^x}\right)^2-1\right)\right] - \ln\left|\frac{\left(\sqrt{1+e^x}\right)-1}{\left(\sqrt{1+e^x}\right)+1}\right|$$

Therefore $\int\frac{xe^x}{\sqrt{1+e^x}}dx = \boxed{2\left(\sqrt{1+e^x}\right)\left[\ln\left(\left(\sqrt{1+e^x}\right)^2-1\right)\right] - \ln\left|\frac{\left(\sqrt{1+e^x}\right)-1}{\left(\sqrt{1+e^x}\right)+1}\right|}$

Answer 78E.

Given $\int\frac{1+\sin x}{1-\sin x}dx$

$$= \int\frac{1+\sin x}{1-\sin x}\times\frac{(1+\sin x)}{(1+\sin x)}dx$$

$$= \int\frac{(1+\sin x)^2}{1-\sin^2 x}dx$$

$$= \int\frac{1+2\sin x+\sin^2 x}{\cos^2 x}dx$$

$$= \int\left(\sec^2 x+2\tan x\sec x+\tan^2 x\right)dx$$

$$= \int\left(\sec^2 x+2\tan x\sec x+\sec^2 x-1\right)dx$$

$$= \int\left(2\sec^2 x-1+2\sec x\tan x\right)dx$$

$$= 2\tan x - x + 2\sec x + C$$

Answer 79E.

We have to evaluate $\int x\sin^2 x \cos x\, dx$

Substitute $\sin x = t$

Then $\cos x\, dx = dt$

Therefore $\int x\sin^2 x \cos x\, dx = \int \left(\sin^{-1} t\right).t^2.dt$

Integrating by parts, we have

$$\int x\sin^2 x \cos x\, dx = \sin^{-1} t.\frac{t^3}{3} - \int \frac{1}{\sqrt{1-t^2}}.\frac{t^3}{3}dt$$

$$= \frac{t^3}{3}.\sin^{-1} t - \frac{1}{3}\int \frac{t^2.t}{\sqrt{1-t^2}}dt$$

Again substitute $t^2 = u$ in integral parts then $2t\,dt = du$

So $\int x\sin^2 x \cos x\, dx = \frac{t^3}{3}.\sin^{-1} t - \frac{1}{3}\int \frac{u}{\sqrt{1-u}}\frac{du}{2}$

$$= \frac{t^3}{3}.\sin^{-1} t + \frac{1}{6}\int \frac{-1+1-u}{\sqrt{1-u}}du$$

$$= \frac{t^3}{3}.\sin^{-1} t + \frac{1}{6}\int \left(-\frac{1}{\sqrt{1-u}} + \frac{1-u}{\sqrt{1-u}}\right)du$$

$$= \frac{t^3}{3}.\sin^{-1} t + \frac{1}{6}\int \left(-(1-u)^{-1/2} + (1-u)^{1/2}\right)du$$

$$= \frac{t^3}{3}.\sin^{-1} t + \frac{1}{6}\left(\frac{(1-u)^{1/2}}{1/2} - \frac{(1-u)^{3/2}}{3/2}\right) + C$$

$$= \frac{t^3}{3}.\sin^{-1} t + \frac{1}{3}\sqrt{1-t^2} - \frac{1}{9}\left(1-t^2\right)^{3/2} + C$$

$$= \frac{1}{3}.\left(\sin^3 x\right).x + \frac{1}{3}\sqrt{1-\sin^2 x} - \frac{1}{9}\left(1-\sin^2 x\right)^{3/2} + C$$

$$= \boxed{\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C}$$

Answer 80E.

Consider the integral $\int \frac{\sec x \cos(2x)}{\sin x + \sec x}dx$ ...... (1)

Multiply and divide the integrand in (1) with $\cos(x)$.

$$\int \frac{\sec x \cos(2x)}{\sin x + \sec x}dx = \int \frac{\cos x\left(\sec x \cos(2x)\right)}{\cos x\left(\sin x + \sec x\right)}dx$$

$$= \int \frac{\cos x \sec x \cos(2x)}{\cos x \sin x + \cos x \sec x}dx$$

$$= \int \frac{(1)\cos(2x)}{\cos x \sin x + (1)}dx \qquad \left(\text{Since } \sec x = \frac{1}{\cos x}\right)$$

$$= \int \frac{2\cos(2x)}{2(\cos x \sin x + 1)}dx$$

$$= \int \frac{2\cos(2x)}{2\cos x \sin x + 2}dx$$

$$= \int \frac{2\cos(2x)}{\sin 2x + 2}dx \quad ...... (2)$$

Let $\sin 2x + 2 = t$

$\cos 2x dx = dt$

Now

$$\int \frac{\sec x \cos(2x)}{\sin x + \sec x} dx = \int \frac{2\cos(2x)}{\sin 2x + 2} dx$$ From (2).

$$= 2\int \frac{\cos(2x)}{\sin 2x + 2} dx$$

$$= 2\int \frac{dt}{t} dx$$

$$= 2\ln(t) + C$$

$$= 2\ln(\sin 2x + 2) + C$$

Therefore $\int \frac{\sec x \cos(2x)}{\sin x + \sec x} dx = \boxed{2\ln(\sin 2x + 2) + C}$

Answer 81E.

Consider the integral $\int \sqrt{1 - \sin x} dx$ ...... (1)

For any value of θ, $\sin^2\theta + \cos^2\theta = 1$

Therefore $\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1$

The formula for $\sin 2x$ is $2\sin x \cos x$

Therefore $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

Substitute above values in equation (1)

$$\int \sqrt{1 - \sin x} dx = \int \sqrt{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)} dx$$

$$= \int \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2} dx \quad \left(\text{Since } (a-b)^2 = a^2 - 2ab + b^2\right)$$

$$= \int \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) dx$$

$$= -\frac{\cos\frac{x}{2}}{\frac{1}{2}} - \frac{\sin\frac{x}{2}}{\frac{1}{2}} + C$$

$$= \boxed{-2\cos\frac{x}{2} - 2\sin\frac{x}{2} + C}$$

Answer 82E.

We have to evaluate $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Divide numerator and denominator by $\cos^4 x$

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{\sin x}{\cos^3 x}}{\frac{\sin^4 x}{\cos^4 x} + 1} dx$$

$$= \int \frac{\tan x . \sec^2 x}{\tan^4 x + 1} dx$$

Substitute $\tan^2 x = t$

Then $2\tan x \sec^2 x\, dx = dt$

So $$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{dt}{2}}{t^2+1}$$

$$= \frac{1}{2}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{2}\tan^{-1} t + C$$

$$= \boxed{\frac{1}{2}\tan^{-1}\left(\tan^2 x\right) + C}$$

Answer 83E.

Evaluate the integral $\int \left(2x^2+1\right)e^{x^2}\, dx$ ...... (1)

Rewrite the equation (1) as:

$$\int \left(2x^2+1\right)e^{x^2}\, dx = \int 2x^2 e^{x^2}\, dx + \int e^{x^2}\, dx \quad \text{...... (2)}$$

Consider $\int 2x^2 e^{x^2}\, dx$

$$\int 2x^2 e^{x^2}\, dx = \int x\left(2xe^{x^2}\right)dx$$

By using by parts:

$u = x \qquad dv = 2xe^{x^2}\, dx$

$du = dx \qquad v = e^{x^2}$

Now $\int u\,dv = uv - \int v\,du$

$$\int 2x^2 e^{x^2}\, dx = xe^{x^2} - \int e^{x^2}\, dx \quad \text{...... (3)}$$

Plug the equation (3) in (2).

$$\int \left(2x^2+1\right)e^{x^2}\, dx = xe^{x^2} - \int e^{x^2}\, dx + \int e^{x^2}\, dx + C$$

Therefore $\int \left(2x^2+1\right)e^{x^2}\, dx = \boxed{xe^{x^2} + C}$